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Estimates of Eigenvalues and Approximation Numbers for a Class of Degenerate Third-Order Partial Differential Operators

Mussakan Muratbekov [†], Ainash Suleimbekova ^{*,†} and Mukhtar Baizhumanov [†]

Department of Mathematics, M.Kh.Dulaty Taraz Regional University, Taraz 080000, Kazakhstan; m.muratbekov@dulaty.kz (M.M.); mk.baizhumanov@dulaty.kz (M.B.)

* Correspondence: ao.suleimbekova@dulaty.kz; Tel.: +7-707-7187186

[†] These authors contributed equally to this work.

Abstract: In this paper, we study the spectral properties of a class of degenerate third-order partial differential operators with variable coefficients presented in a rectangle. Conditions are found to ensure the existence and compactness of the inverse operator. A theorem on estimates of approximation numbers is proven. Here, we note that finding estimates of approximation numbers, as well as extremal subspaces, for a set of solutions to the equation is a task that is certainly important from both a theoretical and a practical point of view. The paper also obtained an upper bound for the eigenvalues. Note that, in this paper, estimates of eigenvalues and approximation numbers for the degenerate third-order partial differential operators are obtained for the first time.

Keywords: third-order partial differential equations; approximation numbers; singular numbers; eigenvalues; compactness of the resolvent

MSC: 35A21; 35P15



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1. Introduction

Recently, mathematicians' interest in third-order partial differential equations has increased due to the fact that such equations model many real-world phenomena. For example, models of the transmission of electric lines and quantum hydrodynamic models are described by third-order partial differential equations. In papers [1,2], for example, nonlocal boundary value problems for third-order partial differential equations were studied, and in [3–8], the properties of solutions to linearized Korteweg–de Vries equations were studied. In [9], the question of the existence and uniqueness of solutions to the Goursat–Dirichlet problem was studied. Note that, in these papers, for third-order partial differential equations, questions about the existence and smoothness of solutions without degeneracy are studied. However, in such applications, it is often necessary to deal with such cases when the differential operator degenerates, i.e., at some points, the order of the differential operator changes [10,11]. As is known, in this case, at the points where the operator changes the order, the functions from the domain of the operator's definition do not retain their smoothness (solutions of the equation). Therefore, various difficulties arise, which, in turn, affect the spectral characteristics of the operator.

The paper considers the following degenerate third-order differential operator:

$$Lu + \mu u = -k(y) \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} + a(y)u_x + (c(y) + \mu)u, \quad (1)$$

This was initially defined for $C_{0,\pi}^\infty(\overline{\Omega})$, where $\overline{\Omega} = \{(x, y) : -\pi \leq x \leq \pi, 0 \leq y \leq 1\}$, $\mu \geq 0$. $C_{0,\pi}^\infty(\overline{\Omega})$ is a set consisting of infinitely differentiable functions and satisfying the following conditions:

$$u(-\pi, y) = u(\pi, y), \quad u_x(-\pi, y) = u_x(\pi, y), \quad u_{xx}(-\pi, y) = u_{xx}(\pi, y), \quad (2)$$

$$u(x, 0) = u(x, 1) = 0. \quad (3)$$

Further, we assume that the coefficients $k(y), a(y), c(y)$ satisfy the following conditions:

- (i) $k(y) \geq 0$ is a piecewise continuous function in $[0, 1]$ and $k(0) = 0$;
- (ii) $a(y) \geq \delta_0$ and $c(y) \geq \delta > 0$ are continuous functions in $[0, 1]$.

It is easy to see that the operator $L + \mu I$ defined by equality (1) and boundary conditions (2) and (3); closure occurs in $L_2(\Omega)$ and closure is also denoted by $L + \mu I$.

The above operator $L + \mu I$ degenerates along the line $y = 0$, i.e., at these points, the operator $L + \mu I$ changes the order. Consequently, functions from the domain of the operator definition do not retain their smoothness, and these difficulties affect the spectral characteristics of the operator.

In this paper, for the operator $L + \mu$, the following issues will be researched:

- The existence of a resolvent;
- Spectral characteristics, i.e., estimates of singular values (s-numbers) and eigenvalues.

Using $W_2^1(\Omega)$, we denote the Sobolev space with the norm

$$\|u\|_{2,1,\Omega} = [\|u_y\|_2^2 + \|u_x\|_2^2 + \|u\|_2^2]^{\frac{1}{2}},$$

where $\|\cdot\|_2$ is a norm in $L_2(\Omega)$.

