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The Impact of Quasi-Conformal Curvature Tensor on Warped Product Manifolds

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Abstract: This work investigates the effects on the factor manifolds of a singly warped product manifold resulting from the presence of a quasi-conformally flat, quasi-conformally symmetric, or divergence-free quasi-conformal curvature tensor. Quasi-conformally flat warped product manifolds exhibit three distinct scenarios: in one scenario, the base manifold has a constant curvature, while in the other two scenarios, it is quasi-Einstein. Alternatively, the fiber manifold has a constant curvature in two scenarios and is Einstein in one scenario. Quasi-conformally symmetric warped product manifolds present three distinct cases: in the first scenario, the base manifold is Ricci-symmetric and the fiber is Einstein; in the second case, the base manifold is Cartan-symmetric and the fiber has constant curvature; and in the last case, the fiber is Cartan-symmetric, and the Ricci tensor of the base manifold is of Codazzi type. Finally, conditions are provided for singly warped product manifolds that admit a divergence-free quasi-conformal curvature tensor to ensure that the Riemann curvature tensors of the factor manifolds are harmonic.

Keywords: warped product; quasi-conformal curvature tensor; Ricci-symmetric manifold; divergence-free; Einstein-like manifold; Cartan-symmetric manifold

MSC: 53C15; 53C20

1. Introduction

The quasi-conformal curvature tensor, a curvature-like tensor, was initially introduced by Yano and Sawaki in their study of infinitesimal conformal transformations [1]. They provided a condition on the quasi-conformal curvature tensor of a compact orientable manifold N with constant curvature to guarantee that N is isometric to a sphere.

In [2], the authors examined a Sasakian manifold with a quasi-conformal curvature tensor. They demonstrated that a Sasakian manifold is an η-Einstein manifold given that it is quasi-conformally flat. Furthermore, such a manifold is shown to be locally isometric to a unit sphere with a dimension greater than 4. Additionally, they found that a Sasakian manifold is also η-Einstein provided that it is quasi-conformally semi-symmetric. (κ, μ)-contact metric manifolds that are quasi-conformally semi-symmetric or quasi-conformally recurrent are investigated in [3,4]. Detailed characterizations of quasi-conformal flatness and ξ-quasi-conformal flatness on almost Kenmotsu manifolds, along with (κ, μ)-nullity or (κ, μ)′-nullity distributions are given in [5]. Some pseudo-symmetries of generalized Sasakian space-forms admitting a quasi-conformal curvature tensor are considered in [6], while different pseudo-symmetries of the quasi-conformal curvature tensor...
tensor of $N(k)$ quasi-Einstein manifolds are explored in [7]. A non-flat quasi-conformally flat Riemannian manifold whose Ricci tensor satisfies the identity of Ricci simple manifolds is investigated in [8]. Quasi-conformally symmetric manifolds are explored in [9], and these manifolds are demonstrated to be either locally symmetric or conformally flat. In a pseudo-$Z$-symmetric manifold, the associated one-form is shown to be closed if the quasi-conformal curvature tensor is harmonic [10]. Manifolds admitting a divergence-free quasi-conformal curvature tensor are called quasi-conformally conservative manifolds. Sufficient conditions for a manifold admitting a divergence-free quasi-conformal curvature tensor to have constant scalar curvature are discussed in [11]. New findings on invariant submanifolds of para-Sasakian manifolds under the condition of quasi-conformal flatness are given in [12]. Initially, the flatness of the quasi-conformal curvature tensor in para-Sasakian manifolds is examined, and the authors demonstrate that a quasi-conformally flat para-Sasakian manifold is an \( \eta \)-Einstein manifold. They also provide some results concerning the sectional curvature of these manifolds. Secondly, they concentrate on the invariant submanifolds of a quasi-conformally flat para-Sasakian manifold, proving that a totally umbilical submanifold within a para-Sasakian manifold is invariant. Additionally, they investigate the curvature properties of these submanifolds and show that a totally umbilical invariant submanifold of a quasi-conformally flat para-Sasakian manifold is also an \( \eta \)-Einstein manifold. Finally, the authors explore the sectional curvature properties of invariant submanifolds within a quasi-conformally flat para-Sasakian manifold. A comprehensive classification of quasi-conformally recurrent Riemannian manifolds with a harmonic quasi-conformal curvature tensor is provided in [13]. It is demonstrated that such a manifold can be any of the following: quasi-conformally symmetric, conformally flat, a manifold of constant curvature, with vanishing scalar curvature, or Ricci recurrent. The quasi-conformal curvature tensor on a Riemannian submersion is studied in [14], and notable results are explored by examining the relationships between the pseudo-projective curvature tensor and the quasi-conformal curvature tensor. Some pseudo-symmetric quasi-conformal curvature conditions for a generalized quasi-Einstein manifold to be a quasi-Einstein manifold are provided in [15].

The quasi-conformal curvature tensor has significant applications in general relativity, providing valuable insights into the geometric structure and physical properties of space-time. A quasi-conformally flat perfect fluid space-time is both a de Sitter space-time and a Robertson–Walker space-time. This space-time is studied as a solution within \( f(R, G) \)-gravity theory, leading to a relationship among the snap, jerk, and deceleration parameters using the flat Friedmann–Robertson–Walker metric. For three different models of \( f(R, G) \), the weak energy condition, null energy condition, and dominant energy condition are satisfied, while the strong energy condition is violated, aligning well with recent observational studies indicating that the Universe is in an accelerating phase. Perfect fluid space-times with a vanishing quasi-conformal curvature tensor are considered [16]. Also, the divergence of the quasi-conformal curvature tensor is studied in the context of perfect fluids, resulting in the derivation of numerous physical outcomes.

In [17], the authors demonstrate that Einstein’s space-time, which is quasi-conformally flat, maintains a constant curvature, and its energy-momentum tensor, satisfying Einstein’s field equation with a cosmological constant, remains covariantly constant. Furthermore, they demonstrate that such space-time displays constant energy density and isotropic pressure, with the perfect fluid behaving akin to a cosmological constant. In addition, they examine space-times satisfying Einstein’s field equation without a cosmological constant, showing infinitesimal spatial isotropy relative to \( \delta \) if the space-time of the perfect fluid is characterized by a vanishing quasi-conformal curvature tensor. They also investigate scenarios where radiation and extremely hot gases occupy the space-time, resulting in a purely electromagnetic distribution with a quasi-conformal curvature tensor that vanishes. Finally, their research extends to dust-like fluid space-time exhibiting a quasi-conformal curvature tensor that vanishes. The pseudo quasi-conformal curvature tensor and the
generalized quasi-conformal curvature tensors are significant extensions of the quasi-conformal curvature tensor and have been widely studied in the literature [18–20].

