

Article

Stability of the Stochastic Ginzburg–Landau–Newell Equations in Two Dimensions

Jing Wang ^{1,†}  and Yan Zheng ^{2,*,†} 

¹ Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; matjwang@shnu.edu.cn

² College of Sciences, National University of Defense Technology, Changsha 410022, China

* Correspondence: yanzhengyl@163.com

† These authors contributed equally to this work.

Abstract: This paper concerns the 2D stochastic Ginzburg–Landau–Newell equations with a degenerate random forcing. We study the relationship between stationary distributions which correspond to the original and perturbed systems and then prove the stability of the stationary distribution. This suggests that the complexity of stochastic systems is likely to be robust. The main difficulty of the proof lies in estimating the expectation of exponential moments and controlling nonlinear terms while working on the evolution triple $H^2 \subset H^1 \subset H^0$ to obtain a bound on the difference between the original solution and the perturbed solution.

Keywords: stochastic stability; stochastic Ginzburg–Landau–Newell equations; stationary distribution; exponential ergodicity

MSC: 93E15; 37A25; 35Q60



Citation: Wang, J.; Zheng, Y. Stability of the Stochastic Ginzburg–Landau–Newell Equations in Two Dimensions. *Axioms* **2024**, *13*, 412. <https://doi.org/10.3390/axioms13060412>

Academic Editors: Dragana Valjarević, Vladica Stojanović and Aleksandar Dj Valjarević

Received: 13 May 2024

Revised: 13 June 2024

Accepted: 14 June 2024

Published: 19 June 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The study of stability in physical systems is a cornerstone of modern theoretical physics, and at the forefront of this research lie the Ginzburg–Landau–Newell equations. This mathematical framework, an extension of the original Ginzburg–Landau theory, plays a crucial role in understanding phase transitions and pattern formation in a diverse range of systems, from superconductors to fluid dynamics. Ginzburg–Landau–Newell equations are an important model discussed in [1–4] as describing the wave pattern and turbulence phenomenon. In this paper, we are concerned with the 2D stochastic Ginzburg–Landau–Newell Equations (SGLNEs)

$$\begin{cases} du = [\mu u - \eta \nabla u + (1 + i\alpha)\Delta u - (1 + i\beta)|u|^2 u - (r + i\delta)|v|^2 u]dt + Q_1 dW_1(t), \\ dv = [\mu v + \eta \nabla v + (1 + i\alpha)\Delta v - (1 + i\beta)|v|^2 v - (r + i\delta)|u|^2 v]dt + Q_2 dW_2(t), \end{cases} \quad (1)$$

with the zero Dirichlet boundary condition, where $(u(t), v(t))$ is the complex-valued ($\mathbb{C}$ -valued) unknown function defined on $D \subset \mathbb{R}^2$, and $\alpha, \beta, r, \delta, \mu > 0$ and $\eta > 0$ are real numbers. The stochastic forcing is modeled by the white in time Gaussian noise $(Q_1 \dot{W}_1, Q_2 \dot{W}_2)$, which is to be specified in Section 2.

This model has been studied by Huang et al. [5,6], where the existence and uniqueness of H^1 -strong solutions, the unique ergodicity, as well as the exponentially mixing property are proven. Here, we are mainly concerned with the issue of stability from the perspective of statistical mechanics. Roughly speaking, SGLNE (1) is perturbed so that all of its coefficients and even noise terms could be changed; then, the relationship between the original and perturbed systems is investigated by measuring the gap between their corresponding stationary distributions. In this way, the stability of the stationary distribution for SGLNE

(1) is proven in the sense of Hairer et al. [7,8], which is in striking difference with its deterministic analogue and indicates the robustness of equilibrium for statistical hydrodynamics. Our research delves into the intricacies of the stochastic Ginzburg–Landau–Newell equations, unraveling their stability of the stationary distribution in the framework of statistical hydrodynamics, which reveals that the complex interplay between order and disorder in these dynamical systems is likely to be robust under disturbances. By pushing the boundaries of our current understanding, we aim to contribute novel insights that may pave the way for advancements in materials science, quantum computing, and beyond.

To put it more specifically, we continue to use the hypotheses on the coefficients of SGLNE (1) as in [5,6].

(H1) Assume $2\mu + \eta^2 \leq \varrho_1$, where ϱ_1 is the first eigenvalue of the Dirichlet problem

$$-\Delta u = \varrho u \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D.$$

(H2) $r \geq 1$, $4\beta^2 + 3r^2 + 4\delta^2 \leq 5$.

For simplicity, denote $\Xi := (\mu, \eta, \alpha, \beta, r, \delta, Q_1, Q_2)$, and let μ^Ξ be the unique stationary distribution for SGLNE parameterized by Ξ and $\mu^{\hat{\Xi}}$ be any invariant probability measure for SGLNE associated with disturbed parameter $\hat{\Xi}$. Range of Ξ ($\hat{\Xi}$, respectively) will be denoted by Λ_0 (Λ , respectively). Now, we are ready to state our main result.

Theorem 1. Suppose the hypotheses (H1) and (H2) hold. Then, $\mu^{\hat{\Xi}} \rightarrow \mu^\Xi$ as $\hat{\Xi} \rightarrow \Xi$.

This is the simplified version of Theorem 3, and the exact meaning of convergences is explained in Section 2.

It is worth mentioning that Hairer, Mattingly, and Scheutzow [8] attached great importance to the stability of the stationary distribution and proposed a general framework, yet this is not suitable for our need. Hairer and Mattingly [7] obtained a deeper result by establishing the regular dependence of stationary distributions on the parameters for the 2D stochastic Navier–Stokes equations. Regrettably, with the nonlinear term being cubic in the case of SGLNE, we have to work on the Gelfand triple $H^2 \subset H^1 \subset H^0$ to make up for the deficiency of its lacking unique L^2 -solutions to generate transition semigroups. Furthermore compared with [5,6], new nonlinear terms emerge and exponential moments therein need to be estimated more subtly. These difficulties lead us to obtain a weaker sense of stability, as opposed to [7,8].

The rest of the paper is organized as follows: in Section 2, we introduce some necessary notations, along with the degenerate noise construction. Then, we recall from [5,6] some main results related to SGLNE (1) including the well posedness, the existence and uniqueness of stationary distributions, and the spectral gap. In Section 3, we first give a sharp estimate of expectation of exponential moments and then establish the bound on the difference between solutions corresponding to different set of parameters. Finally, in Theorem 3, we obtain the stability of stationary distributions. In the last section, we discuss the application of this model and point out the direction of our future research.

2. Preliminaries

In this section, we introduce some necessary notations, then recall the main result established in [5,6]. Let L^2 and H^1 be the Hilbert spaces on D endowed with the complex inner products

$$\langle u, v \rangle = \int_D u \bar{v} dx, \quad \langle u, v \rangle_1 = \langle u, v \rangle + \int_D \nabla u \nabla \bar{v} dx.$$

For $s \in \mathbb{N}$, let H^s be the usual Sobolev space on D . We denote by $\|\cdot\|_{L^p}$ the L^p norm and by $\|\cdot\|_s$ the H^s Sobolev-norm. L^2 is sometimes represented by H^0 , and $\|\cdot\|_{L^2}$ is denoted

as $\|\cdot\|$ for simplicity. The letter $C(\cdot)$ with parameters or not inside the bracket denotes a positive constant, whose value may change from line to line.