2. Main Results

Theorem 1. Let the coefficients of operator L satisfying conditions (i)–(ii) be fulfilled. Then:

- (a) for operator $L + \mu I$, for $\mu \geq 0$, there is a bounded inverse operator $(L + \mu I)^{-1}$ in the Hilbert space $L_2(\Omega)$;
- (b) the estimate

$$\|u\|_{2,1,\Omega} \leq C \|(L + \mu I)u\|_2 \quad (4)$$

holds for any $u \in D(L)$, where $u \in D(L)$ is the domain of operator L and $C > 0$ is a constant;

- (c) the resolvent of operator L is compact in $L_2(\Omega)$.

Definition 1 ([12]). Let L_n be the collection of all finite-dimensional operators of dimension $\leq n$ and let A be a linear completely continuous operator; then, the numbers

$$S_{n+1}(A) = \min_{k \in L_n} \|A - K\|_{2 \rightarrow 2}, \quad n = 0, 1, 2, \dots$$

are called the approximation numbers of the operator A . Here, $\|\cdot\|_{2 \rightarrow 2}$ is the norm of the operator from $L_2(\Omega)$ to $L_2(\Omega)$.

Theorem 2. Let the coefficients $k(y), a(y), c(y)$ of the operator L satisfy conditions (i)–(ii). Then, for the approximation numbers (s-numbers) and resolvents of operator L , the estimate

$$C_1^{-1} \frac{1}{k^{\frac{3}{2}}} \leq s_k(L + \mu I)^{-1} \leq C \frac{1}{k^{\frac{1}{2}}}, \quad k = 1, 2, 3, \dots \quad (5)$$

holds; here, $C_1 > 0$ and $C > 0$ are constant numbers.

Theorem 3. Let the coefficients $k(y), a(y), c(y)$ of operator L satisfy the conditions of Theorem 2 and ensure that the spectrum of $\sigma(L + \mu I)^{-1}$ is not empty. Then, for the eigenvalues λ_k of the operator $(L + \mu I)^{-1}$ the estimate

$$|\lambda_k| \leq \frac{C}{k^{\frac{1}{2}}}, \quad k = 1, 2, 3, \dots \quad (6)$$

holds; here, $C > 0$ does not depend on k .

Example 1. In space $L_2(\Omega)$, consider the operator defined by the equality

$$(L + \mu I)u(x, y) = -y \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} + (y^2 + 1) \frac{\partial u}{\partial x} + (y + 1)u + \mu u,$$

and boundary conditions (2) and (3).

It is easy to see that the coefficients of the operator $L + \mu I$ satisfy all the conditions of Theorems 1–3; therefore, for $\mu \geq 0$, there is a continuous inverse operator $(L + \mu I)^{-1}$ in the space $L_2(\Omega)$ and for the eigenvalues of operator $(L + \mu I)^{-1}$, the estimate

$$|\lambda_k| \leq \frac{C}{k^{\frac{1}{2}}}, \quad k = 1, 2, 3, \dots \quad (7)$$

holds, where $C > 0$ is independent of k ($k = 1, 2, 3, \dots$).

Now, consider the case without degeneracy.

Example 2. Let the coefficients $k(y), a(y),$ and $c(y)$ of Equation (1) be constant, i.e., $k(y) = 1, a(y) = 1, c(y) = 1, \mu \geq 0$. In the space $L_2(\Omega)$, consider the operator defined by the equality

$$(L + \mu I)u(x, y) = -\frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + u + \mu u,$$

and the boundary conditions (2) and (3).

Since the coefficients of the operator $L + \mu I$ are constant, using direct calculation, we will make sure that the domain of definition of the operator L coincides with the space $L_2^{2,3}(\Omega)$. $L_2^{2,3}(\Omega)$ is the space of functions obtained by completing the set of functions $C^\infty(\overline{\Omega})$ satisfying conditions (2) and (3), relative to the norm:

$$\|u\|_{L_2^{2,3}(\Omega)} = \left(\int_{\Omega} \left(\left| \frac{\partial^3 u}{\partial x^3} \right|^2 + \left| \frac{\partial^2 u}{\partial y^2} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) dx dy \right)^{\frac{1}{2}}.$$

It is easy to verify that $L_2^{2,3}(\Omega) \subset W_2^2(\Omega)$, $W_2^2(\Omega)$ is a Sobolev space.

Now, using the approximation properties of the space $W_2^2(\Omega)$ and repeating the calculations and reasoning that were used in the proof of Theorems 2 and 3 of this paper, we obtain the following:

$$|\lambda_k(L + \mu I)^{-1}| \leq \frac{C}{k}, \quad k = 1, 2, \dots \quad (8)$$

From estimates (7) and (8), it follows that degeneration significantly affects the estimates of the eigenvalues.

