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A Characterization of Normal Injective and Normal Projective Hypermodules

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Abstract: This study is motivated by the recently published papers on normal injective and normal projective hypermodules. We provide a new characterization of the normal injective and normal projective hypermodules by using the splitting of the short exact sequences of hypermodules. After presenting some of their fundamental properties, we show that if a hypermodule is normal projective, then every exact sequence ending with it is splitting. Moreover, if a hypermodule is normal injective, then every exact sequence starting with it is splitting as well. Finally, we investigate the relationships between semisimple, simple, normal injective, and normal projective hypermodules.

Keywords: short exact sequence; normal injective hypermodule; normal projective hypermodule

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1. Introduction

Let H be a nonempty set and $P^*(H)$ be the set of nonempty subsets of H. The map $\circ : H \times H \longrightarrow P^*(H)$ is referred to as *hyperoperation* on H. Thus, for the element $a, b \in H$, $a \circ b$ is not a single element, as the result of a classical operation, yet is a nonempty subset of the set H. Owing to this, the theory of hypercompositional structures (which are algebraic structures having at least one hyperoperation) was presented by Marty [1] as a generalization of the theory of algebraic structures while studying problems in noncommutative algebra. Following [1], Marty introduced hypergroups using the hyperoperation on a set as a first hypercompositional structure. In 1983, Krasner [2] introduced the notion of hyperring, which we now call Krasner hyperring. In that article, Krasner also defined a hypermodule over a hyperring, with the name of the Krasner hypermodule. The contribution of entire these structures is a canonical hypergroup, which has plenty applications in hypercompositional algebra. Additionally, other types of hyperrings and hypermodules were introduced by Rota [3] and Vougiouklis [4], namely, multiplicative hyperrings and generalized hyperrings. In this article, by "hypermodule", we mean Krasner hypermodules.

Massouros studied hypermodules in detail [5], where specific hypermodules were introduced, namely, free and cyclic hypermodules. After that, Massouros investigated more properties and theorems of hypercompositional structures [6,7]. Moreover, the fundamental relationships in hypercompositional structures were studied by Vougiouklis [8]. Recently, researchers have focused on the new aspects of hypercompositional structures and have studied them from different points of view. The category of Krasner hypermodules has been studied [9,10], where researchers investigated the injectivity and projectivity properties of Krasner hypermodules.

In [11], Bordbar et al. focused on a specific type of homomorphism among hypermodules termed normal homomorphisms and, for that reason, on the normal projective; they also gave an equivalent description of these hypermodules via exact chains of Krasner hypermodules and normal epimorphism, while in [12], the supplements in Krasner R hypermodules were investigated in addition to their relationship to projective hypermodules, especially along with normal π-projective hypermodules. Additionally, a novel characterization of normal injective R hypermodules via divisible R hypermodules was studied [13]. In [14], the authors contribute some consequences of normal injective hypermodules with category aspects and define torsion and torsionable elements in hypermodules.

Motivated by the injective modules and projective modules studied with help of short exact sequences in recent years (see [15–19]), in this study, we aimed to acquire some consequences and develop the concept of short exact sequences and their relationship with normal injective and projective hypermodules. After a brief introduction to hypermodules and homomorphisms in hypermodules, in Section 3, we define a type of short exact sequence, namely, the splitting exact sequence, and provide some examples of it. Then, we prove that the canonical hypergroup $\prod_{i\in I} M_i$ is a left $\mathbb{Z}$ hypermodule, where $\{M_i\}_{i\in I}$ is a non empty family of left $\mathbb{Z}$ hypermodules. This theorem is supported by an example. Finally, we show that every exact sequence ending with a normal projective hypermodule must be split. Similarly, every exact sequence starting with a normal injective hypermodule must be split. Then, by splitting short exact sequences, we investigate the relationships between semisimple, simple, normal injective, and normal projective hypermodules as the main theorem in this study.

2. Preliminaries

This part concisely reviews the concepts and consequences pertinent to kinds of hyperrings and hypermodules. To better understand the topic, we begin with some definitions and theorems of hypercompositional algebra introduced in books [20,21] and articles [5–7].