Here, we focus on the degenerate stochastic forcing. To be precise, for $N \in \mathbb{N}$, let $\Omega = C_0(\mathbb{R}^+, \mathbb{R}^{2N})$ be the space of all continuous functions with initial values 0, $\mathbf{P}$, the standard Wiener measure on $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}^+, \mathbb{R}^{2N}))$. Then, the coordinate processes

$$W_1(t, \omega) := (\omega_1(t), \dots, \omega_N(t)), \quad W_2(t, \omega) := (\omega_{N+1}(t), \dots, \omega_{2N}(t)), \quad \omega \in \Omega, \quad (2)$$

are two independent standard Wiener processes on $(\Omega, \mathcal{F}, \mathbf{P})$. We define the linear map $Q_i : \mathbb{R}^N \rightarrow H^1(D)$, $i = 1, 2$ in a natural way by

$$Q_i e_j = q_{i,j} \mathbf{e}_j, \quad q_{i,j} > 0, \quad i = 1, 2, \quad j = 1, \dots, N, \quad (3)$$

where $\{e_i\}_{i=1}^N$ is the canonical basis of $\mathbb{R}^N$, and $\mathbf{e}_i$'s are the normalized eigenfunctions corresponding to the eigenvalues $0 < \varrho_1 \leq \varrho_2 \leq \dots$ of $-\Delta$. Literally, $(Q_1 W_1(t), Q_2 W_2(t))$ is the considered degenerate noise. The noise is degenerate in the sense that it drives the system only in the first N Fourier modes.

This type of 3D SGLNE has been studied in [5] and the existence and uniqueness of strong solutions in $C([0, T]; H^1 \times H^1) \cap L^2([0, T]; H^2 \times H^2)$ with $H^1 \times H^1$ -initial data are obtained. For fixed initial data $\Phi_0 = (u_0, v_0) \in H^1 \times H^1$, we denote this unique solution by $\{u(t, \Phi_0), v(t, \Phi_0)\}$. Then, $\{(u(t, \Phi_0), v(t, \Phi_0)); t \geq 0\}$ generates a strong Markov process on the state space $H^1 \times H^1$. We denote by $\mathcal{P}_t$ the Markov semigroup associated with $\{u(t, \Phi_0), v(t, \Phi_0)\}$, and by $\mathcal{P}_t^*$ the dual semigroup of $\mathcal{P}_t$, which acts on the probability measure space $\mathcal{M}(H^1 \times H^1)$. A distribution $\mu \in \mathcal{M}(H^1 \times H^1)$ is called stationary provided $\mathcal{P}_t^* \mu = \mu$ for each $t \geq 0$. The sufficient yet not excessive conditions for the existence and the uniqueness of stationary distributions are also given in [5].

For $\nu > 0$, we define a family of distances d_ν on $H^1 \times H^1$ by

$$d_\nu(\mathbf{x}, \mathbf{y}) = \inf_{\gamma} \int_0^1 \exp(\nu \|\gamma(t)\|_1^2) \|\dot{\gamma}(t)\|_1 dt, \quad (4)$$

where the infimum runs over all paths γ such that $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$. Immediately, one obtains

$$d_\nu(\mathbf{x}, \mathbf{y}) \leq \|\mathbf{x} - \mathbf{y}\|_1 (\exp(\nu \|\mathbf{x}\|_1^2) + \exp(\nu \|\mathbf{y}\|_1^2)). \quad (5)$$

For $\mu_1, \mu_2 \in \mathcal{M}(H^1 \times H^1)$, we use the notation $\mathcal{C}(\mu_1, \mu_2)$ for the set of all measures Γ on $(H^1 \times H^1) \times (H^1 \times H^1)$ such that $\Gamma(\mathbf{A} \times (H^1 \times H^1)) = \mu_1(\mathbf{A})$ and $\Gamma((H^1 \times H^1) \times \mathbf{A}) = \mu_2(\mathbf{A})$ for any Borel set $\mathbf{A} \subset H^1 \times H^1$. Measure Γ is called a coupling of μ_1 and μ_2 . Define the 1-Wasserstein distance by

$$d_\nu(\mu_1, \mu_2) = \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \int_{(H^1 \times H^1) \times (H^1 \times H^1)} d_\nu(\mathbf{x}, \mathbf{y}) \pi(d\mathbf{x}, d\mathbf{y}). \quad (6)$$

The next theorem establishes the exponential ergodicity.

Theorem 2 (Theorem 1.1 of [6]). *Suppose that (H1), (H2) hold. Let $\{\mathcal{P}_t\}$ be the Markov semigroup associated with (1), then for any N large enough, there exists positive constants $\eta_0 = \frac{1}{4\|Q\|_1^2}$, γ and C such that*

$$d_\eta(\mathcal{P}_t^* \mu_1, \mathcal{P}_t^* \mu_2) \leq C e^{-\gamma t} d_\eta(\mu_1, \mu_2),$$

for any $t > 0$, $\eta \leq \eta_0$ and any two probability distributions μ_1 and μ_2 on $H^1 \times H^1$.

3. Stability of Stochastic Systems

Refined from (Lemma 3.2 of [6]), the following lemma is somewhat technical. We will use it to estimate moments of $\|\cdot\|_1$ and $\|\cdot\|_2$ and in particular guarantee the validity of metric (6).

$(u(t), v(t))$ is the unique solution of Equation (1) corresponding to initial data (u_0, v_0) .

Lemma 1. $(u(t), v(t))$ is the unique solution of Equation (1) corresponding to initial data (u_0, v_0) . Suppose that (H2) holds and $0 < \nu < \frac{1}{4\|Q\|_1^2}$, then there holds

$$\begin{aligned} & \mathbb{E} \exp \left(\nu (\|u(t)\|_1^2 + \|v(t)\|_1^2) + \frac{\nu e^{-\frac{t}{4}}}{4} \int_0^t (\|u(s)\|_2^2 + \|v(s)\|_2^2) ds \right) \\ & \leq C \exp \left(\nu (\|u_0\|_1^2 + \|v_0\|_1^2) e^{-\frac{1}{4}t} \right). \end{aligned} \quad (7)$$