3. The Existence of the Resolvent Proof of Theorem 1

Lemma 1. Let us assume that $k(y), a(y), c(y)$ satisfy conditions (i)–(ii). Then, the inequality

$$\frac{1}{(\delta + \mu)} \|(L + \mu I)u\|_2 \geq \|u\|_2 \quad (9)$$

holds for any $u \in D(L)$.

Proof. Let us consider the scalar product $\langle (L + \mu I)u, u \rangle$, $u \in C_{0,\pi}^\infty(\overline{\Omega})$. Integrating by parts and taking into account that the outside integral terms disappear using force $u \in C_{0,\pi}^\infty(\overline{\Omega})$, we obtain the estimate (9). Due to the continuity of the norm, the estimate (9) is true for all $u \in D(L)$. Lemma 1 is proved. $\square$

Through direct calculation, we will make sure that the operator $(L + \mu I)$ in $L_2(\Omega)$, using the Fourier method, is reduced to the study of the following operator:

$$(l_n + \mu I)z(y) = -z''(y) + (ik(y)n^3 + ina(y) + c(y) + \mu)z(y), n = 0, \pm 1, \pm 2, \dots$$

With the boundary condition $z(0) = z(1) = 0$, where $z(y) \in C_0^\infty[0, 1]$, $C_0^\infty[0, 1]$ is a set consisting of infinitely differentiable functions.

It is easy to verify that the operator $l_n + \mu I$ admits closure in $L_2(\Omega)$, which we will also denote by $l_n + \mu I$.

Lemma 2. Let us assume that the coefficients of operator L satisfy conditions (i)–(ii). Then, the estimate

$$\frac{1}{(\delta + \mu)} \|(l_n + \mu I)z(y)\|_2 \geq \|z(y)\|_2, \quad (10)$$

holds for any $z(y) \in D(l_n + \mu I)$.

Proof. Let $z(y) \in C_0^\infty[0, 1]$; then, the estimate

$$\langle (l_n + \mu I)z, z \rangle = \int_0^1 [|z'|^2 + (c(y) + \mu)|z|^2 + (in^3k(y) + ina(y))|z|^2] dy \quad (11)$$

holds.

From inequality (11), taking into account condition (i) and using the properties of complex numbers, we have:

$$|\langle (l_n + \mu I)z, z \rangle| \geq \int_0^1 [|z'|^2 + (c(y) + \mu)|z|^2] dy \geq \int_0^1 [|z'|^2 + (\delta + \mu)|z|^2] dy \quad (12)$$

Hence, using the Cauchy inequality, and also taking into account condition (i), we find

$$\|(l_n + \mu I)z\|_2 \geq (\delta + \mu)\|z\|_2.$$

From the last inequality, due to the continuity of the norm, the last estimate is true for all $z(y) \in D(l_n)$. Lemma 2 is proved. $\square$

Lemma 3. Let the coefficients of the operator L satisfy the conditions of Lemma 2. Then, there is a bounded inverse operator $(l_n + \mu I)^{-1}$ in the space $L_2(0, 1)$.

Proof. Lemma 3 is proved in the same way as Lemma 2.3 of [13] and Lemma 2.2 of [14]. $\square$

Lemma 4. Let us assume that the conditions of Lemma 1 are satisfied. Then, the following inequalities hold:

$$\frac{1}{\delta + \mu} \|(l_n + \mu I)z\|_2 \geq \|z\|_2; \quad (13)$$

$$\frac{1}{|n| \cdot \delta_0} \|(l_n + \mu I)z\|_2 \geq \|z\|_2, \quad n \neq 0; \quad (14)$$

$$\frac{1}{(\delta + \mu)^{\frac{1}{2}}} \|(l_n + \mu I)z\|_2 \geq \|z'(y)\|_2. \quad (15)$$

Proof. From estimate (10), we have

$$\|(l_n + \mu I)^{-1}\|_{2 \rightarrow 2} \leq \frac{1}{\delta + \mu}.$$

Inequality (13) is proved.