Let H be a nonempty set along with a map and let $+ : H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation on H. Then, $(H, +)$ is denoted as a hypergroupoid. Furthermore, for nonempty subsets X and Y of H, describe

$$X + Y = \{z \in x + y \,|\, x \in X \text{ } and \text{ } y \in Y\} = \bigcup_{x\in X, y\in Y} x + y.$$

We basically write $a + X$ and $X + a$ rather than $\{a\} + X$ and $X + \{a\}$ in sequence for each $a \in H$ and any nonempty subset X of H. A hypergroupoid $(H, +)$ is denoted as a

(1) Semihypergroup if $+$ is associative;
(2) Quasihypergroup if $+$ is reproductive.

Provided that the hypergroupoid $(H, +)$ is a semihypergroup and a quasihypergroup, it is denoted as a hypergroup. A nonempty subset S of a hypergroup $(H, +)$ is termed as a subhypergroup of H if, for every $a \in S$, $a + S = S = S + a$.

A hypergroup $(H, +)$ is called a canonical hypergroup if

(1) $+$ is commutative.
(2) For any $a \in H$, there is a unique element in H, denoted by $-a$, such that $0 \in a + (-a)$.
(3) If $c \in a + b$, then $a \in c + (-b) := c - b$, where $a, b, c \in H$.

Observe from [22] that if $(H, +)$ is a canonical hypergroup, then $a + 0 = a$ for all $a \in H$.

(Krasner) hyperring is a triple $(R, +, .)$ such that

(1) $(R, +)$ is a canonical hypergroup;
(2) $(R, .)$ is a monoid along with a bilaterally absorbing element 0;
(3) The multiplication distributes over the addition on both sides.

A hyperring $(R, +, .)$ is *commutative* provided that $(R, .)$ is commutative.

Let R and $(M, +)$ be a canonical hypergroup. M is said to be a left Krasner R hypermodule if there is a map $R \times M \longrightarrow M$, written $(r, m) \longmapsto rm$, that satisfies the following conditions:

(1) $r(m_1 + m_2) = rm_1 + rm_2$;
(2) $(r + s)m = rm + sm$;
(3) $(r.s)m = r(sm)$;
(4) $1_R m = m$ and $r0_M = 0_R m = 0_M$.

For any m, m_1, $m_2 \in M$ and r, $s \in R$.

For simplicity, in every part of this paper, the term "hypermodule" refers to the left Krasner hypermodule. A nonempty subset N of an R hypermodule M is termed a *subhypermodule* of M, written as $N \leq M$ when under the same hyperoperations of M, N is an R hypermodule. It is obvious that M and $\{0_M\}$ are insignificant subhypermodules of M. Additionally, based on [13], every hyperring R is an R hypermodule.

For a hyperring R, assume that K becomes a subhypermodule of an R hypermodule M. Consider

$$\frac{M}{K} = \{\, a + K \mid a \in M \,\}.$$

Then, $\frac{M}{K}$ is an R hypermodule with the following hyperoperation

$$+ : \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}^*(\frac{M}{K}),$$

and the map $\odot : R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ is defined as $(a + K) + (a' + K) = \{\, b + K \mid b \in a + a' \,\}$ and $r \odot (a + K) = ra + K$ for every $a, a', b \in M$ and $r \in R$. $\frac{M}{K}$ is called a *quotient hypermodule* of the R hypermodule M.

Given two R-= hypermodules M and N, a map $f : M \longrightarrow P^*(N)$ is identified as an R homomorphism on the condition that

(1) $f(m_1 +_M m_2) \subseteq f(m_1) +_N f(m_2)$ for all m_1, $m_2 \in M$;
(2) $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$.

f is termed as a strong homomorphism each time $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$ for every m_1, $m_2 \in M$. A single-valued map $f : M \longrightarrow N$ is called a normal homomorphism if

(1) $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$ for all m_1, $m_2 \in M$;
(2) $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$.

