

Article

Recent Advances in Proximity Point Theory Applied to Fractional Differential Equations

Nabil Mlaiki ^{1,*}, Dur-e-Shehwar Sagheer ^{2,†}, Sana Noreen ^{2,†}, Samina Batul ^{2,†} and Ahmad Alooqaily ^{1,3,†}

¹ Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; maloqaily@psu.edu.sa

² Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan; d.e.shehwar@cust.edu.pk (D.-e.-S.S.); mmt213030@cust.pk or sanahoreenkhan@gmail.com (S.N.); samina.batul@cust.edu.pk (S.B.)

³ School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia

* Correspondence: nmlaiki@psu.edu.sa or nmlaiki2012@gmail.com

† These authors contributed equally to this work.

Abstract: This article introduces the concept of generalized $(\mathcal{F}, \mathcal{b}, \phi)$ contraction in the context of $\mathcal{b}$ -metric spaces by utilizing the idea of $\mathcal{F}$ contraction introduced by Dariusz Wardowski. The main findings of the research focus on the existence of best proximity points for multi-valued $(\mathcal{F}, \mathcal{b}, \phi)$ contractions in partially ordered $\mathcal{b}$ -metric spaces. The article provides examples to illustrate the main results and demonstrates the existence of solutions to a second-order differential equation and a fractional differential equation using the established theorems. Additionally, several corollaries are presented to show that the results generalize many existing fixed-point and best proximity point theorems.

Keywords: best proximity point (BPP); fixed point (FP); banach contraction principle (BCP); partially ordered complete metric space (POCMS); partially ordered complete $\mathcal{b}$ -metric space (POCbMS)

MSC: 47H09; 47H10; 54H25



Citation: Mlaiki, N.; Sagheer, D.-e.-S.; Noreen, S.; Batul, S.; Alooqaily, A. Recent Advances in Proximity Point Theory Applied to Fractional Differential Equations. *Axioms* **2024**, *13*, 395. <https://doi.org/10.3390/axioms13060395>

Academic Editors: Shuechin Huang, Yasunori Kimura and Behzad Djafari-Rouhani

Received: 19 April 2024

Revised: 31 May 2024

Accepted: 6 June 2024

Published: 13 June 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Fixed-point (FP) results play a crucial role in solving non-linear problems by ensuring the existence and uniqueness of FP. The foundation of FP theory was laid by Poincare [1,2]. Later, in 1906, Frechet [3] introduced the concept of metric space, which subsequently formed the basis for research on fixed-point theory. In 1912, Brouwer [4] proved a FP theorem on the unit sphere, which was further generalized by Kakutani [5]. A significant milestone in FP theory was achieved in 1922 by Stefan Banach [6], who presented the Banach Contraction Principle (BCP). This fundamental theorem not only guarantees the existence and uniqueness of FP for a contraction mapping defined on a complete metric space, but it also provides a method for constructing it. Unlike Brouwer's FP theorem, BCP offers a more comprehensive approach. Since 1922, numerous mathematicians have attempted to expand upon this renowned theorem, with their efforts branching out in two primary directions. First, by modifying the established axioms of metric spaces, researchers have introduced a plethora of novel spaces, collectively referred to as generalized metric spaces. Examples of these include $\mathcal{b}$ -metric spaces, partial-metric spaces, metric-like spaces, cone-metric spaces, G -metric spaces, and rectangular-metric spaces, among others see [7–9]. Alternatively, mathematicians have substituted the contraction condition with various alternative conditions that broaden the concept of contraction. BCP was later generalized by Edelstein [10] in 1962, who replaced the contraction condition with continuous mapping on a compact space. Since then, the concept of BCP has been extensively developed and generalized in various directions in mathematics and fundamental sciences.

Bakhtin [11] introduced the concept of b-metric by modifying the triangle inequality in metric spaces. This development has significant implications for FP theory, where estimating the solution of FP problems is a major challenge. The notion of fixed-points for multi-valued mappings is crucial in confirming the presence of solutions of integral inclusions. Numerous researchers are actively developing innovative extensions and techniques for addressing nonlinear problems within the framework of b-metric spaces. Notably, a significant body of work has been contributed by [12,13], whose esteemed extensions have significantly advanced the field. Nadler [14] extended the BCP to multivalued maps in the following manner:

Let $(\mathbb{J}, \varphi)$ be a complete metric space and $T : \mathbb{J} \rightarrow CB(\mathbb{J})$ be a multivalued contraction mapping, then T has an FP.

To this end, the researcher can see notable works in [15–17]. In 2012, Wardowski [18] introduced the concept of $\mathcal{F}$ -contraction, called Wardowski's contraction or π -contraction. He generalized the condition in Banach's theorem. Several fixed-point results have been established by various authors, building on the Wardowski-type concept of $\mathcal{F}$ -contractive mappings, which has been a fruitful approach in the field of fixed-point theory. Klim et al. [19] demonstrated FP theorems involving $\mathcal{F}$ -contractions for dynamic processes. In 2022, Sagheer et al. [20] developed the concept of $(\uparrow, \mathcal{F})$ -contractive multi-valued mappings on uniform spaces and proved certain fixed-point results. In 2010, Basha [21] introduced the notion of the best proximity point (BPP) for nonself mappings on metric spaces, and subsequent research has explored the existence of BPP for non-self mappings on metric spaces (see for example [22,23]). Akbar and Gabeleh [24] established BPP theorems for multivalued contractions, as well as for nonexpansive multivalued mappings in complete metric spaces with appropriate geometric properties. Falahi et al. [25] introduced Banach and Kannan-type integral contractions on partially ordered complete metric space (POCMS) and investigated the existence and uniqueness of BPP for these mappings.

Jain et al. [26] introduced an innovative concept of multi-valued $\mathcal{F}$ -contraction on (POCMS) by modifying the distance function, ensuring the existence of BPP. This study addresses a significant research gap in the existing literature by presenting a new multi-valued $(\uparrow\mathcal{F}, b, \tilde{\varphi})$ contraction on the platform of partially ordered complete b-metric spaces (POCbMS) and provides a new perspective on contraction mappings for the existence of BPP.

2. Preliminaries

Definition 1. Mapping $\mathcal{F} : (0, +\infty) \rightarrow \mathbb{R}$ is known as $\mathcal{F}$ -mapping if:

(F1): $\mathcal{F}$ is increasing,

(F2): For every sequence $\{\mathfrak{S}_n\}$ of positive numbers, $\lim_{n \rightarrow \infty} \mathfrak{S}_n = 0$ iff $\lim_{n \rightarrow \infty} \mathcal{F}(\mathfrak{S}_n) = -\infty$,

(F2): There exists $v \in (0, 1)$ such that $\lim_{\mathfrak{S} \rightarrow 0^+} \mathfrak{S}^v \mathcal{F}(\mathfrak{S}) = 0$.

The class of all $\mathcal{F}$ -functions is denoted by the notation $\check{\mathbb{F}}$. In 2012, Wardowski [18] pioneered the concept of $\mathcal{F}$ -contraction, employing an $\mathcal{F}$ -function as a control function.

Definition 2. Let $(\mathbb{J}, \varphi)$ be a metric space. Mapping $\mathcal{S} : \mathbb{J} \rightarrow \mathbb{J}$ is called a $\mathcal{F}$ -contraction if

$$\varphi(\mathcal{S}a, \mathcal{S}b) > 0 \implies \check{\tau} + \mathcal{F}(\varphi(\mathcal{S}a, \mathcal{S}b)) \leq \mathcal{F}(\varphi(a, b)) \text{ for all } a, b \in \mathbb{J}, \quad (1)$$

where $\check{\tau} > 0$, $\mathcal{F} \in \check{\mathbb{F}}$.

In [18], it has been proven that every $\mathcal{F}$ contraction is a continuous mapping.

Khan et al. [27] introduced the idea of the Altering Distance Function (ADF) and utilized it to establish FP theorems with modified distance metrics.

Definition 3. Function $\check{\varphi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is known as ADF if it meets the following requirements:

(D1): $\check{\varphi}$ is continuous,

(D2): $\check{\varphi}$ is monotonically increasing,

(D3): $\check{\phi}(b) > 0$ for all $b > 0$.