Proof. Notice that the quadratic variation of $Q_i W_i(t)$ in H^1 is given by $\mathcal{E}_i \triangleq \sum_{k=1}^N q_{i,k}^2$, then it follows from the Itô's formula that

$$\|u(t)\|_1^2 = \|u_0\|_1^2 + \sum_{i=1}^3 I_i(t), \quad \|v(t)\|_1^2 = \|v_0\|_1^2 + \sum_{i=1}^3 J_i(t), \quad (8)$$

where

$$\begin{aligned} I_1(t) &= 2 \int_0^t \langle u, (1 + i\alpha) \Delta u \rangle_1 + 2\mu \int_0^t \langle u, u \rangle_1 ds - 2\eta \int_0^t \langle u, \nabla u \rangle_1 ds \\ &\quad - 2 \int_0^t \langle u, (1 + i\beta) |u|^2 u \rangle_1 ds - 2 \int_0^t \langle u, (r + i\delta) |v|^2 u \rangle_1 ds =: I_1^1 + I_1^2, \\ I_2(t) &= 2 \sum_{k=1}^N q_{1,k} \int_0^t \langle u, \mathbf{e}_k \rangle_1 dW_{1,k}(s), \quad I_3(t) = \mathcal{E}_1 t, \\ J_1(t) &= 2 \int_0^t \langle v, (1 + i\alpha) \Delta v \rangle_1 + 2\mu \int_0^t \langle v, v \rangle_1 ds + 2\eta \int_0^t \langle v, \nabla v \rangle_1 ds \\ &\quad - 2 \int_0^t \langle v, (1 + i\beta) |v|^2 v \rangle_1 ds - 2 \int_0^t \langle v, (r + i\delta) |u|^2 v \rangle_1 ds =: J_1^1 + J_1^2, \\ J_2(t) &= 2 \sum_{k=1}^N q_{2,k} \int_0^t \langle v, \mathbf{e}_k \rangle_1 dW_{2,k}(s), \quad J_3(t) = \mathcal{E}_2 t, \end{aligned}$$

where I_1^1 and I_1^2 (J_1^1 and J_1^2) are the first and second lines of I_1 (J_1), respectively. Recalling from (H2) that $4\beta^2 + 3r^2 + 4\delta^2 \leq 5$ and noticing that

$$\begin{aligned} & \operatorname{Re} \langle u, (1 + i\beta) |u|^2 u \rangle_1 = \operatorname{Re} \langle u, (1 + i\beta) |u|^2 u \rangle + \operatorname{Re} \langle \nabla u, (1 + i\beta) \nabla (|u|^2 u) \rangle \\ & \geq 3 \| |u| |\nabla u| \|^2 - 2 \left| \operatorname{Re} \left[(1 - i\beta) \int \nabla u \cdot \bar{u} (\operatorname{Im}(\nabla u \cdot \bar{u}) i) dx \right] \right| + \|u\|_{L^4}^4 \\ & \geq \left(3 - 2\sqrt{1 + \beta^2} \right) \| |u| |\nabla u| \|^2 + \|u\|_{L^4}^4 \geq \|u\|_{L^4}^4, \\ & \operatorname{Re} \langle v, (1 + i\beta) |v|^2 v \rangle_1 = \operatorname{Re} \langle v, (1 + i\beta) |v|^2 v \rangle + \operatorname{Re} \langle \nabla v, (1 + i\beta) \nabla (|v|^2 v) \rangle \\ & \geq 3 \| |v| |\nabla v| \|^2 - 2 \left| \operatorname{Re} \left[(1 - i\beta) \int \nabla v \cdot \bar{v} (\operatorname{Im}(\nabla v \cdot \bar{v}) i) dx \right] \right| + \|v\|_{L^4}^4 \geq \|v\|_{L^4}^4, \\ & \operatorname{Re} \langle u, (r + i\delta) |v|^2 u \rangle_1 = \operatorname{Re} \langle u, (r + i\delta) |v|^2 u \rangle + \operatorname{Re} \langle \nabla u, (r + i\delta) \nabla (|v|^2 u) \rangle \\ & \geq r \| |v| |\nabla u| \|^2 - 2 \left| \operatorname{Re} \left[(r - i\delta) \int_D \nabla u \cdot \bar{u} (\operatorname{Re}(\nabla v \cdot \bar{v})) dx \right] \right| + r \| |u| |v| \|^2 \geq r \| |u| |v| \|^2, \\ & \operatorname{Re} \langle v, (r + i\delta) |u|^2 v \rangle_1 = \operatorname{Re} \langle v, (r + i\delta) |u|^2 v \rangle + \operatorname{Re} \langle \nabla v, (r + i\delta) \nabla (|u|^2 v) \rangle \\ & \geq r \| |u| |\nabla v| \|^2 - 2 \left| \operatorname{Re} \left[(r - i\delta) \int_D \nabla v \cdot \bar{v} (\operatorname{Re}(\nabla u \cdot \bar{u})) dx \right] \right| + r \| |u| |v| \|^2 \geq r \| |u| |v| \|^2, \end{aligned}$$

we have

$$I_1^2 + J_1^2 \leq -\|u\|_{L^4}^4 - \|v\|_{L^4}^4 - 2r\|u\|v\|^2.$$

Interpolation inequality and Hölder's inequality yield that

$$2\mu\|u\|_1^2 \leq \frac{1}{2}\|\nabla u\|_1^2 + C(\mu)\|u\|^2 \leq \frac{1}{2}\|\nabla u\|_1^2 + \frac{1}{2}\|u\|_{L^4}^4 + \frac{C(\mu)^2|D|}{2},$$

and

$$2\eta\left|\langle u, \nabla u \rangle_1\right| \leq \frac{1}{2}\|\nabla u\|_1^2 + C(\eta)\|u\|^2 \leq \frac{1}{2}\|\nabla u\|_1^2 + \frac{1}{2}\|u\|_{L^4}^4 + \frac{C(\eta)^2|D|}{2},$$

and similarly,

$$2\mu\|v\|_1^2 + 2\eta\left|\langle v, \nabla v \rangle_1\right| \leq \|\nabla v\|_1^2 + \|v\|_{L^4}^4 + C(\eta)^2|D|.$$

Consequently, taking real part and using Young's inequality give

$$\begin{aligned} \|u\|_1^2 + \|v\|_1^2 &\leq \|u_0\|_1^2 + \|v_0\|_1^2 - \int_0^t (\|\nabla u\|_1^2 + \|\nabla v\|_1^2) ds + C(\mu, \eta, \varepsilon_1, \varepsilon_2)t \\ &\quad + 2 \sum_{k=1}^N q_{1,k} \int_0^t \langle u, \mathbf{e}_k \rangle_1 dW_{1,k}(s) + 2 \sum_{k=1}^N q_{2,k} \int_0^t \langle v, \mathbf{e}_k \rangle_1 dW_{2,k}(s). \end{aligned} \quad (9)$$

Next, we will adopt the strategy of (Lemma 5.1 of [7]). Denoting $G(t, \omega) = 2\nu \sum_{k=1}^N q_{1,k} \langle u, \mathbf{e}_k \rangle_1 e_k + 2\nu \sum_{k=1}^N q_{2,k} \langle v, \mathbf{e}_k \rangle_1 e_k$, for $0 < \nu < 1/(4\|Q\|_1^2)$, one then writes (9) in a concise way

$$\nu d(\|u(t, \omega)\|_1^2 + \|v(t, \omega)\|_1^2) \leq \nu(C - \|u(t, \omega)\|_2^2 - \|v(t, \omega)\|_2^2) dt + G(t, \omega) dW(t, \omega).$$