For a normal homomorphism $f : M \longrightarrow N$, the set $\{m \in M | f(m) = 0_N\}$ is termed as the kernel of f and is shown with $Ker(f)$. Note that $Ker(f) \leq M$.

3. Short Exact Sequences and Applications to Normal Injectivity (Projectivity)

Let $0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be a short exact sequence of hypermodules (see [11]). $\mathbb{E}$ denotes the short exact sequence $0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$; that is, $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$.

Definition 1. *Let $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be a short exact sequence of hypermodules. $\mathbb{E}$ is said to be* splitting *if $Ker(\phi) = Im(\psi)$ is a direct summand of N.*

First, let us give the following examples.

Example 1. *For any hypermodule M, we pay attention to the short exact sequence*

$$\mathbb{E} : 0 \longrightarrow 0 \xrightarrow{\iota} M \xrightarrow{I_M} M \longrightarrow 0,$$

where $\iota : 0 \longrightarrow M$ is the inclusion map, and $I_M : M \longrightarrow M$ is the identity map. It is obvious that $\iota(0) = 0 = Ker(I_M)$ is a direct summand of M, and so $\mathbb{E}$ is splitting.

Example 2. *Admit* $R = \{0, 1, 2, 3\}$ *together with hyperoperation "+" and operation ".":*

+	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{3\}$	$\{2,3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{2,3\}$	$\{1\}$	$\{0,1\}$

and

·	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

Therefore, R is an R hypermodule. It is straightforward to see that the only proper subhypermodules of R are $M_0 = \{0\}$, $M_1 = \{0,1\}$, *and* $M_2 = \{0,2\}$. *Therefore, the short exact sequence*

$$\mathbb{E} : 0 \longrightarrow M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2 \longrightarrow 0$$

is splitting.

Example 3. *Given the hyperring R with the subsequent tables:*

+	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$
2	$\{2\}$	$\{1, 3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{0, 2\}$	$\{1\}$	$\{0, 2\}$

and

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Let us indicate the R hypermodule R with M. It follows that $N = \{0, 2\}$ *is a subhypermodule of M. For the inclusion map* $\iota : N \longrightarrow M$ *and the canonical projection* $\pi : M \longrightarrow \frac{M}{N}$, *we take into account of the short exact sequence*

$$\mathbb{E} : 0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} \frac{M}{N} \longrightarrow 0.$$

Owing to the fact that N is small in M, $N = Ker(\pi)$ *is not a direct summand of M (see [23]). It means that* $\mathbb{E}$ *is not splitting.*

Example 4. *For a semisimple hypermodule M, let*

$$\mathbb{E} : 0 \longrightarrow M_1 \xrightarrow{\psi} M \xrightarrow{\phi} M_2 \longrightarrow 0$$

be a short exact sequence of hypermodules. Since M is semisimple, it follows from [24], Theorem 9, that $Ker(\phi) = Im(\psi)$ *is a direct summand of M. From this, the short exact sequence* $\mathbb{E}$ *is splitting.*

Remark 1. *In classical algebra, given any two M and N modules, the trivial splitting short exact sequence* $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\iota} M \oplus N \xrightarrow{\pi} N \longrightarrow 0$ *is written, where* $M \oplus N$ *is the external direct sum of these modules M and N. However, since the external direct sum of two hypermodules cannot be defined, there is no trivial splitting short exact sequence for any two given hypermodules over an arbitrary hyperring (see ([6], Theorem 16)).*

Let $(\mathbb{Z}, +, .)$ be the set of integers and let $(M, +)$ be any canonical hypergroup. For each $m \in M$ and $n \in \mathbb{Z}$, describe the set nm as follows:

$$nm = \begin{cases} m + m + \ldots + m \ (n \ times), & n > 0; \\ 0_M, & n = 0; \\ (-m) + (-m) + \ldots + (-m) \ (n \ times), & n < 0. \end{cases}$$

On the set $\mathbb{Z}$, let "$+$" and "$\cdot$" denote the usual additive and multiplication. We identify a singleton set $\{n\}$ by n for all $n \in \mathbb{Z}$. Then, we can think of the triple $(\mathbb{Z}, +, .)$ as a hyperring, say a *trivial hyperring of integers*; therefore, M has the structure of a left $\mathbb{Z}$-hypermodule together with a map $\mathbb{Z} \times M \longrightarrow M$ via $(n, m) \mapsto nm$ for all $n \in \mathbb{Z}$ and $m \in M$.

