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A Conservative Difference Scheme for Solving the Coupled Fractional Schrödinger–Boussinesq System

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Abstract: In this paper, a high-accuracy conservative implicit algorithm for computing the space fractional coupled Schrödinger–Boussinesq system is constructed. Meanwhile, the conservative nature, a priori boundedness, and solvability of the numerical solution are presented. Then, the proposed algorithm is proved to be second-order convergence in temporal and fourth-order spatial convergence using the discrete energy method. Finally, some numerical experiments validate the effectiveness of the conservative algorithm and confirm the accuracy of the theoretical results for different choices of the fractional-order α .

Keywords: fractional Schrödinger–Boussinesq equation; conservative difference scheme; high-order approximation; convergence

MSC: 65M06; 65M12; 35Q51



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1. Introduction

The coupled Schrödinger–Boussinesq system (CSBS) is a basic equation in laser and plasma physics. It describes how coupled Langmuir and dust-acoustic waves propagate nonlinearly in a dusty plasma. Over the years, the theoretical results on the well-posedness and dynamic behaviors of the analytical solution to the CSBS have been widely exploited [1–6]. Several methods for finding the exact solutions to the CSBS have been presented [7–10]. Because the exact solutions to the CSBS often contain certain special functions, scholars have begun utilizing efficient methods to seek numerical solutions for it, such as the multi-symplectic method [11], orthogonal spline collocation method [12], radial basis function-finite difference method [13], cut-off function method [14], scalar auxiliary variable method [15], Adams prediction–correction method [16], finite-element method [17,18], and energy-preserving compact finite difference methods [19].

Laskin, a Canadian scholar, in 2000, enhanced Feynman’s path integral to Lévy paths and formulated a Schrödinger model incorporating a nonlocal Laplacian operator [20,21]. The fractional Laplacian operator is a powerful tool for describing memory and hereditary properties. It is widely used in addressing various nonlinear problems, including the fractional wave equations [22], fractional Schrödinger equation [23], fractional Ginzburg–Landau equation [24], fractional coupled Schrödinger–Boussinesq equation [25], fractional Korteweg–de Vries equation [26–28], and fractional Fornberg–Whitham equation [29]. Additional notable results on this subject are available in [30–33].

This paper primarily focuses on the numerical solution for the fractional coupled Schrödinger–Boussinesq system (FCSBS)

$$iu_t - (-\Delta)^{\alpha/2}u - uv = 0, \quad (1)$$

$$v_{tt} - v_{xx} + \gamma v_{xxxx} - f_{xx}(v) - \omega|u|^2 = 0, \quad (2)$$

where $1 < \alpha \leq 2$, $i = \sqrt{-1}$ and coefficients γ and ω are positive constants. The function $f(v)$ is sufficiently smooth and real, with $f(0) = 0$. The complex function $u(x, t)$ represents the electric field in Langmuir oscillations, while the real function $v(x, t)$ characterizes low-frequency density perturbations.

Assuming $v_t = \phi_{xx}$, then system FCSBS (1) transforms into the following equivalent form:

$$iu_t - (-\Delta)^{\alpha/2}u - uv = 0, \quad x \in \Omega, \quad t \in (0, T], \quad (3)$$

$$v_t = \phi_{xx}, \quad x \in \Omega, \quad t \in (0, T], \quad (4)$$

$$\phi_t - v + \gamma v_{xx} - f(v) - \omega|u|^2 = 0, \quad x \in \Omega, \quad t \in (0, T], \quad (5)$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \phi(x, 0) = \phi_0(x), \quad x \in \Omega, \quad (6)$$

$$u(x, t) = v(x, t) = \phi(x, t) = 0, \quad v_{xx}(x, t) = 0, \quad x \in \partial\Omega, \quad t \in [0, T] \quad (7)$$

where $\Omega = (x_L, x_R)$ and $u_0(x)$, $v_0(x)$, $\phi_0(x)$ are given smooth functions. The fractional Laplacian can be considered as the Riesz fractional derivative

$$-(-\Delta)^{\alpha/2}u(x, t) = \partial_x^\alpha u(x, t) = -\frac{1}{2\cos(\alpha\pi/2)}[x_L D_x^\alpha + {}_x D_{x_R}^\alpha]u(x, t),$$

where ${}_x D_x^\alpha u(x, t)$ represents the left Riemann–Liouville fractional derivative

$${}_x D_x^\alpha u(x, t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{x_L}^x \frac{u(\xi, t)}{(x-\xi)^{\alpha-1}} d\xi,$$

and ${}_x D_{x_R}^\alpha u(x, t)$ represents the right Riemann–Liouville fractional derivative

$${}_x D_{x_R}^\alpha u(x, t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^{x_R} \frac{u(\xi, t)}{(\xi-x)^{\alpha-1}} d\xi.$$

It is noteworthy that the initial-boundary value problem (3)–(7) possesses three conservative laws, namely, the Langmuir Plasmon number

$$I_u(t) = \int_{\Omega} |u|^2 dx = \int_{\Omega} |u_0|^2 dx = I_u(0), \quad t \geq 0, \quad (8)$$

the total perturbed number density

$$I_v(t) = \int_{\Omega} v dx = \int_{\Omega} v_0 dx = I_v(0), \quad t \geq 0, \quad (9)$$

and the total energy

$$I_e(t) = \int_{\Omega} \left(v^2 + \phi_x^2 + 2\omega|(-\Delta)^{\frac{\alpha}{4}}u|^2 + \gamma v_x^2 + 2F(v) + 2\omega v|u|^2 \right) dx = I_e(0), \quad t \geq 0. \quad (10)$$

Here, $F(v) > 0$ is the primitive function of $f(v)$.

Han et al. [25] investigated the local and global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$ for FCSBS, in which the nonlinear term $f(v)$ remains undetermined. Shao and Guo [34] derived local mild solutions to the Schrödinger-damped Boussinesq system and its fractional counterpart in one dimension using the contracting mapping principle. They also presented the precise results concerning the existence and nonexistence of global mild solutions. Given the inherent nonlocality and nonlinearity of FCSBS, obtaining analytical solutions

for the system is an extremely challenging task. Therefore, numerical simulation has emerged as a crucial approach for its study. Ray [35] developed a time-splitting Fourier spectral method, which has been proven to be unconditionally stable. The error norms and graphical solutions are also presented in this work. In [36], Liao et al. developed and rigorously analyzed two efficient conservative difference schemes for FCSBS. Each scheme is demonstrated to preserve two fundamental conservation laws: mass conservation and energy conservation, while converging with an accuracy of $O(\tau^2 + h^2)$. Compared to CSBS, numerical methods for solving FCSBS are quite scarce. Thus, the goal of this paper is to develop a new conservative scheme for solving FCSBS, while also rigorously implementing error estimates for the proposed scheme.

The contributions of this article are summarized as follows: (1) To maintain the same physical properties as the original differential equation, we develop a conservative scheme. We rigorously demonstrate that the scheme can preserve three conservative laws simultaneously, as evidenced by numerical examples. (2) The boundedness and solvability of the numerical solutions obtained from the conservative scheme are established. (3) Importantly, the numerical solutions provided by the conservative scheme unconditionally converges to the exact solutions in the L^2 -norm with a convergence order of $O(\tau^2 + h^4)$.

The subsequent sections of this paper are organized as follows: In Section 2, we introduce relevant notations and auxiliary lemmas, followed by the derivation of the conservative scheme. Theoretical analyses for the proposed scheme are presented in Section 3. Numerical experiments, conducted to validate the proposed scheme, are discussed in Section 4. Finally, concluding remarks are provided in Section 5.

2. Construction of Conservative Difference Scheme

In this section, we initially introduce relevant notations and auxiliary lemmas that will be utilized later. Subsequently, we elaborate on the establishment of a conservative difference scheme for the initial-boundary value problems (3)–(7).

2.1. Notations and Lemmas

Let $h = (x_R - x_L)/J$ and $\tau = T/N$ denote the uniform step sizes in the spatial and temporal directions, respectively, where J and N are positive integers. Define

$$\Omega_h = \{x_j | x_j = x_L + jh, j = 0, 1, 2, \dots, J\}; \quad \Omega_\tau = \{t_n | t_n = n\tau, n = 0, 1, 2, \dots, N\}.$$

For any grid function $u = \{u_j^n | u_j^n = u(x_j, t_n), (x_j, t_n) \in \Omega_h \times \Omega_\tau\}$, we define the following notations:

$$\begin{aligned} u_j^{n+1/2} &= \frac{u_j^{n+1} + u_j^n}{2}, \quad \delta_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \\ \delta_x u_j^n &= \frac{u_{j+1}^n - u_j^n}{h}, \quad \delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}. \end{aligned}$$

Denote the space

$$\mathcal{L}^{n+\alpha}(\mathbb{R}) = \left\{ u(x) \left| \int_{\mathbb{R}} (1 + |\omega|)^{n+\alpha} |\mathcal{F}u(\omega)| d\omega < \infty, \quad u(x) \in L^1(\mathbb{R}) \right. \right\},$$

where $\mathcal{F}u(\omega) = \int_{\mathbb{R}} e^{i\omega x} u(x) dx$ is the Fourier transformation of $u(x)$. We now introduce several lemmas crucial for constructing the conservative difference scheme.