Warped product manifolds play a crucial role in differential geometry and general relativity due to their versatile structure and wide range of applications [21–26]. In differential geometry, they provide a method to construct new manifolds from known ones, preserving certain geometric properties while introducing new ones [23]. The standard model of cosmology, known as the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, is a warped product manifold that describes a homogeneous, isotropic expanding or contracting universe [27,28]. Interest in the study of warped product geometry has been increasing over the past decade. The research focuses on two main areas: one involving the exploration of geometric concepts on warped product manifolds, and the other concerning the generalization of these manifolds [29–31]. In this discussion, we will present some intriguing recent findings in both areas.

In [32], the concircular curvature tensor on warped product manifolds is explored. The study examines the effects of concircular flatness and concircular symmetry on the fiber and base manifolds of a warped product manifold. Specifically, it analyzes concircularly flat and concircularly symmetric warped product manifolds. Additionally, the properties of a divergence-free concircular curvature tensor on warped product manifolds are considered. A generalization of this work to any Pokharial-like curvature tensor is considered in [33]. The work in [34] explores the semi-conformal curvature tensor on singly warped product manifolds. It presents the forms of the Riemann and Ricci tensors for both the base and fiber manifolds of a semi-conformally flat singly warped product manifold. The study shows that the fiber manifold of such a manifold has constant curvature. Furthermore, it specifies the conditions on the warping function necessary for the base manifold to be either a quasi-Einstein or Einstein manifold. In [35], the authors determine the necessary and sufficient conditions for a sequential warped product manifold to qualify as a quasi-Einstein manifold. Additionally, they explore the necessary conditions for a sequential standard static space-time and a sequential generalized Robertson–Walker space-time to be classified as manifolds of quasi-constant curvature. The necessary and sufficient conditions for doubly warped product manifolds to be gradient Ricci-harmonic solitons are provided in [36]. The authors establish the necessary conditions for a Ricci–Bourguignon soliton, structured as a sequential warped product, to be an Einstein manifold when the potential field is regarded as either a Killing or conformal vector field. The primary relations for a gradient Ricci-harmonic soliton on sequential warped product manifolds are examined in [37]. Projective collineations on pseudo-Riemannian warped product manifolds are examined in [38], exploring their relationships with curvature collineations and Ricci curvature collineations. The study focuses on the inheritance properties of these collineations from the warped product manifolds to their factor manifolds. The necessary and sufficient conditions for the lift of a vector field on a factor manifold to be a 2-Killing vector field on doubly and multiply warped product manifolds are provided in [39,40].

Despite the significant interest in the study of warped product manifolds and the quasi-conformal curvature tensor, no research has been conducted on the quasi-conformal curvature tensor in the context of warped product manifolds. We aim to address this gap in the literature. This work explores the impact of a flat, symmetric, or divergence-free quasi-conformal curvature tensor on the factor manifolds of a singly warped product manifold. For quasi-conformally flat warped product manifolds, three scenarios are identified: in one, the base manifold has a constant curvature; in the other two, it is quasi-Einstein. On the other hand, the fiber manifold exhibits a constant curvature in two scenarios and is Einstein in one. For quasi-conformally symmetric warped product manifolds, three distinct cases emerge: the base manifold is Ricci-symmetric and the fiber is Einstein in the first case; the base manifold is Cartan-symmetric and the fiber has a constant curvature in the second case; and the fiber is Cartan-symmetric with the Ricci tensor of the base manifold being of Codazzi type in the third case. Lastly, this work provides conditions under which singly
warped product manifolds with a divergence-free quasi-conformal curvature tensor have harmonic Riemann curvature tensors in their factor manifolds.

2. Preliminaries

The quasi-conformal curvature tensor, denoted as $Q_{ijkl}$, indeed plays a significant role in understanding the geometry of space-time within the framework of Riemannian manifolds. The quasi-conformal curvature tensor arises as a modification of the Weyl conformal curvature tensor $C_{ijkl}$. In local coordinates, the quasi-conformal curvature tensor is expressed as follows:

$$Q_{ijkl} = AR_{ijkl} + B\left[g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}\right] - \frac{R}{n}\left(\frac{A}{n-1} + 2B\right)[g_{ik}g_{jl} - g_{il}g_{jk}],$$

(1)

where $A$ and $B$ are not simultaneously zero. This tensor adheres to the same properties as the Riemann curvature tensor. Tensors belonging to the class that exhibit the symmetries of the Riemann curvature tensor are termed curvature-like tensors. These symmetries are

$$Q_{ijkl} = -Q_{ijkl} = -Q_{ijlk} = -Q_{klij},$$

$$0 = Q_{ijkl} + Q_{ilij} + Q_{ijkl}.$$ 

These symmetries ensure consistency and regularity in the geometric properties described by curvature-like tensors, akin to those encapsulated by the Riemann curvature tensor. Curvature-like tensors are important in many areas of mathematics, such as general relativity and Riemannian geometry. Curvature information can be encoded in a variety of geometric situations using curvature-like tensors, which are sophisticated mathematical constructions. They are essential resources for comprehending the geometry of spaces and connections because of their symmetries and other characteristics. Applications of curvature-like tensors in mathematics appear in various mathematical contexts.

It is noted that the quasi-conformal curvature tensor $Q_{ijkl}$ coincides with the Weyl conformal tensor $C_{ijkl}$ where $A = 1$, $B = -\frac{1}{n-2}$, that is,

$$Q_{ijkl} = R_{ijkl} - \frac{1}{n-2}\left[g_{ik}R_{jl} - g_{il}R_{jk} + g_{jl}R_{ik} - g_{jk}R_{il}\right] + \frac{R}{(n-1)(n-2)}[g_{ik}g_{jl} - g_{il}g_{jk}],$$

$$= C_{ijkl}.$$ 

The elegance of this tensor lies in its remarkable versatility, as it allows for the retrieval of many significant tensors by assigning different values to the parameters $A$ and $B$. By adjusting these values, one can derive a wide range of important tensor forms such as those defined by Pokharial, Mishra, Yano, and many others [41–45], making this tensor a powerful and flexible tool in mathematical and physical applications.

A warped product manifold is an extension of Riemannian product manifolds. It systematically combines various geometries. In general relativity, warped product manifolds provide a framework for modeling space-time geometries with non-trivial fiber structures. These structures are particularly useful in gravitational theories, such as Kaluza–Klein theory and higher dimensional gravity models.

