

Three Existence Results in the Fixed Point Theory

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Abstract: In the present paper, we obtain three results on the existence of a fixed point for nonexpansive mappings. Two of them are generalizations of the result for F -contraction, while third one is a generalization of a recent result for set-valued contractions.

Keywords: complete metric space; contraction; fixed point; graph; set-valued mapping

MSC: 47H09; 47H10; 54E50

1. Introduction

The work referenced here [1] was the starting point of the fixed point theory of nonexpansive maps, which is a growing field of research; see [2–7]. In particular, the convergence of iterated Bregman projections and of the alternating algorithm were studied in [2], fixed point theorems under Mizoguchi–Takahashi-type conditions were obtained in [3], Ćirić-type results were proved in [4], fixed point theory in modular function spaces was discussed in [5] and cyclic contractions were analyzed in [6]. Note that the research on nonexpansive maps in spaces with graphs is of great importance; see [8–12] and the references mentioned therein. In particular, fixed point results on metric spaces with a graph are obtained in [9,11], Reich-type contractions were studied in [8], hybrid methods were studied in [10] and the convergence of fixed points in graphical spaces was considered in [12]. In [13], D. Wardowski introduced an interesting class of mappings which contains Banach contractions and showed the existence of fixed points for these mappings. More precisely, we can assume that (X, ρ) is a complete metric space, $\tau > 0$, $F : (0, \infty) \rightarrow \mathbb{R}^1$ is a strictly increasing function and that $T : X \rightarrow X$ is a mapping such that for every pair of points $u, v \in X$ satisfying $u \neq v$,

$$F(\rho(T(u), T(v))) + \tau \leq F(\rho(u, v)).$$

Assuming two additional assumptions on F , D. Wardowski showed that the mapping of T has a unique fixed point. In the subsequent research it was shown that these two additional assumptions are not necessary. One of the examples of F is the function $\ln(\cdot)$, and in this case the Wardowski contraction is a strict contraction. Wardowski-type contractions were studied in [14–16]. In this work, we obtain three results on the existence of a fixed point for nonexpansive mappings in a complete metric space. Two of them are generalizations of the result by D. Wardowski for F -contraction, while the third one is a generalization of a recent result by S.-H. Cho for set-valued contractions [17].

Assume that (X, ρ) is a complete metric space. Let $\mathbb{N}$ be the set of all natural numbers. We assume that the sum over empty set is zero. For every element $x \in X$ and every real number $r > 0$, set

$$B(x, r) = \{y \in X : \rho(x, y) \leq r\}.$$

For every element $\xi \in X$ and each nonempty set $A \subset X$ set

$$\rho(\xi, A) = \inf\{\rho(\xi, \eta) : \eta \in A\}.$$



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2. The First Result

Assume that (X, ρ) is a complete metric space where ρ is a metric, $K \subset X$ is a nonempty closed set, $\tau > 0$, and $F : (0, \infty) \rightarrow \mathbb{R}^1$ is an increasing function satisfying

$$F(t) \leq F(s)$$

for each $s > t > 0$ and that $T : K \rightarrow X$ is an operator such that for every pair of points $x, y \in K$ satisfying $x \neq y$,

$$F(\rho(T(x), T(y))) + \tau \leq F(\rho(x, y)). \quad (1)$$

Set $T^0(x) = x, x \in K$.

D. Wardowski in [13] proved the existence of a fixed point of T in the case when $K = X_0$, assuming that F is strictly increasing. Here, we assume that F is merely increasing in general.

Since the function F is increasing, the following proposition holds [18].

Proposition 1. *There is a countable set $E \subset (0, \infty)$ such that the function F is continuous at every element $z \in (0, \infty) \setminus E$.*

Equation (1) implies the following proposition.

Proposition 2. *For every pair of elements $x, y \in K$, the relation $\rho(T(x), T(y)) \leq \rho(x, y)$ is true.*

Theorem 1. *Assume that $K_0 \subset X$ is a nonempty bounded set and for each integer $n \geq 1$ there is an element $x_n \in K_0$ for which $T^n(x_n) \in X$ is defined. Then, $x_* \in K$ can be found, satisfying $T(x_*) = x_*$.*

Proof. Fix $\theta \in K_0$. $M_0 > 0$ is found, for which

$$\rho(z, \theta) \leq M_0, z \in K_0. \quad (2)$$

Proposition 2 and (2) imply that for each $n, m \in \mathbb{N}$,

$$\begin{aligned} \rho(x_n, x_m) &\leq 2M_0, \\ \rho(T(x_n), T(\theta)) &\leq \rho(x_n, \theta) \leq M_0, \\ \rho(T(x_n), \theta) &\leq M_0 + \rho(\theta, T(\theta)). \end{aligned} \quad (3)$$

