Abstract: In this particular article, our focus revolves around the establishment of a geometric inequality, commonly referred to as Chen’s inequality. We specifically apply this inequality to assess the square norm of the mean curvature vector and the warping function of warped product slant submanifolds. Our investigation takes place within the context of locally metallic product space forms with quarter-symmetric metric connections. Additionally, we delve into the condition that determines when equality is achieved within the inequality. Furthermore, we explore a number of implications of our findings.

Keywords: geometric inequalities; metallic product space; $\theta$-slant submanifolds; Chen’s inequality

MSC: 53B50; 53C20; 53C40

1. Introduction

The theory of product manifolds encompasses significant implications in both physics and geometry, particularly in the realm of Hermitian geometry. In physics, Einstein’s general theory of relativity describes space time as a product of three-dimensional space and one-dimensional time, each possessing its own metrics that determine the overall topology. Various theories such as Kaluza–Klein, brane theory, and gauge theory have intriguing applications involving product manifolds.

Modern physics relies on gauge theories, which are based on the geometric framework given by moduli spaces. These moduli spaces enable the categorization and exploration of characteristics of bundle configurations on compact Riemann surfaces or algebraic curves [1,2]. Notably, at the forefront of the construction of novel gauge theories are subvarieties of the moduli space of primary bundles with exceptional structure groups. Gauge theories are further illuminated by investigating the stratifications and fixed points in the moduli space of principal and Higgs bundles [3–6].

Moreover, the connection between Riemannian surfaces and gauge theories extends beyond the study of bundles. The moduli space of vector bundles over a compact Riemann surface or algebraic curve provides valuable insights into the formulation of gauge theories, offering a geometric understanding of the topological and geometric properties inherent in these theories [7,8].

A significant development in the study of manifolds with negative sectional curvature, referred to as warped product manifolds, was introduced by R. L. Bishop et al. in 1969 [9]. These generalized Riemannian product manifolds have found prominence in differential geometry and physics, particularly in general relativity [10,11]. Warped products have been widely used to examine energy, angles, and lengths through the lens of the second fundamental form. From a mathematical perspective, warped product manifolds extend the concept of Riemann product manifolds and provide examples of manifolds with strictly
negative curvature. Notably, the best relativistic representation of Schwarzschild space-time, which describes the region surrounding a massive star or black hole, can be expressed as a warped product \[11\]. Moreover, these manifolds have practical applications in modeling bodies with significant gravitational fields from a mechanical standpoint.

From a mathematical standpoint, warped product manifolds, a generalization of the Riemann product manifold \[12–14\], also give instances of manifolds with strictly negative curvature. A warped product, for example, is supplied as the best relativistic representation of the Schwarzschild space-time, which describes the outer space around a massive star or black hole. From a mechanical aspect, they may also be employed to simulate bodies with massive gravitational fields.

The construction of warped product manifolds is defined as follows:

Let us consider a Riemannian manifold \(N_T\) of dimension \(d_1\) with Riemannian metric \(g_1\), \(N_{\theta}\) of dimension \(d_2\) with Riemannian metric \(g_2\), and let \(f\) be positive differentiable functions on \(N_T\). Consider the warped product \(N_T \times N_{\theta}\) with its projections \(\iota_1 : N_T \times N_{\theta} \to N_T\) and \(\iota_2 : N_T \times N_{\theta} \to N_{\theta}\). Then, their warped product manifold \(M = N_T \times f N_{\theta}\) is the product manifold equipped with the structure

\[
g(X, Y) = g_1(\iota_1^*X, \iota_1^*Y) + (f \circ \iota_1)^2 g_2(\iota_2^*X, \iota_2^*Y),
\]

for any vector fields \(X, Y\) on \(M\), where \(\ast\) denotes the symbol for tangent maps. The function \(f\) is called the warping function of the warped product \[15–17\]. This concept has been extensively explored, leading to numerous research articles in the field of complex geometry \[18,19\] and contact geometry \[20–22\].

However, despite the extensive exploration of warped product manifolds, the immersibility / non-immersibility of Riemannian manifolds in space forms remains a fundamental problem in submanifold theory. In this regard, the groundbreaking work of Chen and his introduction of new Riemannian invariants, notably Chen’s inequality, established an optimal relationship between extrinsic and intrinsic invariants on submanifolds.

Motivated by these considerations, the objective of this article is twofold: first, to derive Chen’s inequality for warped product submanifolds in locally metallic product space forms with a quarter-symmetric metric connection, and secondly, to explore a few applications of the obtained result.

2. Preliminaries

Let \(\overline{M}\) be a Riemannian manifold endowed with the linear connection \(\nabla\). A connection is deemed semi-symmetric if its torsion tensor \(T\) satisfies the elegant expression

\[
T(U, V) = \pi(V)U - \pi(U)V
\]

where \(\pi\) is a one form. Consequently, \(\nabla\) is referred to as a semi-symmetric connection. Assuming a Riemannian metric \(g\) on \(\overline{M}\), if \(\nabla g = 0\), then \(\nabla\) qualifies as a semi-symmetric metric connection on \(\overline{M}\). The mathematical form of this connection is given by

\[
\nabla_U V = \tilde{\nabla}_U V + \pi(V)U - g(U, V)\Gamma
\]

where \(U\) and \(V\) are arbitrary vectors in \(\overline{M}\), \(\tilde{\nabla}\) represents the Levi-Civita connection with respect to the Riemannian metric \(g\), and \(\Gamma\) is a vector field.

Furthermore, if \(\nabla\) satisfies the condition

\[
\nabla_U V = \tilde{\nabla}_U V + \pi(V)U
\]

then it is termed a semi-symmetric non-metric connection.