A warped product manifold $\tilde{N} \times_f \tilde{N}$ refers to a Riemannian or pseudo-Riemannian manifold represented as the product manifold $\tilde{N} = \tilde{N} \times \tilde{N}$ with its natural projections $\pi : \tilde{N} \times \tilde{N} \rightarrow \tilde{N}$ and $\eta : \tilde{N} \times \tilde{N} \rightarrow \tilde{N}$. Here $(\tilde{N}, \tilde{g})$ and $(\tilde{N}, \tilde{G})$ denote two pseudo-Riemannian manifolds with dimensions $\tilde{n}$, $\tilde{n}$ and the metric $g$ tensor is determined accordingly as

$$g = \tilde{g} \oplus F\tilde{g},$$

(2)
In such definition, the function $F : \mathbb{N} \rightarrow (0, \infty)$ is smooth and positive on $\bar{N}$. The indices $i, j, \cdots \in \{1, \ldots, \bar{n}\}$, $a, b, \cdots \in \{1, \ldots, \bar{n}\}$, and $\alpha, \beta, \cdots \in \{\bar{n} + 1, \ldots, \bar{n} + \tilde{n}\}$ where $n = \bar{n} + \tilde{n}$. Locally,

$$g_{ij} = \begin{cases} \bar{g}_{ab} & i = a, j = b, \\ F \bar{g}_{\alpha\beta} & i = a, j = \beta, \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (3)

The Ricci curvature tensor $R_{ij}$ of the warped product $\mathbb{N}$ can be described in detail through its local components, which are as follows:

$$R_{ab} = \bar{R}_{ab} - \frac{\tilde{n}}{2F} T_{ab},$$ \hspace{1cm} (4)

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \frac{1}{2} [T + \frac{\tilde{n} - 1}{2F} \Delta F] \bar{g}_{\alpha\beta},$$ \hspace{1cm} (5)

$$R_{a\alpha} = 0.$$ \hspace{1cm} (6)

The Riemannian curvature tensor $R_{ijkl}$ can be described locally as

$$R_{\alpha\beta\gamma\delta} = F \bar{R}_{\alpha\beta\gamma\delta} - \frac{1}{4} \Delta F \bar{G}_{\alpha\beta\gamma\delta},$$ \hspace{1cm} (7)

$$R_{ab\beta} = -\frac{1}{2} T_{ab} \bar{g}_{\alpha\beta},$$ \hspace{1cm} (8)

$$R_{ab\gamma} = R_{ab\gamma}. \hspace{1cm} (9)$$

Here, $\bar{G}$ and $\Delta F$ are specified as follows:

$$\bar{G}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta},$$

$$\Delta F = \bar{g}^{ab} F_a F_b,$$

$$F_a = \partial_a F = \partial \frac{F}{\partial x^a}.$$  

Also, the tensor $T_{ab}$ is of type $(0, 2)$ and is given as

$$T_{ab} = \nabla_b F_a - \frac{1}{2F} F_a F_b,$$

$$T_{a\beta} = T_{\alpha a} = 0.$$  

The scalar curvature $R$ is expressed as

$$R = \bar{R} + \frac{1}{F} \bar{R} - \frac{\tilde{n}}{F} [T + \frac{\tilde{n} - 1}{4F} \Delta F]$$ \hspace{1cm} (10)

where $T = g^{ab} T_{ab}$. The non-zero covariant derivatives on the warped product manifolds are

$$\nabla_c R_{ab} = \nabla_c \bar{R}_{ab} - \frac{\tilde{n}}{2} \nabla_c \left( \frac{1}{F} T_{ab} \right),$$

$$\nabla_\gamma R_{ab\beta} = \nabla_\gamma \bar{R}_{ab\beta},$$

$$\nabla_\beta R_{ab} = \frac{F_a}{2F} \left( \bar{R}_{ab} - \frac{1}{2} \left( T + \frac{\tilde{n} - 1}{2F} \Delta F \right) \bar{g}_{\alpha\beta} \right)$$

$$+ \frac{1}{2} F_e \left( \bar{R}_{ae} + \frac{\tilde{n} - 1}{2F} \bar{g}_{ae} \right) \bar{g}_{ab} \beta,$$

$$\nabla_a R_{ab\beta} = \frac{F_a}{F} \bar{R}_{ab\beta} \left(T + \frac{\tilde{n} - 1}{2F} \Delta F \right) \bar{g}_{ab} \beta$$

$$+ \frac{1}{2} \nabla_a \left(T + \frac{\tilde{n} - 1}{2F} \Delta F \right) \bar{g}_{ab} \beta.$$
3. Quasi-Conformally Flat Warped Product Manifolds

Lorentzian warped product manifolds offer a versatile representation for diverse space-times. These relativistic configurations captivate attention in mathematical and physical realms due to their intriguing geometric attributes. Exploring these space-times, the quasi-conformal curvature tensor emerges as a pivotal instrument, facilitating deeper insights into their intricate geometry.

**Theorem 1.** Let $N$ be a non-trivial quasi-conformally flat manifold. Then,

1. the Riemann tensor of $N$ is given by

\[
R_{ijkl} = \frac{1}{n-2} [g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il}]
- \frac{\mathcal{R}}{n-1)(n-2)} [g_{ik} g_{jl} - g_{il} g_{jk}].
\]  

(11)

given that $A + (n-2)B = 0$.

2. $N$ is of constant curvature given that $A \neq 0$ and $A \neq -(n-2)B$.

3. $N$ is Einstein given that $A = 0$ and $A \neq (n-2)B$.

**Proof.** A manifold $N$ is termed quasi-conformally flat if the quasi-conformal curvature tensor vanishes identically on $N$. In this case,

\[
0 = A R_{ijkl} + B[g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il}]
- \frac{\mathcal{R}}{n-1)+2B) [g_{ik} g_{jl} - g_{il} g_{jk}].
\]

Therefore, the Riemann tensor of a quasi-conformally flat manifold is expressed as:

\[
AR_{ijkl} = -B[g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il}]
+ \frac{\mathcal{R}}{n-1)+2B) [g_{ik} g_{jl} - g_{il} g_{jk}].
\]  

(12)

Through the contraction of the aforementioned equation using $g^{il}$, one obtains

\[
-AR_{jk} = -B[(2-n)R_{jk} - g_{jk} \mathcal{R}]
+ \frac{\mathcal{R}}{n-1)+2B) [(1-n)g_{jk}]
= -B(2-n)R_{jk} + BRg_{jk} - \frac{RA}{n} g_{jk} + \frac{2BR}{n} (1-n)g_{jk}
= -B(2-n)R_{jk} + \left( \frac{2-n}{n}B - A \right) Rg_{jk}.
\]  

(13)

Thus, one may obtain

\[
[A + (n-2)B] R_{jk} = [A + (n-2)B] \frac{\mathcal{R}}{n} g_{jk}.
\]  

(14)

Therefore, it is possible to obtain

\[
[A + (n-2)B] \left( R_{jk} - \frac{\mathcal{R}}{n} g_{jk} \right) = 0.
\]  

(15)
Therefore, \( A + (n - 2)B = 0 \) or the Ricci curvature satisfies
\[
R_{jk} = \frac{R}{n} \delta_{jk}. \tag{16}
\]

Let us begin by considering the first scenario where \( A + (n - 2)B = 0 \). A direct substitution in Equation (12) results in
\[
-(n - 2)B R_{ijkl} = \frac{B}{n} \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right]
+ \frac{R B}{n - 1} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right]. \tag{17}
\]