For the trivial hyperring $(\mathbb{Z}, +, .)$, let $\{M_i\}_{i \in I}$ be a nonempty family of canonical hypergroups (or $\mathbb{Z}$-hypermodules) and let $\prod_{i \in I} M_i$ be the direct product of the canonical hypergroups M_i.

$$\textstyle\prod_{i \in I} M_i = \{\alpha \mid \alpha : I \longrightarrow \bigcup_{i \in I} M_i \ via \ \alpha(i) \in M_i \ for \ all \ i \in I\}.$$

For every $i \in I$, let us denote $\alpha(i) := m_i$ and $\alpha = (m_i)_{i \in I}$. Here, $(i \in I)$ m_i is called the ith component of α. The function $\alpha = (m_i)_{i \in I} = (m_1, m_2, \ldots, m_i, \ldots)$ in case I is a countable set.

Let $\alpha = (m_i)_{i \in I}$, $\beta = (m'_i)_{i \in I} \in \prod_{i \in I} M_i$. Functioning by the definition of equality, $\alpha = \beta$ if and only if $m_i = m'_i$ for every $i \in I$.

Theorem 1. *Let $\{M_i\}_{i \in I}$ become a nonempty family of left $\mathbb{Z}$-hypermodules, where $(\mathbb{Z}, +, .)$ is the trivial hyperring. Then, the canonical hypergroup $\prod_{i \in I} M_i$ is a left $\mathbb{Z}$ hypermodule with the map $n(m_i)_{i \in I} = (nm_i)_{i \in I}$.*

Proof. From [6], Theorem 16, it is sufficient to show that the axiom (2) of the hypermodule is satisfied. Let $m = (m_i)_{i \in I} \in \{M_i\}_{i \in I}$ and $r, s \in \mathbb{Z}$. Since the set $r + s$ contains the only element $r + s$, then

$$\begin{aligned} (r+s)m &= (r+s)(m_i)_{i \in I} = ((r+s)m_i)_{i \in I} \\ &= (rm_i + sm_i)_{i \in I} \\ &= (rm_i)_{i \in I} + (sm_i)_{i \in I} \\ &= rm + sm. \end{aligned}$$

Hence, the canonical hypergroup $\prod_{i \in I} M_i$ is a left $\mathbb{Z}$ hypermodule. □

We say that $\mathbb{Z}$ hypermodule $\prod_{i \in I} M_i$ is the *direct product* of the family $\{M_i\}_{i \in I}$. Consider the following subset of the $\mathbb{Z}$ hypermodule $\prod_{i \in I} M_i$:

$$\textstyle\bigoplus_{i \in I}^{w} M_i = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid m_i \neq 0_{M_i} \ only \ for \ finitely \ many \ i \in I\}.$$

Since the empty set $\emptyset$ is finite, we have $(0_{M_i})_{i \in I} \in \bigoplus_{i \in I}^{w} M_i \neq \emptyset$. If the union of two finite sets is finite, it can be seen that $\bigoplus_{i \in I}^{w} M_i$ is a subhypermodule of $\prod_{i \in I} M_i$. We say the hypermodule $\bigoplus_{i \in I}^{w} M_i$ is *an external direct sum* of the family $\{M_i\}_{i \in I}$. If the index set $I = \{1, 2, \ldots, n\}$, then $\prod_{i \in I} M_i = \bigoplus_{i \in I}^{w} M_i$ is clear.