Example 1. Define $\mathcal{F} : (0, +\infty) \rightarrow \mathbb{R}$ as $\mathcal{F}(a) = \ln a + a$, with $a > 0$, and constant $v \in (\frac{1}{2}, 1)$. The contraction condition (1) takes the following form:

$$\frac{\wp(\mathcal{S}a, \mathcal{S}b)}{\wp(a, b)} e^{\wp(\mathcal{S}a, \mathcal{S}b) - \wp(a, b)} \leq e^{-\check{\tau}},$$

for all $a, b \in \mathbb{R}^+$, such that $\mathcal{S}a \neq \mathcal{S}b$.

The notion of the $\mathcal{P}$ -property, introduced in [28], has been utilized to develop a broader and more comprehensive extension of the BCP in order to enhance its versatility and range of applications.

Definition 4. Consider a pair $(\mathbb{K}, \mathbb{L})$ of non-empty subsets of a metric space $\mathbb{J}$. Then, the pair $(\mathbb{K}, \mathbb{L})$ possesses the $\mathcal{P}$ -property if and only if the following implication holds:

$$\left. \begin{array}{l} \wp(a_1, b_1) = \wp(\mathbb{K}, \mathbb{L}) \\ \wp(a_2, b_2) = \wp(\mathbb{K}, \mathbb{L}) \end{array} \right\} \implies \wp(a_1, a_2) = \wp(b_1, b_2),$$

where $a_1, a_2 \in \mathbb{K}_0 = \{a \in \mathbb{K} : \wp(a, b) = \wp(\mathbb{K}, \mathbb{L}) \text{ for some } b \in \mathbb{L}\}$ and $b_1, b_2 \in \mathbb{L}_0 = \{b \in \mathbb{L} : \wp(a, b) = \wp(\mathbb{K}, \mathbb{L}) \text{ for some } a \in \mathbb{K}\}$ and $\wp(\mathbb{K}, \mathbb{L}) = \inf\{\wp(a, b) : a \in \mathbb{K} \text{ and } b \in \mathbb{L}\}$, with $\mathbb{K}_0$ being non-empty.

Definition 5. Let $\uparrow : \mathbb{K} \times \mathbb{K} \rightarrow [0, \infty)$ be a mapping. Mapping $\mathcal{S} : \mathbb{K} \rightarrow \mathcal{CB}(\mathbb{K})$ is said to be $\uparrow$ -admissible if

$$\uparrow(a, b) \geq 1 \implies \uparrow(v, w) \geq 1,$$

for $v \in \mathcal{S}a$ and $w \in \mathcal{S}b$ [29].

3. Main Results

Building on the work of Jain et al. [26], who introduced multivalued $\mathcal{F}$ contractions on Partially Ordered Complete Metric Spaces (POCMS) using ADF to establish BPP theorems, this paper presents a novel concept of BPP for multivalued $\uparrow\mathcal{F}$ -contractions on Partially Ordered Complete b -Metric Spaces (POCbMS), incorporating an ADF to further generalize and extend the existing results.

Definition 6. The three values $(\mathbb{J}, \preceq, \wp_b)$ are called partially ordered b -metric space, if $(\mathbb{J}, \preceq)$ is a partially ordered set and $(\mathbb{J}, \wp_b)$ is b -metric space. For two non-empty subsets $\mathbb{L}, \mathbb{K}$ of $\mathbb{J}$, the following notations are crucial for the following.

$\mathcal{D}(a, \mathbb{L}) = \inf\{\wp_b(a, b) : b \in \mathbb{L}\}$
$\delta(\mathbb{K}, \mathbb{L}) = \sup\{\wp_b(a, b) : a \in \mathbb{K} \wedge b \in \mathbb{L}\}$
$\wp_b(\mathbb{K}, \mathbb{L}) = \inf\{\wp_b(a, b) : a \in \mathbb{K} \wedge b \in \mathbb{L}\}$
$\mathbb{K}_0 = \{a \in \mathbb{K} : \wp_b(a, b) = \wp_b(\mathbb{K}, \mathbb{L}) \text{ for some } b \in \mathbb{L}\}$
$\mathbb{L}_0 = \{b \in \mathbb{L} : \wp_b(a, b) = \wp_b(\mathbb{K}, \mathbb{L}) \text{ for some } a \in \mathbb{K}\}$

Definition 7. Let $\mathbb{K}$ and $\mathbb{L}$ be two non-empty subsets of $(\mathbb{J}, \preceq, \wp_b)$ and $\mathcal{S} : \mathbb{K} \rightarrow 2^{\mathbb{L}}$ is a multivalued mapping. Then, point $a \in \mathbb{J}$ is called BPP for $\mathcal{S}$ if:

$$\mathcal{D}(a, \mathcal{S}a) = \wp_b(\mathbb{K}, \mathbb{L}).$$

Note, if we consider a map $\mathcal{S} : \mathbb{K} \rightarrow \mathbb{K}$ in the above definition, the concept of BPP effectively becomes an FP.

Definition 8. Let $\mathbb{K}$ and $\mathbb{L}$ be two non-empty closed subsets of a $\text{POCbMS}(\mathbb{J}, \preceq, \wp_b)$, such that $\mathcal{S}a_0 \subset \mathbb{L}_0$, $a_0 \in \mathbb{K}_0$. An $\uparrow$ -admissible mapping $\mathcal{S} : \mathbb{K} \rightarrow \mathcal{CB}(\mathbb{L})$ is called a $(\uparrow\mathcal{F}, b, \check{\phi})$ contraction if it satisfies

$$\check{\tau} + \mathcal{F}\left(\uparrow(a, b)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b)))\right) \leq \mathcal{F}(\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(\mathbb{L}, \mathbb{M}))) \quad \text{for } (a \preceq b), \quad (2)$$

where $\mathcal{N}(a, b) = \max\left\{\wp_b(a, b), \mathcal{D}(a, \mathcal{S}a), \mathcal{D}(b, \mathcal{S}b), \frac{\mathcal{D}(a, \mathcal{S}b) + \mathcal{D}(b, \mathcal{S}a)}{2b}\right\}$, $\uparrow : \mathbb{J} \times \mathbb{J} \rightarrow [0, \infty)$ and $\check{\phi}$ is an ADF with $\check{\phi}(1 + m) \leq \check{\phi}(1) + \check{\phi}(m)$ for all $1, m \in [0, +\infty)$.

If $\mathcal{F}(a) = \ln a$, then the contraction condition (2) takes the following form:

$$\begin{aligned} \check{\tau} + \ln\left(\uparrow(a_0, a_1)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b)))\right) &\leq \ln(\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(\mathbb{L}, \mathbb{M}))) \\ \Leftrightarrow \ln e^{\check{\tau}} + \ln\left(\uparrow(a_0, a_1)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b)))\right) &\leq \ln(\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(\mathbb{L}, \mathbb{M}))) \\ \Leftrightarrow \ln(\uparrow(a_0, a_1)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b))) &\leq \ln\left\{\frac{\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(a, b))}{e^{\check{\tau}}}\right\} \\ \Leftrightarrow (\uparrow(a_0, a_1)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b))) &\leq \frac{1}{e^{\check{\tau}}}(\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(\mathbb{L}, \mathbb{M}))) \\ \Leftrightarrow (\uparrow(a_0, a_1)(b^2(\check{\phi}(\delta(\mathcal{S}a, \mathcal{S}b))) &\leq \nu(\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp_b(\mathbb{L}, \mathbb{M}))) \quad \text{where } \frac{1}{e^{\check{\tau}}} = \nu. \end{aligned}$$

Theorem 1. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty closed subset of $\text{POCbMS}(\mathbb{J}, \preceq, \wp_b)$ and possesses a $\mathcal{P}$ property. Let $\mathcal{S} : \mathbb{K} \rightarrow \mathcal{CB}(\mathbb{L})$ be an $(\uparrow\mathcal{F}, b, \check{\phi})$ contraction, such that the following conditions are satisfied:

- (Q1): There exist $a_0 \in \mathbb{K}$, $a_1 \in \mathcal{S}a_0$, such that $\uparrow(a_0, a_1) \geq 1$ with $a_0 \preceq a_1$.
- (Q2): There exist $a_0, a_1 \in \mathbb{K}_0$ and $b_0 \in \mathcal{S}a_0$, such that $\wp_b(a_1, b_0) = \wp_b(\mathbb{K}, \mathbb{L})$ with $a_0 \preceq a_1$.
- (Q3): For all $a, b \in \mathbb{K}_0$ if $a \preceq b$ then $\mathcal{S}a \subset \mathcal{S}b$.
- (Q4): If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$, such that $a_n \rightarrow a$, then $a_n \preceq a$ for all n .