It is important to note that \( B \) cannot be zero because, in such a scenario, \( A \) would also be zero, resulting in a trivial case. Then,
\[
R_{ijkl} = \frac{1}{n - 2} \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right]
- \frac{R}{(n - 1)(n - 2)} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right]. \tag{18}
\]

The subsequent scenario involves Equation (16)
\[
R_{jk} = \frac{R}{n} \delta_{jk}. \tag{19}
\]

that is, the manifold \( N \) is Einstein. In this situation, where \( A + (n - 2)B \neq 0 \) and \( R_{jk} = \frac{R}{n} \delta_{jk} \), Equation (12) leads to
\[
A \left( R_{ijkl} - \frac{R}{n(n-1)} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right] \right) = 0.
\]

That is, \( N \) is of constant curvature. It remains to consider the scenario where \( A = 0 \) and \( A \neq -(n - 2)B \). In this case, \( B \) cannot be zero and Equation (12) leads to
\[
0 = - \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right]
+ \frac{2R}{n} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right]. \tag{20}
\]

A contraction over \( i \) and \( k \) yields
\[
R_{jl} = \frac{R}{n} \delta_{jl}. \tag{21}
\]

\( N \) is Einstein. \( \square \)

The results of this section are summarized in the following table:

<table>
<thead>
<tr>
<th>A quasi-conformally flat manifold ( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([A + (n - 2)B] \left( R_{jk} - \frac{R}{n} \delta_{jk} \right) = 0)</td>
</tr>
<tr>
<td>(A + (n - 2)B = 0)</td>
</tr>
<tr>
<td>(A = 0)</td>
</tr>
<tr>
<td>(A \neq 0)</td>
</tr>
<tr>
<td>(N) is of constant curvature</td>
</tr>
</tbody>
</table>

**Corollary 1.** The Riemann tensor of a conformally flat manifold \( N \) is given by
\[
R_{ijkl} = \frac{1}{n - 2} \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right]
- \frac{R}{(n - 1)(n - 2)} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right]. \tag{22}
\]
Proof. It is noted that the $A + (n - 2)B = 0$ holds for the special values $A = 1$, $B = -\frac{1}{n-2}$. Thus,

$$Q_{ijkl} = C_{ijkl}$$

$$= R_{ijkl} - \frac{1}{n-2} \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right] + \frac{R}{(n-1)(n-2)} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].$$

$$= 0.$$

The result is a direct consequence of the above result. □

Based on the preceding results, we have identified three cases for the examination of quasi-conformally flat warped product manifolds. In the remainder of this section, we will delve into the geometry of warped product manifolds and their factor manifolds across these three cases.

The case $A + (n - 2)B = 0$:

Theorem 2. In a non-trivial quasi-conformally flat warped product manifold $N$ where $A + (n - 2)B = 0$, then

1. $N$ is conformally flat and the scalar curvature of $N$ is given by

$$\frac{R}{n-1} = \tilde{n} T \frac{\tilde{n}}{\tilde{n} F}.$$  

2. The Ricci curvature of the base manifold is given by

$$\tilde{R}_{bd} = \frac{1}{\tilde{n}} \left( R - \frac{\tilde{n}(n-2)T}{2nF} \right) \tilde{g}_{bd} + \frac{\tilde{n}}{2nF} (\tilde{n} - 2) T_{bd}.$$  

3. The fiber manifold is of constant curvature.

Proof. It is worth noting that $A$ and $B$ are not simultaneously zero. In this scenario, the warped product manifold is conformally flat, and

$$R_{ijkl} = \frac{1}{n-2} \left[ g_{ik} R_{jl} - g_{il} R_{jk} + g_{jl} R_{ik} - g_{jk} R_{il} \right]$$

$$- \frac{R}{(n-1)(n-2)} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].$$  

(23)

Employing the Ricci tensor equations on warped product manifolds implies

$$R_{abcd} = \tilde{R}_{abcd}$$

$$= \frac{1}{n-2} \left[ g_{ac} \tilde{R}_{bd} - g_{ad} \tilde{R}_{bc} + g_{bd} \tilde{R}_{ac} - g_{bc} \tilde{R}_{ad} \right]$$

$$+ \frac{\tilde{n}}{2(n-2)F} \left[ g_{ac} T_{bd} - g_{ad} T_{bc} + g_{bd} T_{ac} - g_{bc} T_{ad} \right]$$

$$- \frac{R}{(n-1)(n-2)} \left[ g_{ac} \tilde{g}_{bd} - g_{ad} \tilde{g}_{bc} \right].$$

A contraction over $a$ and $c$ infers

$$R_{bd} = \frac{\tilde{n} - 2}{n-2} R_{bd} + \frac{1}{n-2} \left( R - \frac{n-1}{n-1} R \right) \tilde{g}_{bd}$$

$$+ \frac{\tilde{n}}{2(n-2)F} \left[ (\tilde{n} - 2) T_{bd} + \tilde{g}_{bd} T \right].$$
A simple form of the above equation is

\[ R_{bd} = \frac{1}{\hat{n}} \left( R - \frac{\hat{n}(n-2)T}{2\hat{n}F} \right) \delta_{bd} + \frac{\hat{n}}{2\hat{n}F} (n-2)T_{bd}. \]

One more time, a contraction over the indices \( a \) and \( d \) implies

\[ \frac{R}{n-1} = \frac{\hat{n}T}{\hat{n}F}. \]

Back substitution in the equation of the Ricci tensor gives

\[ R_{bd} = \frac{1}{\hat{n}} \left( R - \frac{(n-2)R}{2(n-1)} \right) \delta_{bd} + \frac{\hat{n}}{2\hat{n}F} (n-2)T_{bd}. \]

Now, we consider the next case as

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{n-2} \left[ \delta_{\alpha\gamma} R_{\beta\delta} - \delta_{\alpha\delta} R_{\beta\gamma} + \delta_{\beta\delta} R_{\alpha\gamma} - \delta_{\beta\gamma} R_{\alpha\delta} \right] \]

\[ - \frac{R}{(n-1)(n-2)} \left[ \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} \right]. \]

Therefore, utilizing the identities of the Riemann tensor on the warped product manifold infers

\[ T_{ab} \delta_{\alpha\beta} = \frac{2}{n-2} \left[ F \delta_{\alpha\beta} R_{ab} + \delta_{\alpha\beta} R_{ab} \right] - \frac{2RF}{(n-1)(n-2)} \delta_{\alpha\beta} \delta_{ab}. \]

A simple form of the above equation is deduced by contraction as

\[ 2\hat{n} \delta_{\alpha\beta} = \left( (n-2)T - 2F \hat{R} + 4\hat{n}T + \hat{n} - 1 \right) \delta_{\alpha\beta}. \]