Lemma 1 (see [22,37]). Suppose the function $u \in \mathcal{L}^{4+\alpha}(\mathbb{R})$, $1 < \alpha \leq 2$ and let

$$\Delta_h^\alpha u(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} u(x - kh)$$

be the fractional centered difference. Then, we have

$$\mathcal{A}[\partial_x^\alpha u(x)] = \delta_x^{(\alpha)} u(x) + O(h^4), \quad (11)$$

where

$$\begin{aligned} \delta_x^{(\alpha)} u(x) &= -h^{-\alpha} \Delta_h^\alpha u(x), \\ g_k^{(\alpha)} &= \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1) \Gamma(\frac{\alpha}{2} + k + 1)}, \quad k \in \mathbb{Z}, \\ \mathcal{A}u(x) &= \frac{\alpha}{24} u(x - h) + (1 - \frac{\alpha}{12}) u(x) + \frac{\alpha}{24} u(x + h), \end{aligned}$$

and $\mathbb{Z}$ denotes the set of all integers.

Here, the $\mathcal{A}$ we define is consistent with the $\mathcal{A}^\alpha$ defined in [22,37], just with a simplified notation.

Remark 1. If $u^*(x)$ is defined by

$$u^*(x) = \begin{cases} u(x), & x \in (x_L, x_R), \\ 0, & x \notin (x_L, x_R), \end{cases}$$

and suppose $u^* \in \mathcal{L}^{4+\alpha}(\mathbb{R})$. Then, for $x \in (x_L, x_R)$, we have

$$\mathcal{A}[\partial_x^\alpha u(x)] = -\frac{1}{h^\alpha} \sum_{k=-\lceil \frac{x_R-x}{h} \rceil}^{\lceil \frac{x-x_L}{h} \rceil} g_k^{(\alpha)} u(x - kh) + O(h^4).$$

The classical fourth-order compact approximation for standard second-order derivatives is obtained when $\alpha = 2$ in Formula (11).

Lemma 2 (see [38]). Suppose $u(x) \in C^6(\mathbb{R})$; then, we have

$$\mathcal{B}u_{xx}(x) = \delta_x^2 u(x) + O(h^4),$$

where

$$\mathcal{B}u_j = \frac{1}{12} (u_{j+1} + 10u_j + u_{j-1}), \quad 1 \leq j \leq J-1.$$

2.2. Derivation of the Conservative Difference Scheme

Let $u_j^n = u(x_j, t_n)$, $\phi_j^n = \phi(x_j, t_n)$, and $v_j^n = v(x_j, t_n)$. Meanwhile, U_j^n , Φ_j^n , and V_j^n represent the numerical approximations of u_j^n , ϕ_j^n , and v_j^n at the point (x_j, t_n) , respectively.

By considering Equation (3) at both (x_j, t_n) and (x_j, t_{n+1}) , and then combining them, from Taylor expansion, we can derive

$$i\delta_t u_j^n - (-\Delta)^{\alpha/2} u_j^{n+1/2} - u_j^{n+1/2} v_j^{n+1/2} + O(\tau^2) = 0, \quad (12)$$

Then, acting the operator $\mathcal{A}$ on both sides of (12), we have

$$i\mathcal{A}\delta_t u_j^n - \mathcal{A}(-\Delta)^{\alpha/2} u_j^{n+1/2} - \mathcal{A}(u_j^{n+1/2} v_j^{n+1/2}) + O(\tau^2) = 0, \quad (13)$$

Using Lemma 1, we obtain

$$i\mathcal{A}\delta_t u_j^n + \delta_x^{(\alpha)} u_j^{n+1/2} + O(h^4) - \mathcal{A}(u_j^{n+1/2} v_j^{n+1/2}) + O(\tau^2) = 0. \quad (14)$$

Considering Equations (4) and (5) at (x_j, t_n) and (x_j, t_{n+1}) , respectively, and then combining them, from Taylor expansion, we can obtain

$$\delta_t v_j^n + O(\tau^2) = (\phi_j^{n+1/2})_{xx}, \quad (15)$$

$$\delta_t \phi_j^n + O(\tau^2) = v_j^{n+1/2} - \gamma(v_j^{n+1/2})_{xx} + \frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n} + \frac{\omega}{2}(|u_j^{n+1}|^2 + |u_j^n|^2). \quad (16)$$

Applying the operator $\mathcal{B}$ to both sides of Equations (15) and (16) and utilizing Lemma 2, we have

$$\mathcal{B}\delta_t v_j^n + O(\tau^2) = \delta_x^2 \phi_j^{n+1/2} + O(h^4), \quad (17)$$

$$\mathcal{B}\delta_t \phi_j^n + O(\tau^2) = \mathcal{B}v_j^{n+1/2} - \gamma\delta_x^2 v_j^{n+1/2} + \mathcal{B}\frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n} + \frac{\omega}{2}\mathcal{B}(|u_j^{n+1}|^2 + |u_j^n|^2) + O(h^4). \quad (18)$$

Then, considering the discretizations (6) and (7), we have

$$U_j^0 = u_0(jh), \quad V_j^0 = v_0(jh), \quad \Phi_j^0 = \phi_0(jh), \quad 0 \leq j \leq J, \quad (19)$$

$$U_0^n = U_J^n = 0, \quad V_0^n = V_J^n = 0, \quad \Phi_0^n = \Phi_J^n = 0, \quad 0 \leq n \leq N \quad (20)$$

Neglecting the small terms in Equations (14), (17) and (18), and considering Equations (19) and (20), we can derive the following numerical scheme for (3)–(7):

$$i\mathcal{A}\delta_t U_j^n + \delta_x^\alpha U_j^{n+1/2} - \mathcal{A}(U_j^{n+1/2} V_j^{n+1/2}) = 0, \quad (21)$$

$$\mathcal{B}\delta_t V_j^n = \delta_x^2 \Phi_j^{n+1/2}, \quad (22)$$

$$\mathcal{B}\delta_t \Phi_j^n = \mathcal{B}V_j^{n+1/2} - \gamma\delta_x^2 V_j^{n+1/2} + \mathcal{B}\frac{F(V_j^{n+1}) - F(V_j^n)}{V_j^{n+1} - V_j^n} + \frac{\omega}{2}\mathcal{B}(|U_j^{n+1}|^2 + |U_j^n|^2), \quad (23)$$

$$U_j^0 = u_0(jh), \quad V_j^0 = v_0(jh), \quad \Phi_j^0 = \phi_0(jh), \quad (24)$$

$$U_0^n = U_J^n = V_0^n = V_J^n = \Phi_0^n = \Phi_J^n = 0. \quad (25)$$

Define

$$U^n = (U_1^n, U_2^n, \dots, U_{J-1}^n)^T, \\ U^{n+1/2} V^{n+1/2} = (U_1^{n+1/2} V_0^{n+1/2}, U_2^{n+1/2} V_1^{n+1/2}, \dots, U_{J-1}^{n+1/2} V_{J-1}^{n+1/2}), \\ P_1 = \begin{pmatrix} 1 - \frac{\alpha}{12} & \frac{\alpha}{24} & 0 & \cdots & 0 \\ \frac{\alpha}{24} & 1 - \frac{\alpha}{12} & \frac{\alpha}{24} & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & \frac{\alpha}{24} & 1 - \frac{\alpha}{12} & \frac{\alpha}{24} \\ 0 & \cdots & 0 & \frac{\alpha}{24} & 1 - \frac{\alpha}{12} \end{pmatrix}, \quad P_2 = \frac{1}{12} \begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{pmatrix},$$

where P_1 and P_2 are square matrices with order $J - 1$. Denote

$$G_1 = P_1^{-1}, \quad G_2 = P_2^{-1},$$

it is easy to verify that G_1 and G_2 are symmetric positive definite matrixes. Thus, the vector forms of the fourth-order difference scheme (21)–(25) and can be written as

$$i\delta_t U^n + G_1 \delta_x^\alpha U^{n+1/2} - U^{n+1/2} V^{n+1/2} = 0, \quad (26)$$

$$\delta_t V^n = G_2 \delta_x^2 \Phi^{n+1/2}, \quad (27)$$

$$\delta_t \Phi^n = V^{n+1/2} - \gamma G_2 \delta_x^2 V^{n+1/2} + \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n} + \frac{\omega}{2} (|U^{n+1}|^2 + |U^n|^2), \quad (28)$$

$$U^0 = u_0, \quad V^0 = v_0, \quad \Phi^0 = \phi_0, \quad (29)$$

$$U_0^n = U_j^n = 0, \quad V_0^n = V_j^n = 0, \quad \Phi_0^n = \Phi_j^n = 0. \quad (30)$$

3. Theoretical Analysis

Define the grid function spaces on Ω_h as $\mathbb{V}_h^0 := \{U | U = (U_0, U_1, \dots, U_J)^T\} \subseteq \mathbb{C}^{J+1}$ and $\mathbb{V}_h := \{U | U = (U_1, U_2, \dots, U_{J-1})^T\} \subseteq \mathbb{C}^{J-1}$. For any grid function $U \in \mathbb{V}_h^0$, the discrete inner product is defined as follows:

$$\langle \delta_x U^n, \delta_x U^n \rangle = h \sum_{j=0}^{J-1} |\delta_x U_j^n|^2.$$