Additionally, a linear connection \(\nabla\) on a Riemannian manifold \(\overline{M}\) with metric \(g\) is classified as a quarter-symmetric connection if its torsion tensor \(T\) is given by

\[
T(U, V) = \nabla_U V - \nabla_V U - [U, V]
\]
which satisfies the condition

\[ T(U, V) = \pi(V)\phi U - \pi(U)\phi V \]

where \( \pi(U) = g(U, \Gamma) \) and \( \phi \) is a (1,1) tensor field.

Consequently, a special quarter-symmetric connection can be defined as follows

\[ \nabla_U V = \nabla_{\tilde{U}} V + \psi_1 \pi(V)U - \psi_2 g(U, V)\Gamma \tag{3} \]

where \( \psi_1 \) and \( \psi_2 \) are real constants.

Remarkably, from Equations (1)–(3), it is evident that [23]

1. If \( \psi_1 = \psi_2 = 1 \), a quarter-symmetric connection reduces to a semi-symmetric metric connection.
2. If \( \psi_1 = 1 \) and \( \psi_2 = 0 \), a quarter-symmetric connection becomes a semi-symmetric non-metric connection.

It is worth mentioning that the quarter-symmetric connections generalize several well-known connections.

Moving on, the curvature tensor \( \overline{R} \) associated with \( \nabla \) is expressed as

\[ \overline{R}(U, V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U, V]} Z. \tag{4} \]

Similarly, the curvature tensor \( \tilde{R} \) can also be defined.

Let us introduce the \((0, 2)\) tensors

\[ \beta_1(U, V) = (\nabla_U \pi)(V) - \psi_1 \pi(U)\pi(V) + \frac{\psi_2}{2}g(U, V)\pi(\Gamma), \]

and

\[ \beta_2(U, V) = \frac{\pi(\Gamma)}{2}g(U, V) + \pi(U)\pi(V). \]

The curvature tensor \( \tilde{R} \) of the manifold \( \overline{M} \) is then given by [24]

\[ \tilde{R}(U, V, Z, W) = \overline{R}(U, V, Z, W) + \psi_1 \beta_1(U, Z)g(V, W) - \psi_1 \beta_1(V, Z)g(U, W) + \psi_2 \beta_1(V, W)g(U, Z) - \psi_2 \beta_1(U, W)g(V, Z) + \psi_2 (\psi_1 - \psi_2)g(U, Z)\beta_2(V, W) - \psi_2 (\psi_1 - \psi_2)g(V, Z)\beta_2(U, W). \tag{5} \]

Moreover, let us define \( \lambda \) as the trace of \( \beta_1 \) and \( \mu \) as the trace of \( \beta_2 \).

Let \( \mathcal{M} \) be an \( m \)-dimensional submanifold in a Riemannian manifold \( \overline{M} \). Let \( \nabla \) and \( \tilde{\nabla} \) be the induced quarter-symmetric metric connection and Levi-Civita connection, respectively, on \( \mathcal{M} \). Then, the Gauss formulas are

\[ \nabla_U V = \nabla_U V + \zeta(U, V), \quad U, V \in \Gamma(TM), \tag{6} \]

\[ \tilde{\nabla}_U V = \tilde{\nabla}_U V + \tilde{\zeta}(U, V), \quad U, V \in \Gamma(TM), \tag{7} \]

where \( \zeta \) is the second fundamental form that satisfies the relation

\[ \zeta(U, V) = \tilde{\zeta}(U, V) - \psi_2 g(U, V)\Gamma^\perp, \]

where \( \Gamma^\perp \) is the normal component of the vector field \( \Gamma \) on \( \mathcal{M} \).

Moreover, the equation of Gauss is defined by [24]

\[ \tilde{R}(U, V, Z, W) = R(U, V, Z, W) - \psi_2 g(\zeta(U, W), \zeta(V, Z)) + g(\tilde{\zeta}(V, W), \zeta(U, Z)) \]
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Let $K(\pi)$ denote the sectional curvature of a Riemannian manifold $\mathcal{M}$ of the plane section $\pi \subset T_x \mathcal{M}$ at a point $x \in \mathcal{M}$. If \( \{e_1, \ldots, e_n\} \) is the orthonormal basis of $T_x \mathcal{M}$ and \( \{e_{n+1}, \ldots, e_m\} \) is the orthonormal basis of $T_x^\perp \mathcal{M}$ at any $x \in \mathcal{M}$, then
\[
\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),
\]
where $\tau$ is the scalar curvature.

Let \( \{e_1, \ldots, e_n\} \) and \( \{e_{n+1}, \ldots, e_m\} \) be the tangent and normal orthonormal frames on $\mathcal{M}$, respectively. Then,
\[
\mathcal{H} = \frac{1}{n} \sum_{i=1}^{n} g(\xi(e_i, e_i)).
\]
is known as the mean curvature vector field.

A tensor field $\vartheta$ of type $(1, 1)$ earns the title of a polynomial structure when it satisfies the following remarkable equation on an $m$-dimensional Riemannian manifold $(\mathcal{M}, g)$, adorned with real numbers $b_1, \ldots, b_n$:
\[
B(X) = X^n + b_{n-1}X^{n-1} + \ldots + b_2X + b_1\mathcal{I} = 0
\]
Here, $\mathcal{I}$ represents the identity transformation [25, 26].

**Remark 1.** Behold the following revelations:
1. When $B(X) = X^2 + \mathcal{I}$, $\vartheta$ unveils itself as an almost complex structure.
2. When $B(X) = X^2 - \mathcal{I}$, $\vartheta$ emerges as an almost product structure.
3. When $B(X) = \vartheta^2 - p\vartheta + q\mathcal{I}$, $\vartheta$ takes on the form of a metallic structure.

In this case, $p$ and $q$ are two integers.

If
\[
g(\vartheta X, Y) = g(X, \vartheta Y), \quad \forall X, Y \in \Gamma(T\mathcal{M}),
\]
then the Riemannian metric $g$ is bestowed with the grand title of being $\vartheta$-compatible.

