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# Fractional-Order Sequential Linear Differential Equations with Nabla Derivatives on Time Scales

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**Abstract:** In this paper, we present a general theory for fractional-order sequential differential equations with Riemann–Liouville nabla derivatives and Caputo nabla derivatives on time scales. The explicit solution, in the case of constant coefficients, for both the homogeneous and the non-homogeneous problems, are given using the  $\nabla$ -Mittag-Leffler function, Laplace transform method, operational method and operational decomposition method. In addition, we also provide some results about a solution to a new class of fractional-order sequential differential equations with convolutional-type variable coefficients using the Laplace transform method.

**Keywords:** time scales;  $\nabla$ -Mittag-Leffler function; sequential differential equations; fractional-order nabla derivatives

**MSC:** 34K42



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## 1. Introduction

Continuous fractional calculus theory has been studied for more than 300 years; now, it has evolved in a respected mathematical discipline with numerous applications in many fields of science and engineering (see, for example, [1–4], and the references contained therein). However, the study of discrete problems and discrete models is still a relatively new field. The pioneering works [5,6] about fractional operators were introduced for  $q$ -calculus and difference calculus, respectively. It was not until the early 21st century that this topic further developed. The papers on this topic [7–10], which are required reading in order to deeply understand the background of discrete dynamic behaviors, obtained some interesting results by applying discrete fractional calculus to discrete chaos behaviors. In [11–14], the delta-type discrete fractional calculus was studied. In [15,16], the nabla-type discrete fractional calculus was studied. In order to unify differential equations and difference equations, Hilger [17] proposed was the first to propose the time scale. P.R. Williams [18] provided a definition of fractional integrals and derivatives on time scales to unify three cases of specific time scales, which improved the results in [19]. N.R.D.O. Bastos provided a definition of fractional  $\Delta$ -integrals and  $\Delta$ -derivatives on time scales in [20]. The application of delta fractional calculus and Laplace transform on some specific discrete time scales were also discussed in [21–23]. In light of the above work, we further studied the theory of fractional integrals and derivatives on general time scales in [24–26], where the  $\nabla$ -Laplace transform, fractional  $\nabla$ -power function,  $\nabla$ -Mittag-Leffler function, Riemann–Liouville-type, and Caputo-type fractional  $\nabla$ -integrals and fractional  $\nabla$ -differentials were studied. The  $\Delta$ -power function and fractional  $\Delta$ -integrals and fractional  $\Delta$ -differentials on time scales were, respectively, defined. Some of their properties were discussed in detail. After that, through using the Laplace transform method, the existence of the solution and the dependency of the solution upon the initial value for a Cauchy-type problem with Riemann–Liouville fractional  $\nabla$ -derivatives and  $\Delta$ -derivatives were studied. With the rise of interdisciplinary research, an increasing number of researchers are shifting their focus

from the application of fractional equations to practical problems in time scales [3,27–32]. However, the theory of fractional linear differential equations is the foundation of the study of nonlinear fractional differential equation theory; thus, deeper theoretical research, such as [33–36], should not stagnate.

Inspired by these results, in this paper, we will continue the work we started in [24,25] by presenting a general theory for sequential differential equations of a fractional order on time scales. The aim of this paper is to present a general and unified theory to study the continuous and discrete sequential differential equations of a fractional order on time scales. Specifically, we will provide the explicit solution, in the case of constant coefficients, for both the homogeneous and the non-homogeneous problem using the  $\nabla$ -Mittag-Leffler function, Laplace transform method, operational method and operational decomposition method. In addition, we studied a class of new fractional sequential differential equations with convolutional-type variable coefficients; in addition, some results about the solutions for this kind of fractional sequential differential equation are given using the Laplace transform method and convolution method.

The structure of this paper is as follows: In Section 2, we provide some preliminaries about time scales, convolution, etc., and we also detail definitions of nabla generalized power functions, fractional nabla integrals and fractional nabla derivatives on time scales, as well as a definition of the  $\nabla$ -Mittag-Leffler function, which is an important tool for solving fractional differential equations. Then, in Section 3, the basic conception of sequential linear differential equations of a fractional order on time scales and the structure of a general solution are presented. In Section 4, we introduce a method, one that is independent of the Laplace transform, for obtaining a fundamental system of solutions for the equation  $\mathbb{L}_{\nabla\alpha}y(t) = 0$ , which yields an explicit expression for the general solution to the non-homogeneous equation  $\mathbb{L}_{\nabla\alpha}y(t) = f(t)$ . In Section 5, we first prove a fact that the nabla generalized power function is positive, which is also an open problem (see [37]); then, we introduce the concept of  $\alpha$ -analyticity on time scales and detail some results about a solution for fractional sequential differential equations with convolution-type variable coefficients using the Laplace transform method.

## 2. Preliminaries

First, readers can refer to reference [38] for some well-known definitions and theorems about equations on time scale  $\mathbb{T}$ , where  $\mathbb{T}$  is a nonempty, closed subset of real numbers. Next, we will briefly introduce some of the results of our previous research [24,25], which are important for the research in this article.

**Definition 1.** (see [24]) On time scales, we define the fractional generalized  $\nabla$ -power function as

$$\hat{h}_{\alpha}(t, t_0) = \mathcal{L}_{\nabla, t_0}^{-1} \left\{ \frac{1}{z^{\alpha+1}} \right\} (t) (\alpha > -1),$$

where  $z \in \mathbb{C} \setminus \{0\}$ ,  $t \geq t_0$ . If  $\alpha > 0$ , then

$$\hat{h}_{\alpha}(t_0, t_0) = 0. \quad (1)$$

**Definition 2.** (see [24]) Let  $f : [t_0, \infty)_{\mathbb{T}} \rightarrow \mathbb{C}$ , then the solution of the shifting problem is

$$\begin{aligned} u^{\nabla t}(t, \rho(\tau)) &= -u^{\nabla \tau}(t, \tau), t, \tau \in \mathbb{T}, t \geq \tau \geq t_0, \\ u(t, t_0) &= f(t), t \in \mathbb{T}, t \geq t_0 \end{aligned}$$

which is called the shift of  $f$  and is denoted by  $\tilde{f}$ .

**Definition 3.** (see [24]) Let  $f, g : \mathbb{T} \rightarrow \mathbb{R}$ , then  $f * g$  is defined by

$$(f * g)(t) = \int_{t_0}^t \tilde{f}(t, \rho(\tau)) g(\tau) \nabla \tau, t \in \mathbb{T},$$

where  $\tilde{f}$  is the shift that was introduced in Definition 9.

**Definition 4.** (see [24]) On time scales, the fractional generalized  $\nabla$ -power function  $\hat{h}_\alpha(t, s)$  is defined as the shift of  $\hat{h}_\alpha(t, t_0)$ , which is

$$\hat{h}_\alpha(t, s) = \widetilde{\hat{h}_\alpha(\cdot, t_0)}(t, s) (t \geq s \geq t_0).$$

**Lemma 1.** (see [37]) Let  $\mathbb{T}$  be an isolated time scale and  $r > -1$  be a rational number. Then, there exists a unique power function  $\hat{h}_r(t, s)$  on a time scale such that  $\hat{h}_r(t, s) > 0$  for  $t > s$  and  $\hat{h}_r(t, t_0) = \mathcal{L}_{\nabla, t_0}^{-1} \left\{ \frac{1}{z^{r+1}} \right\}$ .

Further, in accordance with the results in [24], we have the following lemma.

**Lemma 2.** Let  $\mathbb{T}$  be an isolated time scale, and let  $\alpha > -1$  be a real number. Then, on a time scale, there exists a unique power function  $\hat{h}_\alpha(t, s)$  such that  $\hat{h}_\alpha(t, s) \geq 0$  for  $t > s$  and  $\hat{h}_\alpha(t, t_0) = \mathcal{L}_{\nabla, t_0}^{-1} \left\{ \frac{1}{z^{\alpha+1}} \right\}$ .

In this paper, we always denote  $\Omega := [t_0, t_1]_{\mathbb{T}}$  as a finite interval on a time scale  $\mathbb{T}$  ( $\sup \mathbb{T} = \infty$ ).

**Definition 5.** (see [24]) Assume that  $t, t_0 \in \Omega$ . Define the Riemann–Liouville fractional  $\nabla$ -integral  $I_{\nabla, t_0}^\alpha f$  of order  $\alpha > 0$  as

$$\begin{aligned} I_{\nabla, t_0}^\alpha f(t) &:= \hat{h}_{\alpha-1}(t, t_0) * f(t) \\ &= \int_{t_0}^t \widetilde{\hat{h}_{\alpha-1}(\cdot, t_0)}(t, \rho(\tau)) f(\tau) \nabla \tau \\ &= \int_{t_0}^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla \tau \quad (t > t_0). \end{aligned}$$

**Definition 6.** (see [24]) Assume that  $t, t_0 \in \Omega$ . Define the Riemann–Liouville fractional  $\nabla$ -derivative  $D_{\nabla, t_0}^\alpha f$  of order  $\alpha \geq 0$  as

$$D_{\nabla, t_0}^\alpha f(t) = D_{\nabla}^m I_{\nabla, t_0}^{m-\alpha} f(t) \quad (m = [\alpha] + 1; t > t_0).$$

**Definition 7.** (see [25]) Assume that  $t, t_0 \in \Omega$ . The Caputo fractional derivative of order  $\alpha \geq 0$  is defined via the Riemann–Liouville fractional derivative by

$${}^C D_{\nabla, t_0}^\alpha f(t) := D_{\nabla, t_0}^\alpha [f(t) - \sum_{k=0}^{m-1} \hat{h}_k(t, t_0) f^{\nabla^k}(t_0)] (t > t_0), \quad (2)$$

where

$$m = [\alpha] + 1 \quad \text{for } \alpha \notin \mathbb{N}; \quad m = \alpha \quad \text{for } \alpha \in \mathbb{N}. \quad (3)$$

**Definition 8.** (see [18]) A subset  $I \subset \mathbb{T}$  is called a time scale interval if it is of the form  $I = A \cap \mathbb{T}$  for some real interval  $A \subset \mathbb{R}$ . On a time scale interval  $I$ ,  $f : I \rightarrow \mathbb{R}$  is said to be left dense and absolutely continuous if for all  $\varepsilon > 0$  there exist  $\delta > 0$  such that  $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$  whenever a disjoint finite collection of sub-time scale intervals  $(a_k, b_k] \cap \mathbb{T} \subset I$  for  $1 \leq k \leq n$  satisfies  $\sum_{k=1}^n |b_k - a_k| < \delta$ . This was then denoted by  $f \in AC_{\nabla}$ . If  $f^{\nabla^{m-1}} \in AC$ , then we denoted it as  $f \in AC_{\nabla}^m$ .

The following properties can be deduced from the above definitions.

**Property 1.** (see [25]) Let  $\alpha \geq 0$  and  $m$  be given by (3). If  $f(t) \in AC_{\nabla}^m(\Omega)$ , then the Caputo fractional derivative  ${}^C D_{\nabla, t_0}^\alpha f(t)$  exists almost everywhere on  $\Omega_{k^m}$ .

(a) If  $\alpha \notin \mathbb{N}$ ,  ${}^C D_{\nabla, t_0}^\alpha f(t)$  is represented by

$${}^C D_{\nabla, t_0}^\alpha f(t) = \widehat{h}_{m-\alpha-1}(t, t_0) * f^{\nabla^m}(t) =: I_{\nabla, t_0}^{m-\alpha} D_{\nabla}^m f(t), \quad (4)$$

where  $m = [\alpha] + 1$ , then  $\alpha \notin \mathbb{N}$ ,  ${}^C D_{\nabla, t_0}^\alpha f(t_0) = 0$  (where the notation  ${}^C D_{\nabla, t_0}^\alpha f(t_0)$  denotes the limit of  ${}^C D_{\nabla, t_0}^\alpha f(t)$  as  $t \rightarrow t_0^+$ ). In particular, when  $0 < \alpha < 1$  and  $f(t) \in AC_{\nabla}(\Omega)$ , we have

$${}^C D_{\nabla, t_0}^\alpha f(t) = \widehat{h}_{-\alpha}(t, t_0) * f^{\nabla}(t) =: I_{\nabla, t_0}^{1-\alpha} f^{\nabla}(t). \quad (5)$$

(b) If  $\alpha = m \in \mathbb{N}$ , then  ${}^C D_{\nabla, t_0}^\alpha f(t)$  is represented by  ${}^C D_{\nabla, t_0}^m f(t) = f^{\nabla^m}(t)$  ( $m \in \mathbb{N}$ ). In particular, we have

$${}^C D_{\nabla, t_0}^0 f(t) = f(t). \quad (6)$$

Throughout this paper, we denote  $f^{\nabla^n} = D_{\nabla}^n f = D_{\nabla, t_0}^n f$ ,  $n \in \mathbb{N}$ .

