Symmetric Identities Involving the Extended Degenerate Central Fubini Polynomials ARISING FROM THE FERMIONIC $p$-ADIC INTEGRAL ON $\mathbb{Z}_p$

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1. Introduction

Special functions have an importance role in many areas of engineering, physics, mathematics, and other linked disciplines involving areas like quantum mechanics, mathematical physics, functional analysis, numerical analysis, differential equations, and so on (cf. [1–26]). In the family of special functions, special polynomials possess are also an area of special interest. Recently, central-type special polynomials (and their corresponding numbers), including central Bell, central Fubini, and central factorial polynomials (and numbers), are some the most studied families of special polynomials (cf. [2,3,6,10,15–20,24,25]). For example, the central factorial numbers in [3], the degenerate central factorial numbers in [17,20], the central Fubini polynomials, and numbers in [2,6,15,19,24], the central degenerate Fubini polynomials and numbers in [25], the central Bell numbers and polynomials in [18], and the degenerate central Bell polynomials and numbers in [16] have been considered, and many of their relations and properties have been examined and investigated. Motivated by these types of studies, here, we consider the two variable degenerate $w$-torsion central Fubini polynomials and we research some of their symmetric relations and properties. To do this, we start by reviewing the relevant definitions and notations.

1. Special Functions

In order to define the central-type special polynomials, we need to introduce the central Bell polynomials, central Fubini polynomials, and central factorial polynomials, and their corresponding numbers. These types of polynomials and numbers have been studied extensively in recent years due to their applications in various areas of mathematics and physics.

1.1. Central Bell Polynomials

The central Bell polynomials $B_n^{(c)}(x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{B_n^{(c)}}{n!} x^n = e^{e^x - 1}
$$

1.2. Central Fubini Polynomials

The central Fubini polynomials $F_n^{(c)}(x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{F_n^{(c)}}{n!} x^n = e^{e^x - 1 - \frac{x}{e^{e^x-1}}}
$$

1.3. Central Factorial Polynomials

The central factorial polynomials $F_n^{(c)}(x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{F_n^{(c)}}{n!} x^n = e^{e^x - 1 - \frac{x^{n+1}}{(n+1)!}}
$$

1.4. Central Degenerate Bell Polynomials

The central degenerate Bell polynomials $B_n^{(c)}(w, x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{B_n^{(c)}(w, x)}{n!} x^n = (1 + w)^{e^x - 1}
$$

1.5. Central Degenerate Fubini Polynomials

The central degenerate Fubini polynomials $F_n^{(c)}(w, x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{F_n^{(c)}(w, x)}{n!} x^n = (1 + w)^{e^x - 1 - \frac{x}{e^{(1+w)^x-1}}}
$$

1.6. Central Degenerate Factorial Polynomials

The central degenerate factorial polynomials $F_n^{(c)}(w, x)$ are defined by the following exponential generating function:

$$
\sum_{n=0}^\infty \frac{F_n^{(c)}(w, x)}{n!} x^n = (1 + w)^{e^x - 1 - \frac{x^{n+1}}{(n+1)!}}
$$

By using these central-type special polynomials and numbers, we can derive many interesting symmetric identities.
The two variable (abbreviated with “t.w.”) central Fubini polynomials are defined (cf. [15,19,24]) as follows

\[
\frac{1}{1 - y(e^{\frac{1}{2}} - e^{-\frac{1}{2}})} e^{xt} = \sum_{j=0}^{\infty} F_{j,c}(x;y) \frac{t^j}{j!}
\]  

(1)

where \(|t| < \log|y^{-1}||\) for \(y \neq 1\) and \(|t| < 2\pi\) for \(y = 1\). Upon setting \(x = 0, F_{j,c}(0;y) = F_{j,c}(y)\) becomes central Fubini polynomials. Also, upon setting \(y = 1, F_{j,c}(0;1) := F_{j,c}\) becomes the usual central Fubini numbers (cf. [2,6,10,14,19,24]).