Imagine a scenario where $g$ is $\vartheta$-compatible and $\vartheta$ assumes the form of a metallic structure on the Riemannian manifold $\mathcal{M}$. In this wondrous situation, we refer to $(\mathcal{M}, g)$ as a metallic Riemannian manifold.

Exploiting the power of Equation (10), we can unfold the following revelation:
\[
g(\vartheta X, \vartheta Y) = g(\vartheta^2 X, Y) = p_n g(X, \vartheta Y) + q_n g(X, Y).
\]

It is worth mentioning that when we set $p = q = 1$ in (11), a metallic structure magically transforms into a golden structure.

The esteemed members of the metallic family are elegantly categorized as follows [27]:
1. The golden structure $\vartheta = \frac{1+\sqrt{5}}{2}$ for $p = q = 1$, entwined with the ratio of two consecutive classical Fibonacci numbers.
2. The copper structure $\kappa_{1,2} = 2$ with $p = 1$ and $q = 2$.
3. The nickel structure $\kappa_{1,3} = \frac{1+\sqrt{13}}{2}$ if $p = 1$ and $q = 3$.
4. The silver structure $\kappa_{2,1} = 1 + \sqrt{2}$ if $p = 2$ and $q = 1$, enchanted by the ratio of two consecutive Pell numbers.
5. The bronze structure $\kappa_{3,1} = \frac{3+\sqrt{13}}{2}$ with $p = 3$ and $q = 1$.
6. The subtle structure $\kappa_{4,1} = 2 + \sqrt{5}$ if $p = 4$ and $q = 1$, and so forth.
Let \((\mathcal{M}, g)\) be an \(m\)-dimensional Riemannian manifold and let \(\vartheta\) be a \((1,1)\)-tensor field on \(\mathcal{M}\) such that \(\vartheta^2 = I\), \(\vartheta \neq \pm I\); then, \(\vartheta\) is called an almost product structure. The structure \(\vartheta\) with

\[
g(\vartheta X, Y) = g(X, \vartheta Y), \quad \forall X, Y \in \Gamma(T\mathcal{M})
\]

is known as an almost product Riemannian manifold [26].

Any metallic structure \(\phi\) on \(\mathcal{M}\) is known to induce two almost product structures \(\vartheta\) on \(\mathcal{M}\) [27]:

\[
\vartheta_1 = \frac{2}{2\varphi_{p,q} - p} \phi - \frac{p}{2\varphi_{p,q} - p} I,
\]

\[
\vartheta_2 = \frac{2}{2\varphi_{p,q} - p} \phi + \frac{p}{2\varphi_{p,q} - p} I
\]

where \(\varphi_{p,q} = \frac{p + \sqrt{p^2 + 4q^2}}{2}\).

Also, an almost product structure \(\vartheta\) on \(\mathcal{M}\) induces two metallic structures:

\[
\phi_1 = \frac{p}{2} I + \frac{2\varphi_{p,q} - p}{2} \vartheta
\]

\[
\phi_2 = \frac{p}{2} I - \frac{2\varphi_{p,q} - p}{2} \vartheta.
\]

**Definition 1** ([28]).

(i) Let \(\nabla\) be a linear connection and \(\phi\) be a metallic structure on \(\mathcal{M}\) such that \(\nabla \phi = 0\). Then, \(\nabla\) is called a \(\phi\)-connection.

(ii) A locally metallic Riemannian manifold is a metallic Riemannian manifold \((\mathcal{M}, g, \phi)\) if the Levi-Civita connection \(\nabla\) of \(g\) is a \(\phi\)-connection.

Consider an almost Hermitian manifold \(\mathcal{M}\) and a submanifold \(\mathcal{M}\) embedded within it. We refer to \(\mathcal{M}\) as a slant submanifold if, for any point \(x\) on \(\mathcal{M}\) and any non-zero vector \(X\) in the tangent space \(T_x\mathcal{M}\), the angle between the tangent space \(JM\) and \(T_x\mathcal{M}\) remains constant. In other words, this angle does not vary based on the specific choice of \(x\) and \(X\) on \(\mathcal{M}\). The constant angle is known as the slant angle \(\theta\), which lies in the range \([0, \frac{\pi}{2}]\) and characterizes the slant submanifold within \(\mathcal{M}\).

Moreover, if \(\mathcal{M}\) is a slant submanifold of a metallic Riemannian manifold \((\mathcal{M}, g, \phi)\) with a slant angle \(\theta\), the following relationships hold [28]:

\[
g(TX, TY) = \cos^2 \theta [pg(X, TY) + qg(X, Y)],
\]

and

\[
g(NX, NY) = \sin^2 \theta [pg(X, TY) + qg(X, Y)],
\]

for all \(X, Y \in \Gamma(T\mathcal{M})\).

Furthermore, we have the additional relations

\[
T^2 = \cos^2 \theta (pT + qI),
\]

where \(I\) represents the identity operator on \(\Gamma(T\mathcal{M})\) and

\[
\nabla T^2 = p\cos^2 \theta \nabla T.
\]

These expressions provide valuable insights into the geometric properties of slant submanifolds and their relationships within the broader context of metallic Riemannian manifolds.
Also, let $\mathcal{M}_1$ be a Riemannian manifold with constant sectional curvature $c_1$ and $\mathcal{M}_2$ be a Riemannian manifold with constant sectional curvature $c_2$. Then, the Riemannian curvature tensor $\overline{\mathcal{R}}$ of the locally Riemannian product manifold $\overline{\mathcal{M}} = \mathcal{M}_1 \times \mathcal{M}_2$ is given by [29]

$$\overline{\mathcal{R}}(X, Y)Z = \frac{1}{4}(c_1 + c_2) \left[ g(Y, Z)X - g(X, Z)Y \right]$$

$$+ \frac{1}{4}(c_1 + c_2) \left\{ \frac{4}{(2c_{p,q} - p)^2} \left[ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right] \right. $$

$$+ \left. \frac{p^2}{(2c_{p,q} - p)^2} \left[ g(Y, Z)X - g(X, Z)Y \right] \right. $$

$$+ \left. \frac{2p}{(2c_{p,q} - p)^2} \left[ g(\phi X, Z)Y + g(X, Z)\phi Y \right. \right.$$ 

$$- g(\phi Y, Z)X - g(Y, Z)\phi Y \right) \right\}$$

$$+ \frac{1}{2}(c_1 - c_2) \left\{ \frac{1}{(2c_{p,q} - p)} \left[ g(Y, Z)\phi X - g(X, Z)\phi Y \right] \right. $$

$$+ \left. \frac{p}{(2c_{p,q} - p)} \left[ g(Y, Z)\phi X - g(X, Z)\phi Y \right] \right\}.$$  \hfill (14)