Let $\epsilon \in (0, 1)$. We show that the following property holds:

(P1) There exists a natural number n_0 such that for each integer $n > n_0$ and each integer $i \in [n_0, n)$, we have

$$\rho(T^i(x_n), T^{i+1}(x_n)) \leq \epsilon.$$

Choose an integer:

$$n_0 > 1 + \tau^{-1}(F(2M_0 + \rho(\theta, T(\theta))) - F(\epsilon)). \quad (4)$$

Let $n > n_0$ be an integer. In order to prove that (P1) holds in view of Proposition 2, it is enough to prove that

$$\rho(T^{n_0}(x_n), T^{n_0+1}(x_n)) \leq \epsilon.$$

Assume the contrary. Then, according to Proposition 2,

$$\rho(T^{n_0}(x_n), T^{n_0+1}(x_n)) \geq \epsilon, i = 0, \dots, n_0. \quad (5)$$

Equations (1) and (5) imply that for every $i \in \{0, \dots, n_0 - 1\} \setminus \{n_0 - 1\}$,

$$\begin{aligned} F(\rho(T^{i+1}(x_n), T^{i+2}(x_n))) + \tau &\leq F(\rho(T^i(x_n), T^{i+1}(x_n))), \\ F(\rho(T^{n_0}(x_n), T^{n_0+1}(x_n))) &\leq \tau(1 - n_0) + F(\rho(T(x_n), x_n)). \end{aligned} \quad (6)$$

It follows from (3), (5) and (6) that

$$\begin{aligned} F(\epsilon) &\leq F(\rho(T^{n_0}(x_n), T^{n_0+1}(x_n))) \leq \tau(1 - n_0) + F(2M_0 + \rho(\theta, T(\theta))), \\ \tau(n_0 - 1) &\leq F(2M_0 + \rho(\theta, T(\theta))) - F(\epsilon), \\ n_0 &\leq 1 + \tau^{-1}(F(2M_0 + \rho(\theta, T(\theta))) - F(\epsilon)). \end{aligned}$$

This contradicts inequality (4) and proves property (P1).

We show that the following property holds:

(P2) $n_0 \in \mathbf{N}$ is found, such that for each triplet of integers i, n_1, n_2 satisfying $n_1, n_2, i \geq n_0$, $i \leq n_1, n_2$,

$$\rho(T^i(x_{n_1}), T^i(x_{n_2})) \leq \epsilon.$$

Choose an integer:

$$n_0 > 1 + \tau^{-1}(F(2M_0) - F(\epsilon)). \quad (7)$$

Let $n_1, n_2 \geq n_0$ be integers. In view of Proposition 2, in order to show that property (P2) holds, it is enough to show that

$$\rho(T^{n_0}(x_{n_1}), T^{n_0}(x_{n_2})) \leq \epsilon.$$

Assume the contrary. Then,

$$\rho(T^{n_0}(x_{n_1}), T^{n_0}(x_{n_2})) > \epsilon. \quad (8)$$

Proposition 2 and (1), (2) and (8) imply that for each $i \in \{0, \dots, n_0 - 1\}$,

$$\begin{aligned} \rho(T^i(x_{n_1}), T^i(x_{n_2})) &> \epsilon, \\ F(\rho(T^{i+1}(x_{n_1}), T^{i+1}(x_{n_2}))) + \tau &\leq F(\rho(T^i(x_{n_1}), T^i(x_{n_2}))), \\ F(\rho(T^{n_0}(x_{n_1}), T^{n_0}(x_{n_2}))) &\leq (1 - n_0)\tau + F(\rho(x_{n_1}, x_{n_2})) \\ &\leq (1 - n_0)\tau + F(2M_0). \end{aligned} \quad (9)$$

According to (8) and (9),

$$F(\epsilon) \leq \tau(1 - n_0) + F(2M_0), \quad n_0 \leq 1 + \tau^{-1}(F(2M_0) - F(\epsilon)).$$

This contradicts (7). Therefore, (P2) holds.

We will prove that the following property is fulfilled:

(P3) $n_* \in \mathbf{N}$ is found, such that for each triplet $i_1, i_2, n \in \mathbf{N}$ satisfying $n_* \leq i_1, i_2 \leq n$,

$$\rho(T^{i_1}(x_n), T^{i_2}(x_n)) \leq \epsilon.$$

According to Proposition 1, we may assume that F is continuous at ϵ . Assume that property (P3) does not hold. Then, for any $k \in \mathbf{N}$, there are integers $i_{k,1}, i_{k,2}, n_k$ for which

$$k \leq i_{k,1} < i_{k,2} \leq n_k, \quad (10)$$

$$\rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k})) > \epsilon. \quad (11)$$