Fixing $t > 0$ and $\forall s \in [0, t]$, set

$$Y(s) = \nu e^{\frac{1}{4}(s-t)} (\|u(s)\|_1^2 + \|v(s)\|_1^2) + \frac{\nu}{4} \int_0^s e^{\frac{1}{4}(r-t)} (\|u(r)\|_2^2 + \|v(r)\|_2^2) dr.$$

It is straightforward that $Y(0) = \nu e^{-\frac{1}{4}t} (\|u_0\|_1^2 + \|v_0\|_1^2)$ and

$$Y(t) \geq \nu (\|u(t)\|_1^2 + \|v(t)\|_1^2) + \frac{\nu e^{-t/4}}{4} \int_0^t (\|u(s)\|_2^2 + \|v(s)\|_2^2) ds.$$

Applying Itô's formula to $Y(s)$ yields that

$$\begin{aligned} dY(s) &= \frac{\nu}{4} e^{\frac{1}{4}(s-t)} (\|u(s)\|_1^2 + \|v(s)\|_1^2) + e^{\frac{1}{4}(s-t)} \nu d(\|u(s)\|_1^2 + \|v(s)\|_1^2) \\ &\quad + \frac{\nu}{4} e^{\frac{1}{4}(s-t)} (\|u(s)\|_2^2 + \|v(s)\|_2^2) \\ &\leq \nu \exp\left(\frac{1}{4}(s-t)\right) \left(C - \frac{1}{2}(\|u(s)\|_2^2 + \|v(s)\|_2^2)\right) ds + \exp\left(\frac{1}{4}(s-t)\right) G(s) dW(s). \end{aligned}$$

The last inequality holds from the concise form of (9). Since $s \in [0, t]$, we have

$$Y(s) \leq Y(0) + 4\nu C - \frac{\nu}{2} \int_0^s (\|u(r)\|_2^2 + \|v(r)\|_2^2) dr + \int_0^s \exp\left(\frac{1}{4}(r-t)\right) G(r) dW(r).$$

Noting that $G(t)^2 \leq 4\nu^2 \|Q\|_1^2 (\|u(t)\|_2^2 + \|v(t)\|_2^2)$, one obtains for any K that

$$\begin{aligned} & \mathbf{P} \left(\nu (\|u(t)\|_1^2 + \|v(t)\|_1^2) + \frac{\nu e^{-\frac{t}{4}}}{4} \int_0^t (\|u(s)\|_2^2 + \|v(s)\|_2^2) ds - \nu (\|u_0\|_1^2 + \|v_0\|_1^2) e^{-\frac{t}{4}} - 4\nu C > K \right) \\ & \leq \mathbf{P}(Y(t) - Y(0) - 4\nu C > K) \\ & \leq \mathbf{P} \left(\int_0^t \exp \left(\frac{1}{4}(r-t) \right) G(r) dW(r) - \frac{\nu}{2} \int_0^t (\|u(r)\|_2^2 + \|v(r)\|_2^2) dr > K \right) \\ & \leq \mathbf{P} \left(\int_0^t \exp \left(\frac{1}{4}(r-t) \right) G(r) dW(r) - \frac{1}{8\nu \|Q\|_1^2} \langle M \rangle(t) > K \right) \leq \exp \left(- \frac{1}{4\nu \|Q\|_1^2} K \right), \end{aligned}$$

where $M(s) = \int_0^s \exp \left(\frac{1}{4}(r-t) \right) G(r) dW(r)$ and exponential martingale inequality is applied in the last inequality. Since $\mathbf{E}e^X = \int_0^\infty \mathbf{P}(e^X > y) dy$ holds for any random variable X , one has

$$\begin{aligned} & \mathbf{E} \exp \left(\nu (\|u(t)\|_1^2 + \|v(t)\|_1^2) + \frac{\nu e^{-\frac{t}{4}}}{4} \int_0^t (\|u(s)\|_2^2 + \|v(s)\|_2^2) ds \right) \\ & \leq \frac{4\nu \|Q\|_1^2 \exp(4\nu C)}{1 - 4\nu \|Q\|_1^2} \exp \left(\nu (\|u_0\|_1^2 + \|v_0\|_1^2) e^{-\frac{t}{4}} \right), \end{aligned}$$

the claim then follows from $0 < \nu < 1/(4\|Q\|_1^2)$. $\square$

Remark 1. It is straightforward from Lemma 1 that

$$\mathbf{E} \exp \left(\nu \sup_{0 \leq s \leq t} (\|u(s)\|_1^2 + \|v(s)\|_1^2) \right) \leq C \exp \left(\nu (\|u_0\|_1^2 + \|v_0\|_1^2) \right). \quad (10)$$

We will use this bound to apply the Lebesgue dominated convergence theorem.

In order to consider the perturbation of stochastic systems, we introduce more notations. Recall that $\Xi := (\mu, \eta, \alpha, \beta, r, \delta, Q_1, Q_2)$, and Λ is the parameter space, which contains all Ξ 's that ensure the existence of stationary distributions for 2D SGLNE, which are not necessarily unique. We equip Λ with the natural distance by

$$\begin{aligned} d(\Xi, \hat{\Xi})^2 &= |\mu - \hat{\mu}|^2 + |\eta - \hat{\eta}|^2 + |\alpha - \hat{\alpha}|^2 + |\beta - \hat{\beta}|^2 + |r - \hat{r}|^2 + |\delta - \hat{\delta}|^2 \\ &\quad + \|Q_1 - \hat{Q}_1\|_1^2 + \|Q_2 - \hat{Q}_2\|_1^2. \end{aligned}$$

Λ_0 is the subset of Λ that ensures the existence of a unique stationary distribution for 2D SGLNE. Then, Λ and Λ_0 are not empty, see [5]. Then, for any $\Xi \in \Lambda_0$, we denote by μ^Ξ the unique stationary distribution for SGLNE with parameter Ξ and by $\mathcal{P}_t^\Xi$ the corresponding semigroup. For $\hat{\Xi} \in \Lambda$, $\mu^{\hat{\Xi}}$ will simply denote any stationary probability distribution for $\mathcal{P}_t^{\hat{\Xi}}$. For simplicity, we will simply write $\mathcal{P}_t^{\Xi*}$ for $(\mathcal{P}_t^\Xi)^*$.

Let $(u_0, v_0) \in H^1 \times H^1$, and for any two sets of parameters Ξ and $\hat{\Xi}$, denote by (u_t, v_t) ($(\hat{u}_t, \hat{v}_t)$, respectively) the unique solution to (1) starting from (u_0, v_0) with respect to parameter Ξ ($\hat{\Xi}$, respectively). For $R > 0$, define the stopping time

$$\tau_R := \inf \{ t \geq 0 : \|u(t, u_0)\|_1 \vee \|\hat{u}(t, u_0)\|_1 \vee \|v(t, v_0)\|_1 \vee \|\hat{v}(t, v_0)\|_1 \geq R \}.$$

The following result establishes the bound on the difference between solutions corresponding to different set of parameters, by which we will estimate the closeness of the time t transition distribution in the final theorem.