Then, $\mathcal{S}$ has a BPP.

Proof. By using (Q1),

$$\uparrow(a_n, a_{n+1}) \geq 1 \text{ for all } a_n \preceq a_{n+1}, \quad n = 0, 1, 2, \dots$$

For (Q2), there exists $a_0, a_1 \in \mathbb{K}_0$ and $b_0 \in \mathcal{S}a_0$, such that $\wp_b(a_1, b_0) = \wp_b(\mathbb{K}, \mathbb{L})$ and $a_0 \preceq a_1$.

Utilizing (Q3), $\mathcal{S}a_0 \subset \mathcal{S}a_1$, so there exist $b_1 \in \mathcal{S}a_1$ with $\wp_b(a_2, b_1) = \wp_b(\mathbb{K}, \mathbb{L})$, such that $a_1 \preceq a_2$.

Therefore, for each $n \in \mathbb{N}$, $a_{n+1} \in \mathbb{K}_0$ and $b_n \in \mathcal{S}a_n$, such that $\wp_b(a_{n+1}, b_n) = \wp_b(\mathbb{K}, \mathbb{L})$. Thus,

$$\wp_b(a_{n+1}, b_n) = \mathcal{D}(a_{n+1}, \mathcal{S}a_n) = \wp_b(\mathbb{K}, \mathbb{L}), \quad \forall n \in \mathbb{N} \quad (3)$$

with $a_0 \preceq a_1 \preceq a_3 \preceq \dots \preceq a_n \preceq a_{n+1} \dots$. If there exist n_0 , such that $a_{n_0} = a_{n_0+1}$, then $\wp_b(a_{n_0+1}, b_{n_0}) = \mathcal{D}(a_{n_0}, \mathcal{S}a_{n_0}) = \wp_b(\mathbb{K}, \mathbb{L})$, then a_{n_0} is the best proximity point for $\mathcal{S}$. Assume that $a_n \neq a_{n+1}$ for all n , as

$$\begin{aligned} \wp_b(a_{n+1}, b_n) &= \wp_b(\mathbb{K}, \mathbb{L}) \text{ and } \wp_b(a_n, b_{n-1}) = \wp_b(\mathbb{K}, \mathbb{L}) \\ \Rightarrow \wp_b(a_{n+1}, a_n) &= \wp_b(b_n, b_{n-1}), \quad \forall n \in \mathbb{N}. \quad (\text{by } \mathcal{P} \text{ - property}) \end{aligned} \quad (4)$$

Given $a_{n-1} \prec a_n$, so

$$\begin{aligned} \mathcal{F}(b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq \check{\tau} + \mathcal{F}(\uparrow(a_n, a_{n+1})(b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) \\ &= \check{\tau} + \mathcal{F}(\uparrow(a_n, a_{n+1})(b^2(\check{\phi}(\wp_b(b_{n-1}, b_n)))) \\ &\leq \mathcal{F}(\uparrow(a_n, a_{n+1})(b^2(\check{\phi}(\delta(\mathcal{S}a_{n-1}, \mathcal{S}a_n)))) \\ &\leq \mathcal{F}(\check{\phi}(\mathcal{N}(a_{n-1}, a_n)) - \check{\phi}(\wp_b(\mathbb{K}, \mathbb{L}))) - \check{\tau}. \end{aligned} \quad (5)$$

Now

$$\begin{aligned} \mathcal{N}(a_{n-1}, a_n) &= \max \left\{ \wp_b(a_{n-1}, a_n), \mathcal{D}(a_{n-1}, \mathcal{S}a_{n-1}), \mathcal{D}(a_n, \mathcal{S}a_n), \frac{\mathcal{D}(a_{n-1}, \mathcal{S}a_n) + \mathcal{D}(a_n, \mathcal{S}a_{n-1})}{2b} \right\} \\ &\leq \max \left\{ \wp_b(a_{n-1}, a_n), \wp(a_{n-1}, b_{n-1}), \wp(a_n, b_n), \frac{\wp(a_{n-1}, b_n) + \wp(a_n, b_{n-1})}{2b} \right\} \\ &\leq \max \left\{ \wp_b(a_{n-1}, a_n), (b\wp_b(a_{n-1}, b_{n-2}) + \wp_b(b_{n-2}, b_{n-1})), (b\wp_b(a_n, b_{n-1}) + \wp_b(b_{n-1}, b_n)), \right. \\ &\quad \left. \frac{b\wp_b(a_{n-1}, b_{n-2}) + b^2\wp_b(b_{n-2}, b_{n-1}) + b^2\wp_b(b_{n-1}, b_n) + \wp_b(a_n, b_{n-1})}{2b} \right\} \\ &\leq \max \left\{ \wp_b(a_{n-1}, a_n), (b\wp_b(a_{n-1}, b_{n-2}) + \wp_b(b_{n-2}, b_{n-1})), (b\wp_b(a_n, b_{n-1}) + \wp_b(b_{n-1}, b_n)), \right. \\ &\quad \left. \frac{b\wp_b(a_{n-1}, b_{n-2}) + b^2\wp_b(b_{n-2}, b_{n-1}) + b^2\wp_b(b_{n-1}, b_n) + b\wp_b(a_n, b_{n-1})}{2b} \right\} \\ &\leq \max \left\{ \wp_b(a_{n-1}, a_n), b(\wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_{n-1}, a_n)), b(\wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_n, a_{n+1})), \right. \\ &\quad \left. \frac{b(\wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_{n-1}, a_n)) + b(\wp_b(a_n, a_{n+1}) + \wp_b(\mathbb{K}, \mathbb{L}))}{2} \right\} \\ &\leq \max \left\{ \wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_{n-1}, a_n), \wp_b(\mathbb{K}, \mathbb{L}) + b(\wp_b(a_n, a_{n+1})) \right\}. \end{aligned}$$

Therefore, (5) becomes

$$\begin{aligned} \mathcal{F}(b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq \mathcal{F}(\check{\phi} \max \left\{ b(\wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_{n-1}, a_n)), b(\wp_b(\mathbb{K}, \mathbb{L}) + \wp_b(a_n, a_{n+1})) \right\} \\ &\quad - \check{\phi}(b(\wp_b(\mathbb{K}, \mathbb{L}))) - \check{\tau}. \end{aligned} \quad (6)$$

If $\wp_b(a_n, a_{n+1}) > \wp_b(a_{n-1}, a_n)$ then

$$\begin{aligned} \mathcal{F}(b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq \mathcal{F}(b(\check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) + (b(\wp_b(a_n, a_{n+1}))) - (b(\check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})))) - \check{\tau} \\ \mathcal{F}(b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq \mathcal{F}(b(\check{\phi}(\wp_b(a_n, a_{n+1})))) - \check{\tau} \leq \mathcal{F}(b(\check{\phi}(\wp_b(a_n, a_{n+1})))) \end{aligned}$$

which leads to a contradiction. Therefore,

$$\wp_b(a_n, a_{n+1}) \leq \wp_b(a_{n-1}, a_n). \quad (7)$$

Hence

$$\begin{aligned} (b^2(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq (b(\check{\phi}(\wp_b(a_{n-1}, a_n)))) - \check{\tau} \\ \mathcal{F}(b(\check{\phi}(\wp_b(a_n, a_{n+1})))) &\leq \mathcal{F}(\check{\phi}(\wp_b(a_{n-1}, a_n))) - \check{\tau}. \end{aligned}$$

If $(\check{\phi}(\wp_b(a_n, a_{n+1}))) = \psi_n$, the above inequality becomes

$$\mathcal{F}(b(\psi_n)) \leq \mathcal{F}(\psi_{n-1}) - \check{\tau}.$$

Iteratively,

$$\begin{aligned} \mathcal{F}(b^n(\psi_n)) &\leq \mathcal{F}(b^{n-1}(\psi_{n-1}) - \check{\tau}) \leq \mathcal{F}(b^{n-2}(\psi_{n-2}) - 2\check{\tau}) \leq \dots \leq \mathcal{F}(\psi_0) - n\check{\tau}. \quad (8) \\ &\implies \lim_{n \rightarrow \infty} \mathcal{F}(b^n(\psi_n)) = -\infty \\ &\implies \lim_{n \rightarrow \infty} b^n(\psi_n) = 0. \quad \text{by}(F2) \end{aligned}$$