In view of property (P1), we may assume that for every integer $k \geq 1$,

$$\rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,1}+1}(x_{n_k})) \leq \epsilon. \quad (12)$$

Let $k \geq 1$. According to (11) and (12),

$$i_{k,1} + 1 < i_{k,2}$$

and we may assume that

$$\rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}-1}(x_{n_k})) \leq \epsilon. \quad (13)$$

Equations (1) and (13) imply that

$$\begin{aligned} \epsilon &< \rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k})) \\ &\leq \rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}-1}(x_{n_k})) + \rho(T^{i_{k,2}-1}(x_{n_k}), T^{i_{k,2}}(x_{n_k})) \\ &\leq \epsilon + \rho(T^{i_{k,2}-1}(x_{n_k}), T^{i_{k,2}}(x_{n_k})). \end{aligned} \quad (14)$$

Property (P1) and (10), (14) imply that

$$\lim_{k \rightarrow \infty} \rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k})) = \epsilon. \quad (15)$$

Clearly, for each $k \in \mathbf{N}$,

$$\begin{aligned} \rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k})) &\leq \rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,1}+1}(x_{n_k})) \\ &+ \rho(T^{i_{k,1}+1}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k})) + \rho(T^{i_{k,2}}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k})). \end{aligned} \quad (16)$$

According to (1) and (11), for each integer $k \geq 1$,

$$F(\rho(T^{i_{k,1}+1}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k}))) \leq F(\rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k}))) - \tau. \quad (17)$$

In view of Proposition 2 and (15), we may assume that there exists

$$\Delta = \lim_{k \rightarrow \infty} \rho(T^{i_{k,1}+1}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k})). \quad (18)$$

It follows from (15), (17) and (18) that

$$0 \leq \Delta \leq \epsilon. \quad (19)$$

Property (P1) and (15), (18) and (19) imply that

$$\Delta = \epsilon \quad (20)$$

and

$$\epsilon = \lim_{k \rightarrow \infty} \rho(T^{i_{k,1}+1}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k})). \quad (21)$$

Since the function F is continuous at ϵ , Equations (15) and (21) imply that

$$\begin{aligned} F(\epsilon) &= \lim_{k \rightarrow \infty} F(\rho(T^{i_{k,1}+1}(x_{n_k}), T^{i_{k,2}+1}(x_{n_k}))) \\ &= \lim_{k \rightarrow \infty} F(\rho(T^{i_{k,1}}(x_{n_k}), T^{i_{k,2}}(x_{n_k}))). \end{aligned}$$

This contradicts (17) and proves property (P3).

Let $\epsilon > 0$. Property (P3) implies that there exists $n_0 \in \mathbf{N}$ such that for any triplet of integers i_1, i_2, n satisfying $n_0 \leq i_1, i_2 \leq n$,

$$\rho(T^{i_1}(x_n), T^{i_2}(x_n)) \leq \epsilon/4. \quad (22)$$

Property (P2) implies that there exists a natural number $m_0 \geq n_0$ such that for any triplet of integers i, n_1, n_2 satisfying $n_1, n_2, i \geq m_0, i \leq n_j, j = 1, 2$ we have

$$\rho(T^i(x_{n_1}), T^i(x_{n_2})) \leq \epsilon/4. \quad (23)$$

Assume that integers n_1, n_2, i_1, i_2 satisfy

$$m_0 \leq i_1 \leq n_1, m_0 \leq i_2 \leq n_2. \quad (24)$$

According to (22)–(24),

$$\begin{aligned} \rho(T^{i_1}(x_{n_1}), T^{i_2}(x_{n_1})) &\leq \epsilon/4, \\ \rho(T^{i_1}(x_{n_1}), T^{i_2}(x_{n_2})) &\leq \epsilon/4, \\ \rho(T^{i_1}(x_{n_1}), T^{i_1}(x_{n_2})) &\leq \epsilon/4, \rho(T^{i_2}(x_{n_1}), T^{i_2}(x_{n_2})) \leq \epsilon/4. \end{aligned}$$

These relations imply that

$$\rho(T^{i_1}(x_{n_1}), T^{i_2}(x_{n_2})) \leq \rho(T^{i_1}(x_{n_1}), T^{i_2}(x_{n_1})) + \rho(T^{i_2}(x_{n_1}), T^{i_2}(x_{n_2})) \leq \epsilon/2. \quad (25)$$

Thus, we have shown that the following property holds:

For (P4), if integers i_1, i_2, n_1, n_2 satisfy (24), then (25) holds.