For any two grid functions $U, V \in \mathbb{V}_h$, the discrete inner product and the associated l_h^2 -norm are defined as follows:

$$\langle U, V \rangle = h \sum_{j=1}^{J-1} U_j \bar{V}_j, \quad \|U\|^2 = \langle U, U \rangle = h \sum_{j=1}^{J-1} |U_j|^2.$$

where $\bar{V}_j$ represents the complex conjugate of V_j . We also define the discrete l_h^p -norm as

$$\|U\|_{l_h^p}^p = h \sum_{j=1}^{J-1} |U_j|^p, \quad 1 \leq p < \infty,$$

and the discrete maximum norm (l_h^∞ -norm) as

$$\|U^n\|_{l_h^\infty} = \max_{1 \leq j \leq J-1} |U_j^n|.$$

Provided the constant $0 \leq \sigma \leq 1$, the fractional Sobolev norm $\|U\|_{H^\sigma}$ and semi-norm $|U|_{H^\sigma}$ can be defined as [39]

$$\|U\|_{H^\sigma}^2 = h \int_{-\pi}^{\pi} (1 + h^{-2\sigma} |k|^{2\sigma}) |\hat{U}(k)|^2 dk, \quad |U|_{H^\sigma}^2 = h \int_{-\pi}^{\pi} h^{-2\sigma} |k|^{2\sigma} |\hat{U}(k)|^2 dk,$$

where

$$\hat{U}(k) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} U_j e^{-ijk}.$$

Obviously, we can obtain

$$\|U\|_{H^\sigma}^2 = \|U\|^2 + |U|_{H^\sigma}^2. \quad (31)$$

For convenience, we denote a general constant as C , which may vary across different contexts. Next, we introduce several useful auxiliary Lemmas.

Lemma 3 ([23]). For any two grid functions $U, V \in \mathbb{V}_h$, a linear operator Λ^α exists such that

$$\langle \delta_x^2 U, V \rangle = -\langle \delta_x U, \delta_x V \rangle, \quad \langle \delta_x^\alpha U, V \rangle = -\langle \Lambda^\alpha U, \Lambda^\alpha V \rangle.$$

Lemma 4 ([23]). For any complex grid function $U^n \in \mathbb{V}_h$, $0 \leq n \leq N$, we have

$$\operatorname{Im}\langle G_1 \delta_x^\alpha U^{n+1/2}, U^{n+1/2} \rangle = 0, \quad (32)$$

$$\operatorname{Re}\langle G_1 \delta_x^\alpha U^{n+1/2}, \delta_t U^{n+1/2} \rangle = \frac{1}{2\tau} (\langle G_1 \delta_x^\alpha U^{n+1}, U^{n+1} \rangle - \langle G_1 \delta_x^\alpha U^n, U^n \rangle). \quad (33)$$

3.1. The Conservative Property

Considering (26)–(30), we have following discrete conservative laws.

Theorem 1. The scheme (26)–(30) is conservative in the sense

$$\begin{aligned} I_U^n &= \|U^n\|^2 = I_U^0, \\ I_V^n &= \langle V^n, I \rangle = h \sum_{j=1}^{J-1} V_j^n = I_V^0 \\ I_E^n &= -2\omega \langle G_1 \delta_x^\alpha U^n, U^n \rangle - \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle + \|V^n\|^2 - \langle \gamma G_2 \delta_x^2 V^n, V^n \rangle \\ &\quad + 2 \langle F(V^n), I \rangle + 2\omega \langle |U^n|^2, V^n \rangle = I_E^0, \end{aligned}$$

where $I = (1, 1, \dots, 1)^T$ with $(J-1)$ components.

Proof. Computing the inner product of (26) with $(U^{n+1} + U^n)$ and taking the imaginary part, we have

$$\begin{aligned} \operatorname{Im}\langle i\delta_t U^n, U^{n+1} + U^n \rangle + \operatorname{Im}\langle G_1 \delta_x^\alpha U^{n+1/2}, U^{n+1} + U^n \rangle \\ - \operatorname{Im}\langle U^{n+1/2} V^{n+1/2}, U^{n+1} + U^n \rangle = 0. \end{aligned} \quad (34)$$

By virtue of the first identity (32) of Lemma 4 and direct computation, we can deduce

$$\begin{aligned} \operatorname{Im}\langle i\delta_t U^n, U^{n+1} + U^n \rangle &= \operatorname{Re}\langle \delta_t U^n, U^{n+1} + U^n \rangle = \frac{1}{\tau} (\|U^{n+1}\|^2 - \|U^n\|^2), \\ \operatorname{Im}\langle G_1 \delta_x^\alpha U^{n+1/2}, U^{n+1} + U^n \rangle &= 0, \\ \operatorname{Im}\langle U^{n+1/2} V^{n+1/2}, U^{n+1} + U^n \rangle &= 2\operatorname{Im}(h \sum_{j=0}^{M-1} |U_j^{n+1/2}|^2 V_j^{n+1/2}) = 0. \end{aligned}$$

Then, substituting the above equalities into (34), one can obtain

$$I_U^n = \|U^n\|^2 = I_U^0$$

Making the inner product of (27) with $I = (1, 1, \dots, 1)^T$, we have

$$\langle \delta_t V^n, I \rangle = \langle G_2 \delta_x^2 \Phi^{n+1/2}, I \rangle.$$

Then, according to the first identity of Lemma 3, we obtain that

$$\langle G_2 \delta_x^2 \Phi^{n+1/2}, I \rangle = -\langle G_2 \delta_x \Phi^{n+1/2}, \delta_x I \rangle = 0,$$

Thus,

$$\langle \delta_t V^n, I \rangle = \frac{1}{\tau} (\langle V^{n+1}, I \rangle - \langle V^n, I \rangle) = 0.$$

By computing the inner product of (26) with $2\tau\delta_t U^n$ and taking the real part, we have

$$\operatorname{Re}\langle i\delta_t U^n, 2\tau\delta_t U^n \rangle + \operatorname{Re}\langle G_1 \delta_x^\alpha U^{n+1/2}, 2\tau\delta_t U^n \rangle - \operatorname{Re}\langle U^{n+1/2} V^{n+1/2}, 2\tau\delta_t U^n \rangle = 0.$$

Calculating directly, we have

$$\begin{aligned} \operatorname{Re}\langle i\delta_t U^n, 2\tau\delta_t U^n \rangle &= 2\tau \operatorname{Im}\langle \delta_t U^n, \delta_t U^n \rangle = 0, \\ \operatorname{Re}\langle U^{n+1/2} V^{n+1/2}, 2\tau\delta_t U^n \rangle &= \operatorname{Re}\langle (U^{n+1} + U^n) V^{n+1/2}, U^{n+1} - U^n \rangle \\ &= \frac{1}{2} \langle V^{n+1} + V^n, |U^{n+1}|^2 - |U^n|^2 \rangle. \end{aligned}$$

According to (33) of Lemma (32), we obtain

$$\begin{aligned} \operatorname{Re}\langle G_1 \delta_x^\alpha U^{n+1/2}, 2\tau\delta_t U^n \rangle &= \operatorname{Re}\langle G_1 \delta_x^\alpha (U^{n+1} + U^n), (U^{n+1} - U^n) \rangle \\ &= \langle G_1 \delta_x^\alpha U^{n+1}, U^{n+1} \rangle - \langle G_1 \delta_x^\alpha U^n, U^n \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \langle G_1 \delta_x^\alpha U^{n+1}, U^{n+1} \rangle - \langle G_1 \delta_x^\alpha U^n, U^n \rangle - \frac{1}{2} \langle V^{n+1} + V^n, |U^{n+1}|^2 - |U^n|^2 \rangle &= 0, \\ -2\omega \langle G_1 \delta_x^\alpha U^{n+1}, U^{n+1} \rangle + 2\omega \langle G_1 \delta_x^\alpha U^n, U^n \rangle + \omega \langle V^{n+1} + V^n, |U^{n+1}|^2 - |U^n|^2 \rangle &= 0 \quad (35) \end{aligned}$$

Next, taking the inner product of (27) with $2\tau\delta_t \Phi^n$ can yield

$$\langle \delta_t V^n, 2\tau\delta_t \Phi^n \rangle = \langle G_2 \delta_x^2 \Phi^{n+1/2}, 2\tau\delta_t \Phi^n \rangle,$$

and using the relation

$$\begin{aligned} \langle G_2 \delta_x^2 \Phi^{n+1/2}, 2\tau\delta_t \Phi^n \rangle &= \langle G_2 \delta_x^2 (\Phi^{n+1} + \Phi^n), \Phi^{n+1} - \Phi^n \rangle \\ &= \langle G_2 \delta_x^2 \Phi^{n+1}, \Phi^{n+1} \rangle - \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle, \end{aligned}$$

we obtain

$$2\tau \langle \delta_t V^n, \delta_t \Phi^n \rangle = \langle G_2 \delta_x^2 \Phi^{n+1}, \Phi^{n+1} \rangle - \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle. \quad (36)$$

By making the inner product of (28) with $2\tau\delta_t V^n$, we have

$$\begin{aligned} \langle \delta_t \Phi^n, 2\tau\delta_t V^n \rangle &= \langle V^{n+1/2}, 2\tau\delta_t V^n \rangle - \langle \gamma G_2 \delta_x^2 V^{n+1/2}, 2\tau\delta_t V^n \rangle \\ &+ \left\langle \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n}, 2\tau\delta_t V^n \right\rangle + \frac{\omega}{2} \langle |U^{n+1}|^2 + |U^n|^2, 2\tau\delta_t V^n \rangle. \end{aligned}$$