From Equality (11) and using the properties of complex numbers, we find

$$|\langle (l_n + \mu I)z, z \rangle| \geq \left| \int_0^1 (in^3k(y) + ina(y))|z|^2 dy \right|.$$

By virtue of conditions (i)–(ii), we note that functions $k(y)$ and $a(y)$ do not change signs. Therefore, from the last inequality, we can obtain

$$|\langle (l_n + \mu I)z, z \rangle| \geq \int_0^1 |in^3k(y) + ina(y)| \cdot |z|^2 dy.$$

Hence, and given that $a(y) \geq \delta_0 > 0$, we have

$$\|(l_n + \mu I)z\|_2 \geq |n|\delta_0\|z\|_2. \quad (16)$$

From inequality (16), and according to the definition of the operator norm, we obtain

$$\|(l_n + \mu I)^{-1}\|_{2 \rightarrow 2} \leq \frac{1}{|n| \cdot \delta_0}, \quad n \neq 0.$$

Inequality (14) is proved.

From inequality (12), using inequality (10), we have

$$\frac{1}{\delta + \mu} \|(l_n + \mu I)z\|_{2 \rightarrow 2}^2 \geq \|z'\|_2^2.$$

Hence, and according to the definition of the operator norm, we find

$$\left\| \frac{d}{dy} (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2} \leq \frac{1}{(\delta + \mu)^{\frac{1}{2}}}.$$

Inequality (15) is proved. Lemma 4 is proved. $\square$

Proof of Theorem 1. First, let us prove point (a) of Theorem 1. Lemma 3 implies that

$$u_k(x, y) = \sum_{n=-k}^k (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \quad (17)$$

is a solution to Problem (18)–(20):

$$(L + \mu I)u_k(x, y) = f_k(x, y), \quad (18)$$

$$u_{k,x}^{(i)}(-\pi, y) = u_{k,x}^{(i)}(\pi, y), \quad i = 0, 1, 2, \quad (19)$$

$$u_k(x, 0) = u_k(x, 1) = 0, \quad (20)$$

here, $f_k(x, y) \rightarrow f(x, y)$, $f_k(x, y) = \sum_{n=-k}^k f_n(y) \cdot e^{inx}$, $i^2 = -1$.

From Lemma 1, and given the fundamentality of the sequence $\{f_k(x, y)\}$, we have

$$\|u_k(x, y) - u_m(x, y)\|_2 \rightarrow 0 \text{ as } k, m \rightarrow \infty.$$

Hence, and by virtue of the completeness of the space $L_2(\Omega)$, we find

$$u_k(x, y) \xrightarrow{L_2(\Omega)} u(x, y). \quad (21)$$

Now, using the equalities (18) and (21) we can obtain that, for any $f(x, y) \in L_2(\Omega)$,

$$u(x, y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \quad (22)$$

is a strong solution to the following problem:

$$(L + \mu I)u = f, \quad u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \quad i = 0, 1, 2, \quad u(x, 0) = u(x, 1) = 0. \quad (23)$$

Definition 2. Let us call function $u \in L_2(\Omega)$ a strong solution to problem (23) if there is a sequence $\{u_k(x, y)\}_{k=1}^{\infty} \subset C_{0,\pi}^{\infty}(\overline{\Omega})$, such that

$$\|u_k - u\|_2 \rightarrow 0, \quad \|(L + \mu I)u_k - f\|_2 \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Hence, it is not difficult to verify that formula (22) is the inverse operator to the closed operator $L + \mu I$. According to Lemma 1 and equality (23), we can obtain that point a) of Theorem 1 holds for all $\mu \geq 0$. Point a) of Theorem 1 is proved.

Now, we prove the inequality (4) of Theorem 1 From (22), we have

$$\begin{aligned} \|u\|_2^2 &= 2\pi \sum_{n=-\infty}^{\infty} \left\| (l_n + \mu I)^{-1} f_n(y) \right\|_{L_2(0,1)}^2 \leq \\ &\leq \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2}^2 \cdot 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 \leq \\ &\leq \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2}^2 \cdot \|f(x, y)\|_2^2, \end{aligned}$$

where it is easy to see that, by virtue of the orthonormality of the system $\{e^{inx}\}$, it follows that

$$\|f(x, y)\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx} \right\|_2^2 = 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_2^2.$$

Hence, and from the estimate (13), we can obtain

$$\|u\|_2^2 \leq \left(\frac{1}{\delta + \mu} \right)^2 \|f(x, y)\|_2^2 = C_1^2 \|f(x, y)\|_2^2.$$

It follows from the last estimate that

$$\|u\|_2 \leq C_1 \|(L + \mu I)u\|_2 \quad (24)$$

Here, $C_1 = \frac{1}{\delta + \mu}$, $(L + \mu I)u = f(x, y)$.