Example 5. *For the trivial hyperring $(\mathbb{Z}, +, .)$, let M and K be two $\mathbb{Z}$ hypermodules. Let $N = M \oplus^w K$. Define these sets $M_1 = \{(m, 0_K) | m \in M\}$ and $M_2 = \{(0_M, k) | k \in K\}$. It follows that M_1 and M_2 are subhypermodules of N.*

Next, we show that N is the internal direct sum of subhypermodules M_1 and M_2. Let $a = (m, k) \in N = M \oplus^w K$. Therefore, we can write

$$a = (m, k) = (m, 0_K) + (0_M, k) \in M_1 + M_2$$

and so $M = M_1 + M_2$. For uniqueness, let $a \in (m_1, 0_K) + (0_M + k_1)$ and $a \in (m_2, 0_K) + (0_M + k_2)$, where m_1, $m_2 \in M_1$ and k_1, $k_2 \in K$. Therefore,

$$\begin{array}{rcl}(0_M, 0_K) = 0_N & \in & a - a \\ & = & [(m_1, 0_K) + (0_M + k_1)] - [(m_2, 0_K) + (0_M + k_2)] \\ & = & (m_1 - m_2, 0_K) + (0_M, k_1 - k_2) \\ & = & (m_1 - m_2, k_1 - k_2)\end{array}$$

so $0_M \in m_1 - m_2$ and $0_K \in k_1 - k_2$. Thus, $m_1 = m_2$ and $k_1 = k_2$. It means that the sum $M_1 + M_2$ is direct according to [24], Theorem 1.

For the canonical map $\iota_1 : M \longrightarrow M_1 \subseteq N$ and for the canonical projection $\pi_2 : N \longrightarrow K$, we have that $Im(\iota_1) = M_1 = Ker(\pi_2)$ is a direct summand of N. Hence, the short exact sequence of $\mathbb{Z}$ hypermodules

$$\mathbb{E} : 0 \longrightarrow M \xrightarrow{\iota_1} N \xrightarrow{\pi_2} K \longrightarrow 0.$$

is splitting.

Now let us start by giving the lemma, which has a significant key role in our research.

Lemma 1. *Let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be normal homomorphisms of hypermodules with $g \circ f = I_M$. Then, N has the decomposition $N = Im(f) \oplus Ker(g)$.*

Proof. Let $a \in N$. Then, $g(a - (f \circ g)(a)) = g(a) - ((g \circ f) \circ g)(a) = g(a) - g(a)$ and so $0_N \in g(a) - g(a) = g(a - (f \circ g)(a)) = \bigcup_{x \in a - (f \circ g)(a)} g(x)$. There is an element $x_0 \in a - (f \circ g)(a)$ such that $g(x_0) = 0_N$. Since N is a canonical hypergroup, $x_0 \in a - (f \circ g)(a)$ implies $a \in x_0 + (f \circ g)(a) \subseteq Im(f) + Ker(g)$, and thus the sum $Im(f) + Ker(g)$ is N.

If $f(a) \in Im(f) \cap Ker(g)$ for some element $a \in M$, then $g(f(a)) = 0_N$, and so $a = I_M(a) = (g \circ f)(a) = g(f(a)) = 0$. It follows that $Im(f) \cap Ker(g) = 0_N$. Hence, we obtain that $N = Im(f) \oplus Ker(g)$. □

Corollary 1. *For hypermodules $U \subseteq M$, let $\iota : U \longrightarrow M$ be the inclusion map. Provided that there is a normal homomorphism $g : M \longrightarrow U$ with $g \circ \iota = I_U$, U is a direct summand of M.*

Proof. It follows from Lemma 1. □

Corollary 2. *If $f : M \longrightarrow M$ is an R-normal homomorphism such that $f \circ f = f$, then $M = Im(f) \oplus Ker(f)$.*

Proof. Assume that $f : M \longrightarrow M$ is an R-normal homomorphism such that $f \circ f = f$. Let $m \in M$. From the first part of the proof of Lemma 1, there is an element $m_0 \in Ker(f)$ such that $m_0 \in m - f(m)$. If condition (3) for M, which is a canonical hypergroup, is used again, we obtain $m \in m_0 + f(m) \subseteq Ker(f) + Im(f) = Im(f) + Ker(f)$, so $M = Im(f) + Ker(f)$. From the last part of the proof of Lemma 1, we deduce that $M = Im(f) \oplus Ker(f)$, as required. □

The following theorem characterizes the short exact sequence that is splitting.