Using (F3), there exists $\gamma \in (0, 1)$, such that

$$\begin{aligned} \lim_{n \rightarrow 0} (\mathbf{b}^n \psi_n)^\gamma \mathcal{F}(\mathbf{b}^n \psi_n) &= 0 \quad \text{for all } n \in \mathbb{N}. \\ (8) \implies \lim_{n \rightarrow \infty} (\mathbf{b}^n \psi_n)^\gamma (\mathcal{F}(\mathbf{b}^n \psi_n) - \mathcal{F}(\psi_0)) &\leq \lim_{n \rightarrow \infty} (\mathbf{b}^n \psi_n)^\gamma n \check{\tau} \leq 0. \end{aligned} \quad (9)$$

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} (\mathbf{b} \psi_n)^\gamma n \check{\tau} \leq 0. \\ &\implies \lim_{n \rightarrow \infty} (\mathbf{b} \psi_n)^\gamma n = 0. \end{aligned}$$

So, there exist $n_1 \in \mathbb{N}$, such that

$$\begin{aligned} (\mathbf{b}^n \psi_n)^\gamma n &\leq 1 \quad \text{for all } n \geq n_1 \\ \implies \mathbf{b}^n \psi_n &\leq \frac{1}{n^{\frac{1}{\gamma}}}. \end{aligned} \quad (10)$$

To show that $\{a_n\}$ is a Cauchy sequence, assume $n, m \in \mathbb{N}$, such that $n > m \geq n_1$.

$$\begin{aligned} \check{\phi}(\wp_{\mathbf{b}}(a_n, a_m)) &\leq \mathbf{b}(\check{\phi}(\wp_{\mathbf{b}}(a_n, a_{n+1}))) + \mathbf{b}(\check{\phi}(\wp_{\mathbf{b}}(a_{n+1}, a_m))) \\ &\leq \mathbf{b}(\check{\phi}(\wp_{\mathbf{b}}(a_n, a_{n+1}))) + \mathbf{b}^2(\check{\phi}(\wp_{\mathbf{b}}(a_{n+1}, a_{n+2}))) + \mathbf{b}^2(\check{\phi}(\wp_{\mathbf{b}}(a_{n+2}, a_m))) \\ &\leq \mathbf{b}(\check{\phi}(\wp_{\mathbf{b}}(a_n, a_{n+1}))) + \mathbf{b}^2(\check{\phi}(\wp_{\mathbf{b}}(a_{n+1}, a_{n+2}))) + \dots + \mathbf{b}^{m-n}(\check{\phi}(\wp_{\mathbf{b}}(a_{m-1}, a_m))). \end{aligned} \quad (11)$$

Therefore,

$$\begin{aligned} \check{\phi}(\wp_{\mathbf{b}}(a_n, a_m)) &\leq \mathbf{b} \psi_n + \mathbf{b}^2 \psi_{n+1} + \mathbf{b}^3 \psi_{n+2} \dots + \mathbf{b}^{m-n} \psi_m \\ &= \sum_{i=n}^{m-1} \mathbf{b}^{i-n+1} (\check{\phi}(\wp_{\mathbf{b}}(a_i, a_{i+1}))) \\ &\leq \sum_{i=n}^{\infty} \mathbf{b}^i (\psi_i) \\ \implies \check{\phi}(\wp_{\mathbf{b}}(a_n, a_m)) &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\gamma}}}. \end{aligned}$$

As $\gamma \in (0, 1)$, by using the P-series test $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\gamma}}}$ is convergent. Therefore, $\{a_n\}$ is a Cauchy sequence in $\mathbb{K}$. Given that $\mathbb{K}$ is complete, $\mathbf{t} \mathbf{a} \in \mathbb{K}$ exists, such that

$$\lim_{n \rightarrow \infty} a_n = \mathbf{a} \text{ or } a_n \rightarrow \mathbf{a}.$$

As $\wp_{\mathbf{b}}(a_n, a_{n+1}) = \wp_{\mathbf{b}}(b_{n-1}, b_n)$. Thus $\{b_n\}$ in $\mathbb{K}$ is a Cauchy sequence and, hence, convergent, this implies $b_n \rightarrow \mathbf{b} \in \mathbb{K}$. Hence, the relation $\wp_{\mathbf{b}}(a_{n+1}, b_n) = \wp_{\mathbf{b}}(\mathbb{K}, \mathbb{L})$ for all n , $\implies \wp_{\mathbf{b}}(\mathbf{a}, \mathbf{b}) = \wp_{\mathbf{b}}(\mathbb{K}, \mathbb{L})$.

To show that $\mathbf{b} \in \mathcal{S}a$, proceed as follows.

Given that $\{a_n\}$ is an increasing sequence in $\mathbb{K}$ and $a_n \rightarrow \mathbf{a} \implies a_n \preceq \mathbf{a}$ for all $n \in \mathbb{N}$ by (Q3).

Suppose $b \notin Sa$. Now

$$\begin{aligned} \mathcal{F}(b^2(\check{\phi}(\mathcal{D}(b_n, Sa)))) &\leq \mathcal{F}(b^2(\check{\phi}(\delta(Sa_n, Sa)))) \\ &\leq \mathcal{F}\left(\check{\phi}\left(\max\left\{\wp_b(a_n, a), \mathcal{D}(a_n, Sa_n), \mathcal{D}(a, Sa), \frac{\mathcal{D}(a_n, Sa) + \mathcal{D}(a, Sa_n)}{2b}\right\}\right)\right) - \\ &\quad \check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) - \check{\tau} \\ &\leq \mathcal{F}\left(\check{\phi}\left(\max\left\{(\wp_b(a_n, a)), \mathcal{D}(a_n, Sa_n), \mathcal{D}(a, Sa), \frac{\mathcal{D}(a_n, Sk) + \wp(a, Sa_n)}{2b}\right\}\right)\right) - \\ &\quad \check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) - \check{\tau} \end{aligned}$$

Taking limit as $n \rightarrow \infty$ to obtain

$$\begin{aligned} \mathcal{F}(b^2(\check{\phi}(\mathcal{D}(b, Sa)))) &\leq \mathcal{F}\left(\check{\phi}\left(\max\left\{0, (\wp(a_n, a)), (\mathcal{D}(a_n, Sa_n)), (\mathcal{D}(a, Sa)), \frac{\mathcal{D}(a_n, Sa) + \mathcal{D}(a, Sa_n)}{2b}\right\}\right)\right) - \\ &\quad \check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) - \check{\tau} \\ &\leq \mathcal{F}(\check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) + (\mathcal{D}(b, Sa))) - \check{\phi}(\wp_b(\mathbb{K}, \mathbb{L})) - \check{\tau} \\ \implies \mathcal{F}(b^2(\check{\phi}(\mathcal{D}(b, Sa)))) &\leq \mathcal{F}(\check{\phi}(\mathcal{D}(b, Sa))) - \check{\tau}, \end{aligned}$$

which is a contradiction.

This means that $b \in Sa$, and, hence, $\mathcal{D}(a, Sa) = \wp_b(\mathbb{K}, \mathbb{L})$. This implies a is the BPP of S . $\square$

If we consider $\mathbb{K} = \mathbb{L}$ in Theorem 1, the following corollaries are obtained.

Corollary 1. Consider a non-empty closed subset $\mathbb{K}$ of $\text{POCbMS}(\mathbb{J}, \preceq, \wp_b)$. Let $S : \mathbb{K} \rightarrow \mathcal{CB}(\mathbb{K})$ be a multi-valued $(\uparrow\mathcal{F}, b, \check{\phi})$ contraction, such that the conditions provided below are satisfied:

- (A1): There exist $a_0 \in \mathbb{K}, a_1 \in Sa_0$, such that $\uparrow(a_0, a_1) \geq 1$ with $a_0 \preceq a_1$.
- (A2): There exist $a_0, a_1 \in \mathbb{K}$ and $b_0 \in Sa_0$ such that $\wp_b(a_1, b_0) = 0$ and $a_0 \preceq a_1$.
- (A3): For all $a, b \in \mathbb{K}, a \preceq b \implies Sa \preceq Sb$.
- (A4): If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$, such that $a_n \rightarrow a$ then $a_n \preceq a$ for all $n \in \mathbb{N}$.

Then, S has a FP.