Taking into account the relations

$$\begin{aligned} \langle V^{n+1/2}, 2\tau\delta_t V^n \rangle &= \langle V^{n+1} + V^n, V^{n+1} - V^n \rangle \\ &= \|V^{n+1}\|^2 - \|V^n\|^2, \\ -\langle \gamma G_2 \delta_x^2 V^{n+1/2}, 2\tau\delta_t V^n \rangle &= -\gamma \langle G_2 \delta_x^2 (V^{n+1} + V^n), V^{n+1} - V^n \rangle \\ &= -\gamma \langle G_2 \delta_x^2 V^{n+1}, V^{n+1} \rangle + \gamma \langle G_2 \delta_x^2 V^n, V^n \rangle, \\ \left\langle \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n}, 2\tau\delta_t V^n \right\rangle &= 2 \left\langle \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n}, V^{n+1} - V^n \right\rangle \\ &= 2 \langle F(V^{n+1}) - F(V^n), I \rangle, \\ \frac{\omega}{2} \langle |U^{n+1}|^2 + |U^n|^2, 2\tau\delta_t V^n \rangle &= \omega \langle |U^{n+1}|^2 + |U^n|^2, V^{n+1} - V^n \rangle, \end{aligned}$$

we obtain

$$\begin{aligned} 2\tau \langle \delta_t \Phi^n, \delta_t V^n \rangle &= \|V^{n+1}\|^2 - \|V^n\|^2 - \gamma \langle G_2 \delta_x^2 V^{n+1}, V^{n+1} \rangle + \gamma \langle G_2 \delta_x^2 V^n, V^n \rangle \\ &+ 2 \langle F(V^{n+1}) - F(V^n), I \rangle + \omega \langle |U^{n+1}|^2 + |U^n|^2, V^{n+1} - V^n \rangle. \end{aligned}$$

Summing this equation and (35) and (36), we arrive at the formula

$$\begin{aligned}
& -2\omega \langle G_1 \delta_x^\alpha U^{n+1}, U^{n+1} \rangle + 2\omega \langle G_1 \delta_x^\alpha U^n, U^n \rangle - \langle G_2 \delta_x^2 \Phi^{n+1}, \Phi^{n+1} \rangle + \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle \\
& + \|V^{n+1}\|^2 - \|V^n\|^2 - \gamma \langle G_2 \delta_x^2 V^{n+1}, V^{n+1} \rangle + \gamma \langle G_2 \delta_x^2 V^n, V^n \rangle + 2 \langle F(V^{n+1}), I \rangle - 2 \langle F(V^n), I \rangle \\
& + \omega \langle V^{n+1} + V^n, |U^{n+1}|^2 - |U^n|^2 \rangle + \omega \langle |U^{n+1}|^2 + |U^n|^2, V^{n+1} - V^n \rangle = 0.
\end{aligned}$$

Noting that

$$\begin{aligned}
& \omega \langle V^{n+1} + V^n, |U^{n+1}|^2 - |U^n|^2 \rangle + \omega \langle |U^{n+1}|^2 + |U^n|^2, V^{n+1} - V^n \rangle \\
& = 2\omega \langle |U^{n+1}|^2, V^{n+1} \rangle - 2\omega \langle |U^n|^2, V^n \rangle.
\end{aligned}$$

This implies $I_E^n = I_E^{n-1} = \dots = I_E^0$. $\square$

3.2. A Priori Bound

Lemma 5 ([23]). For any nonsingular matrix G and $U \in \mathbb{V}$, two positive integers C_0 and C_1 exist such that

$$C_0 \|U\|^2 \leq \langle GU, U \rangle \leq C_1 \|U\|^2,$$

where $C_0 = \min\{\lambda_j\}$, $C_1 = \max\{\lambda_j\}$, λ_j is the singular value of the nonsingular matrix G .

Lemma 6 ((Young's inequality) [40]). If $a \geq 0$, $b \geq 0$, then

$$ab \leq \frac{\epsilon a^p}{p} + \frac{b^q}{\epsilon q},$$

where $\epsilon > 0$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$.

Lemma 7 ([24]). For any $1/4 < \sigma_0 \leq \sigma \leq 1$ and $U \in l_h^2$, a constant $C = C(\sigma_0) > 0$ independent of $h > 0$ exists such that

$$\|U\|_{l_h^4} \leq C \|U\|_{H^{\sigma_0/\sigma}}^{\sigma_0/\sigma} \|U\|^{1-\sigma_0/\sigma}.$$

Lemma 8 ([24]). For any $1 < \alpha \leq 2$ and $U \in l_h^2$, it holds that

$$|\frac{2}{\pi}|^\alpha |U|_{H^{\frac{\alpha}{2}}}^2 \leq \langle -\delta_x^\alpha U, U \rangle \leq |U|_{H^{\frac{\alpha}{2}}}^2.$$

Lemma 9 ([41]). For any $1 \leq \alpha \leq 2$ and $U \in l_h^2$, a constant $C = C(\delta) > 0$ independent of $h > 0$ exists such that

$$\|U\|_{l_h^\infty} \leq C \|U\|_{H_h^{\alpha/2}}.$$

Lemma 10 ([42,43]). For any grid functions $U \in l_h^2$. Given $\epsilon > 0$, a constant $C(\epsilon)$ dependent on ϵ exists such that

$$\|\delta_x^k U\|_{l_h^p}^2 \leq \epsilon \|\delta_x^n U\|^2 + C(\epsilon) \|U\|^2.$$

where $p \in [2, \infty]$, $0 \leq k < n$.

Theorem 2. The difference solution of the scheme (26)–(30) satisfies

$$\begin{aligned}
& \|U^n\| \leq C, \quad \|\Lambda^\alpha U^n\| \leq C, \quad \|U^n\|_{l_h^\infty} \leq C, \\
& \|V^n\| \leq C, \quad \|\delta_x V^n\| \leq C, \quad \|V^n\|_{l_h^\infty} \leq C, \\
& \|\Phi^n\| \leq C, \quad \|\delta_x \Phi^n\| \leq C, \quad \|\Phi^n\|_{l_h^\infty} \leq C.
\end{aligned}$$

Proof. Theorem 1 shows that

$$\|U^n\| \leq C, \quad (37)$$

$$\|V^n\|_{l_h^\infty} \leq C, \quad (38)$$

$$\begin{aligned} & -2\omega \langle G_1 \delta_x^\alpha U^n, U^n \rangle - \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle + \|V^n\|^2 - \langle \gamma G_2 \delta_x^2 V^n, V^n \rangle \\ & \leq C + 2|\langle F(V^n), I \rangle| + 2\omega |\langle |U^n|^2, V^n \rangle|. \end{aligned} \quad (39)$$

It follows from Lemmas 3 and 5 that three positive constants exist such that

$$-\langle G_1 \delta_x^\alpha U^n, U^n \rangle = \|M_1 \Lambda^\alpha U^n\|^2 \geq C_1 \|\Lambda^\alpha U^n\|^2, \quad (40)$$

$$-\langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle = \|M_2 \delta_x \Phi^n\|^2 \geq C_2 \|\delta_x \Phi^n\|^2, \quad (41)$$

$$-\langle G_2 \delta_x^2 V^n, V^n \rangle = \|M_2 \delta_x V^n\|^2 \geq C_3 \|\delta_x V^n\|^2. \quad (42)$$

Using Lemma 8, we have

$$\|\Lambda^\alpha U^n\|^2 = C_\alpha |U^n|_{H^{\frac{\alpha}{2}}}^2, \quad (43)$$

where $C_\alpha \in [1, (2/\pi)^\alpha]$. Considering $\|V^n\|_{l_h^\infty} \leq C$, we have

$$|\langle F(V^n), I \rangle| \leq C. \quad (44)$$

Using Lemma 6 with parameters $p = q = 2$ and $\epsilon = 2\omega$ yields

$$\langle |U^n|^2, V^n \rangle \leq \omega \|U^n\|_{l_h^4}^4 + \frac{1}{4\omega} \|V^n\|^2. \quad (45)$$

When $\frac{1}{4} < \sigma_0 < \frac{\alpha}{4}$ using Lemma 7 with $\sigma = \frac{\alpha}{2}$, Lemma 8 with $p = \frac{\alpha}{4\sigma_0} > 1$ and (31) lead to

$$\begin{aligned} \|U^n\|_{l_h^4}^4 & \leq C \|U^n\|_{H^{\frac{\alpha}{2}}}^{\frac{8\sigma_0}{\alpha}} \|U^n\|^{4-\frac{8\sigma_0}{\alpha}} \\ & \leq C \left(\epsilon \|U^n\|_{H^{\frac{\alpha}{2}}}^2 + C(\epsilon) \right) \\ & = C \left(\epsilon |U^n|_{H^{\frac{\alpha}{2}}}^2 + \epsilon \|U^n\|^2 + C(\epsilon) \right), \end{aligned} \quad (46)$$

where ϵ is an arbitrary positive constant. Let $\epsilon = \frac{C_1 C_\alpha}{2C\omega}$, and considering (39)–(46), we have

$$\omega C_1 C_\alpha |U^n|_{H^{\frac{\alpha}{2}}}^2 + C_2 \|\delta_x \Phi^n\|^2 + \frac{1}{2} \|V^n\|^2 + \gamma C_3 \|\delta_x V^n\|^2 \leq C,$$

which implies that

$$|U^n|_{H^{\frac{\alpha}{2}}}^2 \leq C, \quad \|\delta_x \Phi^n\|^2 \leq C, \quad \|V^n\|^2 \leq C, \quad \|\delta_x V^n\|^2 \leq C. \quad (47)$$

It follows from Lemmas 9 and 10 that

$$\|U^n\|_{l_h^\infty}^2 \leq C, \quad \|V^n\|_{l_h^\infty}^2 \leq C.$$

This completes the proof. $\square$

3.3. Solvability

In this section, we discuss the solvability of the finite difference scheme (26)–(30).