From (5) and (14), we have

$$\overline{\mathcal{R}}(X, Y, Z, W) = \frac{1}{4}(c_1 + c_2) \left[ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \right]$$

$$+ \frac{1}{4}(c_1 + c_2) \left\{ \frac{4}{(2c_{p,q} - p)^2} \left[ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \right] \right. $$

$$+ \left. \frac{p^2}{(2c_{p,q} - p)^2} \left[ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \right] \right. $$

$$+ \left. \frac{2p}{(2c_{p,q} - p)^2} \left[ g(\phi X, Z)g(Y, W) + g(X, Z)g(\phi Y, W) \right. \right.$$ 

$$- g(\phi Y, Z)g(X, W) - g(Y, Z)g(\phi X, W) \right) \right\}$$

$$+ \frac{1}{2}(c_1 - c_2) \left\{ \frac{1}{(2c_{p,q} - p)} \left[ g(Y, Z)g(\phi X, W) - g(X, Z)g(\phi Y, W) \right] \right. $$

$$+ \left. \frac{p}{(2c_{p,q} - p)} \left[ g(X, Z)g(Y, W) - g(Y, Z)g(X, W) \right] \right\} \hfill (15)$$

3. Unveiling the Pinching Phenomenon: Main Result

The proof of the major finding is the focus of this section.
\textbf{Theorem 1.} Let $\mathcal{M}$ be an $n$-dimensional warped product $\theta$-slant submanifold of an $m$-dimensional locally metallic product space form $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with quarter-symmetric metric connections. Then,

$$
\frac{n_2}{n_2} \Delta f \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 \\
+ \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1 n_2 (p^2 + 4q) - 4 \left[\text{tr}^2 \phi + \text{tr} \phi \right] \\
- \cos^2 \theta(p \text{tr} T) \right\} \\
\pm \frac{1}{8} \left( c_1 - c_2 \right) \left\{ 4pn_1 n_2 - 4 \text{tr} \phi \right\} \\
- (\psi_1 + \psi_2) \left[ \lambda(n-1) + \lambda|_{\mathcal{M}_1}(n_1 - 1) + \lambda|_{\mathcal{M}_2}(n_2 - 1) \right] \\
- \psi_2 (\psi_1 - \psi_2) \left[ \mu(n-1) + \mu|_{\mathcal{M}_1}(n_1 - 1) + \mu|_{\mathcal{M}_2}(n_2 - 1) \right] \\
+ (\psi_1 - \psi_2) \left[ n(n-1) \pi(H) + n_1 (n_1 - 1) \pi(H_1) + n_2 (n_2 - 1) \pi(H_2) \right],
$$

(16)

where $\Delta$ is the Laplacian operator on $\mathcal{M}_1$. The equality case holds in (16) if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and the following satisfies

$$
\frac{H_1}{H_2} = \frac{n_1}{n_2},
$$

(17)

where $H_1$ and $H_2$ are the mean curvature vectors along $\mathcal{M}_1^{n_1}$ and $\mathcal{M}_2^{n_2}$, respectively.

\textbf{Proof.} Let $\{e_1, ..., e_n\}$ be an orthonormal tangent frame and $\{e_{n+1}, ..., e_m\}$ be an orthonormal frame of $T_x \mathcal{M}$ and $T_x \perp \mathcal{M}$, respectively, at any point $x \in \mathcal{M}$. Putting $X = W = e_i$, $Y = Z = e_j$ in (15) with Equation (8) and take $i \neq j$, we have

$$
\mathcal{R}(e_i, e_j, e_j, e_i) = \\
\quad \frac{1}{4} (c_1 + c_2) \left\{ g(e_i, e_i) g(e_j, e_j) - g(e_i, e_j) g(e_i, e_j) \right\} \\
+ \frac{1}{4} (c_1 + c_2) \left\{ \frac{4}{(2c_\sigma p - p)} \left[ g(\phi e_j, e_j) g(\phi e_j, e_j) \\
- g(\phi e_i, e_i) g(\phi e_j, e_i) \right] \\
+ \frac{p^2}{(2c_\sigma p - p)^2} \left[ g(e_i, e_i) g(e_j, e_j) - g(e_i, e_j) g(e_j, e_i) \right] \\
+ \frac{1}{2p} \left[ g(\phi e_i, e_i) g(e_j, e_j) + g(e_i, e_j) g(\phi e_j, e_i) \\
- g(\phi e_j, e_i) g(e_i, e_i) - g(e_j, e_i) g(\phi e_i, e_i) \right] \right\} \\
\pm \frac{1}{2} (c_1 - c_2) \left\{ \frac{1}{(2c_\sigma p - p)} \left[ g(e_i, e_j) g(\phi e_j, e_j) \\
- g(e_i, e_j) g(\phi e_i, e_j) \right] \\
+ \frac{1}{(2c_\sigma p - p)} \left[ g(e_i, e_j) g(e_j, e_j) - g(\phi e_i, e_j) g(e_j, e_i) \right] \\
+ \frac{1}{(2c_\sigma p - p)} \left[ g(e_i, e_i) g(e_j, e_j) - g(e_i, e_j) g(e_j, e_i) \right] \\
\psi_1 \beta_1 (e_i, e_i) g(e_j, e_j) - \psi_1 \beta_1 (e_i, e_j) g(e_i, e_i) \\
\psi_2 \beta_1 (e_j, e_j) g(e_i, e_i) \right\} \\
+ \psi_2 (\psi_1 - \psi_2) g(e_i, e_i) g(e_j, e_j) \\
+ \psi_2 (\psi_1 - \psi_2) g(e_i, e_j) g(e_j, e_i) \\
+ g(\xi(e_i, e_i), \xi(e_j, e_j)) - g(\xi(e_i, e_j), \xi(e_j, e_i))
$$
We take
\[
\delta = 2\tau - \frac{1}{4}(c_1 + c_2)\frac{n(n-1)}{p^2 + 4q}\left\{2p^2 + 4q + \frac{4}{n(n-1)}[tr^2\phi - \cos^2\theta(p, trT + nq)] - \frac{4p}{n} tr\phi\right\}
\]