That is, the fiber manifold is Einstein. The next case is

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{n-2} \left[ \delta_{\alpha\gamma} R_{\beta\delta} - \delta_{\alpha\delta} R_{\beta\gamma} + \delta_{\beta\delta} R_{\alpha\gamma} - \delta_{\beta\gamma} R_{\alpha\delta} \right] \]

\[ - \frac{R}{(n-1)(n-2)} \left[ \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} \right]. \]

The fiber manifold is Einstein and consequently

\[ R_{\alpha\beta} = \left( \frac{\hat{R}}{\hat{n}} - \frac{1}{2} \left( T + \frac{\hat{n} - 1}{2F} \Delta F \right) \right) \delta_{\alpha\beta} = H \delta_{\alpha\beta}. \]

where \( H = \frac{\hat{R}}{\hat{n}} - \frac{1}{2} \left( T + \frac{\hat{n} - 1}{2F} \Delta F \right) \). Thus, we have

\[ R_{\alpha\beta\gamma\delta} = \frac{F}{n-2} \left( 2H - \frac{FR}{n-1} \right) \left[ \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} \right]. \]
The use of the identity
\[
R_{a'b'c'd'} = F \tilde{R}_{a'b'c'd'} - \frac{1}{4} \nabla F \tilde{G}_{a'b'c'd'},
\]
infers that the fiber manifold is of constant curvature. \(\square\)

Conformally flat manifolds belong to a category of Riemannian manifolds where, via a conformal transformation, the metric tensor can be altered to resemble a constant multiple of the Euclidean metric tensor at each point.

The following outcomes directly stem from the aforementioned result. The first result gives a good characterization of a class of warped product manifolds of constant scalar curvature as follows.

**Corollary 2.** In a non-trivial quasi-conformally flat warped product manifold \(N\) where \(A + (n - 2)B = 0\), \(N\) has constant (resp. zero) scalar curvature if and only if \(T\) is constant (resp. zero).

**Corollary 3.** In a non-trivial quasi-conformally flat warped product manifold \(N\) where \(A + (n - 2)B = 0\), the Ricci curvature of the base manifold is

1. Einstein, given that \(T_{ab}\) is proportional to the metric tensor;
2. Quasi-Einstein manifold, given that \(T_{ab} = \lambda u^a u^b\) for some constant \(\lambda\) and a 1-form \(u\).

**The case \(A \neq 0\) and \(A \neq -(n-2)B\):**

**Theorem 3.** In a non-trivial quasi-conformally flat warped product manifold where \(A \neq 0\) and \(A \neq -(n-2)B\), the factor manifolds are of constant curvature and

\[
\begin{align*}
R_{a'b'c'd'} &= \left( \frac{FR}{n(n-1)} + \frac{1}{4F} \nabla F \right) G_{a'b'c'd'}, \\
T_{ab} &= -\frac{2FR}{n(n-1)} \delta_{ab}, \\
R_{abcd} &= \frac{R}{n(n-1)} G_{abcd}
\end{align*}
\]

The sectional curvatures of the factor manifolds are

\[
\begin{align*}
\kappa &= \frac{\tilde{R}}{n(n-1)} = \frac{R}{n(n-1)} = \kappa, \\
\tilde{\kappa} &= \frac{\tilde{R}}{n(n-1)} = FK + \frac{1}{4F} \Delta F, \\
T &= -2nFK.
\end{align*}
\]

**Proof.** The quasi-conformally flat manifolds where \(A \neq 0\) and \(A \neq -(n-2)B\) are of constant curvature. In this case, the warped product manifold satisfies

\[
\begin{align*}
R_{ijkl} &= \frac{R}{n(n-1)} G_{ijkl} \\
R_{ij} &= \frac{R}{n} \delta_{ij}
\end{align*}
\]
Here, \( G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl} \). The local components of the Riemannian curvature tensor \( R_{ijkl} \) become

\[
\begin{align*}
R_{n(n-1)} G_{\alpha\beta\gamma\delta} &= \frac{1}{4} \frac{\Delta F}{F} G_{\alpha\beta\gamma\delta} , \\
T_{ab} &= -\frac{2}{n(n-1)} \Delta g_{ab} , \\
\bar{R}_{abcd} &= \frac{R}{n(n-1)} G_{abcd} .
\end{align*}
\]

The definition of the tensor \( G \) infers

\[
\begin{align*}
\bar{R}_{\alpha\beta\gamma\delta} &= \left( \frac{FR}{n(n-1)} + \frac{1}{4F} \Delta F \right) G_{\alpha\beta\gamma\delta} , \\
T_{ab} &= -\frac{2FR}{n(n-1)} g_{ab} , \\
\bar{R}_{abcd} &= \frac{R}{n(n-1)} G_{abcd} .
\end{align*}
\]

The consequences of these equations are

\[
\begin{align*}
\hat{\kappa} &= \frac{\bar{R}}{\bar{n}(\bar{n}-1)} = \frac{R}{n(n-1)} = \kappa , \\
\bar{\kappa} &= \frac{\bar{R}}{\bar{n}(\bar{n}-1)} = F \kappa + \frac{1}{4F} \Delta F , \\
T &= -2\bar{n} F \kappa .
\end{align*}
\]

The proof is complete. \( \square \)

**Corollary 4.** In a non-trivial quasi-conformally flat warped product manifold where \( A \neq 0 \), \( A \neq (n-2)B \), and \( T = 0 \), the warped product and the base manifolds are flat and the fiber manifold has constant sectional curvature \( \frac{1}{4F} \Delta F \).

A significant body of research has been dedicated to examining eigenvalues and their interplay with manifold geometry. This research focuses on exploring the broader spectrum of upper and lower bounds associated with these eigenvalues. Additionally, it delves into a detailed investigation into the asymptotic behavior of eigenvalues within manifold structures.

**Corollary 5.** In a non-trivial quasi-conformally flat warped product manifold where \( A \neq 0 \), \( A \neq -(n-2)B \), and \( T = 0 \), the warping function is an eigenfunction of the Laplacian operator of the base manifold with eigenvalue \( 4 \tilde{\kappa} \).