Corollary 2. Let $\mathbb{K}$ be a non-empty closed subset of $\text{POCbMS}(\mathbb{J}, \preceq, \wp_b)$. Suppose that $S : \mathbb{K} \rightarrow \mathbb{K}$ be a $(\uparrow\mathcal{F}, b, \check{\phi})$ contraction, such that the following conditions are satisfied:

- (A1): There exist $a_0 \in \mathbb{K}, a_1 \in Sa_0$, such that $\uparrow(a_0, a_1) \geq 1$ with $a_0 \preceq a_1$.
- (A2): There exist a_0, a_1 in $\mathbb{K}$ and $b_0 \in Sa_0$, such that $\wp_b(a_1, Sa_0) = 0$ and $a_0 \preceq a_1$.
- (A3): For all $a, b \in \mathbb{K}, a \preceq b \implies Sa \preceq Sb$.
- (A4): If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$, such that $a_n \rightarrow a$, with $a_n \preceq a$ for all $n \in \mathbb{N}$.

Then, S has a FP.

The subsequent corollary is obtained by further choosing $\check{\phi}$ as an identity function.

Corollary 3. Suppose that $\mathbb{K}$ is a non-empty closed subset of $\text{POCbMS}(\mathbb{J}, \preceq, \wp_b)$ and $S : \mathbb{K} \rightarrow \mathbb{K}$ be a $(\uparrow\mathcal{F}, b)$ contraction satisfying the following axioms:

- (A1): There exist $a_0 \in \mathbb{K}, a_1 \in Sa_0$, such that $\uparrow(a_0, a_1) \geq 1$ with $a_0 \preceq a_1$;
- (A2): There exist a_0, a_1 in $\mathbb{K}$ and $b_0 \in Sa_0$, such that $\wp_b(a_1, Sa_0) = 0$ and $a_0 \preceq a_1$;
- (A3): For all $a, b \in \mathbb{K}, a \preceq b \implies Sa \preceq Sb$;
- (A4): If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$, such that $a_n \rightarrow a$, then $a_n \preceq a$ for all $n \in \mathbb{N}$.

Then, S has a FP.

Example 2. Consider $\mathbb{J} = \mathbb{R}^2$ and assume the order $(a, b) \preceq (m, q) \iff a \leq m$ and $b \leq q$, here, $\leq$ is the usual order within $\mathbb{R}$. Define $\wp_b : \mathbb{J} \times \mathbb{J} \rightarrow \mathbb{R}$ as:

$$\wp_b((a_1, b_1), (a_2, b_2)) = (|a_1 - a_2|)^2 + (|b_1 - b_2|)^2.$$

It can be verified that $(\mathbb{J}, \wp_b)$ is a POCbMS, with $b = 2$. Define $\uparrow$ -admissible $\uparrow : \mathbb{K} \times \mathbb{K} \rightarrow [0, \infty)$ by:

$$\uparrow(a, b) = \{(a_1 + b_1) + (a_2 + b_2) + 3\}$$

Let $\mathbb{K} = \{(-7, 0), (0, -7), (0, 5)\}$ and $\mathbb{L} = \{(-2, 0), (0, -2), (0, 0), (-2, 2), (2, 2)\}$ be closed subsets of $\mathbb{J}$. Define $\mathcal{S} : \mathbb{K} \rightarrow \mathcal{CB}(\mathbb{L})$ as

$$\mathcal{S}(a, b) = \begin{cases} \{(0, -2), (0, 0)\} & \text{if } (a, b) = (-7, 0), \\ \{(2, 2), (-2, 2)\} & \text{if } (a, b) = (0, -7), \\ \{(-2, 2), (0, 0), (0, -2), (2, 2)\} & \text{if } (a, b) = (0, 5). \end{cases}$$

For $\mathcal{F}(\uparrow) = \ln \uparrow + \uparrow$, $\check{\tau} = 1$, $b = 2$ and $\check{\phi}(t) = 2t$, we get

$$\begin{aligned} & \frac{\check{\phi}(b(\uparrow(a, b)(\delta(\mathcal{S}a, \mathcal{S}b)))}{\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp(\mathbb{K}, \mathbb{L}))} e^{\check{\phi}(b(\uparrow(a, b)(\delta(\mathcal{S}a, \mathcal{S}b))) - (\check{\phi}(\mathcal{N}(a, b)) - \check{\phi}(\wp(\mathbb{K}, \mathbb{L})))} \\ &= \frac{80}{122} (e^{80-122}) = \frac{40}{61} e^{-42} < e^{-1}. \end{aligned}$$

Hence, all the conditions of Theorem 1 are satisfied, which assures that $(0, 5)$ is BPP of $\mathcal{S}$.

Example 3. Suppose $\mathbb{J} = \{0, 1, 2, 3, \dots\}$ is a partially ordered set, with a usual order and define $\wp : \mathbb{J} \times \mathbb{J} \rightarrow \mathbb{R}$

$$\wp_b(a, b) = \begin{cases} 0 & a = b, \\ (a + b)^2 & a \neq b. \end{cases}$$

Then, $(\mathbb{J}, \wp_b)$ is a POCbMS. Suppose $\mathcal{S} : \mathbb{J} \rightarrow \mathbb{J}$ is defined by

$$\mathcal{S}(a) = \begin{cases} 0 & \text{if } a = 0, \\ a - 1 & \text{if } a \neq 0. \end{cases}$$

Define $\uparrow : \mathbb{J} \times \mathbb{J} \rightarrow [0, \infty)$ as,

$$\uparrow(a_0, a_1) = \begin{cases} 2 & \text{if } a_0, a_1 \in \{0, 1\}, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then, $\mathcal{S}$ is an $\uparrow$ -admissible mapping.

To prove that $\mathcal{S}$ is $(\uparrow, \mathcal{F}, b, \check{\phi})$ -contraction with $\mathcal{F}(\uparrow) = \ln \uparrow + \uparrow$, $\check{\tau} = 1$, the following five cases will arise:

Case 1: Let $a > b$ and $b \neq 0$, then

$$\begin{aligned} \wp(\mathcal{S}a, \mathcal{S}b) &= \wp(a - 1, b - 1) = (a + b - 2)^2 \\ \mathcal{N}(a, b) &= \max \left\{ \wp_b(a, b), \wp_b(a, \mathcal{S}k), \wp_b(b, \mathcal{S}b), \frac{\wp_b(a, \mathcal{S}a) + \wp_b(b, \mathcal{S}b)}{2b} \right\} \\ &= (2a - 1)^2 \\ \Rightarrow \frac{\text{bff}(a_0, a_1) \wp(\mathcal{S}a, \mathcal{S}b)}{\mathcal{N}(a, b)} e^{b(\text{ff}(a_0, a_1) \wp(\mathcal{S}a, \mathcal{S}b) - \mathcal{N}(a, b))} &= \frac{(a + b - 2)^2}{(2a - 1)^2} e^{(a + b - 2)^2 - (2a - 1)^2} \\ &= \frac{(a + b - 2)^2}{(2a - 1)^2} e^{-3a^2 + b^2 + 2ab - 3 + 4a} < e^{-1}. \end{aligned}$$

Case 2: If $b > a$ and $a \neq 0$, then

$$\begin{aligned}\wp_b(Sa, Sb) &= \wp_b(a-1, b-1) = (a+b-2)^2 \\ \mathcal{N}(a, b) &= \max \left\{ \wp_b(a, b), \wp_b(a, Sa), \wp_b(b, Sb), \frac{\wp_b(a, Sa) + \wp_b(b, Sb)}{2b} \right\} \\ &= \max \left\{ (a+b)^2, (2a-1)^2, (2b-1)^2, (a+b-1)^2 \right\} \\ &= (2a-1)^2 \\ \Rightarrow \frac{\text{bff}(a_0, a_1) \wp_b(Sa, Sb)}{\mathcal{N}(a, b)} e^{\text{bff}(a_0, a_1) \wp_b(Sa, Sb) - \mathcal{N}(a, b)} &= \frac{(a+b-2)^2}{(2a-1)^2} e^{(a+b-2)^2 - (2b-1)^2} \\ &= \frac{a+b-2}{2b-1} e^{-3b^2+a^2+2ab+4b-3} < e^{-1}.\end{aligned}$$