Proposition 1. *There exists a $v_0 > 0$ so that, for every positive $v \leq v_0$, there exist $C > 0$ and $C(R) > 0$ so that*

$$\mathbf{E}\|u_{t \wedge \tau_R} - \hat{u}_{t \wedge \tau_R}\|_1^2 + \mathbf{E}\|v_{t \wedge \tau_R} - \hat{v}_{t \wedge \tau_R}\|_1^2 \leq Ce^{C(R)t + v(\|u_0\|_1^2 + \|u_0\|_1^2)} d(\Xi, \hat{\Xi})^2.$$

Proof. Set

$$\begin{aligned} \tilde{u}_t &= u_t - \hat{u}_t, \quad \tilde{v}_t = v_t - \hat{v}_t, \quad \tilde{\mu} = \mu - \hat{\mu}, \quad \tilde{\eta} = \eta - \hat{\eta}, \quad \tilde{\alpha} = \alpha - \hat{\alpha}, \\ \tilde{\beta} &= \beta - \hat{\beta}, \quad \tilde{r} = r - \hat{r}, \quad \tilde{\delta} = \delta - \hat{\delta}, \quad \tilde{Q}_1 = Q_1 - \hat{Q}_1, \quad \tilde{Q}_2 = Q_2 - \hat{Q}_2. \end{aligned}$$

Immediately, one has

$$\begin{aligned} d\tilde{u} &= [\mu\tilde{u} + \tilde{\mu}\hat{u} - \eta\nabla\tilde{u} - \tilde{\eta}\nabla\hat{u} + (1+i\alpha)\Delta\tilde{u} + i\tilde{\alpha}\Delta\hat{u} - (1+i\beta)|u|^2u \\ &\quad + (1+i\beta)|\hat{u}|^2\hat{u} - (r+i\delta)|v|^2u + (\hat{r}+i\hat{\delta})|\hat{v}|^2\hat{u}]dt + \tilde{Q}_1dW_1, \end{aligned}$$

and

$$\begin{aligned} d\tilde{v} &= [\mu\tilde{v} + \tilde{\mu}\hat{v} + \eta\nabla\tilde{v} + \tilde{\eta}\nabla\hat{v} + (1+i\alpha)\Delta\tilde{v} + i\tilde{\alpha}\Delta\hat{v} - (1+i\beta)|v|^2v \\ &\quad + (1+i\beta)|\hat{v}|^2\hat{v} - (r+i\delta)|u|^2v + (\hat{r}+i\hat{\delta})|\hat{u}|^2\hat{v}]dt + \tilde{Q}_2dW_2. \end{aligned}$$

Introduce stochastic convolutions

$$\Psi_1(t) = \int_0^t e^{(1+i\alpha)\Delta(t-s)} \tilde{Q}_1 dW_1(s), \quad \Psi_2(t) = \int_0^t e^{(1+i\alpha)\Delta(t-s)} \tilde{Q}_2 dW_2(s),$$

and set $\tilde{u}_t^{\Psi_1} = \tilde{u}_t - \Psi_1(t)$, $\tilde{v}_t^{\Psi_2} = \tilde{v}_t - \Psi_2(t)$. The Ornstein–Uhlenbeck transformation [9,10] yields that

$$\begin{aligned} d\tilde{u}^{\Psi_1} &= [\mu\tilde{u}^{\Psi_1} + \mu\Psi_1 + \tilde{\mu}\hat{u} - \eta\nabla\tilde{u}^{\Psi_1} - \eta\nabla\Psi_1 - \tilde{\eta}\nabla\hat{u} + (1+i\alpha)\Delta\tilde{u}^{\Psi_1} + i\tilde{\alpha}\Delta\hat{u} \\ &\quad - (1+i\beta)|u|^2u + (1+i\beta)|\hat{u}|^2\hat{u} - (r+i\delta)|v|^2u + (\hat{r}+i\hat{\delta})|\hat{v}|^2\hat{u}]dt, \end{aligned} \quad (11)$$

and

$$\begin{aligned} d\tilde{v}^{\Psi_2} &= [\mu\tilde{v}^{\Psi_2} + \mu\Psi_2 + \tilde{\mu}\hat{v} + \eta\nabla\tilde{v}^{\Psi_2} + \eta\nabla\Psi_2 + \tilde{\eta}\nabla\hat{v} + (1+i\alpha)\Delta\tilde{v}^{\Psi_2} + i\tilde{\alpha}\Delta\hat{v} \\ &\quad - (1+i\beta)|v|^2v + (1+i\beta)|\hat{v}|^2\hat{v} - (r+i\delta)|u|^2v + (\hat{r}+i\hat{\delta})|\hat{u}|^2\hat{v}]dt. \end{aligned} \quad (12)$$

Multiplying (11) with $(I - \Delta)\tilde{u}^{\Psi_1}$, integrating by parts and taking the real part give that

$$\begin{aligned} \frac{1}{2}\partial_t\|\tilde{u}^{\Psi_1}\|_1^2 &= \mu\|\tilde{u}^{\Psi_1}\|_1^2 + \mu\langle\Psi_1, (I - \Delta)\tilde{u}^{\Psi_1}\rangle + \tilde{\mu}\langle\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle \\ &\quad - \eta\langle\nabla\tilde{u}^{\Psi_1}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle - \eta\langle\nabla\Psi_1, (I - \Delta)\tilde{u}^{\Psi_1}\rangle - \tilde{\eta}\langle\nabla\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle \\ &\quad + \langle\Delta\tilde{u}^{\Psi_1}, \tilde{u}^{\Psi_1}\rangle_1 + \Re(i\tilde{\alpha})\langle\Delta\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle \\ &\quad - \Re(1+i\beta)\langle|u|^2u - |\hat{u}|^2\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle - \Re(i\tilde{\beta})\langle|\hat{u}|^2\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle \\ &\quad - \Re(r+i\delta)\langle|v|^2u - |\hat{v}|^2\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle - \Re(\tilde{r}+i\tilde{\delta})\langle|\hat{v}|^2\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle, \end{aligned}$$

where we use $\Re$ to take the real part of the subsequent items. The linear terms are dealt with readily as follows