**The case \( A = 0 \) and \( A \neq (n-2)B \):** In the current case, the warped product manifold is Einstein and, consequently, the \( R_{ij} \) of the warped product \( N \) are:

\[
\begin{align*}
\bar{R}_{ab} &= \frac{R}{\bar{n}} \bar{g}_{ab} + \frac{\bar{n}}{2F} T_{ab} , \\
\bar{R}_{\alpha\beta} &= \left( \frac{R}{\bar{n}} + \frac{1}{2} \left[ T + \frac{\bar{n} - 1}{2F} \Delta F \right] \right) \bar{g}_{\alpha\beta} .
\end{align*}
\]
These equations imply
\[
\begin{align*}
\bar{R} &= \frac{\bar{n}}{n} R + \frac{\bar{n}}{2F} T, \quad (37) \\
\bar{\bar{\bar{n}}} &= \frac{\bar{n} R}{n} + \frac{1}{2} \frac{\bar{n} - 1}{4F} \nabla F. \quad (38)
\end{align*}
\]

4. Quasi-Conformally Symmetric Manifolds

The covariant derivative of the above equation is
\[
\nabla_m Q_{ijkl} = A \nabla_m R_{ijkl} + B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right] - \frac{\nabla_m R}{n} \left( \frac{A}{n-1} + 2B \right) \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].
\]

Let us consider quasi-conformally flat manifolds. A manifold \( N \) is referred to as quasi-conformally symmetric if its quasi-conformal curvature tensor \( Q \) is symmetric, meaning that:
\[
\nabla_m Q_{ijkl} = 0.
\]

**Theorem 4.** In a non-trivial quasi-conformally symmetric manifold \( N \), the scalar curvature is constant. Thus, we have the following cases:

1. If \( A = 0 \), then \( N \) is Ricci-symmetric.
2. If \( A = (n-2)B \), then \( N \) is Cartan-symmetric.
3. If \( A \notin \{0,(n-2)B\} \), then the Weyl tensor is harmonic, the Ricci tensor is of Codazzi type, the Riemann tensor is divergence-free, and
\[
\nabla_m R_{ijkl} = -\frac{1}{n-2} \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right].
\]

**Proof.** Performing the covariant derivative on both sides of the above equation leads to the following
\[
\nabla_m Q_{ijkl} = A \nabla_m R_{ijkl} + B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right] - \frac{\nabla_m R}{n} \left( \frac{A}{n-1} + 2B \right) \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].
\]

Given that the manifold \( N \) is quasi-conformally symmetric, then
\[
0 = \nabla_m Q_{ijkl}
\]
\[
0 = A \nabla_m R_{ijkl} + B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right] - \frac{\nabla_m R}{n} \left( \frac{A}{n-1} + 2B \right) \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].
\]

Therefore, we can obtain the covariant derivative of the Riemann tensor as follows:
\[
A \nabla_m R_{ijkl} = -B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right] + \frac{\nabla_m R}{n} \left( \frac{A}{n-1} + 2B \right) \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].
\]
A contraction over the indices $i$ and $l$ implies

\[
A \nabla_m R_{jk} = -B \left[ \nabla_m R_{jk} - n \nabla_m R_{jk} + \nabla_m R_{jk} - g_{jk} \nabla_m R \right] \\
+ \frac{n}{n-1} \left( A + 2B \right) [g_{jk} - n g_{jk}]
\]

\[
= (n-2) B \nabla_m R_{jk} + B g_{jk} \nabla_m R - \frac{1}{n} (A + 2(n-1)B) g_{jk} \nabla_m R. \tag{40}
\]

Once more, a direct simplification results in:

\[
n(A - (n-2)B) \nabla_m R_{jk} = -(A + (n-2)B) g_{jk} \nabla_m R. \tag{41}
\]

A contraction over $j$ and $k$ implies

\[
2A \nabla_m R = 0.
\]

Let us consider the case where $\nabla_k R \neq 0$. In this case, $A = 0$ and, consequently, Equation (39) reveals

\[
0 = g_{lk} \nabla_m R_{jl} - g_{ll} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} + \frac{2n}{n-2} \nabla_m R \left[ g_{lk} g_{jl} - g_{ll} g_{jk} \right]. \tag{42}
\]

A contraction of the above identity reveals

\[
\nabla_m R_{jl} = - \frac{3n-2}{n(n-2)} g_{jl} \nabla_m R.
\]

A second contraction of the above equation implies that $\nabla_k R = 0$, which is a contradiction. Thus,

\[
A \nabla_m R_{ijkl} = -B \left[ g_{lk} \nabla_m R_{jl} - g_{ll} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right]. \tag{43}
\]

The case $A = 0$: In this case, $B$ cannot also be zero. Equation (43) becomes

\[
 g_{lk} \nabla_m R_{jl} - g_{ll} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} = 0
\]

and so $\nabla_m R_{ik} = 0$.

The case $A \neq 0$: Consequently, Equation (43) becomes

\[
\nabla_m R_{ijkl} = - \frac{B}{A} \left[ g_{lk} \nabla_m R_{jl} - g_{ll} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right].
\]

Substituting back into Equation (40) yields

\[
(A - (n-2)B) \nabla_m R_{jk} = 0.
\]

Therefore, $A = (n-2)B$, or $N$ is Ricci-symmetric. The former case implies

\[
\nabla_m R_{ijkl} = - \frac{1}{n-2} \left[ g_{lk} \nabla_m R_{jl} - g_{ll} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right]. \tag{44}
\]

The manifold $N$ is Ricci-symmetric if and only if $N$ is Cartan-symmetric or locally symmetric. The divergence of the Riemann tensor is given by

\[
\nabla_m R_{ijkl}^m = \nabla_k R_{jl} - \nabla_l R_{jk}.
\]

The contracted second Bianchi identity implies

\[
\nabla_m R_{ijkl}^m = \nabla_k R_{jl} - \nabla_l R_{jk}.
\]
The use of both equations reveals
\[ \nabla_m R_{ijkl}^m = 0. \]

This condition is equivalent to the following conditions: (1) The Ricci curvature is of Codazzi type. (2) The Weyl tensor tensor is harmonic and the scalar curvature is constant. The next case is that \( N \) is Ricci-symmetric, that is, \( \nabla_k R_{jl} = 0. \)

In a Ricci-symmetric manifold, the scalar curvature is constant, leading to \( \nabla_m R = 0. \) Substituting this result back into Equation (39) yields the following implication:
\[ \nabla_m R_{ijkl} = 0. \]

Manifolds characterized by this local property are referred to as locally symmetric or Cartan-symmetric manifolds. The manifold \( N \) is Cartan-symmetric. \[ \square \]

The results of this section are summarized in the following table:

<table>
<thead>
<tr>
<th>( A ) non-trivial quasi-conformally symmetric manifold ( N )</th>
<th>( \nabla_m R = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = 0 )</td>
<td>( N ) is Ricci-symmetric</td>
</tr>
<tr>
<td>( A \neq 0 )</td>
<td>( A = (n - 2)B )</td>
</tr>
<tr>
<td>( A \neq 0 )</td>
<td>( A \neq (n - 2)B )</td>
</tr>
</tbody>
</table>

**Theorem 5.** In a non-trivial quasi-conformally symmetric warped product manifold \( N \), the scalar curvature is constant. Assume that \( A = 0 \). Thus, we have the following:
1. The base manifold is Ricci-symmetric if and only if \( \frac{1}{F} T_{ab} \) is parallel.
2. The fiber manifold is Einstein.