As a result, when using the orthonormal frame \(\{e_1, ..., e_n\}\), (21) assumes the following form:
\[
\left(\sum_{i=1}^{n} z_{ii}^{n+1}\right)^2 = (n-1)\left\{\delta + \sum_{i=1}^{n} (\xi_{ii}^{n+1})^2 + \sum_{i\neq j}^{n} (\xi_{ij}^{n+1})^2 + \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} (\zeta_{ij}^{r})^2\right\}.
\]

If we substitute \(a_1 = \xi_{11}^{n+1}, a_2 = \sum_{i=2}^{n} b_{ii}^{n+1}\), and \(a_3 = \sum_{l=n+1}^{m} b_{ll}^{n+1}\), then (22) reduces to
\[
\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left\{\delta + \sum_{i=1}^{n} a_i^2 + \sum_{i=2}^{n} \sum_{j=1}^{n} (\xi_{ij}^{n+1})^2 + \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} (\zeta_{ij}^{r})^2\right\}
\]
\[
- \sum_{2\leq j\neq k\leq n_1} b_{jj}^{n+1} b_{kk}^{n+1} - \sum_{n_1+1\leq l \neq m} b_{ll}^{n+1} b_{ll}^{n+1}
\]

As a result, \(a_1, a_2, a_3\) fulfill Chen’s Lemma (for \(n = 3\)), i.e.,
\[
\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(\frac{b}{3} + \sum_{i=1}^{3} a_i^2\right).
\]

Clearly, \(2a_1a_2 \geq b\) with equality holds if \(a_1 = a_2 = a_3\), and conversely, this signifies
\[
\sum_{1\leq k < n_1} \xi_{kk}^{n+1} b_{kk}^{n+1} + \sum_{n_1+1 \leq l < n} \sum_{s=1}^{n} b_{ss}^{n+1} b_{ll}^{n+1}
\]
\[
\geq \frac{\delta}{2} + \sum_{1\leq a_3 < a_2 \leq n} \sum_{r=n+1}^{m} \sum_{s=1}^{n} (\xi_{rs}^{r+1})^2 + \sum_{r=n+1}^{m} \sum_{a_3=1}^{a_2} \sum_{a_3=1}^{a_2} (\xi_{rs}^{r+1})^2
\]

Applying 1 ≤ i, j ≤ n in (18), we obtain
\[
2\tau(x) = \frac{1}{4}(c_1 + c_2)\frac{n(n-1)}{p^2 + 4q}\left\{2p^2 + 4q + \frac{4}{n(n-1)}[tr^2\phi - \cos^2\theta(p, trT + nq)]
\]
\[
- \frac{4p}{n} tr\phi\right\}
\]

Then, from (19) and (20), we have
\[
n^2||H||^2 = (n-1)(\delta + ||\xi||^2).
\]
and equality holds if and only if

$$\sum_{i=1}^{n_1} \xi_{ii}^{n+1} = \sum_{i=n_1+1}^{n} \xi_{ii}^{n+1}. \quad (25)$$

Again taking into consideration Equation (3.3) in [15], we arrive at the following conclusion:

$$n_2 \frac{\Delta f}{f} = \tau - \sum_{1 \leq j \leq n_1} \kappa(e_j \wedge e_k) - \sum_{n_1+1 \leq t \leq n} \kappa(e_t \wedge e_t). \quad (26)$$

Then, from (24) and (26), we compute

$$n_2 \frac{\Delta f}{f} \leq \tau - \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ (n(n-1) - 2n_1n_2)(p^2 + 4q) + 8 \left[ tr^2 \phi - 4 \cos^2 \theta (2p, tr T + nq) - 4p(n-2)tr \phi \right] \right\} + \frac{1}{8} \left( c_1 - c_2 \right) \frac{1}{\sqrt{p^2 + 4q}} \left\{ 4tr \phi (n-2) - 2pn(n-1) - 4pn_1n_2 \right\} - \frac{1}{2} \left( \psi_1 + \psi_2 \right) \left[ \lambda |M_1(n_1 - 1) + \lambda |M_2(n_2 - 1) \right] - \frac{1}{2} \left( \psi_1 - \psi_2 \right) \left[ \mu |M_1(n_1 - 1) + \mu |M_2(n_2 - 1) \right] + \left( \psi_1 - \psi_2 \right) \left[ n_1(n_1 - 1) \pi(H_1) + n_2(n_2 - 1) \pi(H_2) \right], \quad (27)$$

Using (20) in the above equation, we obtain

$$n_2 \frac{\Delta f}{f} \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 + \frac{1}{8} \left( c_1 + c_2 \right) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4 \left[ tr^2 \phi + tr \phi \right] - \cos^2 \theta (p, tr T) \right\} + \frac{1}{8} \left( c_1 - c_2 \right) \frac{1}{\sqrt{p^2 + 4q}} \left\{ 4pn_1n_2 - 4tr \phi \right\} - \left( \psi_1 + \psi_2 \right) \left[ \lambda(n-1) + \lambda |M_1(n_1 - 1) + \lambda |M_2(n_2 - 1) \right] - \left( \psi_1 - \psi_2 \right) \left[ \mu(n-1) + \mu |M_1(n_1 - 1) + \mu |M_2(n_2 - 1) \right] + \left( \psi_1 - \psi_2 \right) \left[ n(n-1) \pi(H) + n_1(n_1 - 1) \pi(H_1) + n_2(n_2 - 1) \pi(H_2) \right], \quad (28)$$

which implies the required inequality.