Case 3: If $a > b$ and $b = 0$, then

$$\begin{aligned}\wp_b(Sa, Sb) &= \wp_b(a-1, 0) = (a-1)^2 \\ \mathcal{N}(a, b) &= \max \left\{ \wp_b(a, b), \wp_b(a, Sa), \wp_b(b, Sb), \frac{\wp_b(a, Sa) + \wp_b(b, Sb)}{2b} \right\} \\ &= (2a-1)^2 \\ \Rightarrow \frac{\text{bff}(a_0, a_1) \wp_b(Sa, Sb)}{\mathcal{N}(a, b)} e^{\text{bff}(a_0, a_1) \wp_b(Sa, Sb) - \mathcal{N}(a, b)} &= \frac{(a-1)^2}{(2a-1)^2} e^{(a-1)^2 - (2b-1)^2} \\ &= \frac{(a-1)^2}{(2a-1)^2} e^{-3a^2+2a} < e^{-1}.\end{aligned}$$

Case 4: If $b > a$ and $a = 0$, then

$$\begin{aligned}\wp_b(0, Sb) &= \wp_b(0, b-1) = (b-1)^2 \\ \mathcal{N}(a, b) &= \max \left\{ \wp_b(a, b), \wp_b(a, Sa), \wp_b(b, Sb), \frac{\wp_b(a, Sa) + \wp_b(b, Sb)}{2b} \right\} \\ &= (2b-1)^2 \\ \Rightarrow \frac{\text{bff}(a_0, a_1) \wp_b(Sa, Sb)}{\mathcal{N}(a, b)} e^{\text{bff}(a_0, a_1) \wp_b(Sa, Sb) - \mathcal{N}(a, b)} &= \frac{(b-1)^2}{(2b-1)^2} e^{(a-1)^2 - (2b-1)^2} \\ &= \frac{(b-1)^2}{(2b-1)^2} e^{-3b^2+2b} < e^{-1}.\end{aligned}$$

Case 5: If $b = a$, then

$$\begin{aligned}\wp_b(Sa, Sb) &= \wp_b(a-1, a-1) = 0 \\ \mathcal{N}(a, b) &= \max \left\{ \wp_b(a, b), \wp_b(a, Sa), \wp_b(b, Sb), \frac{\wp_b(a, Sa) + \wp_b(b, Sb)}{2b} \right\} \\ &= (2a-1)^2 \\ \Rightarrow \frac{\text{bff}(a_0, a_1) \wp_b(Sa, Sb)}{\mathcal{N}(a, b)} e^{\text{bff}(a_0, a_1) \wp_b(Sa, Sb) - \mathcal{N}(a, b)} &= \frac{0}{(2a-1)^2} e^{0 - (2a-1)^2} < e^{-1}.\end{aligned}$$

Hence, all the hypotheses of Corollary 3 are satisfied and 0 is a FP of $\mathcal{S}$.

4. Applications

4.1. Solution to an Equation of Motion

A body with mass m started its motion at time $\mathfrak{t} = 0$ and $x = 0$. A force $\mathfrak{f}$ acts on it in the direction of the x -axis and its velocity increases from 0 to 1 instantly after $\mathfrak{t} = 0$. The problem aims to explore a function for a position in terms of time $\mathfrak{t}$.

The governing equation for this problem is

$$m \frac{d^2 x}{dt^2} = f(t, x(t)) \text{ together with } x(0) = 0, x'(1) = 0. \quad (12)$$

To incorporate the axioms of Theorem 1, consider $\wp_b : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ as

$$\wp_b(a, b) = \|a - b\|_\infty = \sup_{t \in [0, 1]} |a(t) - b(t)|^2,$$

it is trivial to show that $(\mathcal{C}[0, 1], \wp_b)$ is a complete bMS. In (12), f is a real valued function on $[0, 1] \times \mathbb{R}$. Green's function for (12), which is defined as

$$\mathcal{G}(t, \xi) = \begin{cases} (-1 + \xi)t, & t \leq \xi, \\ -\xi(1 - t), & t \geq \xi. \end{cases}$$

Assume the following constraints:

1. $|f(t, q) - f(t, r)|^2 \leq \frac{|q - r|}{\alpha(q, r)^{\frac{1}{2}}}$ for all $t \in [0, 1]$ and $q, r \in \mathbb{R}$ with $\check{\phi}(q, r) \geq 0$. Here, $\uparrow : \mathbb{J} \times \mathbb{J} \rightarrow [0, \infty)$.
2. there exist $x_0 \in \mathcal{C}[0, 1]$, such that $\check{\phi}(x_0(t), Sx_0(t)) \geq 0$ for all $t \in [0, 1]$, where S is self-map on $\mathcal{C}[0, 1]$, and $\check{\phi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function.

Theorem 2. Consider mapping $S : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$, which is defined as:

$$Sx(t) = \int_0^1 \mathcal{G}(t, \xi) f(\xi, x(\xi)) d\xi, \quad t \in [0, 1],$$

satisfying the assumptions 1 and 2. Then, (12) has a solution.

Proof. The solution of Equation (12) is

$$x(t) = \int_0^1 \mathcal{G}(t, \xi) f(\xi, x(\xi)) d\xi, \quad t \in [0, 1].$$

Assume $x, y \in \mathcal{C}[0, 1]$, such that $\check{\phi}(x(t), y(t)) \geq 0$ for all $t \in [0, 1]$. Now,

$$\begin{aligned} |S(x(t)) - S(y(t))|^2 &= \left| \int_0^1 \mathcal{G}(t, \xi) f(\xi, x(\xi)) d\xi - \int_0^1 \mathcal{G}(t, \xi) f(\xi, y(\xi)) d\xi \right|^2 \\ |S(x(t)) - S(y(t))|^2 &\leq \left(\int_0^1 \mathcal{G}(t, \xi) |f(\xi, x(\xi)) - f(\xi, y(\xi))| d\xi \right)^2 \\ &\leq \int_0^1 \mathcal{G}(t, \xi)^2 \left(\frac{|x(\xi) - y(\xi)|}{\alpha(x, y)^{\frac{1}{2}}} \right)^2 d\xi. \end{aligned}$$

Taking supremum on both sides to obtain

$$\sup_{t \in [0, 1]} |S(x(t)) - S(y(t))|^2 \leq \sup_{t \in [0, 1]} \frac{|x(\xi) - y(\xi)|^2}{\alpha(x, y)} \sup_{t \in [0, 1]} \left\{ \int_0^1 \mathcal{G}(t, \xi) d\xi \right\}^2.$$

Now,

$$\begin{aligned}\int_0^1 (\mathcal{G}(\mathfrak{t}, \xi) d\xi)^2 &= \int_0^1 ((\mathfrak{t} - 1)\xi)^2 d\xi + \int_{\mathfrak{t}}^1 ((\xi - 1)\mathfrak{t})^2 d\xi \\ &= (\mathfrak{t} - 1)^2 \xi^2 d\xi|_0^{\mathfrak{t}} + \mathfrak{t}(\xi - 1)^2|_{\mathfrak{t}}^1 \\ &= (\mathfrak{t} - 1) \frac{\mathfrak{t}^3}{3} + \left(\mathfrak{t}^2 \frac{(\mathfrak{t} - 1)^3}{3} \right) \\ &= \frac{\mathfrak{t}^4}{3} - \frac{2}{3} \mathfrak{t}^3 + \frac{\mathfrak{t}^2}{3} \quad \forall \mathfrak{t} \in [0, 1].\end{aligned}$$

Hence,

$$\sup_{\mathfrak{t} \in [0, 1]} \left\{ \int_0^1 \mathcal{G}(\mathfrak{t}, \xi) d\xi \right\}^2 = \frac{1}{4}.$$

Therefore,

$$\begin{aligned}\|\mathcal{S}\mathbf{x} - \mathcal{S}\mathbf{y}\|_{\infty} &\leq \frac{1}{4} \frac{\|\mathbf{x} - \mathbf{y}\|_{\infty}}{\alpha(\mathbf{x}, \mathbf{y})}. \\ \implies \alpha(\mathbf{x}, \mathbf{y}) \|\mathcal{S}\mathbf{x} - \mathcal{S}\mathbf{y}\|_{\infty} &\leq \frac{1}{4} \|\mathbf{x} - \mathbf{y}\|_{\infty} \\ \implies \tau + \mathcal{F}(\alpha(\mathbf{x}, \mathbf{y}) \wp_{\mathbf{b}}(\mathcal{S}\mathbf{x}, \mathcal{S}\mathbf{y})) &\leq \mathcal{F}(\mathcal{N}(\mathbf{x}, \mathbf{y})),\end{aligned}$$

with $\mathcal{F}(\mathbf{x}) = \ln \mathbf{x}$ and $\tau = \ln 4$. Hence, all the conditions of Corollary 3 are satisfied. This assures that $\mathcal{S}$ has a fixed point $\mathbf{x}$ in $\mathcal{C}^2([0, 1])$, which is the solution to (12). $\square$