**Lemma 3.** (see [24]) Assume that  $\alpha > 0$ ,  $m - 1 < \alpha \leq m$  ( $m \in \mathbb{N}$ ) and  $f : \Omega \rightarrow \mathbb{R}$  for  $t_0, t \in \Omega_{k^m}$  with  $t_0 < t$ . Then, we have the following:

(1) If  $f \in L_{\nabla, p}(\Omega)$ , then

$$\mathcal{L}_{\nabla, t_0} \{I_{\nabla, t_0}^\alpha f(t)\}(z) = \frac{1}{z^\alpha} \mathcal{L}_{\nabla, t_0} \{f(t)\}(z), \quad (7)$$

(2) and if  $f \in AC_{\nabla}^m(\Omega)$ , then

$$\mathcal{L}_{\nabla, t_0} \{D_{\nabla, t_0}^\alpha f(t)\}(z) = z^\alpha \mathcal{L}_{\nabla, t_0} \{f(t)\}(z) - \sum_{j=1}^m z^{j-1} D_{\nabla, t_0}^{\alpha-j} f(t_0) \quad (8)$$

for those regressive  $z \in \mathbb{C}$  satisfying  $\lim_{t \rightarrow \infty} \{D_{\nabla}^j I_{\nabla, t_0}^{m-\alpha} f(t) \widehat{e}_{\ominus, z}(t, t_0)\} = 0$ ,  $j = 0, 1, \dots, m - 1$ .

**Definition 9.** (see [24]) The  $\nabla$ -Mittag-Leffler function is defined as

$${}_{\nabla} F_{\alpha, \beta}(\lambda; t, t_0) = \sum_{j=0}^{\infty} \lambda^j \widehat{h}_{\alpha j + \beta - 1}(t, t_0) \quad (9)$$

provided the right hand series is convergent, where  $\alpha, \beta > 0$ ,  $\lambda \in \mathbb{R}$ .

### 3. Fundamental Conceptions and the Structure of General Solutions on Sequential Linear Differential Equations of a Fractional Order

In this section, let us first introduce a few necessary definitions.

**Definition 10.** For  $n \in \mathbb{N}$ , we defined a linear sequential fractional differential equation of order  $n\alpha$  with equations of the form

$$\sum_{k=0}^n b_k(t) y^{(k\alpha)}(t) = f(t), \quad (10)$$

where  $b_k(t)$  are given real functions,  $y^{(0)}(t) := y(t)$ , and  $y^{(k\alpha)}(t) := \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t)$  ( $k = 1, 2, \dots, n$ ) (which represents a fractional sequential derivative). For example, for the Riemann–Liouville derivative,  $\mathcal{D}_{\nabla, t_0}^\alpha y$  is

$$\begin{aligned} \mathcal{D}_{\nabla, t_0}^\alpha y &:= D_{\nabla, t_0}^\alpha y, \\ \mathcal{D}_{\nabla, t_0}^{k\alpha} y &:= \mathcal{D}_{\nabla, t_0}^\alpha \mathcal{D}_{\nabla, t_0}^{(k-1)\alpha} y \quad (k = 2, 3, \dots, n); \end{aligned}$$

for the Caputo derivative,  $\mathcal{D}_{\nabla, t_0}^\alpha y$  is

$$\begin{aligned}\mathcal{D}_{\nabla, t_0}^\alpha y &:= {}^C D_{\nabla, t_0}^\alpha y, \\ \mathcal{D}_{\nabla, t_0}^{k\alpha} y &:= \mathcal{D}_{\nabla, t_0}^\alpha \mathcal{D}_{\nabla, t_0}^{(k-1)\alpha} y (k = 2, 3, \dots, n).\end{aligned}$$

In the following section, we mainly discuss Riemann–Liouville derivatives since the study of them is similar to the study of Caputo derivatives.

If  $\forall t \in [t_0, t_1]_{\mathbb{T}}$ ,  $b_n(t) \neq 0$ , then Equation (10) may be expressed in its normal form for the Riemann–Liouville derivative as follows:

$$\mathbb{L}_{\mathbb{M}\alpha} y(t) := \mathcal{D}_{\nabla, t_0}^{n\alpha} y(t) + \sum_{k=0}^{n-1} a_k(t) \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t) \equiv y^{(n\alpha)}(t) + \sum_{k=0}^{n-1} a_k(t) y^{(k\alpha)}(t) = f(t). \quad (11)$$

Note that Equation (11) reduces to

$$D_{\nabla, t_0}^\alpha Y(t) = A(t)Y(t) + B(t), \quad (12)$$

where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ f(t) \end{pmatrix}; Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ \vdots \\ y_n(t) \end{pmatrix}$$

apply by just changing the variables

$$y_1(t) = y(t); D_{\nabla, t_0}^\alpha y_j(t) = y_{j+1}(t) (j = 1, 2, \dots, n-1).$$

**Definition 11.** For  $\alpha > 0$ , we call the  $\alpha$ -Wronskian of  $n$  functions  $u_j(t)$  ( $j = 1, \dots, n$ ) those that have fractional sequential derivatives up to order  $(n-1)\alpha$  in interval  $V \subset (t_0, t_1]_{\mathbb{T}}$  with the following determinant:

$$|W_\alpha(u_1, \dots, u_n)(t)| = \begin{vmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \\ u_1^{(\alpha)}(t) & u_2^{(\alpha)}(t) & \cdots & u_n^{(\alpha)}(t) \\ u_1^{(2\alpha)}(t) & u_2^{(2\alpha)}(t) & \cdots & u_n^{(2\alpha)}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ u_1^{((n-1)\alpha)}(t) & u_2^{((n-1)\alpha)}(t) & \cdots & u_n^{((n-1)\alpha)}(t) \end{vmatrix}.$$

To simplify the notation, the above will be presented as  $|W_\alpha(t)|$ .

We will use  $W_\alpha(t)$  for the corresponding Wronskian matrix.

**Definition 12.** We defined a fundamental system of solutions of Equation  $\mathbb{L}_{\mathbb{M}\alpha} y(t) = 0$  in  $V \subset [t_0, t_1]_{\mathbb{T}}$  as a family of  $n$  functions, which are the solutions of this equation and are linearly independent in  $V$ .

**Definition 13.** The term general solution of the fractional differential equation  $\mathbb{L}_{\alpha}y(t) = f(t)$  refers to any solution to this equation (which only depend on  $n$ -independent constants).

Through using the similar iterative approximation method used in [24], we can prove the existence and uniqueness of the global solutions of (12). From these, we can obtain the following two theorems on the existence and uniqueness of global solutions to Equation  $\mathbb{L}_{\alpha}y(t) = f(t)$  with certain initial conditions.

**Theorem 1.** Assume that  $y_0^k \in \mathbb{R}$  ( $k = 0, 1, \dots, n-1$ ) has been given real numbers, and let  $a_j(t) \in C_{ld}[t_0, t_1]_{\mathbb{T}}$  ( $j = 0, 1, \dots, n-1$ ) and  $f(t) \in C_{ld}[t_0, t_1]_{\mathbb{T}}$  be given left dense and continuous functions on  $[t_0, t_1]_{\mathbb{T}}$ . Then, there exists a unique left dense and continuous solution  $y(t) \in C_{ld}[t_0, t_1]_{\mathbb{T}}$  of the Cauchy problem

$$\begin{aligned} \mathbb{L}_{\alpha}y(t) &= f(t), \\ \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t) &:= y^{(k\alpha)}(t_0) = y_0^k \quad (k = 0, 1, \dots, n-1). \end{aligned}$$

**Theorem 2.** The initial value problem

$$\begin{aligned} \mathbb{L}_{\alpha}y(t) &= 0, \\ \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t) &:= y^{(k\alpha)}(t_0) = 0 \quad (k = 0, 1, \dots, n-1) \end{aligned}$$

has only the trivial solution  $y(t) = 0$ .

**Proposition 1.** Any linear combination of solutions of the homogeneous equation

$$\mathbb{L}_{\alpha}y(t) = 0 \quad (13)$$

is also a solution to this equation.

**Proof.** The result is immediately evident when keeping in mind the linearity of  $\mathbb{L}_{\alpha}$ .  $\square$

**Proposition 2.** If  $u_j(t)$  ( $j = 1, 2, \dots, n$ ) are linearly dependent in  $[t_0, t_1]_{\mathbb{T}}$ , then  $|W_{\alpha}(t)| = 0$ .

**Proof.** Since  $u_j(t)$  ( $j = 1, 2, \dots, n$ ) are linearly dependent in  $[t_0, t_1]_{\mathbb{T}}$ , then there exists  $n$  constants  $\{c_j\}_{j=1}^n$ , of which not all are zero, such that

$$W_{\alpha}(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \quad (14)$$

has a nontrivial solution; thus,  $|W_{\alpha}(t)| = 0$ .  $\square$

It follows from Proposition 2 that we can obtain the following Proposition 3.

**Proposition 3.** If there exists a  $t^*$  such that  $|W_{\alpha}(t^*)| \neq 0$ , then  $u_j(t)$  ( $j = 1, 2, \dots, n$ ) are linearly independent in  $[t_0, t_1]_{\mathbb{T}}$ .

Assume  $u_j(t)$  ( $j = 1, 2, \dots, n$ ) are  $n$  solutions to Equation (13) and that the linear dependency of them are closely connected with the corresponding Wronskian matrix.

**Proposition 4.**  $u_j(t)$  ( $j = 1, 2, \dots, n$ ) are linearly dependent if and only if there exists a  $t^*$  such that  $|W_{\alpha}(t^*)| = 0$ .

**Proof.** As the necessity was proven in Proposition 2, we will now prove the sufficiency. Suppose  $t^* \neq t_0$ , as well as consider the equation

$$W_\alpha(t^*)C = 0, \quad (15)$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq 0.$$

Then, as  $|W_\alpha(t^*)| = 0$ , the equation has a nontrivial solution  $c_j (j = 1, 2, \dots, n)$ . As such, we will construct a function using the following constants:

$$u(t) = \sum_{j=1}^n c_j u_j(t),$$

and we obtain that  $u(t)$  is a solution to Equation (13). From (15), we obtain that  $u(t)$  satisfies the initial value condition

$$u(t^*) = u^{(\alpha)}(t^*) = u^{((n-1)\alpha)}(t^*) = 0.$$

However,  $u(t) \equiv 0$  is also a solution to the equation, satisfying the initial value condition. Via the uniqueness of the solution, we have

$$\sum_{j=1}^n c_j u_j(t) = 0;$$

thus,  $u_j(t) (j = 1, 2, \dots, n)$  are linearly dependent.  $\square$

**Proposition 5.** If  $\{u_j(t)\}_{j=1}^n$  is a fundamental system of solutions to the equation  $\mathbb{L}_{\alpha} y(t) = 0$  in a certain interval  $V \subset (t_0, t_1]_{\mathbb{T}}$ , then the general solution to this differential equation in  $V$  is given by

$$y_g(t) = \sum_{k=1}^n c_k u_k(t),$$

where  $\{c_k\}_{k=1}^n$  are arbitrary constants.