The central factorial numbers of the second kind \(T(j,k)\), for \(j,k \geq 0\), are provided as follows (cf. [3])

\[
x^j = \sum_{k=0}^{\infty} T(j,k)x^k, (j \geq 0),
\]

(2)

where \(x^0 = 1\) and \(x^k = x(x + \frac{k}{2} - 1)(x + \frac{k}{2} - 2)\cdots(x + \frac{k}{2} + 1)\) for \(k \geq 1\). The generating function of \(T(j,k)\) is presented by:

\[
\frac{1}{k!} \left(e^x - e^{-\frac{x}{2}}\right)^k = \sum_{j=0}^{\infty} T(j,k) \frac{t^j}{j!}.
\]

The degenerate exponential function is defined for \(\lambda \in \mathbb{R}\), by (cf. [4,16,17,20,21,23,25])

\[
e_{\lambda}^x(t) = (1 + \lambda t)^x \quad \text{with} \quad e_{\lambda}^1(t) := (1 + \lambda t)^{\frac{t}{\lambda}}.
\]

(3)

The series representations of the function \(e_{\lambda}^x(t)\) is presented as follows:

\[
e_{\lambda}^x(t) = \sum_{u=0}^{\infty} (x)_{u,\lambda} \frac{t^u}{u!},
\]

(4)

where \((x)_{0,\lambda} := 1\) and \((x)_{u,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(u-1)\lambda)\) for \(u \geq 1\).

For \(k \in \mathbb{N}\), the degenerate form of the central factorial numbers and polynomials of the second kind are provided as follows (cf. [17,20])

\[
\frac{1}{k!} \left(e_{\lambda}^1(t) - e_{\lambda}^{-\frac{1}{2}}(t)\right)^k = \sum_{j=k}^{\infty} T_{\lambda}(j,k) \frac{t^j}{j!}
\]

(5)

and

\[
\frac{1}{k!} \left(e_{\lambda}^1(t) - e_{\lambda}^{-\frac{1}{2}}(t)\right)^k = \sum_{j=k}^{\infty} T_{\lambda}(j,k : x) \frac{t^j}{j!},
\]

(6)

respectively. It is obvious that \(\lim_{\lambda \to 0} T_{\lambda}(j,k) = T(j,k)\).

The type 2 degenerate central Fubini polynomials of t.w. are defined as follows (see [25])

\[
\sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(x;y) \frac{t^u}{u!} = \frac{e_{\lambda}^x(t)}{1 - y(e_{\lambda}^1(t) - e_{\lambda}^{-\frac{1}{2}}(t))},
\]

(7)

where \(|t| < \log|y^{-1}||\) for \(y \neq 1\) and \(|t| < 2\pi\) for \(y = 1\). Upon setting \(x = 0\) in (7), \(F_{0,\lambda}^{(C)}(0;y) := F_{0,\lambda}^{(C)}(y)\) are termed the type 2 degenerate central Fubini polynomials and upon letting \(x + 1 = y = 1\) in (7), \(F_{u,\lambda}^{(C)}(0;1) = F_{u,\lambda}^{(C)}\) are termed the degenerate central Fubini numbers. So, we can write that

\[
\sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(y) \frac{t^u}{u!} = \frac{1}{1 - y(e_{\lambda}^1(t) - e_{\lambda}^{-\frac{1}{2}}(t))} \quad \text{and} \quad \sum_{u=0}^{\infty} F_{u,\lambda}^{(C)}(y) \frac{t^u}{u!} = \frac{1}{1 - (e_{\lambda}^1(t) - e_{\lambda}^{-\frac{1}{2}}(t))}.
\]
Note that \( \lim_{x \to 0} F^{(C)}_{\mu, \lambda}(x; y) = F^{(C)}_{\mu}(x; y) \).