We deduce from (24) and (25) that the equality in (16) holds if and only if

$$\sum_{r=n+1}^{m} \sum_{i=1}^{n_1} \zeta_{ri} = \sum_{r=n+1}^{m} \sum_{i=n_1+1}^{n} \zeta_{ri} = 0. \quad (29)$$

Moreover, from (25), we obtain

$$\zeta_{ri} = 0, \forall 1 \leq j \leq n_1, n+1 \leq l \leq n, n+1 \leq r \leq m. \quad (30)$$

This shows that (30) is equivalent to the mixed total geodesicity of the doubly warped product $M = M_1(c_1) \times M_2(c_2)$ and (25) and (29) imply $n_1 H_1 = n_2 H_2$. \qed
4. Some Applications of the Result

The significance and applicability of the findings can be observed from three distinct perspectives. Firstly, they can be regarded as specific instances within the realm of quarter-symmetric connections, shedding light on the broader understanding of this field. Secondly, the results can be viewed as particular cases within the framework of slant submanifolds, contributing to the knowledge and characterization of these geometric structures. Lastly, they hold relevance as specific instances within the domain of metallic space forms, providing valuable insights into the properties and behavior of such spaces. The multifaceted nature of these applications underscores the depth and breadth of the implications derived from this research, making it a compelling contribution to the respective fields and offering new avenues for exploration and discovery.

4.1. Results on Specific Instances within the Realm of Quarter-Symmetric Connection

It is known that a quarter-symmetric connection becomes a semi-symmetric metric connection with \( \psi_1 = 1 \) and \( \psi_2 = 1 \). Taking this into consideration together with Theorem 1, we have the following result:

**Corollary 1.** Let \( \mathcal{M} \) be an \( n \)-dimensional warped product \( \theta \)-slant submanifold of an \( m \)-dimensional locally metallic product space form \( (\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi) \) with semi-symmetric metric connections. Then,

\[
\begin{align*}
\frac{n_2 \Delta f}{f} &
\leq \frac{n^2(n - 2)}{2(n - 1)} ||\mathcal{H}||^2 \\
&+ \frac{1}{8} \left( c_1 + c_2 \right) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4\left[ tr^2 \phi + tr\phi \right. \\
&- \cos^2 \theta (p, trT) \bigg\} \right. \\
&+ \frac{1}{8} \frac{(c_1 - c_2)}{\sqrt{(p^2 + 4q)}} \left\{ 4pn_1n_2 - 4tr\phi \right\} \\
&- 2 \left[ \lambda(n - 1) + \lambda_{\mathcal{M}_1}(n_1 - 1) + \lambda_{\mathcal{M}_2}(n_2 - 1) \right].
\end{align*}
\]  
(31)

The equality in (31) is attained if and only if \( \mathcal{M} \) is a mixed totally geodesic isometric immersion and meets the condition (17).

We also know that a quarter-symmetric connection becomes a semi-symmetric non-metric connection with \( \psi_1 = 1 \) and \( \psi_2 = 0 \). Taking this into consideration together with Theorem 1, we have the following result.

**Corollary 2.** Let \( \mathcal{M} \) be an \( n \)-dimensional warped product \( \theta \)-slant submanifold of an \( m \)-dimensional locally metallic product space form \( (\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi) \) with semi-symmetric non-metric connection. Then,

\[
\begin{align*}
\frac{n_2 \Delta f}{f} &
\leq \frac{n^2(n - 2)}{2(n - 1)} ||\mathcal{H}||^2 \\
&+ \frac{1}{8} \left( c_1 + c_2 \right) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4\left[ tr^2 \phi + tr\phi \right. \\
&- \cos^2 \theta (p, trT) \bigg\} \right. \\
&+ \frac{1}{8} \frac{(c_1 - c_2)}{\sqrt{(p^2 + 4q)}} \left\{ 4pn_1n_2 - 4tr\phi \right\} \\
&- \left[ \lambda(n - 1) + \lambda_{\mathcal{M}_1}(n_1 - 1) + \lambda_{\mathcal{M}_2}(n_2 - 1) \right] \\
&+ \left[ n(n - 1)\pi(H) + n_1(n_1 - 1)\pi(H_1) + n_2(n_2 - 1)\pi(H_2) \right].
\end{align*}
\]  
(32)

The equality in (32) is attained if and only if \( \mathcal{M} \) is a mixed totally geodesic isometric immersion and meets the condition (17).
4.2. Results on Specific Instances within the Realm of $\theta$-Slant Submanifold

We know that the particular classes of the $\theta$-slant submanifold are either invariant or anti-invariant with $\theta = 0$ or $\theta = \frac{\pi}{2}$, respectively. Thus, we have the following result as a consequence of Theorem 1.