**Proof.** Since  $y_g(t)$  is a solution to (13), we need only show that any other solution is a special case of  $y_g(t)$ . Let  $u(t)$  be a solution to Equation (13), which satisfies, for a given  $\theta \in (t_0, t_1]_{\mathbb{T}}$ , the following conditions:

$$u^{(k\alpha)}(\theta) = u_0^k \in \mathbb{R} \quad (k = 0, 1, \dots, n-1).$$

Since  $\{u_j(t)\}_{j=1}^n$  is a fundamental system of solutions,  $|W_\alpha(\theta)| \neq 0$ , as well as the equation

$$\begin{cases} c_1 u_1(\theta) + c_2 u_2(\theta) + \dots + c_n u_n(\theta) = u_0^0 \\ c_1 u_1^{(\alpha)}(\theta) + c_2 u_2^{(\alpha)}(\theta) + \dots + c_n u_n^{(\alpha)}(\theta) = u_0^1 \\ \dots \\ c_1 u_1^{((n-1)\alpha)}(\theta) + c_2 u_2^{((n-1)\alpha)}(\theta) + \dots + c_n u_n^{((n-1)\alpha)}(\theta) = u_0^{n-1}, \end{cases}$$

has a unique solution of  $c_1, c_2, \dots, c_n$ . It is evident that  $\sum_{k=1}^n c_k u_k(t)$  satisfies the initial condition  $u^{(k\alpha)}(\theta) = u_0^k$  ( $k = 0, 1, \dots, n-1$ ). Then, through using Theorem 1, we have

$$u(t) = \sum_{k=1}^n c_k u_k(t),$$

which proves the theorem.  $\square$

**Proposition 6.** *The set of all solutions of the differential equation  $\mathbb{L}_{\alpha} y(t) = 0$ , in a certain interval  $V \subset (t_0, t_1]_{\mathbb{T}}$ , is a vector space of  $n$  dimensions.*

**Proof.** The result follows directly from Proposition 5.  $\square$

**Proposition 7.** *If  $y_p(t)$  is a particular solution to the equation  $\mathbb{L}_{\alpha} y(t) = f(t)$ , then the general solution to this equation is given by*

$$y_g(t) = y_h(t) + y_p(t),$$

where  $y_h(t)$  is the general solution to the associated homogeneous equation  $\mathbb{L}_{\alpha} y(t) = 0$ .

**Proof.** The result is evident.  $\square$

#### 4. The Solution of Sequential Linear Differential Equations with Constant Coefficients

This section introduces a general theory to solve Riemann–Liouville sequential linear fractional differential equations for fractional cases. Some general methods were developed for the purpose of obtaining the general solution to linear sequential fractional differential equations with constant coefficients, and this was achieved using the roots of the characteristic polynomial of the corresponding homogeneous equation, or in finding a certain Green function in the non-homogeneous case.

##### 4.1. The Solution of Integral-Order Differential Equations in the Homogeneous Case

To provide a method or inspiration to solve linear sequential fractional differential equations, we firstly and predominantly undertook a discussion on integral-order differential equations with constant coefficients in the homogeneous case. First, we studied the following second-order differential equation:

$$D_{\nabla}^2 y(t) + a D_{\nabla} y(t) + b y(t) = 0, \quad (16)$$

which is denoted as  $\mathbb{L} y(t) = 0$ .

We then sought a solution of the form

$$y(t) = \hat{e}_{\lambda}(t, t_0),$$

where we can derive

$$D_{\nabla} \hat{e}_{\lambda}(t, t_0) = \lambda \hat{e}_{\lambda}(t, t_0)$$

and

$$D_{\nabla}^2 \hat{e}_{\lambda}(t, t_0) = \lambda^2 \hat{e}_{\lambda}(t, t_0).$$

By substituting them into (16), we can obtain the characteristic equation:

$$P(\lambda) = \lambda^2 + a\lambda + b = 0.$$

If we suppose the characteristic roots are  $\lambda_1, \lambda_2$ , then there the following two cases present themselves:

- (1) When  $\lambda_1 \neq \lambda_2$ , the linearly independent solutions of (16) are

$$y_1(t) = \widehat{e}_{\lambda_1}(t, t_0), y_2(t) = \widehat{e}_{\lambda_2}(t, t_0).$$

- (2) When  $\lambda_1 = \lambda_2 = \lambda$ , it is evident that  $y_1(t) = \widehat{e}_{\lambda}(t, t_0)$  is one solution of (16). But, how does one derive another solution  $y_2(t)$  that is independent of  $y_1(t)$ ?

If we take the derivative of  $y_1(t) = \widehat{e}_{\lambda}(t, t_0)$  for  $\lambda$ , then there is

$$\mathbb{L}\left[\frac{\partial \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda}\right] = \frac{\partial}{\partial \lambda} \mathbb{L}[\widehat{e}_{\lambda}(t, t_0)] = \frac{\partial}{\partial \lambda} [P(\lambda) \widehat{e}_{\lambda}(t, t_0)] = P'(\lambda) \widehat{e}_{\lambda}(t, t_0) + P(\lambda) \frac{\partial \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda}.$$

Since  $\lambda = \lambda_1 = \lambda_2$  is a double root of  $P(\lambda) = 0$ ,  $P(\lambda) = P'(\lambda) = 0$ , and thus

$$y_2(t) = \frac{\partial \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda}$$

is the solution of (16), then apparently  $y_1(t)$  and  $y_2(t)$  are linearly independent and therefore we have the following calculations:

**Theorem 3.** For differential Equation (16), if  $\lambda_1, \lambda_2$  are characteristic roots of

$$P(\lambda) = \lambda^2 + a\lambda + b = 0,$$

then we have the following:

- (1) When  $\lambda_1 \neq \lambda_2$ , the linearly independent solutions of (16) are

$$y_1(t) = \widehat{e}_{\lambda_1}(t, t_0), y_2(t) = \widehat{e}_{\lambda_2}(t, t_0).$$

- (2) When  $\lambda_1 = \lambda_2 = \lambda$ , the linearly independent solutions of (16) are

$$y_1(t) = \widehat{e}_{\lambda}(t, t_0), y_2(t) = \frac{\partial \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda}.$$

For a general linear homogeneous differential equation, we can utilize the following:

$$\mathbb{L}[y(t)] = D_{\nabla}^m y(t) + a_1 D_{\nabla}^{m-1} y(t) + \cdots + a_{m-1} D_{\nabla} y(t) + a_m y(t) = 0, \quad (17)$$

**Theorem 4.** For the differential Equation (17), its characteristic equation is

$$P(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m = 0. \quad (18)$$

- (1) If there are  $m$  distinct roots  $\lambda_1, \lambda_2, \dots, \lambda_m$  for the characteristic Equation (18), then

$$y_1(t) = \widehat{e}_{\lambda_1}(t, t_0), y_2(t) = \widehat{e}_{\lambda_2}(t, t_0), \dots, y_m(t) = \widehat{e}_{\lambda_m}(t, t_0)$$

are the fundamental solution system of Equation (17).

- (2) If  $\lambda$  is a root of multiplicity  $k$  of the characteristic Equation (18), then the functions

$$y_1(t) = \widehat{e}_{\lambda}(t, t_0), y_2(t) = \frac{\partial \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda}, y_3(t) = \frac{\partial^2 \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda^2}, \dots, y_k(t) = \frac{\partial^k \widehat{e}_{\lambda}(t, t_0)}{\partial \lambda^k}$$

are linearly independent solutions of Equation (17).

**Proof.** The proof of (1) is evident.

(2) We only need to prove (for  $l = 1, 2, \dots, k$ ,  $\mathbb{L}[y_l(t)] = 0$ , where  $y_l(t) = \frac{\partial^{l-1}\hat{e}_\lambda(t, t_0)}{\partial \lambda^{l-1}}$ ), as  $\lambda$  is a root of multiplicity  $k$  of the characteristic Equation (18),

$$P(\lambda) = P'(\lambda) = \dots = P^{(k-1)}(\lambda) = 0, P^{(k)}(\lambda) \neq 0, \quad (19)$$

as well as

$$\begin{aligned} \mathbb{L}[y_l(t)] &= \mathbb{L}\left[\frac{\partial^{l-1}\hat{e}_\lambda(t, t_0)}{\partial \lambda^{l-1}}\right] \\ &= \frac{\partial^{l-1}}{\partial \lambda^{l-1}} \mathbb{L}[\hat{e}_\lambda(t, t_0)] \\ &= \frac{\partial^{l-1}}{\partial \lambda^{l-1}} [P(\lambda)\hat{e}_\lambda(t, t_0)] \\ &= \sum_{j=0}^{l-1} \binom{l-1}{j} P^{(j)}(\lambda) \frac{\partial^{l-1-j}\hat{e}_\lambda(t, t_0)}{\partial \lambda^{l-1-j}}. \end{aligned}$$

From (19), we have  $\mathbb{L}[y_l(t)] = 0$ ,  $l = 1, 2, \dots, k$ , that is,  $y_l(t)$ ,  $l = 1, 2, \dots, k$ , which are linearly independent solutions of Equation (17).  $\square$

In general, when the characteristic root is a multiple, we have the following theorem:

**Theorem 5.** Let  $\lambda_1, \lambda_2, \dots, \lambda_r$  be  $r$  distinct roots of multiplicity  $k_1, k_2, \dots, k_r$  ( $\sum_{j=1}^r k_j = n$ ) of the characteristic Equation (18). Then, the functions

$$\begin{aligned} &\hat{e}_{\lambda_1}(t, t_0), \frac{\partial \hat{e}_{\lambda_1}(t, t_0)}{\partial \lambda_1}, \frac{\partial^2 \hat{e}_{\lambda_1}(t, t_0)}{\partial \lambda_1^2}, \dots, \frac{\partial^{k_1} \hat{e}_{\lambda_1}(t, t_0)}{\partial \lambda_1^{k_1}}; \\ &\hat{e}_{\lambda_2}(t, t_0), \frac{\partial \hat{e}_{\lambda_2}(t, t_0)}{\partial \lambda_2}, \frac{\partial^2 \hat{e}_{\lambda_2}(t, t_0)}{\partial \lambda_2^2}, \dots, \frac{\partial^{k_2} \hat{e}_{\lambda_2}(t, t_0)}{\partial \lambda_2^{k_2}}; \\ &\dots \\ &\hat{e}_{\lambda_r}(t, t_0), \frac{\partial \hat{e}_{\lambda_r}(t, t_0)}{\partial \lambda_r}, \frac{\partial^2 \hat{e}_{\lambda_r}(t, t_0)}{\partial \lambda_r^2}, \dots, \frac{\partial^{k_r} \hat{e}_{\lambda_r}(t, t_0)}{\partial \lambda_r^{k_r}} \end{aligned}$$

are linearly independent solutions of Equation (17).

#### 4.2. General Solution of Sequential Differential Equations of a Fractional Order in the Homogeneous Case

As we were inspired by the thinking of solving integral-order differential equations, we will discuss sequential linear differential equations of a fractional order in this section.

Let

$$\mathbb{L}_{\mathfrak{M}\alpha} y(t) := \mathcal{D}_{\nabla, t_0}^{n\alpha} y(t) + \sum_{k=0}^{n-1} a_k \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t), \quad (20)$$

where the coefficients  $\{a_j\}_{j=1}^{n-1}$  are real constants.

As in the usual case, we shall seek the solution of (20) in the form  $y(t) = \nabla F_\alpha(\lambda; t, t_0)$ , where  $\nabla F_\alpha(\lambda; t, t_0)$  is defined as

$$\nabla F_\alpha(\lambda; t, t_0) := \nabla F_{\alpha, \alpha}(\lambda; t, t_0) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{\alpha k + \alpha - 1}(t, t_0).$$

In the following, we first give a lemma for  $\nabla F_\alpha(\lambda; t, t_0)$ .