**Proof.** Assume that \( N \) is a quasi-conformally symmetric manifold, that is, \( \nabla_m Q_{ijkl} = 0 \). In this case, the scalar curvature is constant and
\[ A \nabla_m R_{ijkl} = -B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right]. \]

The case \( A = 0 \): The warped product manifold is Ricci-symmetric and so the covariant derivatives on the warped product manifolds are
\[ \nabla_c R_{ab} = \nabla_c \bar{R}_{ab} - \frac{\bar{R}}{2} \nabla_c \left( \frac{1}{F} T_{ab} \right) = 0. \]

The \( \bar{N} \) is Ricci-symmetric if and only if \( \frac{1}{F} T_{ab} \) is parallel. The fiber manifold is also Ricci-symmetric \( \nabla_{\gamma} R_{a\beta} = \nabla_{\gamma} \bar{R}_{a\beta} = 0 \). Moreover, the fiber manifold is Einstein since
\[ 0 = \nabla_{\beta} R_{aa} = -\frac{F_a}{2F} \left( \bar{R}_{a\beta} - \frac{1}{2} \left( T + \frac{\bar{R} - 1}{2F} \Delta F \right) \bar{g}_{a\beta} \right) + \frac{1}{2} F_a \left( \bar{R}_{a} + \frac{\bar{R} - 1}{2F} T_a \right) \bar{g}_{a\beta}. \]

The proof is complete. \[ \square \]

**Theorem 6.** In a non-trivial quasi-conformally symmetric warped product manifold \( N \), the scalar curvature is constant. Assume that \( A \neq 0 \) and \( A = (n - 2)B \). Thus, we have the following:
1. The base manifold is Cartan-symmetric.
2. The fiber manifold is of constant curvature.
3. The tensor \( T_{bc} \) is covariantly constant.
Proof. Assume that \( N \) is a quasi-conformally symmetric manifold, \( A \neq 0 \) and \( A = (n-2)B \). In this case, the scalar curvature is constant, and \( \nabla e R_{ijkl} = 0 \). Thus,

\[
\nabla e R_{abcd} = \nabla e \tilde{R}_{abcd} = 0.
\]

The base manifold is Cartan-symmetric. The next case is

\[
0 = \nabla a R_{a \beta \gamma \delta} = \frac{1}{2} \left( \frac{F_a}{F} \Delta F - \frac{1}{2} \nabla a (\Delta F) \right) \tilde{c}_{a \beta \gamma \delta}.
\]

This result implies that the fiber manifold is of constant sectional curvature. Also, the case

\[
0 = \nabla a R_{abc \delta} = (\nabla a T_{bc}) g_{\alpha \delta}
\]

infers that the tensor \( T_{bc} \) is parallel or covariantly constant. \( \square \)

**Theorem 7.** In a non-trivial quasi-conformally symmetric warped product manifold \( N \), the scalar curvature is constant. Assume that \( A \neq 0 \) and \( A \neq (n-2)B \). Thus, we have the following:

1. The Ricci tensor of the base manifold is of Codazzi type if and only if \( \frac{1}{2} T_{ab} \) is of Codazzi type.
2. The fiber manifold is Cartan-symmetric \( \nabla \lambda \tilde{R}_{a \beta \gamma \delta} = 0 \).
3. The tensor \( T_{bc} \) is covariantly constant.

Proof. Assume that the warped product manifold is a quasi-conformally symmetric manifold, \( A \neq 0 \) and \( A \neq (n-2)B \). In this case, the scalar curvature is constant and the Weyl tensor is harmonic, the Ricci tensor is of Codazzi type, the Riemann tensor is divergence-free, and

\[
\nabla e R_{ijkl} = -\frac{1}{n-2} \left[ g_{ik} \nabla e R_{jl} - g_{il} \nabla e R_{jk} + g_{ji} \nabla e R_{ik} - g_{jk} \nabla e R_{il} \right].
\]

When the Ricci tensor is of Codazzi type, then the use of the covariant derivative of the Ricci tensor

\[
\nabla e R_{ab} = \nabla e \tilde{R}_{ab} - \frac{\tilde{a}}{2} \nabla e \left( \frac{1}{F} T_{ab} \right),
\]

\[
\nabla \gamma R_{a \beta} = \nabla \gamma \tilde{R}_{a \beta},
\]

on the warped product manifold \( N \) yields the Ricci tensor of the fiber manifold, and the Ricci tensor of the base manifold is of Codazzi type if and only if the tensor \( \frac{1}{2} T_{ab} \) is of Codazzi type. The covariant derivative of the Riemann tensor on the fiber manifold is given by

\[
\nabla \lambda R_{a \beta \gamma \delta} = -\frac{1}{n-2} \left[ g_{a \gamma} \nabla \lambda \tilde{R}_{b \delta} - g_{a \delta} \nabla \lambda \tilde{R}_{b \gamma} + g_{b \delta} \nabla \lambda \tilde{R}_{a \gamma} - g_{b \gamma} \nabla \lambda \tilde{R}_{a \delta} \right].
\]

The use of \( \nabla \gamma R_{a \beta} = \nabla \gamma \tilde{R}_{a \beta} \) and \( \nabla \lambda R_{a \beta \gamma \delta} = F \nabla \lambda \tilde{R}_{a \beta \gamma \delta} \) gives

\[
\nabla \lambda \tilde{R}_{a \beta \gamma \delta} = -\frac{1}{n-2} \left[ g_{a \gamma} \nabla \lambda \tilde{R}_{b \delta} - g_{a \delta} \nabla \lambda \tilde{R}_{b \gamma} + g_{b \delta} \nabla \lambda \tilde{R}_{a \gamma} - g_{b \gamma} \nabla \lambda \tilde{R}_{a \delta} \right],
\]

\[
\bar{n} \nabla \lambda \tilde{R}_{b \gamma} = \tilde{g}_{b \gamma} \nabla \lambda \tilde{R}.
\]
A Codazzi-type Ricci tensor has a constant scalar curvature $\bar{R}$ and, consequently, $\bar{\nabla}_i\bar{R}_{\beta\gamma} = 0$. Back substitution in the identity yields $\bar{\nabla}_i\bar{R}_{\alpha\beta\gamma} = 0$. Also

\[
\nabla_e R_{abcd} = -\frac{1}{n-2} \left[ g_{ae} \nabla_e R_{bd} - g_{ad} \nabla_e R_{bc} + g_{bd} \nabla_e R_{ac} - g_{bc} \nabla_e R_{ad} \right].
\]

\[(n-2)\nabla_e R_{abcd} = -g_{ae} \nabla_e R_{bd} + g_{ad} \nabla_e R_{bc} - g_{bd} \nabla_e R_{ac} + g_{bc} \nabla_e R_{ad} + g_{ac} \nabla_e \left( \frac{1}{F} T_{bd} \right) - g_{ad} \nabla_e \left( \frac{1}{F} T_{bc} \right) + g_{bd} \nabla_e \left( \frac{1}{F} T_{ac} \right) - g_{bc} \nabla_e \left( \frac{1}{F} T_{ad} \right) \]