Theorem 2. *Let $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be a short exact sequence of R hypermodules. In this case, the subsequent statements are equivalent:*

(1) There exists a normal homomorphism $\varphi : N \longrightarrow M$ with $\varphi\psi = I_M$.
(2) $\mathbb{E}$ splits; that is, $Im(\psi) = Ker(\phi)$ is a direct summand of N.
(3) There exists a normal homomorphism $\lambda : K \longrightarrow N$ with $\phi\lambda = I_K$.

Proof. (1) ⇒ (2) and (3) ⇒ (2) By Lemma 1.

$(2) \Rightarrow (3)$ Let $N = Im(\psi) \oplus B$ for some subhypermodule B of N. Since $Im(\psi) \cap B = Ker(\phi) \cap B = 0$, the restriction function ϕ_B of the ϕ normal homomorphism to the B hypersubmodule is a normal monomorphism. Let $k \in K$. Due to the certainty of $\mathbb{E}$, there is an element $n \in N$ such that $\phi(n) = k$. For this reason, we can to write $n \in \psi(m) + b$, where $m \in M$ and $b \in B$. Therefore, $k = \phi(n) \in \phi(\psi(m) + b) = (\phi\psi)(m) + \phi(b) = 0_K + \phi(b) = \phi(b)$, and thus $\phi(b) = k$. It means that ϕ_B is a normal isomorphism.

Set $\lambda = \phi_B^{-1}$. It follows that $\lambda : K \longrightarrow N$ is a normal monomorphism, and $\phi\lambda = I_K$. This completes the proof of $(2) \Rightarrow (3)$.

$(2) \Rightarrow (1)$ Assume that $Im(\psi)$ is a direct summand of the hypermodule N. Let $\pi_1 : N \longrightarrow Im(\psi)$ be the canonical projection. Now, we consider the isomorphism $\psi_0 : M \longrightarrow Im(\psi)$. Set $\varphi = \psi_0^{-1}\pi_1$. Now, for all $m \in M$,

$$(\varphi\psi)(m) = ((\varphi\psi)\psi)(m) = \psi_0^{-1}(\pi_1\psi(m)) = \psi_0^{-1}(\psi(m)) = (\psi_0^{-1}\psi)(m) = m = I_M(m)$$

and thus $\varphi\psi = I_M$. □

Proposition 1. *Let K be a normal projective hypermodule. Then, every exact sequence ending K is splitting.*

Proof. Let $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be any short exact sequence of R hypermodules. Pay attention to the subsequent diagram:

$$\begin{array}{ccccc} & & K & & \\ & & \downarrow{\scriptstyle I_K} & & \\ N & \xrightarrow{\phi} & K & \longrightarrow & 0 \end{array}$$

It follows from the presumption that there is a normal homomorphism $\lambda : K \longrightarrow N$ with $\phi\lambda = I_K$. From Theorem 2, we obtain that $\mathbb{E}$ splits. □

Proposition 2. *Let M become a normal injective hypermodule. If so every exact sequence starting M is splitting.*

Proof. Let $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ become any short exact sequence of R hypermodules. Take into account the subsequent diagram::

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\psi} & N \\ & & \downarrow{\scriptstyle I_M} & & \\ & & M & & \end{array}$$

Owing to the fact that M is normal injective, there is a normal homomorphism $\varphi : N \longrightarrow M$ such that $\varphi\psi = I_M$. From Theorem 2, we obtain that $\mathbb{E}$ splits. □

Corollary 3. *Let M be a normal injective hypermodule and N be any hypermodule containing M. Then, M is a direct summand of N.*

Proof. For the inclusion $\iota : M \longrightarrow N$ and the canonical homomorphism $\pi : N \longrightarrow \frac{N}{M}$, we take into account the short exact sequence $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} \frac{N}{M} \longrightarrow 0$. Owing to fact that M is normal injective, it follows from Proposition 2 that $\mathbb{E}$ splits. Hence, M is a direct summand of N according to Theorem 2. □

Lemma 2. *Let M and I be two hypermodules. If M is semisimple, then every diagram with the exact row*

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \xrightarrow{f} & M \\ & & \big\downarrow{\scriptstyle g} & & \\ & & I & & \end{array}$$

can be extended commutatively by a normal homomorphism $M \longrightarrow I$.