**Lemma 4.** For  $\nabla F_\alpha(\lambda; t, t_0)$ , it is valid that

$$\begin{aligned} D_{\nabla, t_0}^\alpha \nabla F_\alpha(\lambda; t, t_0) &= \lambda \nabla F_\alpha(\lambda; t, t_0), \\ \frac{\partial}{\partial \lambda} \nabla F_\alpha(\lambda; t, t_0) &= \widehat{h}_{\alpha-1}(t, t_0) *_{\nabla} F_\alpha(\lambda; t, t_0). \end{aligned}$$

**Proof.** According to the definition of  $\nabla F_\alpha(\lambda; t, t_0)$ , as well as by taking  $D_{\nabla, t_0}^\alpha \widehat{h}_{\alpha-1}(t, t_0) = 0$  (see Formula (85) of Property 1 in [24]) into account, we have

$$\begin{aligned} D_{\nabla, t_0}^\alpha \nabla F_\alpha(\lambda; t, t_0) &= D_{\nabla, t_0}^\alpha \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \\ &= D_{\nabla, t_0}^\alpha \left( \widehat{h}_{\alpha-1}(t, t_0) + \sum_{k=1}^{\infty} \lambda^k \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \right) \\ &= \sum_{k=1}^{\infty} \lambda^k \widehat{h}_{\alpha k - 1}(t, t_0) \\ &= \sum_{k=0}^{\infty} \lambda^{k+1} \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \\ &= \lambda \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \\ &= \lambda \nabla F_\alpha(\lambda; t, t_0). \end{aligned}$$

For the proof of the second formula, through using the convolution property of the generalized polynomial  $\widehat{h}_\alpha(t, t_0)$ , we have

$$\begin{aligned} \frac{\partial}{\partial \lambda} \nabla F_\alpha(\lambda; t, t_0) &= \frac{\partial}{\partial \lambda} \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \\ &= \sum_{k=1}^{\infty} \lambda^{k-1} \widehat{h}_{\alpha k + \alpha - 1}(t, t_0) \\ &= \sum_{k=1}^{\infty} \lambda^{k-1} \widehat{h}_{\alpha-1}(t, t_0) * \widehat{h}_{\alpha k - 1}(t, t_0) \\ &= \widehat{h}_{\alpha-1}(t, t_0) * \sum_{k=1}^{\infty} \lambda^{k-1} \widehat{h}_{\alpha k - 1}(t, t_0) \\ &= \widehat{h}_{\alpha-1}(t, t_0) *_{\nabla} F_\alpha(\lambda; t, t_0). \end{aligned}$$

The proof is thus completed.  $\square$

From Lemma 4, we have

$$\begin{aligned} \mathcal{D}_{\nabla, t_0}^\alpha \nabla F_\alpha(\lambda; t, t_0) &= \lambda \nabla F_\alpha(\lambda; t, t_0), \\ \mathcal{D}_{\nabla, t_0}^{2\alpha} \nabla F_\alpha(\lambda; t, t_0) &= \lambda^2 \nabla F_\alpha(\lambda; t, t_0), \\ &\dots \\ \mathcal{D}_{\nabla, t_0}^{n\alpha} \nabla F_\alpha(\lambda; t, t_0) &= \lambda^n \nabla F_\alpha(\lambda; t, t_0). \end{aligned}$$

$$\frac{\partial^l}{\partial \lambda^l} \nabla F_\alpha(\lambda; t, t_0) = \left( \prod_{i=1}^l \widehat{h}_{\alpha-1}(t, t_0) \right) *_{\nabla} F_\alpha(\lambda; t, t_0),$$

where  $\prod_{i=1}^l \widehat{h}_{\alpha-1}(t, t_0) = \widehat{h}_{\alpha-1}(t, t_0) * \dots * \widehat{h}_{\alpha-1}(t, t_0)$ , that is, the  $m$  time convolution of  $\widehat{h}_{\alpha-1}(t, t_0)$ . It follows from (20) and the equalities detailed above that

$$\mathbb{L}_{\nabla \alpha}[\nabla F_\alpha(\lambda; t, t_0)] = P_n(\lambda) \nabla F_\alpha(\lambda; t, t_0),$$

where

$$P_n(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k \quad (21)$$

is the characteristic polynomial associated with the equation  $\mathbb{L}_{\mathfrak{M}\alpha}[y(t)] = 0$ , and the root of  $P_n(\lambda) = 0$  is called a characteristic root. Thus,  $y(t) = {}_{\nabla}F_{\alpha}(\lambda; t, t_0)$  is the solution to Equation (20) if and only if  $\lambda$  is the root of (21).

**Theorem 6.** For the differential Equation (20), there are the following two cases for a characteristic root that corresponds to the characteristic equation:

- (1) When all of the characteristic roots are simple, that is, if there are  $n$ -distinct roots  $\lambda_1, \lambda_2, \dots, \lambda_n$  for the characteristic polynomial (21), then

$$y_j(t) = {}_{\nabla}F_{\alpha}(\lambda_j; t, t_0) \quad (j = 1, 2, \dots, n)$$

are the fundamental solution system of Equation (20).

- (2) When the characteristic root is a multiple, that is, if  $\lambda$  is a root of multiplicity  $k$  ( $1 < k \leq n$ ) of the characteristic polynomial (21), then the functions

$$\left( \prod_{i=1}^j {}^* \hat{h}_{\alpha-1}(t, t_0) \right) {}_{\nabla}F_{\alpha}(\lambda; t, t_0), \quad (j = 0, 1, 2, \dots, k-1)$$

are linearly independent solutions of Equation (20).

**Proof.** The result of (1) is evident, and so we only provided a proof for (2). Since  $\lambda$  is a root of multiplicity  $k$  of  $P_n(\lambda) = 0$ , we have

$$P_n(\lambda) = P'_n(\lambda) = \dots = P_n^{(k-1)}(\lambda) = 0, P_n^{(k)}(\lambda) \neq 0. \quad (22)$$

In addition, it follows from  $\mathbb{L}_{\mathfrak{M}\alpha}[{}_{\nabla}F_{\alpha}(\lambda; t, t_0)] = P_n(\lambda){}_{\nabla}F_{\alpha}(\lambda; t, t_0)$  and the classical Leibniz rule that, for  $j = 0, 1, \dots, k-1$ ,

$$\begin{aligned} \mathbb{L}_{\mathfrak{M}\alpha}\left[\frac{\partial^j}{\partial \lambda^j} {}_{\nabla}F_{\alpha}(\lambda; t, t_0)\right] &= \frac{\partial^j}{\partial \lambda^j} \mathbb{L}_{\mathfrak{M}\alpha}[{}_{\nabla}F_{\alpha}(\lambda; t, t_0)] \\ &= \frac{\partial^j}{\partial \lambda^j} [P_n(\lambda){}_{\nabla}F_{\alpha}(\lambda; t, t_0)] \\ &= \sum_{m=0}^j \binom{j}{m} P_n^{(m)}(\lambda) [{}_{\nabla}F_{\alpha}(\lambda; t, t_0)]^{(j-m)}, \end{aligned}$$

and, using (22), we have

$$\mathbb{L}_{\mathfrak{M}\alpha}\left[\frac{\partial^j}{\partial \lambda^j} {}_{\nabla}F_{\alpha}(\lambda; t, t_0)\right] \equiv 0, \quad j = 0, 1, 2, \dots, k-1.$$

Since

$$\frac{\partial^j}{\partial \lambda^j} {}_{\nabla}F_{\alpha}(\lambda; t, t_0) = \left( \prod_{i=1}^j {}^* \hat{h}_{\alpha-1}(t, t_0) \right) {}_{\nabla}F_{\alpha}(\lambda; t, t_0),$$

we have

$$\mathbb{L}_{\mathfrak{M}\alpha}\left[\left( \prod_{i=1}^j {}^* \hat{h}_{\alpha-1}(t, t_0) \right) {}_{\nabla}F_{\alpha}(\lambda; t, t_0)\right] \equiv 0, \quad j = 0, 1, 2, \dots, k-1.$$

The result then follows.  $\square$

When there are several multiple roots for the characteristic equation, we have the following results.

**Theorem 7.** Let  $\{\lambda_j\}_{j=1}^k$  be  $k$ -distinct roots of multiplicity  $\{\mu_j\}_{j=1}^k$   $\left(1 \leq \mu_j \leq n, \sum_{j=1}^k \mu_j = n\right)$  of the characteristic polynomial (21). Then, the functions

$$\begin{aligned} & \nabla F_\alpha(\lambda_1; t, t_0), \quad \widehat{h}_{\alpha-1}(t, t_0) * \nabla F_\alpha(\lambda_1; t, t_0), \quad \dots, \quad \left( \prod_{i=1}^{\mu_1-1} * \widehat{h}_{\alpha-1}(t, t_0) \right) * \nabla F_\alpha(\lambda_1; t, t_0); \\ & \nabla F_\alpha(\lambda_2; t, t_0), \quad \widehat{h}_{\alpha-1}(t, t_0) * \nabla F_\alpha(\lambda_2; t, t_0), \quad \dots, \quad \left( \prod_{i=1}^{\mu_2-1} * \widehat{h}_{\alpha-1}(t, t_0) \right) * \nabla F_\alpha(\lambda_2; t, t_0); \\ & \dots \quad \dots \quad \dots \quad \dots \\ & \nabla F_\alpha(\lambda_k; t, t_0), \quad \widehat{h}_{\alpha-1}(t, t_0) * \nabla F_\alpha(\lambda_k; t, t_0), \quad \dots, \quad \left( \prod_{i=1}^{\mu_k-1} * \widehat{h}_{\alpha-1}(t, t_0) \right) * \nabla F_\alpha(\lambda_k; t, t_0) \end{aligned}$$

are linearly independent solutions of Equation (20).

**Proof.** The result follows from Theorem 6.  $\square$

**Example 1.** 1. The fractional differential equation  $y^{(2\alpha)}(t) - y(t) = 0$  has the following fundamental system of solutions:

$$\{\nabla F_\alpha(1; t, t_0), \nabla F_\alpha(-1; t, t_0)\}.$$

This result follows from Theorem 6 if we take into account that  $\lambda_1 = 1$  and  $\lambda_2 = -1$  are the roots of the characteristic polynomial  $P_2(\lambda) := \lambda^2 - 1 = 0$ .

2. The fractional differential equation  $y^{(2\alpha)}(t) - 2y^{(\alpha)}(t) + y(t) = 0$  has the following fundamental system of solutions:

$$\{\nabla F_\alpha(1; t, t_0), \widehat{h}_{\alpha-1}(t, t_0) * \nabla F_\alpha(1; t, t_0)\}.$$

Since  $\lambda_1 = 1$  is the root of multiplicity two of the characteristic polynomial  $P_2(\lambda) := \lambda^2 - 2\lambda + 1 = 0$ , then Theorem 6 yields the above result.

#### 4.3. General Solution of the Sequential Differential Equations of a Fractional Order in the Non-Homogeneous Case

Consider the following non-homogeneous linear differential equation of a fractional order:

$$\mathbb{L}_{\mathfrak{m}\alpha}[y(t)] = f(t), \quad (23)$$

which is defined by (11). As in the case of ordinary differential equations, the general solution to this equation is a sum of a particular solution to Equation (23), as well as a general solution to the corresponding homogeneous Equation (20).

First, we applied the operational method to derive a particular solution  $y_p(t)$  of (23).

**Theorem 8.** Let  $\{\lambda_j\}_{j=1}^k$  be  $k$ -different roots of multiplicity  $\{\mu_j\}_{j=1}^k$   $(\sum_{j=1}^k \mu_j = n)$  of the characteristic polynomial  $P_n(\lambda) = 0$ , which is associated with the equation  $\mathbb{L}_{\mathfrak{m}\alpha}[y(t)] = 0$  (where  $y^{(k\alpha)}(t) = \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t)$ ). Then, the particular solution to Equation (23), with a given function  $f(t)$  on  $(t_0, t_1]_{\mathbb{T}}$ , is given by

$$y_p(t) = \prod_{j=1}^k \left( \sum_{n=0}^{\infty} \lambda_j^n I_{\nabla, t_0}^{(n+1)\alpha} \right)^{\mu_j} f(t) \quad (24)$$

provided that the series in the right-hand side of (24) are uniformly convergent.

**Proof.** We can consider the inverse operator to the left for  $\mathbb{L}_{\mathfrak{m}\alpha}$  in the form

$$\begin{aligned}\mathbb{L}_{\alpha}^{-1} &= \prod_{j=1}^k [(D_{\nabla, t_0}^{\alpha} - \lambda_j)^{-1}]^{\mu_j} \\ &= \prod_{j=1}^k \left\{ \left[ D_{\nabla, t_0}^{\alpha} (1 - \lambda_j I_{\nabla, t_0}^{\alpha}) \right]^{-1} \right\}^{\mu_j} \\ &= \prod_{j=1}^k (\sum_{n=0}^{\infty} \lambda_j^n I_{\nabla, t_0}^{(n+1)\alpha})^{\mu_j},\end{aligned}$$

which yields the formal solution to Equation (23) in (24). Applying the operator

$$\mathbb{L}_{\alpha} = \prod_{j=1}^k (D_{\nabla, t_0}^{\alpha} - \lambda_j)^{\mu_j}$$

to the function  $y_p(t)$  in (24), as well as when using the uniform convergence of the series considered, we derive that  $y_p(t)$  is the solution to the equation  $\mathbb{L}_{\alpha}[y(t)] = f(t)$ .  $\square$

Second, we applied the Laplace transform method to explicitly derive a particular solution for the non-homogenous equation

$$\mathbb{L}_{\alpha}[y(t)] = f(t).$$

Via Lemma 1, the Laplace transform  $\mathcal{L}_{\nabla, t_0}$  of  $D_{\nabla, t_0}^{\alpha} f(t)$ , for a suitable  $f(t)$ , is given by

$$\mathcal{L}_{\nabla, t_0}\{D_{\nabla, t_0}^{\alpha} f(t)\}(z) = z^{\alpha} \mathcal{L}\{f\}(z);$$

therefore,

$$\mathcal{L}_{\nabla, t_0}\{\mathcal{D}_{\nabla, t_0}^{n\alpha} f(t)\}(z) = z^{n\alpha} \mathcal{L}\{f\}(z).$$

**Theorem 9.** A particular solution to the non-homogenous linear fractional differential Equation (23) with  $y^{(k\alpha)}(t) = \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t)$  is given by

$$y_p(t) = \sum_{j=1}^k \sum_{m=1}^{\mu_j} c_{j,m} (g_{j,m} * f)(t), \quad (25)$$

where

$$g_{j,m}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(z^{\alpha} - \lambda_j)^m}\right\}(t) = \frac{1}{(m-1)!} \left( \prod_{i=1}^{m-1} {}^* \hat{h}_{\alpha-1}(t, t_0) \right) {}^* F_{\alpha, \alpha}(\lambda_j; t, t_0) \quad (26)$$

and  $c_{j,m}$  ( $j = 1, 2, \dots, k; m = 1, 2, \dots, \mu_j$ ) are constants that are defined by the following decomposition into simple fractions:

$$\frac{1}{P_n(\lambda)} = \frac{1}{\prod_{j=1}^k (\lambda - \lambda_j)^{\mu_j}} = \sum_{j=1}^k \sum_{m=1}^{\mu_j} \frac{c_{j,m}}{(\lambda - \lambda_j)^m}, \quad (27)$$

where  $(g * f)(t)$  represents the convolution of  $g(t)$  and  $f(t)$ .