Now, calculate the norm $\|u_x\|$:

$$\begin{aligned} \|u_x\|_2^2 &\leq \sup_{\{n\}} \left\| in(l_n + \mu I)^{-1} \right\|_2^2 \cdot 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 \leq \\ &\leq \sup_{\{n\}} |n| \cdot \left\| (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2}^2 \|f(x, y)\|_{L_2(\Omega)}^2. \end{aligned}$$

From the last estimate and (14), we find

$$\|u_x\|_2^2 \leq \sup_{\{n\}} |n|^2 \cdot \left\| (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2}^2 \cdot \|f(x, y)\|_2^2 \leq \sup_{\{n\}} \frac{|n|^2}{|n|^2 \cdot \delta_0^2} \|f(x, y)\|_2^2 \leq C_2^2 \|f(x, y)\|_2^2$$

Here, $C_2 = \frac{1}{\delta_0}$. From here

$$\|u_x\|_2 \leq C_2 \|(L + \mu I)u\|_2. \quad (25)$$

Similarly, using the estimate (15), we have

$$\|u_y\|_2^2 \leq \sup_{\{n\}} \left\| \frac{d}{dy} (l_n + \mu I)^{-1} \right\|_{2 \rightarrow 2}^2 \cdot \|f(x, y)\|_2^2 \leq \frac{1}{\mu + \delta} \|f(x, y)\|_2^2.$$

From here,

$$\|u_y\|_2 \leq C_3 \|(L + \mu I)u\|_2, \quad (26)$$

where $C_3 = \frac{1}{(\mu + \delta)^{\frac{1}{2}}}$.

From inequalities (24)–(26), we finally obtain

$$\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2 \leq C^2 \|(L + \mu I)u\|_2^2,$$

where $C^2 = \max\{C_1^2, C_2^2, C_3^2\}$.

From the last inequality, we find

$$\|u\|_{2,1,\Omega} \leq C \|(L + \mu I)u\|_2^2,$$

here, $\|u\|_{2,1,\Omega}$ is a norm of $W_2^1(\Omega)$.

We prove point (c) of Theorem 1.

From point (a) of Theorem 1, it follows that operator $(L + \mu I)$ has a continuous inverse operator $(L + \mu I)^{-1}$, and from inequality (4) of Theorem 1, it follows that the domain of values of operator $(L + \mu I)^{-1}$ belongs to $W_2^1(\Omega)$. As you know, the space $W_2^1(\Omega)$ is compactly embedded in $L_2(\Omega)$. It follows that operator $(L + \mu I)^{-1}$ is completely continuous in $L_2(\Omega)$. Theorem 1 is proved. $\square$

4. Proof of Theorems 2 and 3

To study the singular numbers of operator $(L + \mu I)^{-1}$, we need the following lemmas. We introduce the following sets:

$$M = \{u \in L_2(\Omega) : \|Lu\|_2 + \|u\|_2 \leq 1\};$$

$$\tilde{M}_{C_1} = \{u \in L_2(\Omega) : \|u_x\|_2 + \|u_y\|_2 + \|u\|_2 \leq C_1\};$$

$$\overset{0}{M}_{C_1^{-1}} = \{u \in L_2(\Omega) : \|u_{xxx}\|_2 + \|u_{yyy}\|_2 + \|u_{xx}\|_2 + \|u_{yy}\|_2 + \|u_x\|_2 + \|u_y\|_2 + \|u\|_2 \leq C_1^{-1}\}$$

Here, C_1 is a constant number independent of $u(x, y)$.

Lemma 5. Let us assume that the coefficients of operator L satisfy the conditions (i)–(ii). Then, the following relation is true:

$$\overset{0}{M}_{C_1^{-1}} \subseteq M \subseteq \tilde{M}_{C_1}.$$

Proof. Let $u(x, y) \in \overset{0}{M}_{C_1^{-1}}$. Then, the following inequalities are true:

$$\|Lu\|_2 + \|u\|_2 \leq \left\| -k(y) \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial y^2} + a(y) \frac{\partial u}{\partial x} + c(y)u \right\|_2 + \|u\|_2 \leq$$

$$\begin{aligned} &\leq \left(\left\| -k(y) \frac{\partial^3 u}{\partial x^3} \right\|_2 + \left\| -\frac{\partial^2 u}{\partial y^2} \right\|_2 + \left\| a(y) \frac{\partial u}{\partial x} \right\|_2 + \|c(y)u\|_2 \right) + \|u\|_2 \leq \\ &\leq \max_{y \in [0,1]} |k(y)| \cdot \left\| \frac{\partial^3 u}{\partial x^3} \right\|_2 + \max_{y \in [0,1]} |a(y)| \cdot \left\| \frac{\partial u}{\partial x} \right\|_2 + \max_{y \in [0,1]} |c(y)| \cdot \|u\|_2 + \|u\|_2 + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_2. \end{aligned}$$

Suppose that $C_0 = \{ \max_{y \in [0,1]} k(y), a(y), c(y) + 1 \}$; then, from the last inequality, we have:

$$\|Lu\|_2 + \|u\|_2 \leq C_1 (\|u_{xxx}\|_2 + \|u_x\|_2 + \|u\|_2 + \|u_{yy}\|_2), \quad (27)$$

where $C_1 = \max\{C_0, 1\}$.