In view of (P4), the sequences $\{T^{n-1}(x_n)\}_{n=1}^\infty$ and $\{T^{n-2}(x_n)\}_{n=2}^\infty$ converge and there exists

$$x_* = \lim_{n \rightarrow \infty} T^{n-1}(x_n) = \lim_{n \rightarrow \infty} T^{n-2}(x_n).$$

In view of Proposition 2, $x_* = T(x_*)$. Theorem 1 is proved. $\square$

The following example illustrates Theorem 1. Assume that X is the collection of all continuous functions on $[0, 1]$,

$$\rho(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in [0, 1]\}, f_1, f_2 \in X,$$

$$K = \{f \in X : f([0, 1]) \subset [0, 1]\}$$

and

$$T(f) = 2^{-1}f + 2/3, f \in K.$$

Clearly, T is a Wardowski contraction with $F(t) = \ln(t)$, $t > 0$ and $\tau = \ln(2)$. The formula above defines T for all $x \in X$, but we consider T to be a mapping from K to X . Evidently, T does not have a fixed point in T . In view of Theorem 1, $n \in \mathbf{N}$ can be found, such that $T^n(x) \notin K$ for any $x \in K$. A direct calculation shows that $n = 3$.

Let us consider the same mapping T with the same F and τ and

$$K = \{f \in X : f([0, 1]) \subset [0, 2]\} \cup \{f \in X : f([0, 1]) \subset [10, 11]\}.$$

It is not difficult to see that

$$T(K) \not\subset K$$

but T has a fixed point in K .

3. The Second Result

Assume that (X, ρ) is a complete metric space, $T : X \rightarrow 2^X \setminus \{\emptyset\}$; for each $x \in X$, the set $T(x)$ is closed and a function

$$\phi : [0, \infty) \rightarrow [0, \infty)$$

satisfies

$$\lim_{t \rightarrow 0} \phi(t) = 0. \quad (26)$$

In [17], it was shown that the set-valued T has a fixed point if for each $(x, y) \in X^2$ and each $u \in T(x)$ there is $v \in T(y)$, such that

$$\rho(u, v) + \phi(\rho(u, v)) \leq \phi(\rho(x, y)).$$

Examples of such mappings are considered in [17]. Here, we show that T possesses a fixed point under a weaker assumption. Namely, we assume that the following assumption holds:

(A) For each $x, y \in X$ and each $u \in T(x)$,

$$\inf\{\rho(u, v) + \phi(\rho(u, v)) : v \in T(y)\} \leq \phi(x, y). \quad (27)$$

Theorem 2. 1. Assume that $\{\epsilon_i\}_{i=0}^\infty \subset (0, \infty)$,

$$\sum_{i=0}^\infty \epsilon_i < \infty, \quad (28)$$

$\{x_i\}_{i=0}^\infty \subset X$, for each $i \in \mathbf{N} \cup \{0\}$,

$$x_{i+1} \in T(x_i), \quad (29)$$

$$\rho(x_{i+1}, x_{i+2}) + \phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_i, x_{i+1})) + \epsilon_i. \quad (30)$$

(This sequence exists according to assumption (A)). Then, the sequence $\{x_i\}_{i=0}^\infty$ converges to a fixed point of T .

2. Let $\epsilon > 0$. Then, $\delta \in (0, \epsilon)$ can be found such that for each $x_0 \in X$ satisfying $\rho(x_0, T(x_0)) < \delta$, there exists $x_* \in B(x_0, \epsilon)$ such that $x_* \in T(x_*)$.

Proof. Let us prove Assertion 1. According to (30), for each integer $i \geq 0$,

$$\rho(x_{i+1}, x_{i+2}) \leq \phi(\rho(x_i, x_{i+1})) - \phi(\rho(x_{i+1}, x_{i+2})) + \epsilon_i, \quad (31)$$

$$\phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_i, x_{i+1})) + \epsilon_i, \quad (32)$$

$$\phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_0, x_1)) + \sum_{j=0}^i \epsilon_j. \quad (33)$$

Let $m > n \geq 1$ be integers. In view of (30),

$$\begin{aligned} \rho(x_n, x_m) &\leq \sum_{i=n-1}^{m-1} \rho(x_{i+1}, x_{i+2}) \\ &\leq \sum_{i=n-1}^{m-1} (\phi(\rho(x_i, x_{i+1})) - \phi(\rho(x_{i+1}, x_{i+2})) + \epsilon_i) \\ &= \phi(\rho(x_{n-1}, x_n)) - \phi(\rho(x_{m-1}, x_m)) + \sum_{i=n-1}^{m-1} \epsilon_i. \end{aligned} \quad (34)$$