**Proof.** In applying the Laplace transform to Equation (23), we have

$$\mathcal{L}\{\mathbb{L}_{\alpha}[y(t)]\} = P_n(z^{\alpha}) \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\}.$$

Using (27), we can obtain

$$\begin{aligned}\mathcal{L}\{y(t)\} &= \sum_{j=1}^k \sum_{m=1}^{\mu_j} \mathcal{L}\left\{\mathcal{L}^{-1}\left(\frac{c_{j,m}}{(z^\alpha - \lambda_j)^m}\right)\right\} \mathcal{L}\{f(t)\} \\ &= \mathcal{L}\left(\sum_{j=1}^k \sum_{m=1}^{\mu_j} \mathcal{L}^{-1}\left(\frac{c_{j,m}}{(z^\alpha - \lambda_j)^m}\right) * f(t)\right) \\ &= \mathcal{L}\left(\sum_{j=1}^k \sum_{m=1}^{\mu_j} c_{j,m} \mathcal{L}^{-1}\left(\frac{1}{(z^\alpha - \lambda_j)^m}\right) * f(t)\right).\end{aligned}$$

By taking the inverse Laplace transform, we find the particular solution to (23) in the form

$$\begin{aligned}y_p(t) &= \sum_{j=1}^k \sum_{m=1}^{\mu_j} c_{j,m} \mathcal{L}^{-1}\left(\frac{1}{(z^\alpha - \lambda_j)^m}\right) * f(t) \\ &= \sum_{j=1}^k \sum_{m=1}^{\mu_j} c_{j,m} (g_{j,m} * f)(t),\end{aligned}$$

where  $g_{j,m}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(z^\alpha - \lambda_j)^m}\right\}(t)$ . Through using Theorem 40 in [24], we obtain

$$\begin{aligned}g_{j,m}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(z^\alpha - \lambda_j)^m}\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \frac{1}{(z^\alpha - \lambda)} \Big|_{\lambda=\lambda_j}\right\}(t) \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \mathcal{L}^{-1}\left\{\frac{1}{(z^\alpha - \lambda)}\right\} \Big|_{\lambda=\lambda_j}(t) \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}} F_{\alpha,\alpha}(\lambda; t, t_0) \Big|_{\lambda=\lambda_j}(t) \\ &= \frac{1}{(m-1)!} \left(\prod_{i=1}^{m-1} * \hat{h}_{\alpha-1}(t, t_0)\right) *_{\nabla} F_{\alpha,\alpha}(\lambda_j; t, t_0).\end{aligned}$$

This yields the result in (26). The proof is thus completed.  $\square$

Lastly, we applied the operational decomposition method to derive a particular solution for some non-homogenous equations.

**Theorem 10.** Let  $f(t)$  be a ld-continuous function. Then, the linear differential equation

$$y^{(\alpha)}(t) - \lambda y(t) = f(t) \quad (28)$$

has the general solution

$$y_g(t) = {}_{\nabla}F_{\alpha}(\lambda; t, t_0) + y_p(t),$$

where  $y^{(\alpha)}(t) = \mathcal{D}_{\nabla, t_0}^{\alpha} y(t)$ ,  $y_p(t) = {}_{\nabla}F_{\alpha}(\lambda; t, t_0) * f(t)$ , which is a particular solution to (28).

**Proof.** Via Theorem 6,  $y(t) = {}_{\nabla}F_{\alpha}(\lambda; t, t_0)$  is the general solution to the corresponding homogeneous equation  $y^{(\alpha)}(t) - \lambda y(t) = 0$ . Therefore, it is sufficient to verify that  $y_p(t)$  is a particular solution to (28).

Let  $y(t)$  be a solution of (28), we then have

$$D_{\nabla, t_0}^{\alpha} (1 - \lambda I_{\nabla, t_0}^{\alpha}) y(t) = f(t);$$

therefore,

$$\begin{aligned} y(t) &= \left(1 - \lambda I_{\nabla, t_0}^\alpha\right)^{-1} I_{\nabla, t_0}^\alpha f(t) \\ &= \sum_{j=0}^{\infty} I_{\nabla, t_0}^{(j+1)\alpha} \lambda^j f(t) \\ &= \sum_{j=0}^{\infty} \lambda^j \left(\widehat{h}_{(j+1)\alpha-1}(t, t_0) * f(t)\right) \\ &= \nabla F_\alpha(\lambda; t, t_0) * f(t) = y_p(t). \end{aligned}$$

Thus,  $y_p(t)$  is a particular solution to (28). The proof is then completed.  $\square$

**Theorem 11.** Let  $\{\lambda_j\}_{j=1}^k$  be  $k$ -distinct roots of multiplicity  $\{\sigma_j\}_{j=1}^k$  of the characteristic polynomial (21) for the following non-homogeneous differential equation

$$\mathbb{L}_{\alpha} [y(t)] = \prod_{j=1}^k (D_{\nabla, t_0}^\alpha - \lambda_j)^{\sigma_j} y(t) = f(t). \quad (29)$$

Then, the particular solution is given by

$$y_p(t) = (G_\alpha * f)(t), \quad (30)$$

where  $G_\alpha(t)$  is given by

$$G_\alpha(t) = \prod_{j=1}^k * \prod_{l=1}^{\sigma_j} * \nabla F_{\alpha, \alpha}(\lambda_j; t, t_0). \quad (31)$$

**Proof.** Let

$$y_1(t) = \prod_{j=2}^k (D_{\nabla, t_0}^\alpha - \lambda_j)^{\sigma_j} y(t),$$

then Equation (29) can be rewritten as

$$(D_{\nabla, t_0}^\alpha - \lambda_1)^{\sigma_1} y_1(t) = f(t).$$

By successively applying the result of Theorem 10, we can obtain

$$y_1(t) = \left( \prod_{l=1}^{\sigma_1} * \nabla F_{\alpha, \alpha}(\lambda_1; t, t_0) \right) * f(t). \quad (32)$$

Let

$$\begin{aligned} y_l(t) &= \prod_{j=l+1}^k (D_{\nabla, t_0}^\alpha - \lambda_j)^{\sigma_j} y(t), l = 2, 3, \dots, k-1, \\ y_k(t) &= y(t), \end{aligned}$$

then

$$(D_{\nabla, t_0}^\alpha - \lambda_l)^{\sigma_l} y_l(t) = y_{l-1}(t), l = 2, 3, \dots, k.$$

As it is similar to the proof of (32), we can prove that

$$y_l(t) = \left( \prod_{l=1}^{\sigma_l} * \nabla F_{\alpha, \alpha}(\lambda_l; t, t_0) \right) * y_{l-1}(t), l = 2, 3, \dots, k.$$

Thus, we have

$$\begin{aligned} y(t) &= \left\{ \prod_{j=1}^k * \prod_{l=1}^{\sigma_j} * \nabla F_{\alpha, \alpha}(\lambda_j; t, t_0) \right\} * f(t) \\ &= (G_{\alpha} * f)(t). \end{aligned}$$

The proof is thus completed.  $\square$

## 5. Solution of Fractional Sequential Differential Equations with Convolution

In this section, we analyze the solutions of the homogeneous equation  $\mathbb{L}_{\mathbf{m}\alpha} y(t) = 0$ , with variable coefficients expressed by a convolution, where

$$\mathbb{L}_{\mathbf{m}\alpha} y(t) := \mathcal{D}_{\nabla, t_0}^{n\alpha} y(t) + \sum_{k=0}^{n-1} a_k(t) * \mathcal{D}_{\nabla, t_0}^{k\alpha} y(t). \quad (33)$$

First, we needed to prove that the power functions are non-negative on general time scales, as well as introduce the concept of  $\alpha$ -analyticity.

Let  $P_n = \{t_0, t_1, \dots, t_n\} \subset [t_0, t]_{\mathbb{T}}$  be a partition of  $[t_0, t]_{\mathbb{T}}$ , where  $t_0 < t_1 < \dots < t_n = t$ . We denoted  $\|P_n\| = \max\{t_i - t_{i-1} : t_{i-1} < \rho(t_i)\}$  for any  $i \in \{1, 2, \dots, n\}$ , as well as either  $t_i - t_{i-1} \leq \|P_n\|$ , or  $t_i - t_{i-1} > \|P_n\|$ , and  $t_{i-1} = \rho(t_i)$ .  $\|P_n\|$  can be viewed as an isolated time scale; thus, we denoted the graininess function on  $P_n$  by  $v_{P_n}$ . The Laplace transform on  $P_n$  is denoted by  $\mathcal{L}_{\nabla, t_0, P_n}$ , and the power function on  $P_n$  is denoted by  $\hat{h}_{\alpha, P_n}(t, t_0)$ .

**Lemma 5.** Let  $\hat{e}_z(t, t_0)$  be the nabla exponential function on time scale  $\mathbb{T}$  and  $\hat{e}_{z, P_n}(t, t_0)$  be the nabla exponential function on time scale  $P_n$ . It then holds that

- (i)  $\lim_{\|P_n\| \rightarrow 0} \hat{e}_{z, P_n}(t, t_0) = \hat{e}_z(t, t_0)$ ,
- (ii) for any  $\alpha > -1$ ,  $\hat{h}_{\alpha}(t, t_0) \geq 0$ .

**Proof.** Via the definition of the nabla exponential function on a time scale (see [38]), we can obtain

$$\hat{e}_z(t, t_0) = e^{\int_{t_0}^t \hat{\xi}_{v(\tau)}(\tau) \nabla \tau} = e^{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \hat{\xi}_{v(\tau)}(\tau) \nabla \tau}$$

and

$$\hat{e}_{z, P_n}(t, t_0) = e^{\int_{t_0}^t \hat{\xi}_{v_{P_n}(\tau)}(\tau) \nabla \tau} = e^{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \hat{\xi}_{v_{P_n}(\tau)}(\tau) \nabla \tau}.$$

If  $t_i$  is left-scattered in  $\mathbb{T}$ , and when  $\|P_n\|$  is small enough, we have  $t_{i-1} = \rho(t_i)$ . This implies that

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \hat{\xi}_{v_{P_n}(\tau)}(\tau) \nabla \tau &= -\frac{1}{v_{P_n}(t_i)} \text{Log}(1 - zv_{P_n}(t_i)) \cdot v_{P_n}(t_i) \\ &= -\frac{1}{v(t_i)} \text{Log}(1 - zv(t_i)) \cdot v(t_i) \\ &= \int_{t_{i-1}}^{t_i} \hat{\xi}_{v(\tau)}(\tau) \nabla \tau. \end{aligned}$$

If  $t_i$  is left-dense in  $\mathbb{T}$ , then  $v(t_i) = 0$  and  $\hat{\xi}_{v(\tau)}(t_i) = z$ . Since  $\lim_{x \rightarrow 0^+} \frac{-1}{x} \text{Log}(1 - xz) = z$ , for any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , such that

$$\left| \frac{-1}{x} \text{Log}(1 - xz) - z \right| < \varepsilon, 0 < x < \delta.$$

Thus, if  $\|P_n\| < \delta$ , then one can infer that

$$\begin{aligned}
\left| \int_{t_0}^t \widehat{\xi}_{v(\tau)}(\tau) \nabla \tau - \int_{t_0}^t \widehat{\xi}_{v_{P_n}(\tau)}(\tau) \nabla \tau \right| &= \left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} [\widehat{\xi}_{v(\tau)}(\tau) - \widehat{\xi}_{v_{P_n}(\tau)}(\tau)] \nabla \tau \right| \\
&\leq \sum_{v_{P_n}(t_i) < \delta} \int_{t_{i-1}}^{t_i} |\widehat{\xi}_{v(\tau)}(\tau) - \widehat{\xi}_{v_{P_n}(\tau)}(\tau)| \nabla \tau \\
&\leq \sum_{v_{P_n}(t_i) < \delta} \left\{ \int_{t_{i-1}}^{t_i} |\widehat{\xi}_{v(\tau)}(\tau) - z| \nabla \tau + \int_{t_{i-1}}^{t_i} |z - \widehat{\xi}_{v_{P_n}(\tau)}(\tau)| \nabla \tau \right\} \\
&< 2\varepsilon(t - t_0).
\end{aligned}$$

This shows that

$$\lim_{\|P_n\| \rightarrow 0} \hat{e}_{z, P_n}(t, t_0) = \hat{e}_z(t, t_0)$$

holds, which means that Conclusion (i) holds.