From (27) and considering that $u(x, y) \in \overset{0}{M}_{C_1^{-1}}$, we have

$$\|Lu\|_2 + \|u\|_2 \leq C_1 (\|u_{xxx}\|_2 + \|u_{yyy}\|_2 + \|u_{xx}\|_2 + \|u_{yy}\|_2 + \|u_x\|_2 + \|u_y\|_2 + \|u\|_2) \leq C_1 \cdot C_1^{-1} \leq 1.$$

It follows from the last inequality that $u(x, y) \in M$, i.e., $\overset{0}{M}_{C_1^{-1}} \subseteq M$. The left inclusion is proved.

Now, let us prove the right inclusion. Let $u(x, y) \in M$; this means that $u(x, y) \in D(L)$. Therefore, by virtue of inequality (4) of Theorem 1, we find

$$\|u_x\|_2 + \|u_y\|_2 + \|u\|_2 \leq C_0 \|(L + \mu I)u\|_2 \leq C_1 (\|Lu\|_2 + \|u\|_2),$$

where $C_1 = C(\mu)$.

Hence, given that $u(x, y) \in M$, we have

$$\|u_x\|_2 + \|u_y\|_2 + \|u\|_2 \leq C_1 (\|Lu\|_2 + \|u\|_2) \leq C_1.$$

It follows from the last inequality that $u(x, y) \in \tilde{M}_{C_1}$, i.e., $M \subseteq \tilde{M}_{C_1}$. The right inclusion is proved. Lemma 5 is proved. $\square$

Definition 3 ([12]). *The magnitude*

$$d_k = \inf_{\{y_k\}} \sup_{u \in M} \inf_{v \in y_k} \|u - v\|_2$$

is called the Kolmogorov k -widths (diameters) of the set M in space $L_2(\Omega)$, where y_k is the set of all subspaces in $L_2(\Omega)$ whose dimensions do not exceed k .

The following properties of the widths follow directly from Definition 3

- (1) $d_0 \geq d_1 \geq d_2 \geq \dots$;
- (2) $d_k(\tilde{M}) \leq d_k(M)$, $\tilde{M} \subset M$, $k = 1, 2, \dots$,
where $d_k(\tilde{M})$ is the widths (diameters) of the set $\tilde{M}$; $d_k(M)$ is the widths (diameters) of the set M .
- (3) $d_k(nM) = nd_k(M)$, $n > 0$, $nM = \{x' = nx, x \in M\}$.

Lemma 6. *Let the conditions of Lemma 5 be satisfied. Then, the estimate*

$$C_1^{-1} d_k \leq d_k(M) \leq C_1 \tilde{d}_k, \quad k = 1, 2, 3, \dots, \quad (28)$$

holds; here, $C_1 > 0$ is a constant number; $\overset{0}{d}_k, d_k, \tilde{d}_k$ are the widths (diameters) of sets $\overset{0}{M}_{C_1^{-1}}$, and M , $\tilde{M}_{C_1}$, respectively.

Proof. It is not difficult to verify that the proved Lemma 6 follows from the properties of (1)–(3) widths. $\square$

Lemma 7. Let the coefficients of operator L satisfy the conditions (i)–(ii). Then, the estimates

$$C_1^{-1} d_k^0 \leq S_{k+1} \leq C_1 \tilde{d}_k, \quad k = 1, 2, 3, \dots,$$

hold; here $C_1 > 0$ is any constant number, S_k is s numbers of the operator $(L + \mu I)^{-1}$, and $\mu \geq 0$, d_k^0, d_k , and $\tilde{d}_k^0$ are the k widths (diameters) of the sets $M, \tilde{M}$, and $\tilde{M}$.