We claim that there exists

$$\lim_{n \rightarrow \infty} \phi(\rho(x_n, x_{n+1})).$$

According to (33), the sequence $\{\phi(\rho(x_n, x_{n+1}))\}_{n=0}^{\infty}$ is bounded. Let

$$r = \liminf_{n \rightarrow \infty} \phi(\rho(x_n, x_{n+1})). \quad (35)$$

We claim that

$$\lim_{n \rightarrow \infty} \phi(\rho(x_n, x_{n+1})) = r.$$

Let $\epsilon > 0$. According to (35), there exists a natural number n_0 such that for each integer $n \geq n_0$,

$$\phi(\rho(x_n, x_{n+1})) \geq r - \epsilon/8. \quad (36)$$

Equations (28) and (35) imply that there exists a natural number $n_1 > n_0$ such that

$$\sum_{i=n_1}^{\infty} \epsilon_i < \epsilon/8, \quad (37)$$

$$\phi(\rho(x_{n_1}, x_{n_1+1})) < r + \epsilon/8. \quad (38)$$

According to (32), (37) and (38), for any integer $n > n_1$,

$$\phi(\rho(x_n, x_{n+1})) \leq \phi(\rho(x_{n_1}, x_{n_1+1})) + \sum_{i=n_1}^{\infty} \epsilon_i < r + \epsilon/8 + \epsilon/8.$$

Thus, for any integer $n > n_1$,

$$|\phi(\rho(x_n, x_{n+1})) - r| < \epsilon/4 \quad (39)$$

and

$$\lim_{n \rightarrow \infty} \phi(\rho(x_n, x_{n+1})) = r. \quad (40)$$

According to (34), (37) and (39), for any pair of integers $m > n > n_1$,

$$\rho(x_n, x_m) \leq \phi(\rho(x_{n-1}, x_n)) - \phi(\rho(x_{m-1}, x_m)) + \sum_{i=n_1}^{\infty} \epsilon_i < \epsilon.$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n \rightarrow \infty} x_n.$$

Assumption (A) and (29) imply that for any $n \in \mathbb{N}$, there exists

$$v_n \in T(x_*)$$

such that

$$\rho(x_{n+1}, v_n) + \phi(\rho(x_{n+1}, v_n)) \leq \phi(\rho(x_n, x_*)) + \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

According to (26), (28) and (40),

$$\lim_{n \rightarrow \infty} \rho(x_{n+1}, v_n) = 0, \quad \lim_{n \rightarrow \infty} v_n = x_*$$

and $x_* \in T(x_*)$. Assertion 1 is proved.

Let us prove Assertion 2. According to (26), there exists $\delta \in (0, \epsilon)$ such that for each $t \in [0, \delta]$,

$$\phi(t) < \epsilon/4. \quad (41)$$

Assume that $x_0 \in X$ satisfies

$$\rho(x_0, T(x_0)) < \delta. \quad (42)$$

Choose $\{\epsilon_i\}_{i=0}^{\infty} \subset (0, 1)$ such that

$$\sum_{i=0}^{\infty} \epsilon_i < \epsilon/4. \quad (43)$$

In view of (42), there is

$$x_1 \in T(x_0)$$

for which

$$\rho(x_0, x_1) < \delta.$$

Assume that a sequence $\{x_i\}_{i=2}^{\infty}$ is as in Assertion 1. Then, there exists

$$x_* = \lim_{n \rightarrow \infty} x_n$$

such that

$$x_* \in T(x_*)$$

and in view of (34) and (41)–(43),

$$\begin{aligned} \sum_{i=1}^{\infty} \rho(x_i, x_{i+1}) &\leq \phi(\rho(x_0, x_1)) + \sum_{i=0}^{\infty} \epsilon_i \\ &\leq \phi(\rho(x_0, x_1)) + \epsilon/4 < \epsilon/2, \\ \sum_{i=1}^{\infty} \rho(x_i, x_{i+1}) &< \epsilon, \quad \rho(x_*, x_0) < \epsilon. \end{aligned}$$

Assertion 2 is proved. $\square$

Theorem 3. Assume that the function ϕ is bounded, $\epsilon \in (0, 1)$,

$$M_1 > \phi(t), \quad t \in [0, \infty), \quad (44)$$

$$n_0 > 2 + 2\epsilon^{-1}M_1 \quad (45)$$

is an integer and that $\{x_i\}_{i=0}^{n_0} \subset X$ satisfies for each integer $i \in \{0, \dots, n_0 - 1\}$,

$$x_{i+1} \in T(x_i) \quad (46)$$

and for any $i \in \{0, \dots, n_0 - 2\}$,

$$\rho(x_{i+1}, x_{i+2}) + \phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_i, x_{i+1})) + \epsilon/2. \quad (47)$$

Then, there exists $j \in \{1, \dots, n_0 - 1\}$, for which

$$\rho(x_j, x_{j+1}) \leq \epsilon, \quad B(x_j, \epsilon) \cap T(x_j) \neq \emptyset.$$