$$\begin{aligned} \mu\langle\Psi_1, (I - \Delta)\tilde{u}^{\Psi_1}\rangle &\leq \frac{1}{8}\|\tilde{u}^{\Psi_1}\|_2^2 + 2\mu^2\|\Psi_1\|^2, \\ \tilde{\mu}\langle\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle &\leq \frac{1}{8}\|\tilde{u}^{\Psi_1}\|_2^2 + 2\tilde{\mu}^2\|\hat{u}\|^2, \\ \eta\langle\nabla\tilde{u}^{\Psi_1}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle &\leq \frac{1}{8}\|\tilde{u}^{\Psi_1}\|_2^2 + 2\eta^2\|\tilde{u}^{\Psi_1}\|_1^2, \\ \eta\langle\nabla\Psi_1, (I - \Delta)\tilde{u}^{\Psi_1}\rangle &\leq \frac{1}{8}\|\tilde{u}^{\Psi_1}\|_2^2 + 2\eta^2\|\Psi_1\|_1^2, \\ \tilde{\eta}\langle\nabla\hat{u}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle &\leq \frac{1}{8}\|\tilde{u}^{\Psi_1}\|_2^2 + 2\tilde{\eta}^2\|\hat{u}\|_1^2, \\ \langle\Delta\tilde{u}^{\Psi_1}, \tilde{u}^{\Psi_1}\rangle_1 &= -\langle(I - \Delta)\tilde{u}^{\Psi_1}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle + \langle\tilde{u}^{\Psi_1}, (I - \Delta)\tilde{u}^{\Psi_1}\rangle = -\|\tilde{u}^{\Psi_1}\|_2^2 + \|\tilde{u}^{\Psi_1}\|_1^2, \end{aligned}$$

and

$$\begin{aligned}\Re(i\tilde{\alpha})\langle \Delta \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle &= -\Re(i\tilde{\alpha})\langle (I - \Delta) \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle + \Re(i\tilde{\alpha})\langle \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle \\ &\leq 2\tilde{\alpha}\|\hat{u}\|_2\|\tilde{u}^{\Psi_1}\|_2 \leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + 16\tilde{\alpha}^2\|\hat{u}\|_2^2.\end{aligned}$$

Then, we tackle with the nonlinear terms. For the simple terms, one has

$$\begin{aligned}|\Re(i\tilde{\beta})\langle |\hat{u}|^2 \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle| &\leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + 4\tilde{\beta}^2\|\hat{u}\|_{L^6}^6 \leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + 4\tilde{\beta}^2C\|\hat{u}\|_1^6, \\ |\Re(\tilde{r} + i\tilde{\delta})\langle |\hat{v}|^2 \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle| &\leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + 4(\tilde{r}^2 + \tilde{\delta}^2)\|\hat{v}\|^2\|\hat{u}\|^2 \\ &\leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + 4(\tilde{r}^2 + \tilde{\delta}^2)C\|\hat{u}\|_1^2\|\hat{v}\|_1^4.\end{aligned}$$

The tricky one can be dealt with through interpolation inequality,

$$\begin{aligned}&|\Re(1 + i\beta)\langle |u|^2 u - |\hat{u}|^2 \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle| \\ &\leq \sqrt{1 + \beta^2}\|\tilde{u}^{\Psi_1} + \Psi_1\|(|u| + |\hat{u}|)^2\|\tilde{u}^{\Psi_1}\|_2 \\ &\leq \frac{1}{32}\|\tilde{u}^{\Psi_1}\|_2^2 + 8(1 + \beta^2)\|\tilde{u}^{\Psi_1} + \Psi_1\|_{L^4}^2\|u\| + \|\hat{u}\|_{L^8}^4 \\ &\leq \frac{1}{32}\|\tilde{u}^{\Psi_1}\|_2^2 + (1 + \beta^2)C_1\|\tilde{u}^{\Psi_1}\|_2\|\tilde{u}^{\Psi_1}\|(\|u\|_1^4 + \|\hat{u}\|_1^4) + (1 + \beta^2)C_2\|\Psi_1\|_1^2(\|u\|_1^4 + \|\hat{u}\|_1^4) \\ &\leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + C(\beta)[\|\tilde{u}^{\Psi_1}\|_2^2(\|u\|_1^8 + \|\hat{u}\|_1^8) + \|\Psi_1\|_1^2(\|u\|_1^4 + \|\hat{u}\|_1^4)],\end{aligned}$$

and similarly,

$$\begin{aligned}&|\Re(r + i\delta)\langle |v|^2 u - |\hat{v}|^2 \hat{u}, (I - \Delta) \tilde{u}^{\Psi_1} \rangle| \\ &\leq \sqrt{r^2 + \delta^2}(\|\tilde{v}\|(\|u\| + \|\hat{u}\|)\|\tilde{u}^{\Psi_1}\|_2 + \|\tilde{u}\|(\|v\| + \|\hat{v}\|)\|\tilde{u}^{\Psi_1}\|_2) \\ &\leq \sqrt{r^2 + \delta^2}(\|\tilde{v}\|(\|u\|^2 + \|v\|^2 + \|\hat{u}\|^2 + \|\hat{v}\|^2)\|\tilde{u}^{\Psi_1}\|_2 + \|\tilde{u}\|(\|v\|^2 + \|\hat{v}\|^2)\|\tilde{u}^{\Psi_1}\|_2) \\ &\leq \frac{1}{32}\|\tilde{u}^{\Psi_1}\|_2^2 + C(r, \delta)[\|\tilde{v}^{\Psi_2}\|_2^2(\|u\|_1^8 + \|\hat{u}\|_1^8 + \|v\|_1^8 + \|\hat{v}\|_1^8) \\ &\quad + \|\Psi_2\|_1^2(\|u\|_1^4 + \|\hat{u}\|_1^4 + \|v\|_1^4 + \|\hat{v}\|_1^4)] + \frac{1}{16}\|\tilde{v}^{\Psi_2}\|_2^2 + \frac{1}{32}\|\tilde{u}^{\Psi_1}\|_2^2 \\ &\quad + C(r, \delta)[\|\tilde{u}^{\Psi_1}\|_2^2(\|v\|_1^8 + \|\hat{v}\|_1^8) + \|\Psi_1\|_1^2(\|v\|_1^4 + \|\hat{v}\|_1^4)] \\ &\leq \frac{1}{16}\|\tilde{u}^{\Psi_1}\|_2^2 + \frac{1}{16}\|\tilde{v}^{\Psi_2}\|_2^2 + C(r, \delta)[\|\tilde{u}^{\Psi_1}\|_2^2(\|u\|_1^8 + \|\hat{u}\|_1^8) + \|\Psi_1\|_1^2(\|u\|_1^4 + \|\hat{u}\|_1^4) \\ &\quad + \|\tilde{v}^{\Psi_2}\|_2^2(\|u\|_1^8 + \|\hat{u}\|_1^8 + \|v\|_1^8 + \|\hat{v}\|_1^8) + \|\Psi_2\|_1^2(\|u\|_1^4 + \|\hat{u}\|_1^4 + \|v\|_1^4 + \|\hat{v}\|_1^4)].\end{aligned}$$

Applying a similar calculation to (12) and putting these bounds altogether, we yield that

$$\begin{aligned}&\partial_t(\|\tilde{u}^{\Psi_1}\|_1^2 + \|\tilde{v}^{\Psi_2}\|_1^2) \\ &\leq C(\mu, \eta, \beta, r, \delta)(1 + \|u\|_1^8 + \|\hat{u}\|_1^8 + \|v\|_1^8 + \|\hat{v}\|_1^8)(\|\tilde{u}^{\Psi_1}\|_1^2 + \|\tilde{v}^{\Psi_2}\|_1^2) \\ &\quad + C(\mu, \eta, \beta, r, \delta)(1 + \|u\|_1^4 + \|\hat{u}\|_1^4 + \|v\|_1^4 + \|\hat{v}\|_1^4)(\|\Psi_1\|_1^2 + \|\Psi_2\|_1^2) \\ &\quad + C(\tilde{\mu}^2 + \tilde{\eta}^2 + \tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{r}^2 + \tilde{\delta}^2)(\|\hat{u}\|_2^6 + \|\hat{v}\|_2^6).\end{aligned}$$

Noting that $(\mu, \eta, \beta, r, \delta)$ inside $C(\cdot)$ is fixed in the current circumstance, we shall omit them afterwards.