It follows from Lemma 2 that  $\hat{h}_{\alpha, P_n}(t, t_0) \geq 0$  applies for any  $\alpha > -1$ . Then, through using Definition 7 and Lemma 5, we have

$$\begin{aligned}
\lim_{\|P_n\| \rightarrow 0} \hat{h}_{\alpha, P_n}(t, t_0) &= \lim_{\|P_n\| \rightarrow 0} \mathcal{L}_{\nabla, t_0, P_n}^{-1} \left\{ \frac{1}{z^{\alpha+1}} \right\} (t) \\
&= \lim_{\|P_n\| \rightarrow 0} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{z^{\alpha+1}} \hat{e}_{z, P_n}(t, t_0) dz \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{z^{\alpha+1}} \hat{e}_z(t, t_0) dz \\
&= \mathcal{L}_{\nabla, t_0}^{-1} \left\{ \frac{1}{z^{\alpha+1}} \right\} (t) \\
&= \hat{h}_\alpha(t, t_0) \geq 0,
\end{aligned}$$

which shows that Conclusion (ii) holds.

The proof is thus completed.  $\square$

**Definition 14.** Let  $\alpha \in (0, 1]$ ,  $f(t)$  be a real function defined on  $\Omega$ . Then,  $f(t)$  is said to be  $\alpha$ -analytic at  $t_0$  if there exists an interval  $N_+(t_0) = [t_0, t_0 + \delta]_{\mathbb{T}}$  such that, for all  $t \in N_+(t_0)$ ,  $f(t)$  can be expressed as  $\sum_{n=1}^{\infty} c_n \hat{h}_{n\alpha-1}(t, t_0)$  ( $c_n \in \mathbb{R}$ ).

Since  $D\hat{h}_{\alpha+1}(t, t_0) = \hat{h}_\alpha(t, t_0)$  and  $\hat{h}_\alpha(t, t_0) \geq 0$ , for any  $\alpha > -1$ , we can obtain  $\hat{h}_\alpha(t, t_0)$ , which increases for any  $\alpha > 0$ . This implies that there exists a  $\rho > 0$ , such that the series  $\sum_{n=1}^{\infty} c_n \hat{h}_{n\alpha-1}(t, t_0)$  is absolutely convergent for any  $t \in [t_0, t_0 + \rho)_{\mathbb{T}}$  and is divergent for any  $t > t_0 + \rho$ . We called  $\rho$  the convergence radius of the series.

**Definition 15.** A point  $t_0$  is said to be an  $\alpha$ -ordinary point of the equation  $\mathbb{L}_{\nabla, \alpha} y(t) = 0$  if the functions  $a_k(t)$  ( $k = 0, 1, \dots, n-1$ ) are  $\alpha$ -analytic at  $t_0$ .

The following properties are clear.

**Property 2.** Let  $\alpha \in (0, 1]$ . If  $f(t)$  is  $\alpha$ -analytic at  $t_0$ , with the convergence radius  $\rho$  ( $\rho > 0$ ), then

$$D_{\nabla, t_0}^\alpha f(t) = D_{\nabla, t_0}^\alpha \left( \sum_{n=1}^{\infty} c_n \hat{h}_{n\alpha-1}(t, t_0) \right) = \sum_{n=1}^{\infty} c_n D_{\nabla, t_0}^\alpha \hat{h}_{n\alpha-1}(t, t_0).$$

**Property 3.** (see also [24,25]) Let  $\alpha \in (0, 1]$ ,  $t \geq t_0$ . Then,

$$D_{\nabla, t_0}^\alpha \hat{h}_\beta(t, t_0) = \hat{h}_{\beta-\alpha}(t, t_0), \forall \beta > \alpha - 1. \quad (34)$$

and

$${}^C D_{\nabla, t_0}^\alpha \hat{h}_\beta(t, t_0) = \hat{h}_{\beta-\alpha}(t, t_0), \forall \beta > \alpha - 1. \quad (35)$$

### 5.1. Solutions Around an Ordinary Point of a Fractional Differential Equation of Order $\alpha$

In this section, we analyze the existence of solutions for the equation

$$\mathbb{L}_\alpha y(t) := y^{(\alpha)}(t) + p(t) * y(t) = 0$$

around an  $\alpha$ -ordinary point  $t_0 \in \Omega$ , with  $p(t)$  defined in  $\Omega$ . We separately consider the cases where  $y^{(\alpha)}(t)$  represents the Riemann–Liouville and Caputo fractional derivatives of an order  $\alpha$  of the function  $y(t)$ . Since  $t_0$  is an  $\alpha$ -ordinary point,  $p(t)$  can be expressed as follows:

$$p(t) = \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0), \quad (36)$$

where this series is convergent for  $t \in [t_0, t_0 + \rho)_{\mathbb{T}}$ , with  $\rho > 0$ .

**Theorem 12.** Let  $\alpha \in (0, 1]$  and  $t_0 \in \Omega$  be an  $\alpha$ -ordinary point for the equation

$$\mathbb{L}_\alpha y(t) := D_{\nabla, t_0}^\alpha y(t) + p(t) * y(t) = 0. \quad (37)$$

Then, there exists a unique function

$$y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0),$$

which is the solution to Equation (37) for  $t \in (t_0, t_0 + \rho)_{\mathbb{T}}$  and satisfies the initial condition  $a_0 = \lim_{t \rightarrow t_0} [y * \hat{h}_{-\alpha}(\cdot, t_0)](t)$ .

**Proof.** We sought a solution of (37) as follows:

$$y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0). \quad (38)$$

If  $y(t)$  is given by (38), then when using (34), (36) and  $D_{\nabla, t_0}^\alpha \hat{h}_{\alpha-1}(t, t_0) = 0$  (see Formula (85) in [24]), we have

$$\begin{aligned} D_{\nabla, t_0}^\alpha y(t) &= D_{\nabla, t_0}^\alpha \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0) \\ &= D_{\nabla, t_0}^\alpha \left( a_0 \hat{h}_{\alpha-1}(t, t_0) + \sum_{n=1}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0) \right) \\ &= \sum_{n=1}^{\infty} a_n \hat{h}_{n\alpha-1}(t, t_0) \\ &= \sum_{n=0}^{\infty} a_{n+1} \hat{h}_{(n+1)\alpha-1}(t, t_0) \end{aligned} \quad (39)$$

and

$$\begin{aligned} p(t) * y(t) &= \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} \hat{h}_{(n-k)\alpha-1}(t, t_0) * a_k \hat{h}_{(k+1)\alpha-1}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_k \hat{h}_{(n+1)\alpha-1}(t, t_0). \end{aligned} \quad (40)$$

Through substituting (39) and (40) into (37), we can obtain

$$\begin{aligned} D_{\nabla, t_0}^{\alpha} y(t) + p(t) * y(t) \\ = \sum_{n=0}^{\infty} a_{n+1} \hat{h}_{(n+1)\alpha-1}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_k \hat{h}_{(n+1)\alpha-1}(t, t_0) = 0. \end{aligned}$$

Thus, we have the following recurrence formula, which allows us to express  $a_n$  ( $n > 0$ ) in terms of  $a_0$  as follows:

$$a_1 = 0, a_{n+1} = - \sum_{k=0}^{n-1} p_{n-k} a_k, n = 1, 2, \dots$$

When the following equation does not apply

$$y * \hat{h}_{-\alpha}(\cdot, t_0) = a_0 + \sum_{n=2}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0),$$

then we have  $a_0 = \lim_{t \rightarrow t_0} [y * \hat{h}_{-\alpha}(\cdot, t_0)](t)$ .

If we select  $a_n$  with the above formula, then (38) is a solution of (37). The proof is thus completed.  $\square$

**Theorem 13.** Let  $\alpha \in (0, 1]$  and  $t_0 \in \Omega$  be an  $\alpha$ -ordinary point for equation

$$\mathbb{L}_{\alpha} y(t) := {}^C D_{\nabla, t_0}^{\alpha} y(t) + p(t) * y(t) = 0, \quad (41)$$

where  $p(t)$  is defined in  $\Omega$ . Then, there exists a unique function  $y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0)$ , which is the solution to Equation (41) for  $t \in (t_0, t_0 + \rho)_{\mathbb{T}}$  and satisfies the initial condition  $y(t_0) = a_0$ .

**Proof.** We sought the series representation for the solution in the form

$$y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0). \quad (42)$$

Using (34) and (36), we obtained

$$\begin{aligned} {}^C D_{\nabla, t_0}^{\alpha} y(t) &= {}^C D_{\nabla, t_0}^{\alpha} \sum_{n=0}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0) \\ &= {}^C D_{\nabla, t_0}^{\alpha} \left( a_0 + \sum_{n=1}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0) \right) \\ &= \sum_{n=1}^{\infty} a_n \hat{h}_{(n-1)\alpha}(t, t_0) \\ &= \sum_{n=0}^{\infty} a_{n+1} \hat{h}_{n\alpha}(t, t_0) \end{aligned} \quad (43)$$

and

$$\begin{aligned} p(t) * y(t) &= \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=0}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} \hat{h}_{(n-k)\alpha-1}(t, t_0) * a_k \hat{h}_{k\alpha}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_k \hat{h}_{n\alpha}(t, t_0). \end{aligned} \quad (44)$$

Through substituting (43) and (44) into (41), we obtained

$$\begin{aligned} & {}^C D_{\nabla, t_0}^{\alpha} y(t) + p(t) * y(t) \\ &= \sum_{n=0}^{\infty} a_{n+1} \hat{h}_{n\alpha}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_k \hat{h}_{n\alpha}(t, t_0) = 0. \end{aligned}$$

Thus, we obtained the following recurrence formula, which allows us to express  $a_n$  ( $n > 0$ ), in terms of  $a_0$ , as follows:

$$a_1 = 0, a_{n+1} = - \sum_{k=0}^{n-1} p_{n-k} a_k, n = 1, 2, \dots$$

If we select  $a_n$  via the above formula, then (42) is a solution of (41). The proof is thus completed.  $\square$

## 5.2. Solutions Around an Ordinary Point of a Fractional Differential Equation of Order $2\alpha$

In this section, we shall consider the solutions around an  $\alpha$ -ordinary point  $t_0 \in \Omega$  to equation

$$\mathbb{L}_{2\alpha} y(t) := y^{(2\alpha)}(t) + p(t) * y^{(\alpha)}(t) + q(t) * y(t) = 0, \quad (45)$$

where  $p(t)$  and  $q(t)$  are defined on  $\Omega$ , and  $y^{(2\alpha)}$  and  $y^{(\alpha)}$  represent the Riemann–Liouville or Caputo sequential derivatives of order  $2\alpha$  and  $\alpha$  of the function  $y(t)$ .