Proof. Assume that the theorem does not hold. Then,

$$\rho(x_j, x_{j+1}) > \epsilon, \quad j \in \{1, \dots, n_0 - 1\}. \quad (48)$$

According to (44), (47) and (48), for each $i \in \{0, \dots, n_0 - 2\}$,

$$\epsilon < \rho(x_{i+1}, x_{i+2}) \leq \phi(\rho(x_i, x_{i+1})) - \phi(\rho(x_{i+1}, x_{i+2})) + \epsilon/2$$

and

$$\begin{aligned} (n_0 - 1)\epsilon &< \sum_{i=0}^{n_0-2} \rho(x_{i+1}, x_{i+2}) \\ &\leq 2^{-1}\epsilon(n_0 - 1) + \sum_{i=0}^{n_0-2} \phi(\rho(x_i, x_{i+1})) - \phi(\rho(x_{i+1}, x_{i+2})) \\ &\leq 2^{-1}\epsilon(n_0 - 1) + \phi(\rho(x_0, x_1)), \\ 2^{-1}\epsilon(n_0 - 1) &\leq M_1 \end{aligned}$$

and

$$n_0 \leq 1 + 2\epsilon^{-1}M_1.$$

This contradicts (45) and proves Theorem 3. $\square$

Applying Theorem 3 by induction, we can obtain the following result.

Corollary 1. Assume that the function ϕ is bounded, $\epsilon \in (0, 1)$,

$$M_1 > \phi(t), \quad t \in [0, \infty),$$

$$n_0 > 2 + 2\epsilon^{-1}M_1$$

is an integer and that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies for any $i \in \mathbf{N} \cup \{0\}$,

$$x_{i+1} \in T(x_i)$$

and

$$\rho(x_{i+1}, x_{i+2}) + \phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_i, x_{i+1})) + \epsilon/2.$$

Then, there exists $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ such that $n_1 \leq n_0$ and that for any $k \in \mathbf{N}$,

$$1 \leq n_{k+1} - n_k, \quad \rho(x_{n_k}, x_{n_{k+1}}) \leq \epsilon.$$

Theorem 4. Assume that the function ϕ is bounded, $\epsilon \in (0, 1)$, $\epsilon_0 \in (0, \epsilon)$,

$$M_1 > \phi(t), \quad t \in [0, \infty),$$

$$\phi(t) \leq \epsilon/2, \quad t \in [0, \epsilon_0), \quad (49)$$

$$n_0 > 2 + 2\epsilon_0^{-1}M_1 \quad (50)$$

is an integer,

$$\delta = (2n_0)^{-1}\epsilon_0 \quad (51)$$

and that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies for any $i \in \mathbf{N} \cup \{0\}$,

$$x_{i+1} \in T(x_i) \quad (52)$$

and

$$\rho(x_{i+1}, x_{i+2}) + \phi(\rho(x_{i+1}, x_{i+2})) \leq \phi(\rho(x_i, x_{i+1})) + \delta. \quad (53)$$

Then, $\rho(x_i, x_{i+1}) \leq \epsilon$ for each integer $i \geq n_0$.

Proof. Corollary 1 implies that there exists $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ such that $n_1 \leq n_0$ and that for any $k \in \mathbf{N}$,

$$1 \leq n_{k+1} - n_k \leq n_0, \quad (54)$$

$$\rho(x_{n_k}, x_{n_{k+1}}) \leq \epsilon_0. \quad (55)$$

Let $k \geq 1$ be an integer. According to (49), (51), (53) and (55),

$$\sum_{i=n_k}^{n_{k+1}} \rho(x_i, x_{i+1}) \leq \sum_{i=n_k}^{n_{k+1}} (\phi(\rho(x_i, x_{i+1})) - \phi(\rho(x_{i+1}, x_{i+2}))) + n_0 \delta$$

and

$$\epsilon > \epsilon/2 + \epsilon_0/2 \geq \phi(\rho(x_{n_k}, x_{n_{k+1}})) + n_0 \delta \geq \sum_{i=n_k+1}^{n_{k+1}} \rho(x_i, x_{i+1}).$$

This completes the proof of Theorem 4. $\square$

Our results from this section can be applied to the following problem, considered in [17]. Assume that $a < b$ are real numbers, X is the space $C([a, b])$ of all real-valued continuous functions and that

$$\rho(f_1, f_2) = \sup\{|f_1(t) - f_2(t)| : t \in [a, b]\}, f_1, f_2 \in X.$$

We consider a Fredholm-type integral inclusion:

$$x(\tau) \in \int_a^b K(\tau, t, x(t))dt + f(\tau), \tau \in [a, b].$$

It was shown in [17] that the study of this problem is reduced to the analysis of a fixed point problem

$$T(z) = \{y \in X : y(\tau) \in \int_a^b K(\tau, s, z(s))ds + f(\tau), \tau \in [a, b]\}, z \in X$$

and that for mapping T , all the assumptions made in this section hold. Therefore, all the results can be applied for T .