Proof. Owing to the fact that M is a semisimple hypermodule, it follows from [24], Theorem 9, that $Im(f)$ is a direct summand of M. Therefore, we can write $M = Im(f) \oplus M_1$ for some subhypermodule M_1 of M. Note that $\lambda = f^{-1} : Im(f) \longrightarrow U$ is a normal isomorphism. Set $h = g\lambda\pi$, where $\pi : M \longrightarrow Im(f)$ is the canonical projection. Now, for all $a \in U$,

$$(hf)(a) = h(f(a)) = (g\lambda\pi)(f(a)) = (g\lambda)(f(a)) = g(\lambda f(a)) = g(a)$$

and thus $hf = g$, as required. □

Lemma 3. *Let M and P be two hypermodules. Assume that M is semisimple. Provided that $f : M \longrightarrow K$ is an epimorphism of hypermodules and $g : P \longrightarrow K$ is a normal homomorphism of hypermodules, there is a normal homomorphism $h : P \longrightarrow M$ together with $fh = g$.*

Proof. Let $f : M \longrightarrow K$ be a normal epimorphism and $g : P \longrightarrow K$ be a normal homomorphism, where K is a hypermodule. Consider the subsequent diagram:

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle g} & & \\ M & \xrightarrow{f} & K & \longrightarrow & 0 \end{array}$$

From the assumption, there exists a subhypermodule M_1 of M such that $M = Ker(f) \oplus M_1$. Therefore, $f_{M_1} : M_1 \longrightarrow K$ is isomorphism. Set $h = \iota f_{M_1}^{-1} g$, where $\iota : M_1 \longrightarrow M$ is the inclusion. Let $a \in P$. Since f is a normal epimorphism, there exists an element $m \in M$ with $g(a) = f(m)$. Now,

$$(fh)(a) = f(\iota f_{M_1}^{-1} g)(a) = (f \iota f_{M_1}^{-1})(f(m)) = f(m) = g(a).$$

Hence, we obtain that the equality $fh = g$. □

We now prove the main theorem.

Theorem 3. *Let R be a hyperring. If so, the subsequent statements are equivalent.*

(1) The left R hypermodule R is semisimple.
(2) All left hyperideals of R are direct summands of R.
(3) All R hypermodules are normal injective.
(4) All R hypermodules are semisimple.
(5) All R hypermodules are normal projective.
(6) All semisimple R hypermodules are normal projective.
(7) All simple R hypermodules are normal projective.

Proof. (1) ⇔ (2) It follows from [24], Theorem 9.

(1) ⇒ (3) Let M be any left R hypermodule. Consider the following diagram with the exact row:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\iota} & R \\ & & \downarrow{\scriptstyle f} & & \\ & & M & & \end{array}$$

where I is any left hyperideal of R, and $\iota : I \longrightarrow R$ is the inclusion map. Since the left R hypermodule R is semisimple, it follows from Lemma 2 that there is a normal homomorphism $h : R \longrightarrow M$ with $h\iota = f$. By ([11], Theorem 4), we obtain that M is normal injective.

(3) $\Rightarrow$ (4) Let M be any left R hypermodule and $U \subseteq M$. Applying (3), we obtain that U is normal injective and, so, from Corollary 3, M has the decomposition $M = U \oplus V$, where V is a subhypermodule of M. Hence, M is semisimple according to [24], Theorem 9.