Therefore, applying Gronwall's inequality yields

$$\begin{aligned}\|\tilde{u}_{t \wedge \tau_R}^{\Psi_1}\|_1^2 + \|\tilde{v}_{t \wedge \tau_R}^{\Psi_2}\|_1^2 &\leq Ce^{C(R)t} \int_0^{t \wedge \tau_R} \left[(1 + \|u\|_1^4 + \|\hat{u}\|_1^4 + \|v\|_1^4 + \|\hat{v}\|_1^4)\|\Psi_s\|_1^2 \right. \\ &\quad \left. + (\tilde{\mu}^2 + \tilde{\eta}^2 + \tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{r}^2 + \tilde{\delta}^2)(\|\hat{u}_s\|_2^6 + \|\hat{v}_s\|_2^6) \right] ds,\end{aligned}$$

where $\Psi_s := (\Psi_{1s}, \Psi_{2s})$. Taking expectation, applying Cauchy–Schwartz inequality and using the fact that $\mathbf{E}\|\Psi_s\|_1^4 \leq C(\mathbf{E}\|\Psi_s\|_1^2)^2$, one obtains

$$\begin{aligned} & \mathbf{E}\|\tilde{u}_{t \wedge \tau_R}^{\Psi_1}\|_1^2 + \mathbf{E}\|\tilde{v}_{t \wedge \tau_R}^{\Psi_2}\|_1^2 \\ & \leq Ce^{C(R)t} \int_0^{t \wedge \tau_R} \left[\mathbf{E}\|\Psi_s\|_1^2 (1 + \mathbf{E}\|u_s\|_1^8 + \mathbf{E}\|\hat{u}_s\|_1^8 + \mathbf{E}\|v_s\|_1^8 + \mathbf{E}\|\hat{v}_s\|_1^8)^{\frac{1}{2}} \right. \\ & \quad \left. + (\tilde{\mu}^2 + \tilde{\eta}^2 + \tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{\gamma}^2 + \tilde{\delta}^2) \mathbf{E}(\|\hat{u}_s\|_2^6 + \|\hat{v}_s\|_2^6) \right] ds. \end{aligned}$$

Using the bound $x^n \leq n!a^{-n}e^{ax}$, applying Lemma 1 and observing that $\mathbf{E}\|\Psi_t\|_1^2 \leq (\|\tilde{Q}_1\|^2 + \|\tilde{Q}_2\|^2)/(2\sqrt{1+\alpha^2})$, we obtain

$$\begin{aligned} & \mathbf{E}\|\tilde{u}_{t \wedge \tau_R}^{\Psi_1}\|_1^2 + \mathbf{E}\|\tilde{v}_{t \wedge \tau_R}^{\Psi_2}\|_1^2 \\ & \leq Ce^{C(R)t + \nu(\|u_0\|_1^2 + \|v_0\|_1^2)} \left(\|\tilde{Q}_1\|^2 + \|\tilde{Q}_2\|^2 + \tilde{\mu}^2 + \tilde{\eta}^2 + \tilde{\alpha}^2 + \tilde{\beta}^2 + \tilde{\gamma}^2 + \tilde{\delta}^2 \right). \end{aligned}$$

The proof is thus finished. $\square$

At this stage, we are ready to establish our main result, which is also the complete version of Theorem 1.

Theorem 3. For any $\Xi \in \Lambda_0$, and positive constant $\nu \leq \frac{1}{12\|\tilde{Q}\|_1^2}$, there holds

$$\lim_{\hat{\Xi} \rightarrow \Xi} d_\nu(\mu^{\Xi}, \mu^{\hat{\Xi}}) = 0,$$

where $\hat{\Xi} \rightarrow \Xi$ is taken inside Λ and measured by the distance $d(\Xi, \hat{\Xi})$.

Proof. By (Theorem 1.1 of [6]), there exists a $t_0 > 0$ so that

$$d_\nu(P_t^{\Xi*} \mu^{\Xi}, P_t^{\Xi*} \mu^{\hat{\Xi}}) \leq \frac{1}{2} d_\nu(\mu^{\Xi}, \mu^{\hat{\Xi}}),$$

for any $t \geq t_0$. Next, we fix a $t \geq t_0$, then

$$\begin{aligned} d_\nu(\mu^{\Xi}, \mu^{\hat{\Xi}}) &= d_\nu(P_t^{\Xi*} \mu^{\Xi}, P_t^{\hat{\Xi}*} \mu^{\hat{\Xi}}) \leq d_\nu(P_t^{\Xi*} \mu^{\Xi}, P_t^{\Xi*} \mu^{\hat{\Xi}}) + d_\nu(P_t^{\Xi*} \mu^{\hat{\Xi}}, P_t^{\hat{\Xi}*} \mu^{\hat{\Xi}}) \\ &\leq \frac{1}{2} d_\nu(\mu^{\Xi}, \mu^{\hat{\Xi}}) + d_\nu(P_t^{\Xi*} \mu^{\hat{\Xi}}, P_t^{\hat{\Xi}*} \mu^{\hat{\Xi}}). \end{aligned} \quad (13)$$

Next, we carry out the estimate on $d_\nu(P_t^{\Xi*} \mu^{\hat{\Xi}}, P_t^{\hat{\Xi}*} \mu^{\hat{\Xi}})$, which is exactly the integral of $d_\nu(P_t^{\Xi*} \delta_{(u_0, v_0)}, P_t^{\hat{\Xi}*} \delta_{(u_0, v_0)})$, with δ being the δ -measure. The upper bound (5) yields that

$$\begin{aligned} & d_\nu(P_t^{\Xi*} \delta_{(u_0, v_0)}, P_t^{\hat{\Xi}*} \delta_{(u_0, v_0)}) \leq \mathbf{E} d_\nu((u_t, v_t), (\hat{u}_t, \hat{v}_t)) \\ & \leq \left(\mathbf{E}(\|u_t - \hat{u}_t\|_1^2 + \|v_t - \hat{v}_t\|_1^2) 2\mathbf{E}(e^{2\nu(\|u_t\|_1^2 + \|v_t\|_1^2)} + e^{2\nu(\|\hat{u}_t\|_1^2 + \|\hat{v}_t\|_1^2)}) \right)^{\frac{1}{2}}. \end{aligned}$$