4. The Third Result

Assume that (X, ρ) is endowed with a graph G . Let $V(G)$ be the set of its vertices, $E(G)$ be the set of its edges and let

$$(x, x) \in E(G), x \in X.$$

Assume that $\tau > 0$, $F : (0, \infty) \rightarrow \mathbb{R}^1$ is an increasing function and that $T : X \rightarrow X$ is a mapping such that for any $(x, y) \in E(G)$ for which $x \neq y$,

$$(T(x), T(y)) \in E(G) \text{ and } F(\rho(T(x), T(y))) + \tau \leq F(\rho(x, y)). \quad (56)$$

Equation (56) implies the following proposition.

Proposition 3. For any $(x, y) \in E(G)$, $\rho(T(x), T(y)) \leq \rho(x, y)$.

Proposition 4. Let $(x, y) \in E(G)$. Then

$$\lim_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) = 0.$$

Proof. We may assume that

$$T^n(x) \neq T^n(y)$$

for each integer $n \geq 0$. According to (56), for any integer $n \in \mathbb{N} \cup \{0\}$,

$$(T^n(x), T^n(y)) \in E(G),$$

$$F(\rho(T^{n+1}(x), T^{n+1}(y))) + \tau \leq F(\rho(T^n(x), T^n(y))),$$

$$F(\rho(T^n(x), T^n(y))) \leq F(\rho(x, y)) - n\tau. \quad (57)$$

Proposition 3 implies that for any $n \in \mathbf{N} \cup \{0\}$,

$$\rho(T^{n+1}(x), T^{n+1}(y)) \leq \rho(T^n(x), T^n(y)).$$

Assume that the proposition does not hold. Then, $\epsilon > 0$ can be found such that for any integer $n \geq 0$,

$$\epsilon \leq \rho(T^n(x), T^n(y)). \quad (58)$$

In view of (57) and (58), for each $n \in \mathbf{N} \cup \{0\}$,

$$F(\epsilon) \leq F(\rho(T^n(x), T^n(y))) \leq F(\rho(x, y)) - n\tau \rightarrow -\infty \text{ as } n \rightarrow \infty.$$

This contradiction proves Proposition 4. $\square$

Proposition 4 implies the following result.

Proposition 5. Let $x, y \in X$, $q \in \mathbf{N}$, $\{x_i\}_{i=0}^q \subset X$,

$$x_0 = x, x_q = y,$$

$$(x_i, x_{i+1}) \in E(G), i = 0, \dots, q-1.$$

Then,

$$\lim_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) = 0.$$

Proposition 5 implies the following result.

Theorem 5. Assume that $x \in X$ and there are $q \in \mathbf{N}$ and points $\{y_i\}_{i=0}^q \subset X$ such that

$$y_0 = x, y_q = T(x),$$

$$(y_i, y_{i+1}) \in E(G), i = 0, \dots, q-1.$$

Then,

$$\lim_{i \rightarrow \infty} \rho(T^i(x), T^{i+1}(x)) = 0.$$

Theorem 6. Assume that $x \in X$ and there exist $q \in \mathbf{N}$ and $\{y_i\}_{i=0}^q$ such that

$$y_0 = x, y_q = T(x),$$

$$(y_i, y_{i+1}) \in E(G), i = 0, \dots, q-1.$$

Assume that there exists $m_0 \in \mathbf{N}$ such that the following property holds:

(P) for any pair of nonnegative integers $i < j$, there is

$$p \in \{j, \dots, j + m_0\}$$

for which

$$(T^i(x), T^p(x)) \in E(G).$$

Then, the sequence $\{T^n(x)\}_{n=0}^\infty$ converges and its limit is a fixed point of T if the graph of T is closed.

Proof. According to Theorem 5,

$$\lim_{n \rightarrow \infty} \rho(T^n(x), T^{n+1}(x)) = 0. \quad (59)$$

Let $\epsilon \in (0, 1)$. We show that for all sufficiently large $i, j \in \mathbb{N}$,

$$\rho(T^i(x), T^j(x)) \leq \epsilon.$$

Since the collection of all points at which F is not continuous is countable, we may assume that the function F is continuous at ϵ . Choose

$$\delta \in (0, \epsilon/4)$$

such that

$$|F(\xi) - F(\epsilon)| \leq \tau/2 \text{ for any } \xi \in [\epsilon - 4\delta, \epsilon + 4\delta] \quad (60)$$

and set

$$\delta_0 = \delta(4m_0)^{-1}. \quad (61)$$

In view of (59), there exists a natural number n_0 for which

$$\rho(T^i(x), T^{i+1}(x)) \leq \delta_0 \text{ for each integer } i \geq n_0. \quad (62)$$