Theorem 3. If $\omega > 0$, and $F(\cdot) \geq 0$, then the difference solution of the conservative difference scheme (26)–(30) exists.

Proof. Define $a = (a_1, a_2, \dots, a_{J-1})^T$, $b = (b_1, b_2, \dots, b_{J-1})^T$, $c = (c_1, c_2, \dots, c_{J-1})$, and $z = (a^T, b^T, c^T)^T$; then, z is a $3(J-1)$ -dimensional vector or a point of $3(J-1)$ -dimensional Euclidean space $R^{3(J-1)}$. Now, we use the Schauder fixed point to prove the existence of the solutions for the finite difference scheme (26)–(30). For this purpose, we construct a mapping $T_\lambda : R^{3(J-1)} \rightarrow R^{3(J-1)}$ of the $3(J-1)$ -dimensional Euclidean space into itself, with a parameter $\lambda \in (0, 1)$

$$a = U^n + i \frac{\lambda \tau}{2} G_1 \delta_x^\alpha (a + U^n) - i \frac{\lambda \tau}{4} (a + U^n)(b + V^n), \quad (48)$$

$$b = V^n + \frac{\lambda \tau}{2} G_2 \delta_x^2 (c + \Phi^n), \quad (49)$$

$$c = \Phi^n + \frac{\lambda \tau}{2} (b + V^n) - \frac{\lambda \gamma \tau}{2} G_2 \delta_x^2 (b + V^n) + \lambda \tau \frac{F(b) - F(V^n)}{b - V^n} + \frac{\lambda \omega \tau}{2} (|a|^2 + |U^n|^2). \quad (50)$$

Obviously, the mapping $T_\lambda(z)$ defined here is continuous and there is a fixed point

$$z_0 = (U_1^n, U_2^n, \dots, U_{J-1}^n, V_1^n, V_2^n, \dots, V_{J-1}^n, \Phi_1^n, \Phi_2^n, \dots, \Phi_{J-1}^n)^T \in \mathbb{R}^{3(J-1)},$$

satisfying $T_0(z_0) = z_0$. Now, we prove the boundedness of all the possible solutions to the mapping.

Making the inner product of (48) with $(a + U^n)$ and taking the real part, by virtue of Lemma 4, we obtain

$$\|a\|^2 = \|U^n\|^2 = C. \quad (51)$$

and thus a is uniformly bounded. Now, we prove the boundedness of b and c .

Computing the inner product of (49) and (50) with $\gamma(b + V^n)$ and $(c + \Phi^n)$, respectively, we obtain

$$\gamma(\|b\|^2 - \|V^n\|^2) = \frac{\gamma \lambda \tau}{2} \langle G_2 \delta_x^2 (c + \Phi^n), b + V^n \rangle, \quad (52)$$

$$\begin{aligned} \|c\|^2 - \|\Phi^n\|^2 &= \frac{\lambda \tau}{2} \langle b + V^n, c + \Phi^n \rangle - \frac{\lambda \gamma \tau}{2} \langle G_2 \delta_x^2 (b + V^n), c + \Phi^n \rangle \\ &\quad + \lambda \tau \left\langle \frac{F(b) - F(V^n)}{b - V^n}, c + \Phi^n \right\rangle + \frac{\lambda \omega \tau}{2} \langle |a|^2 + |U^n|^2, c + \Phi^n \rangle. \end{aligned} \quad (53)$$

The addition of (52) to (53) yields

$$(\gamma\|b\|^2 + \|c\|^2) - (\gamma\|V^n\|^2 + \|\Phi^n\|^2) = I_1 + I_2 + I_3. \quad (54)$$

Using Young's inequality Lemma 6 with $p = q = 2$ and $\epsilon = 1$, we have

$$I_1 = \frac{\lambda \tau}{2} \langle b + V^n, c + \Phi^n \rangle \leq \frac{\lambda \tau}{2} (\|b\|^2 + \|V^n\|^2 + \|c\|^2 + \|\Phi^n\|^2). \quad (55)$$

$$I_3 = \frac{\lambda \omega \tau}{2} \langle |a|^2 + |U^n|^2, c + \Phi^n \rangle \leq \frac{\lambda \omega \tau}{2} (\|a\|_{l_h^4}^4 + \|U^n\|_{l_h^4}^4 + \|c\|^2 + \|\Phi^n\|^2). \quad (56)$$

Applying Taylor's Theorem, Young's inequality Lemma 6 with $p = q = 2$ and $\epsilon = 1$, and Theorem 2, we obtain

$$\begin{aligned}
 I_2 &= \lambda \tau \left\langle \frac{F(b) - F(V^n)}{b - V^n}, c + \Phi^n \right\rangle \\
 &\leq \lambda \tau h \sum_{j=1}^{J-1} \left| \frac{F(b_j) - F(V_j^n)}{b_j - V_j^n} \right| |c_j + \Phi_j^n| \\
 &\leq \lambda \tau h \sum_{j=1}^{J-1} (C|c_j| + C|\Phi_j^n|) \\
 &\leq \lambda \tau h \sum_{j=1}^{J-1} \left(C^2 + \frac{1}{2}|c_j|^2 + \frac{1}{2}|\Phi_j^n|^2 \right) \\
 &\leq \lambda \tau (x_R - x_L) C^2 + \frac{\lambda \tau}{2} (\|c\|^2 + \|\Phi^n\|^2).
 \end{aligned} \tag{57}$$

By substituting (55)–(57) into (54), we have

$$\begin{aligned}
 &(\gamma \|b\|^2 + \|c\|^2) - (\gamma \|V^n\|^2 + \|\Phi^n\|^2) \\
 &\leq \frac{\lambda \tau}{2} (\|b\|^2 + \|V^n\|^2 + \|c\|^2 + \|\Phi^n\|^2) + \frac{\lambda \omega \tau}{2} (\|a\|_h^4 + \|U^n\|_h^4 + \|c\|^2 + \|\Phi^n\|^2) \\
 &\quad + \lambda \tau (x_R - x_L) C^2 + \frac{\lambda \tau}{2} (\|c\|^2 + \|\Phi^n\|^2) \\
 &\leq \frac{\lambda \tau}{2} (\|b\|^2 + \|V^n\|^2) + \frac{\lambda \tau (2 + \omega)}{2} (\|c\|^2 + \|\Phi^n\|^2) \\
 &\quad + \frac{\lambda \omega \tau}{2} (\|a\|_h^4 + \|U^n\|_h^4) + \lambda \tau (x_R - x_L) C^2.
 \end{aligned} \tag{58}$$

It follows from Theorem 2 that we have

$$\gamma \|b\|^2 + \|c\|^2 \leq \frac{\lambda \tau}{2} \|b\|^2 + \frac{\lambda \tau (2 + \omega)}{2} \|c\|^2 + \frac{\lambda \omega \tau}{2} \|a\|_h^4 + C. \tag{59}$$

If τ is sufficiently small, we have

$$\frac{1}{2} (\gamma \|b\|^2 + \|c\|^2) \leq \frac{\lambda \omega \tau}{2} \|a\|_h^4 + C.$$

According to the definition of the norm $\|\cdot\|_{l_h^p}$ and (51), we can obtain the following estimates:

$$\|a\|_h^4 \leq \|a\|^2 \leq C.$$

This implies that

$$\frac{1}{2} (\gamma \|b\|^2 + \|c\|^2) \leq C. \tag{60}$$

Obviously, (51) and (60) imply that $\|a\|$, $\|b\|$, and $\|c\|$ are uniformly bounded. Thus, $\|z\|$ is uniformly bounded. It follows from the Schauder fixed-point theorem [43] that the conclusion of Theorem 3 holds. This completes the proof. $\square$

3.4. Convergence

In this subsection, we will first introduce two important Lemmas and then prove the convergence of the conservative scheme.

Lemma 11 ([43]). Suppose that $g(x) \in C^2[d_1, d_2]$ and $a_1, a_2, b_1, b_2 \in [d_1, d_2]$; then, $\theta \in (-1, 1)$ and $\eta \in [d_1, d_2]$ exist such that

$$\frac{g(a_2) - g(a_1)}{a_2 - a_1} - \frac{g(b_2) - g(b_1)}{b_2 - b_1} = g^{(2)}(\eta) \left(\frac{1 - \theta}{2} (a_1 - b_1) + \frac{1 + \theta}{2} (a_2 - b_2) \right).$$

Lemma 12 ([43]). Suppose that the discrete time sequence $\{w^n | n = 0, 1, \dots, N; N\tau = T\}$ satisfies the recurrence formula

$$w^n - w^{n-1} \leq A\tau w^n + B\tau w^{n-1} + C_n\tau,$$

where A , B , and $C_n (n = 1, 2, \dots, N)$ are nonnegative constants. Then,

$$\max_{1 \leq n \leq N} |w^n| \leq (w^0 + \tau \sum_{l=1}^N C_l) \exp(2(A+B)T),$$

where τ is sufficiently small, such that $(A+B)\tau \leq \frac{N-1}{2N}$, $N \geq 2$.