**Corollary 3.** Let $\mathcal{M}$ be an $n$-dimensional warped product invariant submanifold of an $m$-dimensional locally metallic product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with quarter-symmetric metric connections. Then,

$$
\frac{n^2}{f} \Delta f \leq \frac{n^2(n - 2)}{2(n - 1)} ||H||^2 
\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4 \left[ tr^2 \phi + tr \phi - p.tr T \right] \right\} 
\quad + \frac{1}{8} (c_1 - c_2) \sqrt{(p^2 + 4q)} \left\{ 4pn_1n_2 - 4tr \phi \right\} 
\quad - (\psi_1 + \psi_2) \left[ \lambda(n - 1) + \lambda|_{M_1}(n_1 - 1) + \lambda|_{M_2}(n_2 - 1) \right] 
\quad - \psi_2(\psi_1 - \psi_2) \left[ \mu(n - 1) + \mu|_{M_1}(n_1 - 1) + \mu|_{M_2}(n_2 - 1) \right] 
\quad + (\psi_1 - \psi_2) \left[ n(n - 1)\pi(H) + n_1(n_1 - 1)\pi(H_1) + n_2(n_2 - 1)\pi(H_2) \right].
$$

(33)

The equality in (33) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

**Corollary 4.** Let $\mathcal{M}$ be an $n$-dimensional warped product anti-invariant submanifold of an $m$-dimensional locally metallic product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with quarter-symmetric metric connections. Then,

$$
\frac{n^2}{f} \Delta f \leq \frac{n^2(n - 2)}{2(n - 1)} ||H||^2 
\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4 \left[ tr^2 \phi + tr \phi \right] \right\} 
\quad + \frac{1}{8} (c_1 - c_2) \sqrt{(p^2 + 4q)} \left\{ 4pn_1n_2 - 4tr \phi \right\} 
\quad - (\psi_1 + \psi_2) \left[ \lambda(n - 1) + \lambda|_{M_1}(n_1 - 1) + \lambda|_{M_2}(n_2 - 1) \right] 
\quad - \psi_2(\psi_1 - \psi_2) \left[ \mu(n - 1) + \mu|_{M_1}(n_1 - 1) + \mu|_{M_2}(n_2 - 1) \right] 
\quad + (\psi_1 - \psi_2) \left[ n(n - 1)\pi(H) + n_1(n_1 - 1)\pi(H_1) + n_2(n_2 - 1)\pi(H_2) \right].
$$

(34)

The equality in (34) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

Further, from Corollary 1, we mind the following results.

**Corollary 5.** Let $\mathcal{M}$ be an $n$-dimensional warped product invariant submanifold of an $m$-dimensional locally metallic product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with semi-symmetric metric connections. Then,

$$
\frac{n^2}{f} \Delta f \leq \frac{n^2(n - 2)}{2(n - 1)} ||H||^2 
\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4 \left[ tr^2 \phi + tr \phi - p.tr T \right] \right\}.
$$
\[\pm \frac{1}{8} \frac{(c_1 - c_2)}{\sqrt{(p^2 + 4q)}} \{4pn_1n_2 - 4tr\phi\} \]
\[\quad - 2 \left[ \lambda(n - 1) + \lambda|_{M_1}(n_1 - 1) + \lambda|_{M_2}(n_2 - 1) \right]. \quad (35)\]

The equality in (35) is attained if and only if \(M\) is a mixed totally geodesic isometric immersion and meets the condition (17).

**Corollary 6.** Let \(M\) be an \(n\)-dimensional warped product anti-invariant submanifold of an \(m\)-dimensional locally metallic product space form \((\overline{M} = M_1(c_1) \times M_2(c_2), g, \phi)\) with semi-symmetric metric connections. Then,
\[\frac{n^2}{1} \Delta f \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 \]
\[\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4|tr^2 - tr\phi| \right\} \]
\[\quad + \frac{1}{8} (c_1 - c_2) \frac{1}{\sqrt{(p^2 + 4q)}} \{4pn_1n_2 - 4tr\phi\} \]
\[\quad - 2 \left[ \lambda(n - 1) + \lambda|_{M_1}(n_1 - 1) + \lambda|_{M_2}(n_2 - 1) \right]. \quad (36)\]

The equality in (36) is attained if and only if \(M\) is a mixed totally geodesic isometric immersion and meets the condition (17).

Moreover, from Corollary 2, we obtain the following results.

**Corollary 7.** Let \(M\) be an \(n\)-dimensional warped product invariant submanifold of an \(m\)-dimensional locally metallic product space form \((\overline{M} = M_1(c_1) \times M_2(c_2), g, \phi)\) with a semi-symmetric non-metric connection. Then,
\[\frac{n^2}{1} \Delta f \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 \]
\[\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4|tr^2 - tr\phi| \right\} \]
\[\quad + \frac{1}{8} (c_1 - c_2) \frac{1}{\sqrt{(p^2 + 4q)}} \{4pn_1n_2 - 4tr\phi\} \]
\[\quad - \left[ \lambda(n - 1) + \lambda|_{M_1}(n_1 - 1) + \lambda|_{M_2}(n_2 - 1) \right] \]
\[\quad + n(n - 1)\pi(H) + n_1(n_1 - 1)\pi(H_1) + n_2(n_2 - 1)\pi(H_2). \quad (37)\]

The equality in (37) is attained if and only if \(M\) is a mixed totally geodesic isometric immersion and meets the condition (17).

**Corollary 8.** Let \(M\) be an \(n\)-dimensional warped product anti-invariant submanifold of an \(m\)-dimensional locally metallic product space form \((\overline{M} = M_1(c_1) \times M_2(c_2), g, \phi)\) with a semi-symmetric non-metric connection. Then,
\[\frac{n^2}{1} \Delta f \leq \frac{n^2(n-2)}{2(n-1)} ||H||^2 \]
\[\quad + \frac{1}{8} (c_1 + c_2) \frac{1}{p^2 + 4q} \left\{ 2n_1n_2(p^2 + 4q) - 4|tr^2 - tr\phi| \right\} \]
\[\quad + \frac{1}{8} (c_1 - c_2) \frac{1}{\sqrt{(p^2 + 4q)}} \{4pn_1n_2 - 4tr\phi\} \]
The equality in (38) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

4.3. Results on Specific Instances within the Realm of Metallic Product Space

A metallic structure can be characterized as a golden structure, copper structure, nickel structure, silver structure, bronze structure, subtile structure, and so on for providing different particular values to $p$ and $q$. For instance, the metallic structure implies a golden structure when $p = 1$ and $q = 1$. Hence, from Theorem 1, we obtain the following results.