Similarly, the following notations hold for \( p \) being a fixed odd prime number: \( \mathbb{Z}_p \)
denotes the ring of \( p \)-adic integers, \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers, and \( \mathbb{C}_p \)
denotes the completion of the algebraic closure of \( \mathbb{Q}_p \). The normalized \( p \)-adic norm is provided by \( |p|_p = p^{-1} \). For a continuous function \( f : \mathbb{Z}_p \to \mathbb{C}_p \), the fermionic \( p \)-adic integral of \( f \) is provided (cf. [1,5,8,11–14,21,22,24,26]) as follows:

\[
\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x = 0}^{p^n-1} f(x)(-1)^x, \quad (8)
\]

where \( \mu_{-1}(x + p^n\mathbb{Z}_p) = (-1)^x \). It is apparent from (8) that

\[
2f(0) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x + 1) d\mu_{-1}(x). \quad (9)
\]

With the construction and introduction of the fermionic \( p \)-adic integral (cf. [12,22]), it has been utilized for not only integral representations of many special numbers, polynomials, and functions, but also to provide a deep analysis of many families of special numbers and polynomials, such as Euler, tangent, Boole, Genocchi, Changhee, Frobenius–Euler, and Fubini polynomials and numbers (cf. [1,5,8,11–14,21,22,24,26]). One of the most useful aims of the fermionic \( p \)-adic integral (abbreviated with "f.p.a.i." ) is to acquire more formulas and properties for special numbers and polynomials. In recent years, by utilizing the fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \), the symmetric identities of some special polynomials, such as \( q \)-Frobenius–Euler polynomials under \( S_2 \) in [1], Carlitz’s-type twisted \( (h, q) \)-tangent polynomials in [5], power sums polynomials in [11], and degenerate \( q \)-Euler polynomials in [21], have been studied and investigated in detail. In this study, we focus on a novel extension of the degenerate central Fubini polynomials. First, we introduce the two variable degenerate (abbreviated as "t.w.d.") \( w \)-torsion central Fubini polynomials. Then, we provide a f.p.a.i. representation of the degenerate \( w \)-torsion central Fubini polynomials, through which we acquire diverse novel symmetric identities for the degenerate \( \tilde{w} \)-torsion central Fubini polynomials. Lastly, by using some series manipulation methods, we further obtain the identity of symmetry for the t.w.d. \( \tilde{w} \)-torsion central Fubini polynomials.

Let \( t \in \mathbb{Z}_p \) with \( |t|_p < p^{-\frac{1}{\lambda - 1}} \). The f.p.a.i. representations of the polynomials \( F_{\mu, \lambda}^{(C)}(y) \)
and \( F_{\mu, \lambda}^{(C)}(x; y) \) are presented by (1) and (9), respectively, as follows (cf. [15,19]):

\[
\frac{1}{2} \int_{\mathbb{Z}_p} (1 - xy(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}})) z d\mu_{-1}(x) = \frac{1}{1 - y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}})} = \sum_{u=0}^{\infty} F_{\mu, \lambda}^{(C)}(y) \frac{\lambda^u}{u!}, \quad (10)
\]

and

\[
\frac{1}{2} e^{xt} \int_{\mathbb{Z}_p} (1 - x(y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}}) z) d\mu_{-1}(z) = \frac{e^{xt}}{1 - y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}})} = \sum_{u=0}^{\infty} F_{\mu, \lambda}^{(C)}(x; y) \frac{\lambda^u}{u!}, \quad (11)
\]