From the construction of the scheme, we obtain the following result.

Theorem 4. Suppose that $u(x, t)$, $v(x, t)$, $\phi(x, t)$ are the sufficiently smooth solutions to the problem (3)–(7). Then, the truncation errors of scheme (26)–(30) satisfy

$$|r^n| + |s^n| + |d^n| \leq C(\tau^2 + h^4), \quad n \geq 0.$$

Proof. Replacing U^n , V^n , and Φ^n by u^n , v^n , and ϕ^n in (26)–(30), respectively. Using Taylor expansion, we obtain

$$i\delta_t u^n + G_1 \delta_x^\alpha u^{n+1/2} - u^{n+1/2} v^{n+1/2} = r^n, \quad 0 \leq n \leq N-1, \quad (61)$$

$$\delta_t v^n = G_2 \delta_x^2 \phi^{n+1/2} + s^n, \quad 0 \leq n \leq N-1, \quad (62)$$

$$\begin{aligned} \delta_t \phi^n &= v^{n+1/2} - \gamma G_2 \delta_x^2 v^{n+1/2} + \frac{F(v^{n+1}) - F(v^n)}{v^{n+1} - v^n} \\ &\quad + \frac{\omega}{2} (|u^{n+1}|^2 + |u^n|^2) + d^n, \quad 0 \leq n \leq N-1, \end{aligned} \quad (63)$$

$$u^0 = u_0, \quad v^0 = v_0, \quad \phi^0 = \phi_0, \quad (64)$$

$$u_0^n = u_j^n = 0, \quad v_0^n = v_j^n = 0, \quad \phi_0^n = \phi_j^n = 0, \quad 0 \leq n \leq N, \quad (65)$$

where

$$|r^n| + |s^n| + |d^n| \leq C(\tau^2 + h^4), \quad n \geq 0.$$

We directly obtain the result. $\square$

We present the error estimation of the conservative difference scheme (26)–(30) in the following theorem. Readers can refer to [23,24,32,33] for the same style of analysis method regarding the proof of convergence.

Theorem 5. Suppose that $u(x, t)$, $v(x, t)$, and $\phi(x, t)$ are the sufficiently smooth solutions to the problem (3)–(7). Then, the solutions U_j^n , V_j^n , and Φ_j^n of the scheme (26)–(30) converges to the solutions u_j^n , v_j^n , and ϕ_j^n of the problem (3)–(7) with order $O(\tau^2 + h^4)$.

Proof. Define the errors

$$e_j^n = U_j^n - u_j^n, \quad g_j^n = V_j^n - v_j^n, \quad \rho_j^n = \Phi_j^n - \phi_j^n, \quad j = 0, 1, \dots, J-1, \quad n = 0, 1, \dots, N.$$

The corresponding vectors are defined as

$$e^n = U^n - u^n, \quad g^n = V^n - v^n, \quad \rho^n = \Phi^n - \phi^n, \quad n = 0, 1, \dots, N.$$

By subtracting (26)–(30) from (61)–(65), respectively, we can obtain the following error equations:

$$i\delta_t e^n + G_1 \delta_x^\alpha e^{n+1/2} - (e^{n+1/2} V^{n+1/2} + u^{n+1/2} g^{n+1/2}) = r^n, \quad (66)$$

$$\delta_t g^n - G_2 \delta_x^2 \rho^{n+1/2} = s^n, \quad (67)$$

$$\delta_t \rho^n = g^{n+1/2} - \gamma G_2 \delta_x^2 g^{n+1/2} + \Theta^n + \frac{\omega}{2} (e^{n+1} \bar{U}^{n+1} + u^{n+1} \bar{e}^{n+1} + e^n \bar{U}^n + u^n \bar{e}^n) + d^n, \quad (68)$$

$$e_j^0 = 0, g_j^0 = 0, \rho_j^0 = 0, \quad (69)$$

$$e_0^n = e_J^n = 0, g_0^n = g_J^n = 0, \rho_0^n = \rho_J^n = 0, 0 \leq n \leq N, \quad (70)$$

where

$$\Theta^n = \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n} - \frac{F(v^{n+1}) - F(v^n)}{v^{n+1} - v^n}.$$

By computing the inner product of (66) with $2\tau e^{n+1/2}$ and taking the imaginary part, we obtain

$$\begin{aligned} & \operatorname{Im} \langle i\delta_t e^n, 2\tau e^{n+1/2} \rangle + \operatorname{Im} \langle G_1 \delta_x^\alpha e^{n+1/2}, 2\tau e^{n+1/2} \rangle \\ & - \operatorname{Im} \langle e^{n+1/2} V^{n+1/2} + u^{n+1/2} g^{n+1/2}, 2\tau e^{n+1/2} \rangle \\ & = \operatorname{Im} \langle r^n, 2\tau e^{n+1/2} \rangle. \end{aligned}$$

According to Lemma 4 and Cauchy–Schwarz inequality, we have

$$\begin{aligned} \operatorname{Im} \langle i\delta_t e^n, 2\tau e^{n+1/2} \rangle &= \|e^{n+1}\|^2 - \|e^n\|^2, \\ \operatorname{Im} \langle G_1 \delta_x^\alpha e^{n+1/2}, 2\tau e^{n+1/2} \rangle &= 0, \\ \operatorname{Im} \langle r^n, 2\tau e^{n+1/2} \rangle &\leq C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|r^n\|^2), \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Im} \langle (e^{n+1/2} V^{n+1/2} + u^{n+1/2} g^{n+1/2}), 2\tau e^{n+1/2} \rangle \\ &= \frac{\tau}{4} \operatorname{Im} \langle (e^{n+1} + e^n)(V^{n+1} + V^n), e^{n+1} + e^n \rangle + \frac{\tau}{4} \operatorname{Im} \langle (u^{n+1} + u^n)(g^{n+1} + g^n), e^{n+1} + e^n \rangle \\ &= \frac{\tau}{4} \operatorname{Im} \langle (u^{n+1} + u^n)(g^{n+1} + g^n), e^{n+1} + e^n \rangle \\ &\leq C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|g^{n+1}\|^2 + \|g^n\|^2). \end{aligned}$$

Thus,

$$\|e^{n+1}\|^2 - \|e^n\|^2 \leq C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|g^{n+1}\|^2 + \|g^n\|^2) + C\tau (\tau^2 + h^4)^2. \quad (71)$$

Computing the inner product of (57) with $2\tau \gamma g^{n+1/2}$ can yield

$$\gamma (\|g^{n+1}\|^2 - \|g^n\|^2) = \langle G_2 \delta_x^2 \rho^{n+1/2}, 2\tau \gamma g^{n+1/2} \rangle + \langle s^n, 2\tau \gamma g^{n+1/2} \rangle. \quad (72)$$

By computing the inner product (58) with $2\tau \rho^{n+1/2}$, we have

$$\begin{aligned} \|\rho^{n+1}\|^2 - \|\rho^n\|^2 &= \langle g^{n+1/2}, 2\tau \rho^{n+1/2} \rangle - \langle \gamma G_2 \delta_x^2 g^{n+1/2}, 2\tau \rho^{n+1/2} \rangle \\ &+ \langle \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n} - \frac{F(v^{n+1}) - F(v^n)}{v^{n+1} - v^n}, 2\tau \rho^{n+1/2} \rangle \\ &+ \omega \langle (e^{n+1} \bar{U}^{n+1} + u^{n+1} \bar{e}^{n+1} + e^n \bar{U}^n + u^n \bar{e}^n), \tau \rho^{n+1/2} \rangle \\ &+ \langle d^n, 2\tau \rho^{n+1/2} \rangle. \end{aligned} \quad (73)$$

Denote

$$B^n = \|e^n\|^2 + \gamma \|g^n\|^2 + \|\rho^n\|^2, \quad (74)$$

and $\gamma_0 = \min\{1, \gamma\}$, we have

$$B^n \geq \gamma_0 (\|e^n\|^2 + \|g^n\|^2 + \|\rho^n\|^2). \quad (75)$$

Adding (71)–(73) yields

$$B^{n+1} - B^n \leq I_1 + I_2 + I_3 + I_4. \quad (76)$$

Then, by using Cauchy–Schwarz inequality, we have

$$\begin{aligned} I_1 &= \langle g^{n+1/2}, 2\tau \rho^{n+1/2} \rangle \\ &= \frac{\tau}{2} (\langle g^{n+1}, \rho^{n+1} \rangle + \langle g^{n+1}, \rho^n \rangle + \langle g^n, \rho^{n+1} \rangle + \langle g^n, \rho^n \rangle) \\ &\leq \frac{\tau}{2} (\|g^{n+1}\|^2 + \|g^n\|^2 + \|\rho^{n+1}\|^2 + \|\rho^n\|^2). \end{aligned} \quad (77)$$

It follows from Lemma 11 and the Cauchy–Schwarz inequality that

$$\begin{aligned} I_2 &= \left\langle \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n} - \frac{F(v^{n+1}) - F(v^n)}{v^{n+1} - v^n}, 2\tau \rho^{n+1/2} \right\rangle \\ &\leq C\tau \langle g^{n+1} + g^n, \rho^{n+1} + \rho^n \rangle \\ &\leq C\tau (\|g^{n+1}\|^2 + \|g^n\|^2 + \|\rho^{n+1}\|^2 + \|\rho^n\|^2). \end{aligned} \quad (78)$$