Proof. From estimate (28) and considering equality $S_{k+1} = d_k$ [12], we obtain a proof of Lemma 7. $\square$

Let us introduce the following functions:

$$N(\lambda) = \sum_{S_{k+1} > \lambda} 1 \text{ is the number of } S_{k+1} \text{ greater than } \lambda > 0;$$

$$N^0(\lambda) = \sum_{d_k^0 > \lambda} 1 \text{ is the number of } d_k^0 \text{ greater than } \lambda > 0;$$

$$\tilde{N}(\lambda) = \sum_{\tilde{d}_k^0 > \lambda} 1 \text{ is the number of } \tilde{d}_k^0 \text{ greater than } \lambda > 0.$$

Lemma 8. Let the conditions of Lemma 7 be fulfilled. Then, the inequality is true

$$N^0(C_1 \lambda) \leq N(\lambda) \leq \tilde{N}(C_1^{-1} \lambda). \quad (29)$$

Proof. Using Lemma 7, we find [14] (Lemma 4.4):

$$N(\lambda) = \sum_{S_{k+1} > \lambda} 1 \leq \sum_{C_1 \tilde{d}_k^0 > \lambda} 1 = \sum_{\tilde{d}_k^0 > C_1^{-1} \lambda} 1 = \tilde{N}(C_1^{-1} \lambda).$$

Similarly,

$$N^0(C_1 \lambda) = \sum_{d_k^0 > C_1 \lambda} 1 = \sum_{C_1^{-1} d_k^0 > \lambda} 1 \leq \sum_{S_{k+1} > \lambda} 1 = N(\lambda).$$

Lemma 8 is proved. $\square$

Proof of Theorem 2. It is easy to verify that $\tilde{M} \subset W_2^1(\Omega)$, $M^0 \subset W_2^3(\Omega)$, where $W_2^1(\Omega)$, and $W_2^3(\Omega)$ are Sobolev spaces. Hence, according to the results of [15–17], for the functions $N^0(\lambda)$, $\tilde{N}(\lambda)$ the estimates

$$C_1^{-1} \lambda^{-\frac{2}{3}} \leq N^0(\lambda) \leq C_1 \lambda^{-\frac{2}{3}}, \quad (30)$$

$$C^{-1} \lambda^{-2} \leq \tilde{N}(\lambda) \leq C \lambda^{-2} \quad (31)$$

hold, where C_1, C is independent of $\lambda > 0$.

Now, let $\lambda = d_k^0$; when $N^0(d_k^0) = k, k = 1, 2, 3, \dots$ and from (30), it follows that

$$C^{-1} d_k^{0-\frac{2}{3}} \leq k \leq C d_k^{0-\frac{2}{3}}.$$

From here,

$$C_1^{-1} \frac{1}{k^{\frac{3}{2}}} \leq d_k^0 \leq C_1 \frac{1}{k^{\frac{3}{2}}}, \quad k = 1, 2, 3, \dots \quad (32)$$

Similarly, from (31) we have

$$C^{-1} \frac{1}{k^{\frac{1}{2}}} \leq \tilde{d}_k^0 \leq C \frac{1}{k^{\frac{1}{2}}}, \quad k = 1, 2, 3, \dots \quad (33)$$

Now, using Lemma 7 from (32) and (33), we obtain that

$$C^{-1} \frac{1}{k^{\frac{3}{2}}} \leq S_{k+1} \leq C \frac{1}{k^{\frac{1}{2}}}, \quad k = 1, 2, 3, \dots \quad (34)$$

Estimate (5) is proved, i.e., Theorem 2 is proved. $\square$

Proof of Theorem 3. According to point (c) of Theorem 1, the operator $(L + \mu I)^{-1}$ is completely continuous. As is known, for completely continuous operators, the Weyl's inequality [12]:

$$\prod_{j=1}^k |\lambda_j(A)| \leq \prod_{j=1}^k S_j(A), \quad k = 1, 2, 3, \dots \quad (35)$$

holds, where A is a completely continuous operator, $\lambda_j(A)$ are eigenvalues numbered in non-increasing order of absolute values, and $S_j(A)$ are singular numbers arranged in non-increasing order.

From (34) and (35), we have

$$|\lambda_k|^k \leq \prod_{j=1}^k |\lambda_j| \leq \prod_{j=1}^k S_j \leq C^k (k!)^{-\frac{1}{2}}.$$

Next, using the inequality $e^k k! > k^k$ ($k = 1, 2, \dots$), we find

$$|\lambda_k|^k \leq C^k (k!)^{-\frac{1}{2}} \leq C^k e^{\frac{k}{2}} k^{-\frac{k}{2}}.$$

From here, we finally have

$$|\lambda_k| \leq C \cdot k^{-\frac{1}{2}}, \quad k = 1, 2, 3, \dots$$

Estimate (6) is proved, i.e., Theorem 3 is proved. $\square$

5. Conclusions

In conclusion, here are a few words about the obtained results:

- The existence of the inverse operator was proved for a class of degenerate third-order partial differential operators;
- The coercivity estimate was obtained;
- The compactness of the resolvent was proved;
- Two-sided estimates of approximation numbers were obtained;
- Upper estimates for the eigenvalues were obtained.