Also, the f.p.a.i. representations of polynomials \( F_{\mu, \lambda}^{(C)}(y) \) and \( F_{\mu, \lambda}^{(C)}(x; y) \) are provided by (7) and (9), respectively, as follows (see [25]):

\[
\frac{1}{2} \int_{\mathbb{Z}_p} (1 - x(y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}}))) z d\mu_{-1}(x) = \frac{1}{1 - y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}})} = \sum_{u=0}^{\infty} F_{\mu, \lambda}^{(C)}(y) \frac{\lambda^u}{u!}, \quad (12)
\]

and

\[
\frac{1}{2} e^{xt} \int_{\mathbb{Z}_p} (1 - x(y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}}))) z d\mu_{-1}(z) = \frac{e^{xt}}{1 - y(e^\frac{1}{\lambda} - e^{-\frac{1}{\lambda}})} = \sum_{u=0}^{\infty} F_{\mu, \lambda}^{(C)}(x; y) \frac{\lambda^u}{u!}, \quad (13)
\]
where $\lambda, t \in \mathbb{C}_p$ with $|t\lambda| < p^{(1-p)^{-1}}$.

From (3), (12), and (13), we see that

$$
\frac{e^\lambda_z(t)}{2} \int_{\mathbb{Z}_p} (1)^z(y(e^{1\lambda}(t) - e^{-\lambda}(t)))^z d\mu_{-1}(z)
= \left(\sum_{u=0}^\infty \frac{F_u^{(C)}(y) t^u}{u!}\right) = \sum_{u=0}^\infty \frac{F_u^{(C)}(x; y) t^u}{u!},
$$

which yields $u \geq 0$, such that

$$
P_u^{(C)}(x; y) = \sum_{l=0}^u (x)_{\lambda, \lambda} P_u^{(C)}(y) \left(\frac{u}{l}\right).
$$

Also, it is observed that

$$
\frac{1 - (e^{1\lambda}(t) - e^{-\lambda}(t))y^k}{1 - (e^{1\lambda}(t) - e^{-\lambda}(t))y} = \sum_{i=0}^{k-1} y^i(e^{1\lambda}(t) - e^{-\lambda}(t))^i = \sum_{i=0}^{k-1} \sum_{l=0}^{i} \left(\begin{array}{c} i \\ l \end{array}\right) (-1)^{i-l} y^i e^{-\lambda}(t)
= \sum_{u=0}^\infty \frac{1}{u!} \delta^{(u)}(0)_{\lambda, \lambda} \left(\begin{array}{c} u \\ l \end{array}\right) \sum_{i=0}^{k-1} y^i \left(\sum_{i=0}^{k-1} y^{i} T_2(u, i) \right) \left(\frac{u}{l!}\right),
$$

where $\delta f(x) := f(x + \frac{1}{2}) - f(x - \frac{1}{2})$ and (cf. [19])

$$
\delta^n(0)_{\lambda, \lambda} = n!T_2(\lambda, u).
$$

2. Main Results

In this section, we introduce the two variable degenerate $w$-torsion central Fubini polynomials in (18) by means of their exponential generating function. Thereafter, we provide a fermionic $p$-adic integral representation of these polynomials. Using this representation, we investigate two symmetric identities for these polynomials in Theorems 1 and 2, using some special $p$-adic integral techniques. Moreover, by utilizing some series manipulation methods, we obtain a more symmetric identity (Theorem 3) for the two variable degenerate $w$-torsion central Fubini polynomials.

We first provide our main definition as follows.

**Definition 1.** For $w \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we define the two variable degenerate $w$-torsion central Fubini polynomials as follows

$$
\sum_{u=0}^\infty \frac{F_u^{(C)}(x; y) t^u}{u!} = \frac{e^{\lambda_z}(t)}{1 - y^u(e^{1\lambda}(t) - e^{-\lambda}(t))},
$$

where $|t| < |\log y^{-1}|$ for $y \neq 1$ and $|t| < 2\pi$ for $y = 1$.

We now analyze some special cases of polynomials (18), as follows:

**Remark 1.** In some special cases, $F_{u, w}(y|\lambda) := F_{u, w}(0; y|\lambda)$ and $F_{u, w}(1|\lambda) := F_{u, w}(0; 1|\lambda)$ are called the degenerate $w$-torsion central Fubini polynomials and the degenerate $w$-torsion central Fubini numbers, respectively.
Remark 2. Upon setting \( w = 1 \) in (18), the polynomials \( F_{u,w}^{(C)}(x; y|\lambda) \) become the usual t.w.d. Fubini polynomials \( F_{u}^{(C)}(x; y|\lambda) \) in (13).