Assume that

$$j > i \geq n_0$$

are integers. We show that

$$\rho(T^i(x), T^j(x)) \leq \epsilon.$$

Assume the contrary. Then,

$$\rho(T^i(x), T^j(x)) > \epsilon. \quad (63)$$

According to (61) and (62), we may assume that for any $s \in \{i, \dots, j-1\}$,

$$\rho(T^i(x), T^s(x)) \leq \epsilon. \quad (64)$$

According to Equation (62), for any $s \in \{i+1, \dots, i+m_0\}$,

$$\rho(T^i(x), T^s(x)) \leq \delta_0 m_0 < \delta < \epsilon. \quad (65)$$

According to (63) and (65),

$$j > i + m_0, \quad j - 1 \geq i + m_0. \quad (66)$$

In view of (66),

$$j - m_0 > i$$

and (P) implies that there is

$$p \in \{j - m_0 - 1, \dots, j - 1\} \quad (67)$$

for which

$$(T^i(x), T^p(x)) \in E(G). \quad (68)$$

According to (67) and (68),

$$p \geq i.$$

If

$$T^p(x) = T^i(x),$$

then according to (62) and (67),

$$\rho(T^i(x), T^j(x)) \leq \sum \{\rho(T^s(x), T^{s+1}(x)) : s \in \{p, \dots, j-1\}\} \leq m_0 \delta_0 \leq \epsilon$$

and this contradicts (63). Therefore,

$$T^p(x) \neq T^i(x). \quad (69)$$

According to (56), (64), (67) and (69),

$$F(\rho(T^{i+1}(x), T^{p+1}(x))) + \tau \leq F(\rho(T^i(x), T^p(x))) \leq F(\epsilon). \quad (70)$$

It follows from (60) and (70) that

$$\rho(T^{i+1}(x), T^{p+1}(x)) \leq \epsilon - 4\delta. \quad (71)$$

According to (61), (62), (67) and (71),

$$\begin{aligned} \rho(T^i(x), T^j(x)) &\leq \rho(T^i(x), T^{i+1}(x)) + \rho(T^{i+1}(x), T^{p+1}(x)) \\ &\quad + \rho(T^{p+1}(x), T^j(x)) \leq \delta_0 + \epsilon - 4\delta \\ &\quad + \sum \{\rho(T^s(x), T^{s+1}(x)) : s \in \{p, \dots, j-1\} \setminus \{p\}\} \leq \delta_0 + \epsilon - 4\delta + m_0\delta_0 < \epsilon. \end{aligned}$$

This contradicts (63) and proves Theorem 6. $\square$

Theorem 6 was proved under assumption (P). It holds in the following two cases. In the first case, X is equipped with an order $\leq$ and $(x, y) \in E(G)$ if and only if $x \leq y$. In the second case, m is a natural number $A_i \subset X$, $i = 1, \dots, m$ are nonempty closed sets, $A_{m+1} = A_1$, $\cup_{i=1}^m A_i = X$,

$$T : \cup_{i=1}^m A_i \rightarrow \cup_{i=1}^m A_i$$

and for any $i \in \{1, \dots, m\}$, any $a \in A_i$, any $b \in A_{i+1}$,

$$T(A_i) \subset A_{i+1},$$

$$F(\rho(T(a), T(b))) + \tau \leq F(\rho(a, b))$$

and

$$E(G) = \cup_{i=1}^m (A_i \times A_{i+1}).$$

In this case, T is called a cyclical operator [6].

Let us consider the following example [19]. Assume that $X = [0, \infty)$, $\rho(u, v) = |u - v|$, $u, v \in [0, \infty)$ and that $(u, v) \in E(G)$ if and only if $u \leq v$ and

$$(u, v) \in [0, 1] \times [0, 1] \cup \cup_{n=1}^{\infty} (n, n+1] \times (n, n+1],$$

$$T(0) = 0, \quad T(u) = 2^{-1}u + n/2, \quad u \in (n, n+1], \quad n = 0, 1, \dots$$

It was shown in [19] that for each $(u, v) \in E(G)$,

$$\rho(T(u), T(v)) \leq 2^{-1}\rho(u, v).$$

It is not difficult to see that T is a Wardowski contraction with $F(t) = \ln(t)$, $t > 0$ and $\tau = \ln(2)$.

5. Conclusions

We consider three fixed point problems, and for each of them establish the existence of a fixed point. In the first and the third cases, we consider single-valued Wardowski type contraction, while in the second case we study Cho-type set-valued contractions. In the second case, we also study approximate fixed points.

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