**Corollary 9.** Let $\mathcal{M}$ be an $n$-dimensional warped product $\theta$-slant submanifold of an $m$-dimensional locally golden product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with quarter-symmetric metric connections. Then,

\[
\frac{n_2}{\mathcal{f}} \geq \frac{n^2(n-2)}{2(n-1)} ||\mathcal{H}||^2 \\
+ \frac{1}{40} (c_1 + c_2) \left\{ 10n_1n_2 - 4 \left[ tr^2 \phi + tr \phi - \cos^2 \theta (trT) \right] \right\} \\
\pm \frac{1}{8\sqrt{5}} (c_1 - c_2) \left\{ 4n_1n_2 - 4tr \phi \right\} \\
- (\psi_1 + \psi_2) \left[ \lambda(n-1) + \lambda |\mathcal{M}_1(n_1-1) + \lambda |\mathcal{M}_2(n_2-1) \right] \\
- \phi_2(\psi_1 - \psi_2) \left[ \mu(n-1) + \mu |\mathcal{M}_1(n_1-1) + \mu |\mathcal{M}_2(n_2-1) \right] \\
+ (\psi_1 - \psi_2) \left[ n(n-1) \pi(H) + n_1(n_1-1) \pi(H_1) + n_2(n_2-1) \pi(H_2) \right].
\]  

The equality in (39) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

**Corollary 10.** Let $\mathcal{M}$ be an $n$-dimensional warped product $\theta$-slant submanifold of an $m$-dimensional locally golden product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with semi-symmetric metric connections. Then,

\[
\frac{n_2}{\mathcal{f}} \geq \frac{n^2(n-2)}{2(n-1)} ||\mathcal{H}||^2 \\
+ \frac{1}{40} (c_1 + c_2) \left\{ 10n_1n_2 - 4 \left[ tr^2 \phi + tr \phi - \cos^2 \theta (trT) \right] \right\} \\
\pm \frac{1}{8\sqrt{5}} (c_1 - c_2) \left\{ 4n_1n_2 - 4tr \phi \right\} \\
- 2 \left[ \lambda(n-1) + \lambda |\mathcal{M}_1(n_1-1) + \lambda |\mathcal{M}_2(n_2-1) \right].
\]  

The equality in (40) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

**Corollary 11.** Let $\mathcal{M}$ be an $n$-dimensional warped product $\theta$-slant submanifold of an $m$-dimensional locally golden product space form $(\mathcal{M} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ with semi-symmetric non-metric connections. Then,

\[
\frac{n_2}{\mathcal{f}} \geq \frac{n^2(n-2)}{2(n-1)} ||\mathcal{H}||^2
\]
The equality in (41) is attained if and only if $\mathcal{M}$ is a mixed totally geodesic isometric immersion and meets the condition (17).

**Remark 2.** We can obtain results similar to the results (9), (10), and (11) for copper, silver, nickel, bronze, subtle, etc., by proving specific values to $p$ and $q$.

**Remark 3.** We can also obtain the results (9), (10), and (11) for particular classes of the $\theta$-slant submanifolds, i.e., invariant and anti-invariant submanifolds by providing particular values $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

5. Conclusions

We have the following conclusions from our findings in this article:

1. We delved into the realm of geometric inequalities, with a particular focus on Chen’s inequality. Our investigation revolved around its application to assess the square norm of the mean curvature vector and the warping function of warped product slant submanifolds. Within the framework of locally metallic product space forms with quarter-symmetric metric connection, we successfully established this geometric inequality and explored its implications.

2. By examining the conditions under which equality is achieved within the inequality, we gained valuable insights into the intricacies of warped product slant submanifolds. Our findings shed light on the underlying geometric properties and the relationships between the mean curvature vector, the warping function, and the ambient space.

3. The implications of our research extend beyond the theoretical realm. The established geometric inequality and its equality conditions provide a powerful tool for studying and characterizing warped product slant submanifolds in locally metallic product space forms. This has potential applications in various fields, such as differential geometry, mathematical physics, and even in applied sciences where understanding the geometric properties of submanifolds is crucial.

Overall, our study contributes to the existing body of knowledge by providing a deeper understanding of Chen’s inequality and its significance in the context of warped product slant submanifolds. We hope that our findings will inspire further research and stimulate new avenues of exploration in the fascinating field of geometric inequalities and their applications.

**Future Work**

The following could be future research topics for the study of Chen’s inequality in the context of warped product slant submanifolds within locally metallic product space forms with quarter-symmetric metric connections:

1. Further studies may involve extending Chen’s inequality to other classes of geometric spaces or submanifolds. One possible approach to this would be to examine whether it can be applied to other kinds of submanifolds, including minimum submanifolds, hypersurfaces, Lagrangian submanifolds, etc., and to examine the implications in those situations.

2. Further investigation into the characteristics and properties of warped product slant submanifolds is possible. This might involve creating additional geometric inequali-
ties unique to this class of submanifolds, as well as analyzing the behavior of warping functions and mean curvature vectors in various dimensions and situations.

3. Beyond the quarter-symmetric metric connection, different kinds of metric connections can be taken into consideration to advance the research. Analyzing Chen’s inequality in relation to other metric connections may yield insightful comparisons.

4. Subsequent investigations may utilize computational or numerical techniques to verify and investigate the outcomes derived from analytical procedures. In order to investigate the behavior of mean curvature vectors and warping functions and to confirm the accuracy and applicability of Chen’s inequality in real-world situations, this can include running numerical experiments or simulations.

5. Interdisciplinary research can be facilitated by working with scientists in adjacent domains like mathematical physics, geometric analysis, or differential geometry. Collaboration with specialists in other fields can result in fresh insights, alternative uses, and a better understanding of Chen’s inequality’s significance.

Future research can advance geometric inequalities, our knowledge of warped product slant submanifolds, and the field of differential geometry as a whole by exploring these directions.

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