Remark 3. Letting \( w - 1 = \lambda \to 0 \) in (18), the polynomials \( F_{u,w}^{(C)}(x; y|\lambda) \) become the familiar t.w. central Fubini polynomials \( F_{u}^{(C)}(x; y) \) in (11).

Similar to (12) and (13), for \( w \in \mathbb{N} \), the f.p.a.i. representations of polynomials \( F_{u,w}^{(C)}(y|\lambda) \) and \( F_{u,w}^{(C)}(x; y|\lambda) \) are provided by
\[
\sum_{u=0}^{\infty} F_{u,w}^{(C)}(y|\lambda) \frac{t^{u}}{u!} = \frac{2}{1 - (e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w}y^{w}} \quad \text{and} \quad \sum_{u=0}^{\infty} F_{u,w}^{(C)}(x; y|\lambda) \frac{t^{u}}{u!} = \frac{e_{\lambda}^{1}(t)}{2} \int_{Z_{p}} \left( - \left( e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t) \right)^{w} y^{w} \right) \mu_{-1}(x),
\]
where \( \lambda, t \in \mathbb{C}_{p} \) with \( |\lambda t| < p^{(1-p)^{-1}} \). We observe from (12) and (19) that
\[
\frac{\int_{Z_{p}} \left( - \left( e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t) \right) y \right)^{w} \mu_{-1}(x)}{\int_{Z_{p}} \left( - \left( e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t) \right)^{w} y^{w} \right) \mu_{-1}(x)} = \frac{1 - (e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w} y^{w}}{1 - (e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w}} = \sum_{j=0}^{w-1} \frac{1}{j!} (e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{j} y^{j}.
\]

Now, we provide our first symmetric identity for the two variable degenerate \( w \)-torsion central Fubini polynomials as follows.

**Theorem 1.** The following symmetric identity
\[
\sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_{1}-1} y^{w_{1} i} \delta^{w_{2} i}(0)_{m,\lambda} F_{u-m,w_{1}}^{(C)}(y|\lambda) = \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{w_{2}-1} y^{w_{2} i} \delta^{w_{1} i}(0)_{m,\lambda} F_{u-m,w_{2}}^{(C)}(y|\lambda)
\]
holds for \( 0 \leq u \), where \( w_{1}, w_{2} \) are two odd numbers.

**Proof.** For \( w_{1}, w_{2} \in \mathbb{N} \), we consider that
\[
I = \int_{Z_{p}} (-y^{w_{1}}(e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w_{1}}) y \mu_{-1}(x_{1}) \int_{Z_{p}} (-y^{w_{2}}(e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w_{2}}) y \mu_{-1}(x_{2})
\]
which is invariant under the interchange of \( w_{1} \) and \( w_{2} \). Then, using (23), we find that
\[
I = \int_{Z_{p}} (-y^{w_{1}}(e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w_{1}})^{w_{2}} y \mu_{-1}(x) \times \left( \frac{\int_{Z_{p}} (-y^{w_{2}}(e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w_{2}})^{w_{1}} y \mu_{-1}(x)}{\int_{Z_{p}} (-y^{w_{1}}(e_{\lambda}^{1}(t) - e_{\lambda}^{-1/2}(t))^{w_{1}})^{w_{2}} y \mu_{-1}(x)} \right).
\]
First, we observe that