4.2. Solution to a Fractional Differential Equation

This section establishes the existence of a solution to a nonlinear fractional differential equation, leveraging the framework of $\mathbf{b}$ -metric space and utilizing $(\mathfrak{r}\mathcal{F}, \mathbf{b}, \phi)$ contraction. Fractional calculus has a wide range of applications across various engineering disciplines and scientific fields. Its applications include modeling and analyzing complex phenomena such as heat diffusion, control systems, and signal processing, which are crucial in many engineering applications, enabling the development of innovative solutions and more accurate problem-solving approaches. The authors are referred to [30,31] for the application of fractional differential equations. We adopt the notation from [32,33] to define the Caputo fractional derivative of the $\mathbf{ff}$ order of a continuous function $\mathbf{q} : [0, +\infty) \rightarrow \mathbb{R}$, which is defined as:

$${}^{\mathcal{C}}\mathcal{D}_{\mathfrak{t}}^{\mathbf{ff}}(\mathbf{q}(\mathfrak{t})) = \frac{1}{\Gamma(\mathbf{ff} - n)} \int_a^{\mathfrak{t}} (\mathfrak{t} - \xi)^{n - \mathbf{ff} - 1} \mathbf{q}^{(n)}(\xi) d\xi \quad (n - 1) < \mathbf{ff} < n, \quad n = [\mathbf{ff}] + 1).$$

Here, Γ represents the Gamma function and $[\mathbf{ff}]$ represents the integral component of a real number. Suppose $\wp_{\mathbf{b}} : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ is defined as

$$\wp_{\mathbf{b}}(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_{\infty} = \max_{\mathfrak{t} \in [0, 1]} |\mathbf{a}(\mathfrak{t}) - \mathbf{b}(\mathfrak{t})|^2.$$

Then, $(\mathcal{C}([0, 1]), \wp_{\mathbf{b}})$ is a $\mathbf{b}$ metric space with $\mathbf{b} = 2$.

Consider the following non-linear fractional differential equation:

$${}^{\mathcal{C}}\mathcal{D}^{\mathbf{ff}}(\mathbf{a}(\mathfrak{t}) + \mathbf{f}(\mathfrak{t}, \mathbf{a}(\mathfrak{t}))) = 0 \quad (0 \leq \mathfrak{t} \leq 1, 1 < \mathbf{ff} \leq 2), \quad (13)$$

with $\mathbf{a}(0) = \mathbf{a}(1) = 0$ and $\mathbf{f}$ as a real valued function with domain $[0, 1] \times \mathbb{R}$. Assume the following conditions are satisfied:

1. $|f(\mathfrak{t}, a) - f(\mathfrak{t}, b)|^2 \leq \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)^{\frac{1}{2}}} \mathbb{J}(a, b)$ for all $\mathfrak{t} \in [0, 1]$ also $a, b \in \mathbb{R}$, such that
$$\mathbb{J}(a, b) = \max \left\{ |a - b|, |a - Sa|, |b - Sb|, \frac{|a - Sb| + |b - Sa|}{2b} \right\},$$

Here $\uparrow : \mathbb{J} \times \mathbb{J} \rightarrow [0, \infty)$.

2. there exist $a_0 \in \mathcal{C}[0, 1]$ such that $\check{\varphi}(a_0(\mathfrak{t}), Sa_0(\mathfrak{t})) \geq 0$ for all $\mathfrak{t} \in [0, 1]$.

Green's function for (13), taken from [34], is as follows:

$$\mathcal{G}(\mathfrak{t}, \xi) = \begin{cases} \frac{\mathfrak{t}(1-\xi)^{\mathfrak{ff}-1} - (\mathfrak{t}-\xi)^{\mathfrak{ff}-1}}{\Gamma(\mathfrak{ff})} & 0 \leq \xi \leq \mathfrak{t} \leq 1, \\ \frac{\mathfrak{t}(1-\xi)^{\mathfrak{ff}-1}}{\Gamma(\mathfrak{ff})} & 0 \leq \mathfrak{t} \leq \xi \leq 1. \end{cases}$$

Theorem 3. Consider mapping $\mathcal{S} : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$, which is defined as:

$$\mathcal{S}(a(\mathfrak{t})) = \int_0^1 \mathcal{G}(\mathfrak{t}, \xi) f(\xi, a(\xi)) d\xi,$$

satisfying the above assumptions of 1 and 2. Then, the fractional differential Equation (13) has a solution.

Proof. It is obvious that solution of (13) is

$$a(\mathfrak{t}) = \int_0^1 \mathcal{G}(\mathfrak{t}, \xi) f(\xi, a(\xi)) d\xi \text{ for all } \mathfrak{t} \in [0, 1].$$

Consider

$$\begin{aligned} | \mathcal{S}(a(y)) - \mathcal{S}(b(y)) |^2 &= \left| \int_0^1 \mathcal{G}(y, \xi) f(\xi, a(\xi)) d\xi - \int_0^1 \mathcal{G}(y, \xi) f(\xi, b(\xi)) d\xi \right|^2 \\ &\leq \int_0^1 |\mathcal{G}(y, \xi) (f(\xi, a(\xi)) - f(\xi, b(\xi)))|^2 d\xi \\ &\leq \int_0^1 |\mathcal{G}(y, \xi)|^2 \left(\frac{|f(\xi, a(\xi)) - f(\xi, b(\xi))|}{\mathfrak{ff}(a, b)^{\frac{1}{2}}} \right)^2 d\xi \\ &\leq \int_0^1 |\mathcal{G}(y, \xi)|^2 \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)} \mathbb{J}(a, b) d\xi \\ &\leq \int_0^1 |\mathcal{G}(y, \xi)|^2 \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)} \max \left\{ |a - b|, |a - Sa|, |b - Sb|, \frac{|a - Sb| + |b - Sa|}{2b} \right\} d\xi \\ &\leq \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)} \max \left\{ \wp(a, b), \wp(a, Sa), \wp(b, Sb), \frac{\wp(a, Sb) + \wp(b, Sa)}{2b} \right\} \left(\int_0^1 (\mathcal{G}(y, \xi)) d\xi \right)^2 \\ &\leq \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)} \max \left\{ \wp(a, b), \wp(a, Sa), \wp(b, Sb), \frac{\wp(a, Sb) + \wp(b, Sa)}{2b} \right\} \times \sup_{y \in [0, 1]} \left(\int_0^1 (\mathcal{G}(y, \xi)) d\xi \right)^2. \end{aligned}$$

As

$$\sup_{y \in [0, 1]} \left(\int_0^1 (\mathcal{G}(y, \xi)) d\xi \right) \leq 1.$$

It follows that

$$| \mathcal{S}(a(y)) - \mathcal{S}(b(y)) |^2 \leq \frac{e^{-\mathfrak{t}}}{\mathfrak{ff}(a, b)} \mathcal{N}(a, b)$$

$$\implies \mathfrak{ff}(a, b) | \mathcal{S}(a(y)) - \mathcal{S}(b(y)) |^2 \leq e^{-\mathfrak{t}} \mathcal{N}(a, b),$$

where

$$\mathcal{N}(a, b) = \max \left\{ \wp_b(a, b), \wp_b(a, Sa), \wp_b(b, Sb), \frac{\wp_b(a, Sb) + \wp_b(b, Sa)}{2b} \right\}.$$

Hence, for all $a, b \in \mathbb{J}$ and for all $y \in [0, 1]$,

Hence,

$$\check{\tau} + \mathcal{F}(\text{ff}(a, b) \wp_b(Sa, Sb)) \leq \mathcal{F}(\wp_b(\mathcal{N}(a, b))),$$

where $\mathcal{F}(y) = \ln(y)$. According to the Corollary 3, S has a fixed point which is the solution to (13). $\square$

5. Conclusions

These outlines encapsulate the core findings and implications of our study, highlighting the significance of the research topic.