As is known, third-order partial differential equations are one of the basic equations of wave theory, as well as some macroscopic models of semiconductors that consider quantum effects, for example, models of the transmission of electric lines. Quantum hydrodynamic models are described by third-order partial differential Equations (1)–(3), (7) and (8). In this regard, there is a need to study the spectral and approximation properties of third-order differential operators. The results and methods used in this paper allowed us to study the following aspects of a linearized Korteweg–de Vries operator with variable unbounded coefficients at infinity:

- The existence of an inverse operator;
- The compactness of a resolvent;
- Estimation of approximation numbers and eigenvalues.

Here, we note that finding estimates of approximation numbers, as well as emergency subspaces, for a set of solutions to the equation is certainly an important problem from both a theoretical and a practical point of view.

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References

1. Aliyev, Z.S.; Mehraliyev, Y.T.; Yusifova, E.H. On some nonlocal inverse boundary problem for partial differential equations of third order. *Turk. J. Math.* **2021**, *45*, 1871–1886. [\[CrossRef\]](#)
2. Belakroum, K.; Ashyralyev, A.; Guezane-Lakoud, A. A note on nonlocal boundary value problem for a third-order partial differential equation. *Filomat* **2018**, *32*, 801–808. [\[CrossRef\]](#)
3. Villanueva, A. On Linearized Korteweg-de Vries Equations. *J. Math. Res.* **2012**, *4*, 2–8. [\[CrossRef\]](#)
4. Maher, A.; Utkina, Y.A. On the cases of explicit solvability of a third order partial differential equation. *Iran. J. Sci. Technol. Trans.* **2010**, *4*, 103–112.
5. Saut, J.C.; Temam, R. Remarks on the Korteweg-de Vries equation. *Isr. J. Math.* **1976**, *24*, 78–87. [\[CrossRef\]](#)
6. Gao, P. The stochastic Korteweg-de Vries equation in a bounded domain. *App. Math. Comput.* **2017**, *310*, 97–111. [\[CrossRef\]](#)
7. Xiang, S. Null controllability of a linearized Korteweg-de Vries equation by backstepping approach. *Siam J. Control Optim.* **2019**, *57*, 1493–1515. [\[CrossRef\]](#)
8. Lukina, G.A. Boundary value problems with integral boundary conditions for the linearized Korteweg-de Vries equation. *Bull. SUSU* **2011**, *234*, 52–61. (In Russian)
9. Dzhuraev, T.D.; Zikirov, O.S. On the Goursat and Dirichlet problems for equation of the third-order. *Nonlinear Oscil.* **2008**, *11*, 320–330. [\[CrossRef\]](#)
10. Borsuk, M.B. Degenerate elliptic boundary value problems of the second order in non-smooth domains. *Mod. Math. Fundam. Dir.* **2005**, *13*, 3–137. (In Russian)
11. Triebel, H. *Interpolation Theory, Function Spaces, Differential Operators*; VEB Deutscher Verlag der Wissenschaften: Berlin, Germany, 1978.
12. Gohberg, I.C.; Krein, M.G. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space (Translations of Mathematical Monographs)*; American Mathematical Society: Providence, RI, USA, 1969.
13. Muratbekov, M.B.; Suleimbekova, A.O. Existence, compactness, estimates of eigenvalues and s-numbers of a resolvent for a linear singular operator of the Korteweg-de Vries Type. *Filomat* **2022**, *36*, 3691–3702. [\[CrossRef\]](#)
14. Muratbekov, M.B.; Muratbekov, M.M. Sturm-Liouville operator with a parameter and its usage to spectrum research of some differential operators. *Complex Var. Elliptic Equations* **2019**, *64*, 1457–1476. [\[CrossRef\]](#)
15. Ismagilov, R.S. On the n - dimensional diameters of compacts in Hilbert space. *Funk. Anal. Its Appl.* **1968**, *2*, 32–39. (In Russian) [\[CrossRef\]](#)
16. Ismagilov, R.S. The cross-sections of sets in linear normalized spaces and the approximation of functions by trigonometric sets. *Achiev. Math. Sci.* **1974**, *2*, 161–178. (In Russian)
17. Kusainova, L.K. Estimates of diameters of the unit sphere of the function space $\dot{L}_p^l(\Omega, v)$ in $L_q(\Omega)$. *Dokl. Akad. Nauk SSSR* **1980**, *251*, 791–794. (In Russian)

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