(4) $\Rightarrow$ (5) For any hypermodule M, consider the following diagram with exact row:

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle g} & & \\ N & \xrightarrow{f} & K & \longrightarrow & 0 \end{array}$$

Since M is semisimple, from Lemma 3, the above diagram is commutative. It means that M is normal projective.

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1) From [24], Corollary 4, it is enough to show that whenever $I \trianglerighteq R$, $I = R$. Let us assume the opposite, that is, $I \neq R$. Therefore, there is a maximal left hyperideal P of R such that P contains the essential left hyperideal I. Applying [24], Theorem 3, $\frac{R}{P}$ is a simple R hypermodule. Now, from (7) and Proposition 1, the short exact sequence

$$0 \longrightarrow P \xrightarrow{\iota} R \xrightarrow{\pi} \frac{R}{P} \longrightarrow 0$$

is splitting, so there exists a minimal left hyperideal S of R such that $R = P \oplus S$. $I \cap S \subseteq P \cap S = 0$ is a contradiction. Hence, $I = R$. $\square$

The next result follows from Theorem 3.

Corollary 4. *Let R be a hyperring. Then, the left R hypermodule R is semisimple if and only if every exact sequence $\mathbb{E} : 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ of R hypermodules is splitting.*

Finally, let us construct an example for Theorem 3.

Example 6. *Let $R = \{0,1,2,3,4,5,6,7,8,9\}$. We describe "+" and "·" in the subsequent tables:*

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	$\{0,2\}$	$\{1,3\}$	$\{2,4\}$	$\{3,5\}$	$\{4,6\}$	$\{5,7\}$	$\{6,8\}$	$\{7,9\}$	$\{0,8\}$
2	2	$\{1,3\}$	$\{0,4\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$	$\{0,6\}$	$\{1,9\}$
3	3	$\{2,4\}$	$\{1,5\}$	$\{0,6\}$	$\{1,7\}$	$\{2,8\}$	$\{3,9\}$	$\{0,4\}$	$\{1,5\}$	$\{2,6\}$
4	4	$\{3,5\}$	$\{2,6\}$	$\{1,7\}$	$\{0,8\}$	$\{1,9\}$	$\{0,2\}$	$\{1,3\}$	$\{2,4\}$	$\{3,5\}$
5	5	$\{4,6\}$	$\{3,7\}$	$\{2,8\}$	$\{1,9\}$	0	1	2	3	4
6	6	$\{5,7\}$	$\{4,8\}$	$\{3,9\}$	$\{0,4\}$	1	$\{0,2\}$	$\{1,3\}$	$\{2,4\}$	$\{3,5\}$
7	7	$\{6,8\}$	$\{5,9\}$	$\{0,4\}$	$\{1,3\}$	2	$\{1,3\}$	$\{0,4\}$	$\{1,5\}$	$\{2,6\}$
8	8	$\{7,9\}$	$\{0,6\}$	$\{1,5\}$	$\{2,4\}$	3	$\{2,4\}$	$\{1,5\}$	$\{0,6\}$	$\{1,7\}$
9	9	$\{0,8\}$	$\{1,7\}$	$\{2,6\}$	$\{3,5\}$	4	$\{3,5\}$	$\{2,6\}$	$\{1,7\}$	$\{0,8\}$

and

·	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	1	8	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

Then, R is a commutative hyperring together with identity element 1*. It is straightforward to see that the only hyperideals of R are $I_1 = \{0\}$, $I_2 = \{0,5\}$, $I_3 = \{0,2,4,6,8\}$ and R. Therefore, the sum $I_2 + I_3$ is direct, so $R = I_2 \oplus I_3$ is semisimple. It follows from Theorem 3 that each left R hypermodule is normal injective (normal projective).*

4. Conclusions

The concept of short exact sequences and their relationship with normal injective and projective hypermodules was developed; a type of short exact sequence, namely, the splitting exact sequence, was defined; and some examples were provided. Moreover, we proved that every exact sequence ending with a normal projective hypermodule, and every exact sequence starting with a normal injective hypermodule must be split. Finally, the relationships between semisimple, simple, normal injective, and normal projective hypermodules were investigated.

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