By using Theorem 2 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} I_3 &= \omega \langle (e^{n+1} \bar{U}^{n+1} + u^{n+1} \bar{e}^{n+1} + e^n \bar{U}^n + u^n \bar{e}^n), \tau \rho^{n+1/2} \rangle \\ &\leq C\omega \tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|\rho^{n+1}\|^2 + \|\rho^n\|^2), \end{aligned} \quad (79)$$

and

$$\begin{aligned} I_4 &= C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|g^{n+1}\|^2 + \|g^n\|^2) + C\tau (\tau^2 + h^4)^2 \\ &\quad + \langle s^n, 2\tau \gamma g^{n+1/2} \rangle + \langle d^n, 2\tau \rho^{n+1/2} \rangle \\ &\leq C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|g^{n+1}\|^2 + \|g^n\|^2 + \|\rho^{n+1}\|^2 + \|\rho^n\|^2) + C\tau (\tau^2 + h^4)^2. \end{aligned} \quad (80)$$

By substituting (77)–(80) into (76), we have

$$\begin{aligned} B^{n+1} - B^n &\leq C\tau (\|e^{n+1}\|^2 + \|e^n\|^2 + \|g^{n+1}\|^2 + \|g^n\|^2 + \|\rho^{n+1}\|^2 + \|\rho^n\|^2) + C\tau (\tau^2 + h^4)^2 \\ &\leq C\tau (B^{n+1} + B^n) + C\tau (\tau^2 + h^4)^2. \end{aligned}$$

Using Lemma 12, it follows that

$$B^n \leq e^{2CT} (B^0 + CT(\tau^2 + h^4)^2), \quad 0 \leq n \leq N, \quad (81)$$

for sufficiently small τ . We observe from (69), (75) and (81) that

$$\|e^n\|^2 + \|g^n\|^2 + \|\rho^n\|^2 \leq \frac{e^{2CT}}{\gamma_0} (CT(\tau^2 + h^4)^2), \quad 0 \leq n \leq N. \quad (82)$$

It follows from (82) that

$$\|e^n\| \leq C(\tau^2 + h^4), \quad \|g^n\| \leq C(\tau^2 + h^4), \quad \|\rho^n\| \leq C(\tau^2 + h^4).$$

This completes the proof. $\square$

4. Numerical Experiments

In this section, we will conduct multiple numerical experiments to validate the theoretical findings. The conservative scheme (26)–(30) can be implemented by the Algorithm 1:

Algorithm 1: The conservative scheme (26)–(30) of the FCSBS

- 1 **Given:** U^n, V^n and Φ^n .
 - 2 **Step 1:** Solve V^{n+1} and Φ^{n+1} from (27) and (28).
 - 3 **Step 2:** Solve U^{n+1} from (26).
-

When $\alpha = 2$, $f(v) = \theta v^2$, $3\gamma \neq \theta$, and $4\gamma b_1 \neq d_1$, the FCSBS (3)–(7) has the exact solitary wave solutions

$$\begin{cases} u(x, t) = \sqrt{\frac{6\gamma b_1}{\theta\omega}} (d_1 - 4\gamma b_1) \operatorname{sech}(\mu\xi) e^{i(\frac{M}{2}x + \delta t)}, \\ v(x, t) = -2b_1 \operatorname{sech}^2(\mu\xi), \\ \phi(x, t) = \frac{4Mb_1}{\mu} \left(\frac{x_r - x}{x_r - x_l} - \frac{1}{1 + e^{2\mu\xi}} \right), \quad x \in [x_l, x_r], t \geq 0. \end{cases}$$

Here, $b_1 = \delta + \frac{M^2}{4}$, $d_1 = 1 - M^2$, $\mu = \sqrt{b_1}$, $\xi = x - Mt$, and M, δ are free parameters. In the following simulations, the solutions at $t = 0$ are taken as the initial conditions, and the parameters are chosen as follows:

$$\gamma = 1/2, \theta = 3/2, \omega = 1/12, M = \sqrt{1/3}, \delta = 1/5.$$

Furthermore, we ensure that the computational domain $x \in [x_l, x_r]$ is sufficiently large to minimize errors introduced by the boundary conditions relative to the entire spatial domain.

Firstly, we measure the accuracy of the proposed scheme by computing errors and convergence rates through

$$\begin{aligned} E_1(h, \tau) &= \|U^n - u(\cdot, n\tau)\|, \quad R1_\tau = \log_2 \left(\frac{E_1(h, 2\tau)}{E_1(h, \tau)} \right), \quad R1_h = \log_2 \left(\frac{E_1(2h, \tau)}{E_1(h, \tau)} \right), \\ E_2(h, \tau) &= \|V^n - v(\cdot, n\tau)\|, \quad R2_\tau = \log_2 \left(\frac{E_2(h, 2\tau)}{E_2(h, \tau)} \right), \quad R2_h = \log_2 \left(\frac{E_2(2h, \tau)}{E_2(h, \tau)} \right). \end{aligned}$$

Since exact solutions for $\alpha \in (1, 2)$ are not available, we obtain reference “exact” solutions U and V using the derived scheme with $h = 1/80$ and $\tau = 1/160$. Additionally, we set $-x_l = x_r = 40$ and $T = 2$. Tables 1 and 2 demonstrate that the numerical scheme (26)–(30) achieves second-order accuracy in time and fourth-order accuracy in space for numerical solutions U^n and V^n with $\alpha = 2$. Table 3 presents the l_h^2 -norm errors and convergence orders for different $\alpha \in (1, 2)$. The table indicates that the scheme is second-order accurate in time. Next, we maintain a fixed time step of $\tau = 1/160$ to assess spatial errors and convergence orders for varying $\alpha \in (1, 2)$, with the results summarized in Table 4. It is evident that the scheme achieves fourth-order accuracy in space. These convergence findings align with the theoretical expectations.

Table 1. Errors and temporal convergence rates at $T = 2$ with $h = 1/100$ and $\alpha = 2$.

τ	$E_1(h, \tau)$	$R1_\tau$	$E_2(h, \tau)$	$R2_\tau$
0.2	1.9023×10^{-3}	-	3.0730×10^{-4}	-
0.1	4.6269×10^{-4}	2.0397	7.8856×10^{-5}	1.9624
0.05	1.1443×10^{-4}	2.0156	1.9945×10^{-5}	1.9832
0.025	2.8480×10^{-5}	2.0064	5.0138×10^{-6}	1.9920

Table 2. Errors and spatial convergence rates at $T = 2$ with $\tau = 1/10,000$ and $\alpha = 2$.

h	$E_1(h, \tau)$	$R1_h$	$E_2(h, \tau)$	$R2_h$
0.8	1.3701×10^{-3}	-	2.4057×10^{-3}	-
0.4	8.3255×10^{-5}	4.0406	1.1953×10^{-4}	4.3310
0.2	5.1080×10^{-6}	4.0267	7.5222×10^{-6}	3.9901
0.1	3.2168×10^{-7}	3.9891	4.7002×10^{-7}	4.0004

Table 3. Errors and temporal convergence orders at $T = 2$ with $h = 1/80$ and $1 < \alpha < 2$.

α	τ	$E_1(h, \tau)$	$R1_\tau$	$E_2(h, \tau)$	$R2_\tau$
1.2	0.4	1.0515×10^{-2}	-	1.4519×10^{-3}	-
	0.2	2.6501×10^{-3}	1.9883	3.8539×10^{-4}	1.9135
	0.1	6.6320×10^{-4}	1.9985	9.9573×10^{-5}	1.9525
	0.05	1.6427×10^{-4}	2.0134	2.4988×10^{-5}	1.9945
1.5	0.4	1.0853×10^{-2}	-	1.8775×10^{-3}	-
	0.2	2.7038×10^{-3}	2.0051	5.0057×10^{-4}	1.9072
	0.1	6.5112×10^{-4}	2.0540	1.2739×10^{-4}	1.9744
	0.05	1.5851×10^{-4}	2.0384	3.1749×10^{-5}	2.0044
1.9	0.4	8.4791×10^{-3}	-	2.1617×10^{-3}	-
	0.2	1.8849×10^{-3}	2.1694	5.5049×10^{-4}	1.9734
	0.1	4.5930×10^{-4}	2.0370	1.3870×10^{-4}	1.9888
	0.05	1.1236×10^{-4}	2.0313	3.4447×10^{-5}	2.0095

Table 4. Errors and spatial convergence orders at $T = 2$ with $\tau = 1/160$ and $1 < \alpha < 2$.