Since  $t_0$  is an  $\alpha$ -ordinary point of (45), then

$$p(t) = \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0) (t \in [t_0, t_0 + \rho_1]_{\mathbb{T}}, \rho_1 > 0) \quad (46)$$

and

$$q(t) = \sum_{n=1}^{\infty} q_n \hat{h}_{n\alpha-1}(t, t_0) (t \in [t_0, t_0 + \rho_2]_{\mathbb{T}}, \rho_2 > 0). \quad (47)$$

**Theorem 14.** Let  $\alpha \in (0, 1]$  and  $t_0 \in \Omega$  be an  $\alpha$ -ordinary point of equation

$$\mathbb{L}_{2\alpha} y(t) := D_{\nabla, t_0}^{2\alpha} y(t) + p(t) * D_{\nabla, t_0}^{\alpha} y(t) + q(t) * y(t) = 0. \quad (48)$$

Then, there exists a unique function that is formed as

$$y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0)$$

for  $t \in (t_0, t_0 + \rho)_{\mathbb{T}}$  with  $\rho = \min\{\rho_1, \rho_2\}$ , which is a solution to Equation (48) and satisfies the following initial conditions:

$$\lim_{t \rightarrow t_0} [y * \hat{h}_{-\alpha}(\cdot, t_0)](t) = a_0$$

and

$$\lim_{t \rightarrow t_0} \left[ D_{\nabla, t_0}^{\alpha} y * \hat{h}_{-\alpha}(\cdot, t_0) \right] (t) = a_1.$$

**Proof.** We sought the solution to Equation (48) in the form

$$y(t) = \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0).$$

Through calculating  $D_{\nabla, t_0}^{\alpha} y(t)$  and  $D_{\nabla, t_0}^{2\alpha} y(t)$  and taking (34) into account, we obtained

$$\begin{aligned} D_{\nabla, t_0}^{\alpha} y(t) &= D_{\nabla, t_0}^{\alpha} \left( a_0 \hat{h}_{\alpha-1}(t, t_0) + \sum_{n=1}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0) \right) \\ &= \sum_{n=1}^{\infty} a_n \hat{h}_{n\alpha-1}(t, t_0) \end{aligned}$$

and

$$\begin{aligned} D_{\nabla, t_0}^{2\alpha} y(t) &= D_{\nabla, t_0}^{\alpha} \sum_{n=1}^{\infty} a_n \hat{h}_{n\alpha-1}(t, t_0) \\ &= D_{\nabla, t_0}^{\alpha} \left( a_1 \hat{h}_{\alpha-1} + \sum_{n=2}^{\infty} a_n \hat{h}_{n\alpha-1}(t, t_0) \right) \\ &= \sum_{n=2}^{\infty} a_n \hat{h}_{(n-1)\alpha-1}(t, t_0) \\ &= \sum_{n=0}^{\infty} a_{n+2} \hat{h}_{(n+1)\alpha-1}(t, t_0). \end{aligned}$$

Through using (34), (46) and (47), we obtained

$$\begin{aligned} p(t) * D_{\nabla, t_0}^{\alpha} y(t) &= \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=1}^{\infty} a_n \hat{h}_{n\alpha-1}(t, t_0) \\ &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} p_{n-k} \hat{h}_{(n-k)\alpha-1}(t, t_0) * a_k \hat{h}_{k\alpha-1}(t, t_0) \\ &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} p_{n-k} a_k \hat{h}_{n\alpha-1}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^n p_{n+1-k} a_k \hat{h}_{n\alpha-1}(t, t_0) \end{aligned}$$

and

$$\begin{aligned} q(t) * y(t) &= \sum_{n=1}^{\infty} q_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=0}^{\infty} a_n \hat{h}_{(n+1)\alpha-1}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} \hat{h}_{(n-k)\alpha-1}(t, t_0) * a_k \hat{h}_{(k+1)\alpha-1}(t, t_0) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} a_k \hat{h}_{(n+1)\alpha-1}(t, t_0). \end{aligned}$$

When substituting these values into Equation (48), we have

$$\begin{aligned}
 & D_{\nabla, t_0}^{2\alpha} y(t) + p(t) * D_{\nabla, t_0}^{\alpha} y(t) + q(t) * y(t) \\
 = & \sum_{n=0}^{\infty} a_{n+2} \widehat{h}_{(n+1)\alpha-1}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=1}^n p_{n+1-k} a_k \widehat{h}_{n\alpha-1}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} a_k \widehat{h}_{(n+1)\alpha-1}(t, t_0) \\
 = & 0.
 \end{aligned}$$

Thus, we arrived at the following recurrence formula for the coefficients  $a_n$  :

$$a_{n+2} = - \sum_{k=0}^n (p_{n-k} a_{k+1} + q_{n-k} a_k), a_2 = 0.$$

Through using these  $a_n$ , we can obtain

$$y(t) = \sum_{n=0}^{\infty} a_n \widehat{h}_{(n+1)\alpha-1}(t, t_0),$$

which is a solution of (48). The proof is thus completed.  $\square$

**Theorem 15.** Let  $\alpha \in (0, 1]$  and  $t_0 \in \Omega$  be an  $\alpha$ -ordinary point of equation

$$\mathbb{L}_{2\alpha} y(t) := {}^C D_{\nabla, t_0}^{2\alpha} y(t) + p(t) * {}^C D_{\nabla, t_0}^{\alpha} y(t) + q(t) * y(t) = 0. \quad (49)$$

Then, for any given number  $a_0, a_1 \in \mathbb{R}$ , there exists a unique function that is formed as

$$y(t) = \sum_{n=0}^{\infty} a_n \widehat{h}_{n\alpha}(t, t_0) \quad (50)$$

for  $t \in (t_0, t_0 + \rho)_{\mathbb{T}}$  with  $\rho = \min\{\rho_1, \rho_2\}$ , which is a solution to Equation (49) and satisfies the following initial conditions:

$$a_2 = 0, a_{n+2} = - \sum_{k=0}^{n-1} (p_{n-k} a_{k+1} + q_{n-k} a_k), n = 1, 2, \dots$$

**Proof.** We sought a solution to Equation (49) in the form  $y(t) = \sum_{n=0}^{\infty} a_n \widehat{h}_{n\alpha}(t, t_0)$ .

When calculating  ${}^C D_{\nabla, t_0}^{\alpha} y(t)$  and  ${}^C D_{\nabla, t_0}^{2\alpha} y(t)$  and taking (34) into account, we can obtain

$$\begin{aligned}
 {}^C D_{\nabla, t_0}^{\alpha} y(t) &= {}^C D_{\nabla, t_0}^{\alpha} \left( a_0 + \sum_{n=1}^{\infty} a_n \widehat{h}_{n\alpha}(t, t_0) \right) \\
 &= \sum_{n=1}^{\infty} a_n \widehat{h}_{(n-1)\alpha}(t, t_0)
 \end{aligned}$$

and

$$\begin{aligned}
 {}^C D_{\nabla, t_0}^{2\alpha} y(t) &= {}^C D_{\nabla, t_0}^{\alpha} \left( a_1 + \sum_{n=2}^{\infty} a_n \widehat{h}_{(n-1)\alpha}(t, t_0) \right) \\
 &= \sum_{n=2}^{\infty} a_n \widehat{h}_{(n-2)\alpha}(t, t_0) \\
 &= \sum_{n=0}^{\infty} a_{n+2} \widehat{h}_{n\alpha}(t, t_0).
 \end{aligned}$$

When using (34), (46) and (47), we obtain

$$\begin{aligned}
 p(t) * {}^C D_{\nabla, t_0}^\alpha y(t) &= \sum_{n=1}^{\infty} p_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=1}^{\infty} a_n \hat{h}_{(n-1)\alpha}(t, t_0) \\
 &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} p_{n-k} \hat{h}_{(n-k)\alpha-1}(t, t_0) * a_k \hat{h}_{(k-1)\alpha}(t, t_0) \\
 &= \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} p_{n-k} a_k \hat{h}_{(n-1)\alpha}(t, t_0) \\
 &= \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} p_{n-k-1} a_{k+1} \hat{h}_{(n-1)\alpha}(t, t_0) \\
 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_{k+1} \hat{h}_{n\alpha}(t, t_0)
 \end{aligned}$$

and

$$\begin{aligned}
 q(t) * y(t) &= \sum_{n=1}^{\infty} q_n \hat{h}_{n\alpha-1}(t, t_0) * \sum_{n=0}^{\infty} a_n \hat{h}_{n\alpha}(t, t_0) \\
 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} \hat{h}_{(n-k)\alpha}(t, t_0) * a_k \hat{h}_{k\alpha}(t, t_0) \\
 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} a_k \hat{h}_{n\alpha}(t, t_0).
 \end{aligned}$$

This shows that (50) is a solution of (49) if and only if

$$\begin{aligned}
 & {}^C D_{\nabla, t_0}^{2\alpha} y(t) + p(t) * {}^C D_{\nabla, t_0}^\alpha y(t) + q(t) * y(t) \\
 &= \sum_{n=0}^{\infty} a_{n+2} \hat{h}_{n\alpha}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_{n-k} a_{k+1} \hat{h}_{n\alpha}(t, t_0) + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} q_{n-k} a_k \hat{h}_{n\alpha}(t, t_0) \\
 &= a_2 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} [p_{n-k} a_{k+1} + q_{n-k} a_k] \hat{h}_{n\alpha}(t, t_0) = 0.
 \end{aligned}$$

Thus, we arrived at the following recurrence formula for the coefficients  $a_n$  :

$$a_2 = 0, a_{n+2} = - \sum_{k=0}^{n-1} (p_{n-k} a_{k+1} + q_{n-k} a_k), n = 1, 2, \dots$$

When using the above  $a_n$ , we can obtain (50), which is a solution of (49). The proof is thus completed.  $\square$

## 6. Applications and Conclusions

Fractional differential equations have a wide range of practical applications. For example, in reviewing Equation (20), which we discussed earlier, when  $t \in \mathbb{T} = \mathbb{R}$ , Equation (20) is the fractional-order equation of continuous variables. When  $t \in \mathbb{T} = \mathbb{Z}$ , then Equation (20) degenerates into the usual difference equation. When  $\mathbb{T}$  selects different closed sets in  $\mathbb{R}$ , our equation corresponds to different fractional-order equations or difference equations with different step lengths. Therefore, it can be seen that the results we obtained have a wide range of applications. In addition, fractional-order equations are also used in many specific fields in real life, including population models, infectious disease models, heat conduction models, wave equations, etc.

In recent years, an increasing number of researchers have focused on fractional-order infectious disease models [39–42]. The global epidemic in recent years has made the

research on fractional-order COVID-19 [43–45] an exceptionally hot topic. The traveling wave solution characterizes the problem of the speed of transmission of infectious diseases during their spread. In recent years, the discussion of the spread and speed of transmission of infectious diseases has become a hot topic, especially since the outbreak of COVID-19, which has once again sounded an alarm for humanity. Inspired by [46,47], we present the following wave equation system, which corresponds to a fractional-order SIR infectious disease model:

$$\begin{aligned} cD_{\nabla}^{\alpha} S(t) &= d_1 D_{\nabla}^{2\alpha} S(t) + \Lambda(t) - \frac{\beta S(t)I(t)}{N(t)} - \mu S(t), \\ cD_{\nabla}^{\alpha} I(t) &= d_2 D_{\nabla}^{2\alpha} I(t) + \frac{\beta S(t)I(t)}{N(t)} + \rho R(t) - (\mu + \phi)I(t), \\ cD_{\nabla}^{\alpha} R(t) &= d_3 D_{\nabla}^{2\alpha} R(t) + \phi I(t) - \rho R(t) - \mu R(t), \\ N(t) &= S(t) + I(t) + R(t), \end{aligned} \quad (51)$$

where  $t \in \mathbb{T}, 0 < \alpha \leq 1$ ;  $S$  represents susceptible individuals;  $I$  represents infected individuals;  $R$  represents recovered individuals;  $\Lambda$  represents total recruitment scale;  $c$  represents disease propagation speed;  $d_1, d_2, d_3$  represent diffusion coefficient;  $\beta$  represents the infection rate;  $\rho$  represents the relapse rate;  $\mu$  represents the natural mortality rate;  $\phi$  represents the cure rate; and the total population is represented by  $N$ . If we choose  $d_1 = d_2 = d_3 = d$ , via adding the first three equations of (51), we can obtain

$$cD_{\nabla}^{\alpha} [S(t) + I(t) + R(t)] = dD_{\nabla}^{2\alpha} [S(t) + I(t) + R(t)] + \Lambda(t) - \mu[S(t) + I(t) + R(t)].$$