- This research draws its primary inspiration from Wardowski's groundbreaking work on $\mathcal{F}$ contraction [18,19].
- The article uses the basic set-ups of the fixed-point theory by expanding on the basic concepts and illustrating Wardowski's $\mathcal{F}$ contraction with examples.
- Building upon the foundation laid by Jain et al. [26], the study expands on the notion of multivalued $\mathcal{F}$ contractions to a more general framework, incorporating the concept of b-metric spaces.
- The work of Jain et al. [26] is further extended using the platform of b-metric space. Furthermore, the multivalued $\mathcal{F}$ contraction is generalized to multivalued $(\uparrow, \mathcal{F}, \tilde{\phi})$ contraction.
- The following strategy is adopted:
 - (i) construct a Picard iterative sequence in b-metric space,
 - (ii) prove that this sequence is Cauchy,
 - (iii) the existence of BBP is established.
- The established results generalize many existing results [26,35–39] in the literature. This fact is assured by providing several corollaries.
- To demonstrate the practicality and validity of the presented theorems, the research includes non-trivial examples and applies its findings to the field of differential equations, specifically ordinary differential equations and fractional differential equations based on Caputo fractional operators for proving the existence of solutions using the established results.

Author Contributions: Conceptualization, D.-e.-S.S. and A.A.; methodology, N.M.; investigation, S.N., A.A. and S.B.; writing—original draft preparation, D.-e.-S.S., S.N., S.B., A.A. and N.M.; writing—review and editing, S.N. and S.B.; supervision, D.-e.-S.S. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors N. Mlaiki and A. Aloqaily would like to thank Prince Sultan University for paying the APC and for the support through the TAS Lab.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Poincare, H. Memoire sur les courbes definies par une equation differentielle. *J. Math. Pures Appl.* **1881**, *7*, 375–422.
2. Poincare, H. *Les Methodes Nouvelles de la Mecanique Celeste*; Gauthiers-Villars: Paris, France, 1892–1999; Dover: New York, NY, USA, 1957; Volume 1–3.
3. Fréchet, M.M. Sur quelques points du calcul fonctionnel. *Rend. Circ. Mat. Palermo* **1906**, *22*, 1–72. [[CrossRef](#)]

4. Browder, F.E. Nonlinear operators and nonlinear equations of evolution in Banach spaces. *Proc. Symp. Pure Math. Am. Math. Soc.* **1976**, *18*, 10008783451. [\[CrossRef\]](#)
5. Kakutani, S. A generalization of Brouwer's fixed point theorem. *Duke Math. J.* **1941**, *8*, 457–459. [\[CrossRef\]](#)
6. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [\[CrossRef\]](#)
7. Shatanawi, W.; Shatanawi, T.A. Some fixed point results based on contractions of new types for extended b -metric spaces. *AIMS Math.* **2023**, *8*, 10929–10946. [\[CrossRef\]](#)
8. Shatanawi, W.; Shatanawi, T.A.M. New fixed point results in controlled metric type spaces based on new contractive conditions. *AIMS Math.* **2023**, *8*, 9314–9330. [\[CrossRef\]](#)
9. Rezazgui, A.-Z.; Tallafha, A.A.; Shatanawi, W. Common fixed point results via $\mathcal{A}_\theta$ - α -contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b -metric space. *AIMS Math.* **2023**, *8*, 7225–7241. [\[CrossRef\]](#)
10. Edelstein, M. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.* **1962**, *1*, 74–79. [\[CrossRef\]](#)
11. Bakhtin, I.A. The contraction principle in quasimetric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
12. Mitrovic, Z.D.; Bodaghi, A.; Aloqaily, A.; Mlaiki, N.; George, R. New versions of some results on fixed points in b -metric spaces. *Mathematics* **2023**, *11*, 1118. [\[CrossRef\]](#)
13. Navascues, M.A.; Mohapatra, R.N. Fixed Point Dynamics in a New Type of Contraction in b -Metric Spaces. *Symmetry* **2024**, *16*, 506. [\[CrossRef\]](#)
14. Nadler, S.B., Jr. Multivalued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [\[CrossRef\]](#)
15. Aydi, H.; Bota, M.F.; Karapinar, E.; Mitrovic, S. Fixed point theorem for set-valued quasicontractions in b -metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 88. [\[CrossRef\]](#)
16. Fabiano, N.; Parvaneh, V.; Mirković, D.; Paunović, L.; Radenović, S. On W -contractions of Jungck–Ćirić–Wardowski-type in metric spaces. *Cogent Math. Stat.* **2020**, *7*, 1792699. [\[CrossRef\]](#)
17. Parvaneh, V.; Mohammadi, B.; De La Sen, M.; Alizadeh, E.; Nashine, H.K. On existence of solutions for some nonlinear fractional differential equations via Wardowski–Mizoguchi–Takahashi type contractions. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 893–902.
18. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *1*, 1–6. [\[CrossRef\]](#)
19. Klim, D.; Wardowski, D. Fixed points of dynamic processes of set-valued F -contractions and application to functional equations. *Fixed Point Theory Appl.* **2015**, *1*, 1–9. [\[CrossRef\]](#)
20. Sagheer, D.E.; Anwar, M.; Hussain, N.; Batul, S. Fixed point and common fixed point theorems on (α, f) -contractive multi-valued mappings in uniform spaces. *Filomat* **2022**, *36*, 6021–6036. [\[CrossRef\]](#)
21. Sadiq Basha, S. Extensions of Banach's contraction principle. *Numer. Func. Anal. Opt.* **2010**, *31*, 569–576. [\[CrossRef\]](#)
22. Basha, S.S.; Shahzad, N.; Jeyaraj, R. Common best proximity points: Global optimization of multi-objective functions. *Appl. Math. Lett.* **2011**, *24*, 883–886. [\[CrossRef\]](#)
23. Karapinar, E.; Erhan, I.M. Best proximity point on different types of contractions. *Appl. Math. Inf. Sci.* **2011**, *3*, 342–353.
24. Abkar, A.; Gabeleh, M. The existence of best proximity points for multivalued non-self-mappings. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. Mat.* **2013**, *107*, 319–325. [\[CrossRef\]](#)
25. Fallahi, K.; Ghahramani, H.; Soleimani Rad, G. Integral type contractions in partially ordered metric spaces and best proximity point. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 177–183. [\[CrossRef\]](#)
26. Jain, S.K.; Meena, G.; Singh, D.; Maitra, J.K. Best proximity point results with their consequences and applications. *J. Inequal. Appl.* **2022**, *2022*, 73. [\[CrossRef\]](#)
27. Khan, M.S.; Swaleh, M.; Sessa, S. Fixed point theorems by altering distances between the points. *Bull. Aust. Math. Soc.* **1984**, *30*, 1–9. [\[CrossRef\]](#)
28. Raj, V.S. A best proximity point theorem for weakly contractive non-self-mappings. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 4804–4808.
29. Ali, M.U.; Kamran, T.; Karapinar, E. Fixed point of α - ψ -contractive type mappings in uniform spaces. *Fixed Point Theory Appl.* **2014**, *1*, 1–12. [\[CrossRef\]](#)
30. Agrawal, O.P. Application of fractional derivatives in thermal analysis of disk brakes. *Nonlinear Dyn.* **2004**, *38*, 191–206. [\[CrossRef\]](#)
31. EL Nabulsi, R.A. A fractional action-like variational approach of some classical, quantum and geometrical dynamics. *Int. J. Appl. Math.* **2005**, *17*, 299.
32. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
33. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998.
34. Bai, Z.; Lu, H. Positive solutions for boundary value problem of nonlinear fractional differential equation. *J. Math. Anal. Appl.* **2005**, *311*, 495–505. [\[CrossRef\]](#)
35. Pragadeeswarar, V.; Marudai, M.; Kumam, P. Best proximity point for multivalued mappings for the complete partially ordered metric space. *J. Nonlinear Sci. Appl.* **2016**, *9*, 1911–1921. [\[CrossRef\]](#)
36. Amnuaykarn, K.; Kumam, P.; Sombut K.; Nantadilok, J. Best Proximity Points of Generalized α - ψ -Geraghty Proximal Contractions in Generalized Metric Spaces. *Fixed Point Theory* **2024**, *25*, 15–30.

37. Eldred, A.A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **2006**, *323*, 1001–1006. [[CrossRef](#)]
38. Isik, H.; Aydi, H.; Mlaiki, N.; Radenović, S. Best Proximity Point Results for Geraghty Type $\mathcal{Z}$ -Proximal Contractions with an Application. *Axioms* **2019**, *8*, 81. [[CrossRef](#)]
39. Suzuki, T.; Kikkawa, M.; Vetro, C. The existence of best proximity points in metric spaces with the property UC. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 2918–2926. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.