α	h	$E_1(h, \tau)$	$R1_h$	$E_2(h, \tau)$	$R2_h$
1.2	0.8	6.7973×10^{-3}	-	2.7931×10^{-3}	-
	0.4	3.5251×10^{-4}	4.2692	1.4894×10^{-4}	4.2291
	0.2	2.3469×10^{-5}	3.9088	9.1033×10^{-6}	4.0322
1.5	0.8	4.4944×10^{-3}	-	2.7949×10^{-3}	-
	0.4	2.2684×10^{-4}	4.3084	1.5832×10^{-4}	4.1419
	0.2	1.4070×10^{-5}	4.0109	9.6993×10^{-6}	4.0288
1.9	0.8	1.9728×10^{-3}	-	2.6786×10^{-3}	-
	0.4	1.2055×10^{-4}	4.0326	1.5358×10^{-4}	4.1244
	0.2	7.4847×10^{-6}	4.0095	9.4378×10^{-6}	4.0244

Secondly, we compute the discrete conservation laws. For this test, we set $h = 1/2$, $\tau = 1/100$, $-x_l = x_r = 200$, and $T = 30$, leaving α as the only free parameter. Tables 5–7 present the Langmuir plasmon number I_U^n , the total perturbed number density I_V^n , and the total energy I_E^n , respectively, where

$$I_U^n = \|U^n\|^2, \quad I_V^n = h \sum_{j=1}^{J-1} V_j^n,$$

$$I_E^n = -2\omega \langle G_1 \delta_x^\alpha U^n, U^n \rangle - \langle G_2 \delta_x^2 \Phi^n, \Phi^n \rangle + \|V^n\|^2 - \langle \gamma G_2 \delta_x^2 V^n, V^n \rangle + 2\langle F(V^n), I \rangle + 2\omega \langle |U^n|^2, V^n \rangle.$$

It is observed that schemes (26)–(30) effectively preserve these quantities, making them suitable for long-term simulation. Notably, the Langmuir plasmon number and the total perturbed number density remain unaffected by α , while the total energy varies selectively with α . To enhance conservation accuracy, a smaller iteration tolerance can be applied, albeit at the expense of increased computational cost.

Table 5. The values of I_U^n at different times with $\tau = 1/100$ and $h = 1/2$.

	$\alpha = 2$	$\alpha = 1.5$	$\alpha = 1.2$
$t = 0$	2.554995107627433	2.554995107627433	2.554995107627433
$t = 2$	2.554995107626755	2.554995107629729	2.554995107640402
$t = 4$	2.554995107626069	2.554995107632120	2.554995107656555
$t = 6$	2.554995107625393	2.554995107634486	2.554995107675087
$t = 8$	2.554995107624756	2.554995107636796	2.554995107695591
$t = 10$	2.554995107624175	2.554995107639026	2.554995107716158

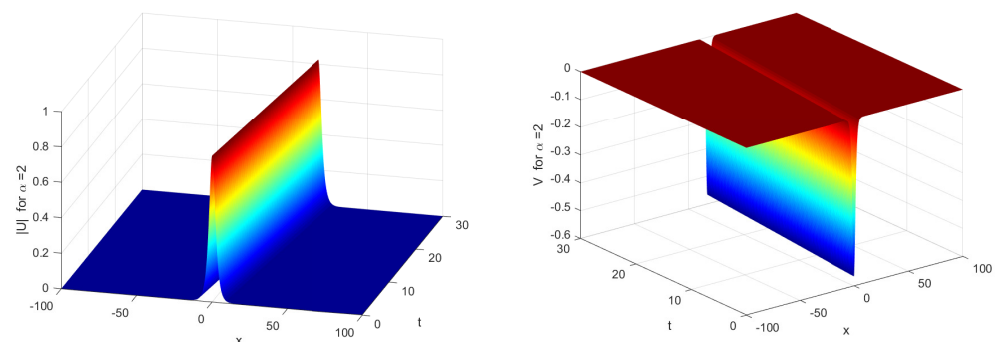
Table 6. The values of I_V^n at different times with $\tau = 1/100$ and $h = 1/2$.

	$\alpha = 2$	$\alpha = 1.5$	$\alpha = 1.2$
$t = 0$	−2.129162589689532	−2.129162589689532	−2.129162589689532
$t = 2$	−2.129162589689426	−2.129162589689919	−2.129162589705912
$t = 4$	−2.129162589689318	−2.129162589694141	−2.129162589842022
$t = 6$	−2.129162589689236	−2.129162589706523	−2.129162590245636
$t = 8$	−2.129162589689183	−2.129162589730667	−2.129162591086285
$t = 10$	−2.129162589689132	−2.129162589768579	−2.129162592542394

Table 7. The values of I_E^n at different times with $\tau = 1/100$ and $h = 1/2$.

	$\alpha = 2$	$\alpha = 1.5$	$\alpha = 1.2$
$t = 0$	0.709988640473137	0.736137911989638	0.760042303573453
$t = 2$	0.709988638543045	0.736137992708215	0.760042445633956
$t = 4$	0.709988636400984	0.736137952537480	0.760042483756246
$t = 6$	0.709988634397767	0.736137815839193	0.760042226275524
$t = 8$	0.709988632497735	0.736137699207129	0.760041902905055
$t = 10$	0.709988630697259	0.736137652809705	0.760041732381828

Next, we simulate the solitary wave solution with $x \in [-100, 100]$, $t \in [0, 30]$, $h = 1/10$, and $\tau = 1/2$. Figures 1–3 depict the waveforms for $|U|$ and V at varying α values. These figures demonstrate that variations in the order α directly impact the shapes of the solitons. As α decreases, small oscillations are observed near the solitary wave of $|U|$ and V , with the amplitudes of these oscillations in V being larger than those in $|U|$. These characteristics mirror the numerical simulations of the space fractional Schrödinger system, which are utilized in physics to alter waveforms without altering nonlinearity and dispersion effects.


Figure 1. The wave forms of the numerical solution for $|U|$ and V with $\alpha = 2$.

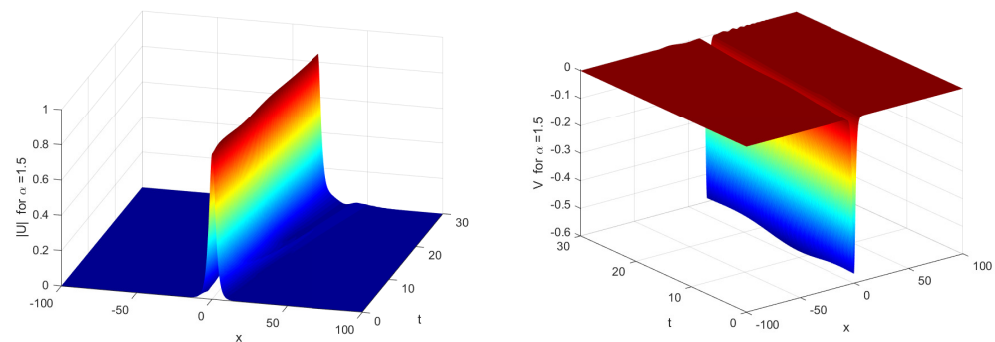


Figure 2. The wave forms of the numerical solution for $|U|$ and V with $\alpha = 1.5$.

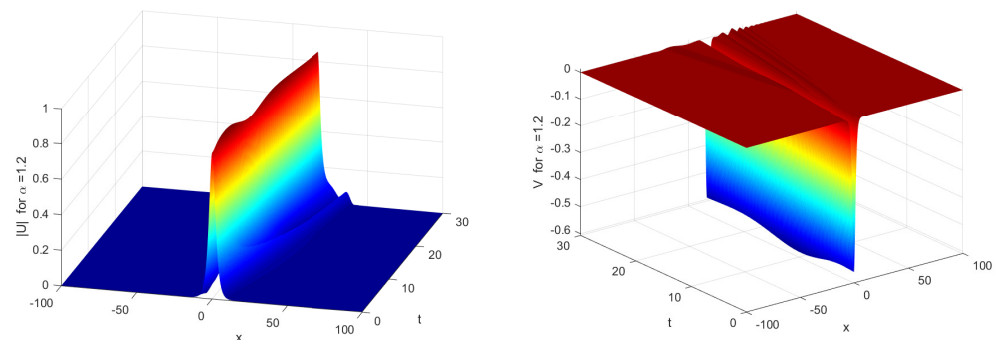


Figure 3. The wave forms of the numerical solution for $|U|$ and V with $\alpha = 1.2$.

Finally, we present a numerical comparison between our scheme (26)–(30) and Scheme I from [36]. Table 8 lists the errors and computational times for both schemes with different values of α . Clearly, our scheme provides a more accurate solution than Scheme I in [36], with only a slightly higher computational time cost.

Table 8. The comparison of errors and CPU time at $T = 2$ with $\tau = 1/40$ and $h = 1/4$.

α	Scheme	$E_1(h, \tau)$	$E_2(h, \tau)$	CPU Time(s)
1.2	Scheme I [36]	4.0005×10^{-3}	2.6840×10^{-3}	1.9330
	Scheme (26)–(30)	6.8842×10^{-5}	3.0626×10^{-5}	2.0220
1.5	Scheme I [36]	3.3001×10^{-3}	2.8382×10^{-3}	2.0680
	Scheme (26)–(30)	4.5795×10^{-5}	3.4216×10^{-5}	2.1207
1.9	Scheme I [36]	2.6803×10^{-3}	2.8846×10^{-3}	2.0837
	Scheme (26)–(30)	1.9252×10^{-5}	3.4442×10^{-5}	2.1573

5. Conclusions

In this paper, based on the compact difference approach, we derived a novel conservative numerical algorithm for solving the space fractional coupled Schrödinger–Boussinesq system. The conservative property, boundedness, solvability, and convergence of the numerical solution are evidenced. Finally, numerical experiments for different fractional-order α illustrated that the derived algorithm can guarantee conservation and convergent with order $O(\tau^2 + h^4)$. The results presented in this paper are applicable for numerical solutions to the classical coupled Schrödinger–Boussinesq system. In further work, we will try to discuss more complex and higher-dimensional fractional partial differential equations.

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