\[
\begin{align*}
&\frac{\int_{\mathbb{Z}} (-y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}} (-y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2)^x d\mu_{-1}(x)} = \frac{1 - y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2}{1 - y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_2}} \\
&= \sum_{i=0}^{w_2-1} y^{w_1i}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1i} \\
&= \sum_{i=0}^{w_2-1} y^{w_2i}(\sum_{l=0}^{w_2i} \frac{1}{i!}(-1)^{i-l}e^{-\frac{1}{\lambda}}(t)) \\
&= \sum_{i=0}^{w_2-1} y^{w_2i}(-1)^{w_2i-i} \frac{1}{i!} e^{x_\lambda(t)} \\
&\sum_{i=0}^{\infty} \left( \sum_{l=0}^{w_2i} \frac{1}{i!} \delta^{w_2i}(0)_{u,\lambda} \right) \int_{\mathbb{Z}} (-y^{w_2}w_2!T_{2,\lambda}(u, w_2i)) \frac{i^\mu}{\mu!}.
\end{align*}
\]

It can be discovered from (24) and (25) that

First, we observe that

\[
I = \int_{\mathbb{Z}} (-y^{w_1}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1})^x d\mu_{-1}(x) \times \left( \frac{\int_{\mathbb{Z}} (-y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}} (-y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2)^x d\mu_{-1}(x)} \right)
\]

Interchanging the roles of \(w_1\) and \(w_2\), following (23), it can be written that

\[
I = \int_{\mathbb{Z}} (-y^{w_2}(e^x_\lambda(t) - 1)^{w_2})^x d\mu_{-1}(x) \times \left( \frac{\int_{\mathbb{Z}} (-y^{w_1}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}} (-y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2)^x d\mu_{-1}(x)} \right).
\]

Note that

\[
\begin{align*}
&\frac{\int_{\mathbb{Z}} (-y^{w_1}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}} (-y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2)^x d\mu_{-1}(x)} = \frac{1 - y^{w_1}y^{w_2}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}w_2}{1 - y^{w_1}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1}} \\
&= \sum_{i=0}^{w_1-1} y^{w_1i}(e^x_\lambda(t) - e^{-1}_\lambda(t))^{w_1i} \\
&= \sum_{i=0}^{w_1-1} y^{w_1i}(\sum_{l=0}^{w_1i} \frac{1}{i!}(-1)^{i-l}e^{-\frac{1}{\lambda}}(t)) \\
&= \sum_{i=0}^{\infty} \left( \sum_{l=0}^{w_1i} \frac{1}{i!} \delta^{w_1i}(0)_{u,\lambda} \right) \int_{\mathbb{Z}} (-y^{w_1}w_1!T_{2,\lambda}(u, w_1i)) \frac{i^\mu}{\mu!}.
\end{align*}
\]

It can be discovered from (27) and (28) that
The following symmetric identity holds for $0 \leq u$:

$$F_{u}^{(C)}(y|\lambda) = \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{u-1} y^{i} T_{2,\lambda}(u, w_{1}) F_{u-m,u_{2}}^{(C)}(y|\lambda).$$

**Corollary 1.** From Theorem 1 and Equation (17), the following relationship

$$\sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{u-1} y^{i} T_{2,\lambda}(u, w_{1}) F_{u-m,u_{2}}^{(C)}(y|\lambda)$$

hold for $0 \leq u$, with $w_{1}, w_{2}$ being two odd numbers.

When selecting $w_{1} = 1$ in (31), we obtain the following relationship.

**Remark 5.** The following relationship holds for $0 \leq u$:

$$F_{u}^{(C)}(y|\lambda) = \sum_{m=0}^{u} \binom{u}{m} \sum_{i=0}^{u-1} y^{i} T_{2,\lambda}(u, i) F_{u-m,u_{2}}^{(C)}(y|\lambda).$$

Our second symmetric identity for the two variable degenerate $w$-torsion central Fubini polynomials is as follows.