By further organizing the fourth equation in (51), it can be concluded that

$$dD_{\nabla}^{2\alpha} N(t) - cD_{\nabla}^{\alpha} N(t) - \mu N(t) = -\Lambda(t). \quad (52)$$

It is evident that the form of (52) is completely consistent with (20). Next, we used the relevant results from Section 4 to solve (52). With (21), we obtained the characteristic polynomial of (52)

$$P_2(\lambda) = \lambda^2 - \frac{c}{d}\lambda - \frac{\mu}{d}.$$

When  $P_2(\lambda) = 0$ , we can obtain the eigenvalues of (52) as follows:

$$\lambda_{1,2} = \frac{\frac{c}{d} \pm \frac{1}{d}\sqrt{c^2 + 4d\mu}}{2}.$$

According to Theorem 8, we can obtain the solution to the wave Equation (52)

$$N(t) = \left( \sum_{n=0}^{\infty} \left( \frac{\frac{c}{d} + \frac{1}{d}\sqrt{c^2 + 4d\mu}}{2} \right)^n I_{\nabla, t_0}^{(n+1)\alpha} \Lambda(t) \right) \cdot \left( \sum_{n=0}^{\infty} \left( \frac{\frac{c}{d} - \frac{1}{d}\sqrt{c^2 + 4d\mu}}{2} \right)^n I_{\nabla, t_0}^{(n+1)\alpha} \Lambda(t) \right). \quad (53)$$

From the practical significance of infectious diseases,  $N(t)$  characterizes the speed of disease transmission in the overall population. We find from (53) that the value of  $N(t)$  only depends on the four parameters  $c, d, \mu$  and  $\Lambda$ . Furthermore, (53) established clear relationships between the four variables, providing a mathematical basis for further quantitative research on the speed and control of disease transmission. Moreover,  $\mathbb{T}$  is a general time scale, and by selecting different types of time scales, it can be used for the study of various intermittent, seasonal and discontinuous long-term epidemics. Due to space limitations, we will provide further research in a separate article.

The theory of fractional-order differential equations is a new topic providing many directions for further research. In this paper, we presented a general theory for sequential differential equations of a fractional order on time scales. We have given the explicit solution, in the case of constant coefficients, for both the homogeneous and the non-homogeneous problem using the  $\nabla$ -Mittag-Leffler function, Laplace transform method, operational method and operational decomposition method. In addition, we proved the fact that the nabla generalized power function is positive, which is notable as this is also an

open problem proposed in [37], which is also important for the studying of fractional-order differential equations on time scales. By introducing the concept of  $\alpha$ -analyticity on time scales and in using the Laplace transform method, we studied the existence of a new class of fractional, sequential differential equations with convolution-type variable coefficients. We hope these results will provide a general and unified theory to study the continuous and discrete sequential differential equations of a fractional order on time scales.

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## References

1. Fan, W.; Cheng, H. A novel numerical inverse technique for multi-parameter time fractional radially symmetric anomalous diffusion problem with initial singularity. *Comput. Math. Appl.* **2024**, *158*, 95–101. [\[CrossRef\]](#)
2. Li, Z.; Zhang, Z. Stabilization Control for a Class of Fractional-Order HIV-1 Infection Model with Time Delays. *Axioms* **2023**, *12*, 695. [\[CrossRef\]](#)
3. Malik, M.; Sajid, M.; Kumar, V. Controllability of singular dynamic systems on time scales. *Asian J. Control* **2022**, *24*, 2771–2777. [\[CrossRef\]](#)
4. Zeng, C.; Chen, Y.Q. Optimal random search, fractional dynamics and fractional calculus. *Fract. Calc. Appl. Anal.* **2014**, *17*, 321–332. [\[CrossRef\]](#)
5. Agarwal, R.P. Certain fractional  $q$ -integrals and  $q$ -derivatives. *Proc. Camb. Philos. Soc.* **1969**, *66*, 365–370. [\[CrossRef\]](#)
6. Diaz, R.; Osler, T.J. Differences of fractional order. *Math. Comput.* **1974**, *28*, 185–202. [\[CrossRef\]](#)
7. Guo, G.C.; Baleanu, D. Chaos synchronization of the discrete fractional logistic map. *Signal Process.* **2014**, *102*, 96–99.
8. Guo, G.C.; Baleanu, D. Discrete chaos in fractional delayed logistic maps. *Nonlinear Dyn.* **2015**, *80*, 1697–1703.
9. Guo, G.C.; Baleanu, D. Discrete fractional logistic map and its chaos. *Nonlinear Dyn.* **2014**, *75*, 283–287.
10. Guo, G.C.; Baleanu, D.; Zeng, S.D. Discrete chaos in fractional sine and standard maps. *Phys. Lett. A* **2014**, *378*, 484–487.
11. Atici, F.M.; Eloe, P.W. A Transform Method in Discrete Fractional Calculus. *Inter. J. Diff. Equ.* **2007**, *2*, 165–176.
12. Atici, F.M.; Eloe, P.W. Initial value problems in discrete fractional calculus. *Proc. Am. Math. Soc.* **2009**, *137*, 981–989. [\[CrossRef\]](#)
13. Atici, F.M.; Şengül, S. Modeling with fractional difference equations. *J. Math. Anal. Appl.* **2010**, *369*, 1–9. [\[CrossRef\]](#)
14. Holm, M. Sum and Difference Compositions in Discrete Fractional Calculus. *CUBO Math. J.* **2011**, *13*, 153–184. [\[CrossRef\]](#)
15. Anastassiou, G.A. Nabla discrete fractional calculus and nabla inequalities. *Math. Comput. Model.* **2010**, *51*, 562–571. [\[CrossRef\]](#)
16. Atici, F.M.; Eloe, P.W. Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I* **2009**, *2009*, 1–12.
17. Hilger, S. Analysis on measure chains a unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [\[CrossRef\]](#)
18. Williams, P.A. Unifying Fractional Calculus with Time Scales. Doctoral Thesis, The University of Melbourne, Melbourne, Australia, 2012.
19. Anastassiou, G.A. Foundations of nabla fractional calculus on time scales and inequalities. *Comput. Math. Appl.* **2010**, *59*, 3750–3762. [\[CrossRef\]](#)
20. Bastos, N. Fractional Calculus on Time Scales. Doctoral Thesis, The University of Aveiro, Aveiro, Portugal, 2012.
21. Anastassiou, G.A. Principles of delta fractional calculus on time scales and inequalities. *Math. Comput. Model.* **2010**, *52*, 556–566. [\[CrossRef\]](#)
22. Bastos, N.; Ferreira, R.; Torres, D. Discrete-time fractional variational problems. *Sign. Proc.* **2011**, *91*, 513–524. [\[CrossRef\]](#)
23. Holm, M.T. The Laplace transform in discrete fractional calculus. *Comput. Math. Appl.* **2011**, *62*, 1591–1601. [\[CrossRef\]](#)
24. Wu, L.; Zhu, J. Fractional Cauchy problem with Riemann-Liouville derivative on time scales. *Abstr. Appl. Anal.* **2013**, *2013*, 795701. [\[CrossRef\]](#)
25. Zhu, J.; Wu, L. Fractional Cauchy Problem with Caputo Nabla Derivative on Time Scales. *Abstr. Appl. Anal.* **2015**, *2015*, 486054. [\[CrossRef\]](#)

26. Zhu, J.; Zhu, Y. Fractional Cauchy Problem with Riemann-Liouville Fractional Delta Derivative on Time Scales. *Abstr. Appl. Anal.* **2013**, *2013*, 401596. [[CrossRef](#)]
27. Duque, C.; Leiva, H.; Tridane, A. Relative asymptotic equivalence of dynamic equations on time scales. *Adv. Contin. Discret. Models* **2022**, *2022*, 4. [[CrossRef](#)]
28. Fang, J.J.; Mou, D.S.; Zhang, H.C.; Wang, Y.Y. Discrete fractional soliton dynamics of the fractional Ablowitz-Ladik model. *Optik* **2021**, *228*, 166186. [[CrossRef](#)]
29. Kisela, T. On dynamical systems with Nabla half derivative on time scales. *Mediterr. J. Math.* **2020**, *17*, 187. [[CrossRef](#)]
30. Liu, M.; Dong, H.; Fang, Y.; Dong, H. Modelling and analysis of dynamic systems on time-space scales and application in burgers equation. *J. Appl. Anal. Comput.* **2022**, *12*, 2555–2577. [[CrossRef](#)]
31. Messina, E.; Raffoul, Y.; Vecchio, A. Qualitative analysis of dynamic equations on time scales using Lyapunov functions. *Diff. Equ. Appl.* **2022**, *14*, 215–226. [[CrossRef](#)]
32. Wu, G.C.; Baleanu, D.; Zeng, S.D.; Deng, Z.G. Discrete fractional diffusion equation. *Nonlinear Dyn.* **2015**, *80*, 281–286. [[CrossRef](#)]
33. Gogoi, B.; Saha, U.K.; Hazarika, B. Existence of solution of a nonlinear fractional dynamic equation with initial and boundary conditions on time scales. *J. Anal.* **2024**, *32*, 85–102. [[CrossRef](#)]
34. Gogoi, B.; Hazarika, B.; Saha, U.K.; Tikare, S. Periodic boundary value problems for fractional dynamic equations on time scales. *Results Math.* **2023**, *78*, 228. [[CrossRef](#)]
35. Gogoi, B.; Saha, U.K.; Hazarika, B.; Torres, D.F.; Ahmad, H. Nabla Fractional Derivative and Fractional Integral on Time Scales. *Axioms* **2021**, *10*, 317. [[CrossRef](#)]
36. Morsy, A.; Nisar, K.S.; Ravichandran, C.; Anusha, C. Sequential fractional order Neutral functional Integro differential equations on time scales with Caputo fractional operator over Banach spaces. *AIMS Math.* **2023**, *8*, 5934–5949. [[CrossRef](#)]
37. Kisela, T. Power functions and essentials of fractional calculus on isolated time scales. *Adv. Diff. Equ.* **2013**, *2013*, 259. [[CrossRef](#)]
38. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
39. Akdim, K.; Ez-Zetouni, A.; Zahid, M. The influence of awareness campaigns on the spread of an infectious disease: A qualitative analysis of a fractional epidemic model. *Model. Earth Syst. Environ.* **2022**, *8*, 1311–1319. [[CrossRef](#)] [[PubMed](#)]
40. Bounkaicha, C.; Allali, K. Dynamics of a time fractional order spatio-temporal SIR with vaccination and temporary immunity. *Partial Diff. Equ. Appl. Math.* **2023**, *7*, 100524. [[CrossRef](#)]
41. Paul, S.; Mahata, A.; Mukherjee, S.; Roy, B. Dynamics of SIQR epidemic model with fractional order derivative. *Partial Diff. Equ. Appl. Math.* **2022**, *5*, 100216. [[CrossRef](#)] [[PubMed](#)]
42. Sadki, M.; Harroudi, S.; Allali, K. Fractional-order SIR epidemic model with treatment cure rate. *Partial Diff. Equ. Appl. Math.* **2023**, *8*, 100593. [[CrossRef](#)]
43. Barros, L.C.; Lopes, M.M.; Pedro, F.S.; Esmi, E.; Santos, J.P.; Sánchez, D.E. The memory effect on fractional calculus: An application in the spread of COVID-19. *Comput. Appl. Math.* **2021**, *40*, 72. [[CrossRef](#)]
44. Hamou, A.A.; Azroul, E.; Alaoui, A.L. Fractional Model and Numerical Algorithms for Predicting COVID-19 with Isolation and Quarantine Strategies. *Int. J. Appl. Comput. Math.* **2021**, *7*, 142. [[CrossRef](#)] [[PubMed](#)]
45. Khalaf, S.L.; Kadhim, M.S.; Khudair, A.R. Studying of COVID-19 fractional model: Stability analysis. *Partial Diff. Equ. Appl. Math.* **2023**, *7*, 100470. [[CrossRef](#)] [[PubMed](#)]
46. Zhu, C.C.; Li, W.T.; Yang, F.Y. Traveling waves in a nonlocal dispersal SIRH model with relapse. *Comput. Math. Appl.* **2017**, *73*, 1707–1723. [[CrossRef](#)]
47. Zhu, C.C.; Li, W.T.; Yang, F.Y. Traveling waves of a reaction-diffusion SIRQ epidemic model with relapse. *J. Appl. Anal. Comput.* **2017**, *7*, 147–171.

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