One derives from Lemma 1 and Proposition 1 that

$$\begin{aligned} d_\nu(P_t^{\Xi*} \delta_{(u_0, v_0)}, P_t^{\hat{\Xi}*} \delta_{(u_0, v_0)}) &\leq Ce^{3\nu(\|u_0\|_1^2 + \|v_0\|_1^2)} (e^{C(R)t} d(\Xi, \hat{\Xi}) + \mathbf{E}\|u_t - u_{t \wedge \tau_R}\|_1 \\ &\quad + \mathbf{E}\|\hat{u}_t - \hat{u}_{t \wedge \tau_R}\|_1 + \mathbf{E}\|v_t - v_{t \wedge \tau_R}\|_1 + \mathbf{E}\|\hat{v}_t - \hat{v}_{t \wedge \tau_R}\|_1). \end{aligned}$$

Taking the integration, one has

$$\begin{aligned} d_\nu(P_t^{\Xi*} \mu^{\hat{\Xi}}, P_t^{\hat{\Xi}*} \mu^{\hat{\Xi}}) &\leq C \int_{H^1 \times H^1} e^{3\nu(\|u\|_1^2 + \|v\|_1^2)} \mu^{\hat{\Xi}}(d(u, v)) (e^{C(R)t} d(\Xi, \hat{\Xi}) + \mathbf{E}\|u_t - u_{t \wedge \tau_R}\|_1 \\ &\quad + \mathbf{E}\|\hat{u}_t - \hat{u}_{t \wedge \tau_R}\|_1 + \mathbf{E}\|v_t - v_{t \wedge \tau_R}\|_1 + \mathbf{E}\|\hat{v}_t - \hat{v}_{t \wedge \tau_R}\|_1). \end{aligned}$$

Since $\widehat{\Xi} \rightarrow \Xi$, $d(\Xi, \widehat{\Xi})$ is bounded. Then, one applies Lemma 1 to obtain uniformly that

$$\begin{aligned} & \int_{H^1 \times H^1} e^{3\nu(\|u\|_1^2 + \|v\|_1^2)} \mu^{\widehat{\Xi}}(d(u, v)) \\ &= \int_{H^1 \times H^1} e^{3\nu(\|u\|_1^2 + \|v\|_1^2)} P_t^{\widehat{\Xi}*} \mu^{\widehat{\Xi}}(d(u, v)) = \mathbf{E} e^{3\nu(\|\hat{u}_t\|_1^2 + \|\hat{v}_t\|_1^2)} \leq C. \end{aligned}$$

Combining (13), gives that

$$\begin{aligned} d_v(\mu^{\Xi}, \mu^{\widehat{\Xi}}) &\leq C e^{C(R)t} d(\Xi, \widehat{\Xi}) + C(\mathbf{E}\|u_t - u_{t \wedge \tau_R}\|_1 \\ &\quad + \mathbf{E}\|\hat{u}_t - \hat{u}_{t \wedge \tau_R}\|_1 + \mathbf{E}\|v_t - v_{t \wedge \tau_R}\|_1 + \mathbf{E}\|\hat{v}_t - \hat{v}_{t \wedge \tau_R}\|_1) \leq \text{I} + \text{II}. \end{aligned}$$

In view of Remark 1, for R sufficiently large, term II can be arbitrarily small after using the dominated convergence theorem. Term I can also be sufficiently small by letting $d(\Xi, \widehat{\Xi}) \rightarrow 0$. Hence, the claim is proven. $\square$

4. Conclusions

Since μ^{Ξ} and $\mu^{\widehat{\Xi}}$ correspond to equilibrium states of original and disturbed systems, respectively, we learn from Theorem 3 that 2D stochastic Ginzburg–Landau–Newell equations are stable under parameter and noise disturbances. This is of great practical significance, for instance, helping to simulate and observe the vortex motion of superconducting materials under strong external magnetic fields. The study of vortex motion will enhance the understanding of complex interactions within superconductors and guide innovation in superconducting technology, such as magnet design and improvement of power transmission systems. However, the stability results we obtained are not yet satisfactory, and in particular, we cannot provide an estimate of convergence speed. In future work, we will consider developing numerical methods and gaining inspiration before moving forward with theoretical research.

Author Contributions: Conceptualization, Y.Z.; methodology, Y.Z.; validation, J.W. and Y.Z.; formal analysis, J.W. and Y.Z.; writing—original draft preparation, Y.Z.; writing—review and editing, J.W.; project administration, J.W. and Y.Z.; funding acquisition, J.W. and Y.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The manuscript is supported by National Natural Science Foundation of China (No. 12071480, No. 11771297, No. 12261047), Scientific Research Program Funds of NUDT (No. 22-ZZCX-016).

Data Availability Statement: No new data were generated or analysed in this paper.

Acknowledgments: The authors are grateful for the reviewers' valuable comments that improved this paper.

Conflicts of Interest: The authors declare that there are no conflict of interests. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Abbreviation

The following abbreviation is used in this manuscript:

SGLNE stochastic Ginzburg–Landau–Newell equations

References

1. Coullet, P.; Elphick, C.; Gil, L.; Lega, J. Topological Defects of Wave Patterns. *Phys. Rev. Lett.* **1987**, *59*, 884–887. [\[CrossRef\]](#)
2. Gil, L.; Lega, L. *Defect in Waves*; Springer Series Synergetics; Springer: Berlin/Heidelberg, Germany, 1988; Volume 41, pp. 164–175.
3. Gao, H. Dynamical behavior of Ginzburg–Landau–Newell model. *Acta Math. Appl. Sin.* **1996**, *1*, 123–134.
4. Joets, A.; Ribotta, R. *Defects in Non-Linear Waves in Convection*; Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 1990; Volume 353.
5. Shen, T.; Huang, J. Ergodicity of 2D stochastic Ginzburg–Landau–Newell equations driven by degenerate noise. *Math. Methods Appl. Sci.* **2017**, *40*, 4812–4831.

6. Zheng, Y.; Huang, J. Exponential mixing of 2D stochastic Ginzburg-Landau-Newell equations driven by degenerate noise. *Stoch. Dyn.* **2020**, *20*, 2050004. [[CrossRef](#)]
7. Harier, M.; Mattingly, J.C. Spectral gaps in Wasserstein distances and the 2D stochastic Navier–Stokes equations. *Ann. Probab.* **2008**, *36*, 2050–2091. [[CrossRef](#)]
8. Hairer, M.; Mattingly, J.C.; Scheutzow, M. Asymptotic coupling and a general form of Harris’ theorem with applications to stochastic delay equations. *Probab. Theory Relat. Fields* **2011**, *149*, 223–259. [[CrossRef](#)]
9. Da Pato, G.; Zabczyk, J. *Ergodicity for Infinite Dimensional Systems*; Cambridge University Press: Cambridge, UK, 1996. Cambridge University Press: Cambridge, UK, 2009.
10. Tomás, C.; Han, X.Y. *Applied Nonautonomous and Random Dynamical Systems: Applied Dynamical Systems*; SpringerBriefs in Mathematics; Springer: Cham, Switzerland, 2016.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.