**Theorem 2.** The following symmetric identity

$$\sum_{i=0}^{u_{1}-1} \sum_{j=0}^{u_{2}-1} \binom{w_{2}i}{l} y^{w_{2}i} (-1)^{l} F_{u_{1},w_{1}}^{(C)}(\frac{w_{2}i}{2} - l, y|\lambda)$$

holds for $0 \leq u$, with $w_{1}, w_{2}$ being two odd numbers.

**Proof.** It can be computed from (23) that
\[ I = \int_{Z_p} (-y^{w_1} (e_{2,1}^t(t) - e_{\lambda, z}^t(t)))^x d\mu_{-1}(x) \times \left( \frac{\int_{Z_p} (-y^{w_2} (e_{2,1}^t(t) - e_{\lambda, z}^t(t)))^x d\mu_{-1}(x)}{\int_{Z_p} (-y^{w_1 w_2} (e_{2,1}^t(t) - e_{\lambda, z}^t(t)))^x d\mu_{-1}(x)} \right) \]

The following relationship holds for \( 0 \leq u \):

\[ F_u(y|\lambda) = \sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2} \left( \frac{w_1^i}{i!} \right) y^{w_1 l} (-1)_{l,\lambda} F_{u,w_2}^{(C)} \left( \frac{w_1^i}{2} i - l, y|\lambda \right). \]
Theorem 3. The polynomials \( F_{m,w}(v; \gamma | \lambda) \) fulfill the following identity for \( m \in \mathbb{Z}_{\geq 0} \) and \( a, b \in \mathbb{R} \) and \( m \geq 0 \):

\[
\sum_{k=0}^{u} \binom{u}{k} F_{u-k,w}^{(C)}(bx; y|\lambda)b^{u-k}F_{k,w}^{(C)}(bx; y|\lambda) = \sum_{k=0}^{u} \binom{u}{k} F_{k,w}^{(C)}(bx; y|\lambda)a^{k}F_{u-k,w}^{(C)}(bx; y|\lambda)b^{u-k}.
\] (37)

Proof. We choose that

\[
\Upsilon = \frac{e^{2\lambda x}(abt)}{(1 - \frac{t}{e^t} - \frac{1}{e^t})^{w}y^{w}}\left(1 - \frac{t}{e^t} - \frac{1}{e^t}\right)^{w}y^{w}
\]

which is symmetric in \( a \) and \( b \). We compute from (18) that

\[
\Upsilon = \sum_{u=0}^{\infty} F_{u,w}^{(C)}(bx; y|\lambda)\frac{(at)^{u}}{u!}
\]

and, in the same way,

\[
\Upsilon = \sum_{u=0}^{\infty} F_{u,w}^{(C)}(bx; y|\lambda)\frac{(bt)^{u}}{u!}
\]

which implies the coveted relation (37). \( \square \)

3. Further Remarks

In this part, we observe further remarks for the two variable degenerate \( w \)-torsion central Fubini polynomials related to the degenerate central factorial polynomials of the second kind. Also, eventually, we provide a representation of the degenerate differential operator on the two variable degenerate \( w \)-torsion central Fubini polynomials.

We observe from (6) and (18) that

\[
\sum_{j=0}^{\infty} F_{ij,w}^{(C)}(x; y|\lambda)\frac{j^{i}}{j!} = \frac{e_{\lambda}^{x}(t)}{1 - y^{w}\left(e_{\lambda}^{x}(t) - e_{\lambda}^{-\frac{1}{2}}(t)\right)^{w}}
\]

\[
= \sum_{k=0}^{\infty} y^{kw}k! \sum_{j=0}^{\infty} T_{\lambda}(j, k : x) \frac{j^{i}}{j!}
\]

which yields \( j \geq 0 \), such that

\[
F_{ij,w}^{(C)}(x; y|\lambda) = \sum_{k=0}^{j} T_{\lambda}(j, k : x)y^{kw}k!,
\] (38)

which is the relation between the two variable degenerate \( w \)-torsion central Fubini polynomials and the degenerate central factorial polynomials of the second kind.
The degenerate differential operator is considered by Kim et al. [23] as follows:

\[
\left( \frac{x d}{dx} \right)_{k,\lambda} = \left( \frac{x d}{dx} \right) \left( \frac{x d}{dx} - \lambda \right) \left( \frac{x d}{dx} - 2\lambda \right) \cdots \left( \frac{x d}{dx} - (k-1)\lambda \right).
\]  

(39)

From (39), we have

\[
\left( \frac{x d}{dx} \right)_{k,\lambda} x^n = (n)_{k,\lambda} x^n.
\]

Let \( f \) be a formal power series written as \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( k \geq 0 \). Then, the degenerate differential operator of this series is given by

\[
\left( \frac{x d}{dx} \right)_{k,\lambda} f(x) = \sum_{n=0}^{\infty} a_n(n)_{k,\lambda} x^n.
\]

Kim et al. [23] found that the degenerate differential operator plays an important role in boson operators. Here, we focus on the representation of the degenerate differential operator on the t.w.d. \( w \)-torsion central Fubini polynomials (18). Here, we provide the following operator formula.

**Theorem 4.** The following relation

\[
\left( \frac{y d}{dy} \right)_{k,\lambda} F^{(\mathcal{C})}_{j,w}(x; y|\lambda) = \sum_{s=0}^{j} s! T_{\lambda}(j, s : x)(sw)_{k,\lambda} y^{sw},
\]

holds for \( w \in \mathbb{N} \) and \( k, j \geq 0 \).

**Proof.** By (38) and (39), we observe that

\[
\left( \frac{y d}{dy} \right)_{k,\lambda} F^{(\mathcal{C})}_{j,w}(x; y|\lambda) = \left( \frac{y d}{dy} \right)_{k,\lambda} \left\{ \sum_{s=0}^{j} s! T_{\lambda}(j, s : x)y^{sw} \right\}
\]

\[
= \sum_{s=0}^{j} s! T_{\lambda}(j, s : x) \left\{ \left( \frac{y d}{dy} \right)_{k,\lambda} y^{sw} \right\}
\]

\[
= \sum_{s=0}^{j} s! T_{\lambda}(j, s : x)(sw)_{k,\lambda} y^{sw}.
\]

\[ \square \]

4. Conclusions

In recent years, after constructions of \( p \)-adic \( q \)-integrals by Teakyun Kim, a Korean mathematician, \( p \)-adic \( q \)-integrals as well as some of their special cases have been utilized not only as integral representations of many special polynomials and functions, but also to deeply analyze many families of special polynomials and numbers, such as central Bell, central Fubini, Bernoulli, and Euler polynomials and numbers.

Also, by means of \( p \)-adic integrals, several special techniques and methods have been utilized to obtain symmetric identities. These identities cover and generalize a lot of well-known special formulas and properties for polynomials, such as Raabe formulas, extended recurrence formulas, Miki identities, and Carlitz identities.

From these motivations, in the presented study, we focused on a novel generalization of degenerate central Fubini polynomials. We first considered two variable degenerate \( w \)-torsion central Fubini polynomials by means of their exponential generating function. Then, we provided a fermionic \( p \)-adic integral representation of the two variable degenerate \( w \)-torsion central Fubini polynomials. From this representation, we investigated...
two symmetric identities (Theorems 1 and 2) for these polynomials, using special $p$-adic integral techniques.

Moreover, by utilizing series manipulation methods, we acquired a more symmetric identity (Theorem 3) for the two variable degenerate $w$-torsion central Fubini polynomials.

Furthermore, we provided a representation of the degenerate differential operator (Theorem 4) on the two variable degenerate $w$-torsion central Fubini polynomials related to the degenerate central factorial polynomials of the second kind.

To the best of our knowledge, the results obtained in this paper are novel and do not seem to be reported in the literature. The results presented here have the potential to be utilized in a lot of branches of statistics, probability, mathematics, engineering, and mathematical physics.


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