A contraction over the first and last indices implies

\[
\nabla_e R_{bc} = \frac{1}{\bar{n}} g_{bc} \nabla_e \left( R - \frac{1}{\bar{F}} T \right) - \left( \frac{n-2}{\bar{n}} \right) \nabla_e \left( \frac{1}{\bar{F}} T_{bc} \right)
\]

Again, we obtain

\[
\left( 1 - \frac{n}{\bar{n}} \right) \nabla_e \bar{R} = -\left( \frac{\bar{n}}{\bar{n}} + \frac{n-2}{\bar{n}} \right) \nabla_e \left( \frac{1}{\bar{F}} T \right)
\]

Utilizing the above two equations, one obtains the last assertion. $\square$

5. Divergence-Free Quasi-Conformal Curvature Tensor

The covariant derivative of the quasi-conformal curvature tensor is given by

\[
\nabla_m Q_{ijkl} = A \nabla_m R_{ijkl} + B \left[ g_{ik} \nabla_m R_{jl} - g_{il} \nabla_m R_{jk} + g_{jl} \nabla_m R_{ik} - g_{jk} \nabla_m R_{il} \right] - \frac{A}{n-1} \left[ g_{ik} g_{jl} - g_{il} g_{jk} \right].
\]

This identity implies that the divergence of the quasi-conformal curvature tensor $\nabla_i Q^i_{kl}$ is given by

\[
\nabla_i Q^i_{kl} = A \nabla_i R^i_{kl} + B \left[ \nabla_k R_{jl} - \nabla_l R_{jk} + g_{jl} \nabla_k R^i_{ik} - g_{jk} \nabla_l R^i_{il} \right] - \frac{1}{n} \left( \frac{A}{n-1} + 2B \right) \left[ (\nabla_k R)_{jl} - (\nabla_l R)_{jk} \right].
\]

(45)

where $\nabla_i$ denotes the covariant derivative and $Q^i_{kl}$ represents the components of the quasi-conformal curvature tensor.

**Theorem 8.** If the manifold $N$ has a quasi-conformal curvature tensor that is divergence-free, then

1. $N$ has constant scalar curvature: In this case the conformal curvature tensor and the Riemann curvature tensor are both harmonic given that $A + B$ is non-zero.
2. $A + (n-2)B = 0$: In this case the conformal curvature tensor is harmonic.

**Proof.** The divergence of the tensor $Q$ is given by Equation (45). Assume that the quasi-conformal curvature tensor is divergence-free; then,

\[
0 = A \nabla_i R^i_{kl} + B \left[ \nabla_k R_{jl} - \nabla_l R_{jk} + g_{jl} \nabla_k R^i_{ik} - g_{jk} \nabla_l R^i_{il} \right] - \frac{1}{n} \left( \frac{A}{n-1} + 2B \right) \left[ (\nabla_k R)_{jl} - (\nabla_l R)_{jk} \right].
\]

(46)

Using the second Bianchi identity, we obtain

\[
0 = (A + B) \left( \nabla_k R_{jl} - \nabla_l R_{jk} \right) + \left( \frac{n-4}{2n} \frac{A}{n(n-1)} \right) \left[ (\nabla_k R)_{jl} - (\nabla_l R)_{jk} \right].
\]

(47)
By multiplying both sides by $g^{il}$, one obtains
\[ 0 = (A + (n - 2)B) \nabla_i R. \]
Thus, the scalar curvature is constant or $A + (n - 2)B = 0$. If the scalar curvature is constant, then
\[ 0 = (A + B) \left( \nabla_i R_{jl} - \nabla_j R_{ik} \right). \]
As a consequence of this discussion, the Ricci tensor is of Codazzi type given that $A + B$ is non-zero. Therefore, the Riemann tensor is harmonic given that $A + B$ is non-zero. Note that in this case, the conformal curvature tensor is also harmonic since the scalar curvature is constant and the Riemann tensor is harmonic.

The second possibility is $A + (n - 2)B = 0$. In such a scenario, the expression simplifies,
\[ 0 = -(n - 3)B \left[ (\nabla_i R_{jl} - \nabla_j R_{ik}) - \frac{1}{2n - 2} (\nabla_k R)g_{jl} - (\nabla_l R)g_{jk} \right]. \]
The case $B = 0$ is trivial and, consequently, for manifolds with $n \geq 4$,
\[ \left( \nabla_i R_{jl} - \nabla_j R_{ik} \right) - \frac{1}{2n - 2} (\nabla_k R)g_{jl} - (\nabla_l R)g_{jk} = 0. \]
The divergence of the Weyl conformal curvature tensor is given as
\[ \nabla C^i_{jkl} = \frac{n - 3}{n - 2} \left( \nabla_i R_{jl} - \nabla_j R_{ik} - \frac{1}{2n - 2} (\nabla_k R)g_{jl} - (\nabla_l R)g_{jk} \right) = 0. \]
Thus, the Weyl conformal curvature tensor is divergence-free or harmonic. \(\square\)

Manifolds that satisfy this last identity are referred to as Einstein-like manifolds of type $I + B$ where the tensor $R_{jl} - \frac{R}{2n - 2}g_{jl}$ acts as a Codazzi tensor.

**Theorem 9.** A manifold $N$ admits a divergence-free quasi-conformal curvature tensor if its Ricci tensor is of Codazzi type. Conversely, $N$ has a Ricci tensor of Codazzi type if the quasi-conformal curvature tensor is divergence-free and $A \notin \{ -B, -(n - 2)B \}$.

**Proof.** The divergence of the quasi-conformal curvature tensor is given as
\[ \nabla_i Q^i_{jkl} = A \nabla_i R^i_{jkl} + B \left[ \nabla_i R_{jl} - \nabla_j R_{ik} + g_{jl} \nabla_i R^i_k - g_{jk} \nabla_i R^i_l \right] - \frac{1}{n} \left( \frac{A}{n - 1} + 2B \right) [ (\nabla_k R)g_{jl} - (\nabla_l R)g_{jk} ] . \] (48)

Now, assume that the Ricci tensor of Codazzi type, that is, $\nabla_i R_{jl} = \nabla_j R_{ik}$ and, consequently, the scalar curvature is zero, that is, $\nabla_i R = 0$. Using the second Bianchi identity, the divergence of the Riemann tensor is given by
\[ \nabla_i R^i_{jkl} = \nabla_k R_{jl} - \nabla_j R_{ik} = 0 \]
Thus, $\nabla_i Q^i_{jkl} = 0$. Conversely, suppose that the manifold $N$ admits a divergence-free quasi-conformal curvature tensor. In this case, $\nabla_i Q^i_{jkl} = 0$ and, consequently,
\[ 0 = (A + B) \left[ \nabla_k R_{jl} - \nabla_j R_{ik} \right] + \left( \frac{n - 4}{2n} B - \frac{A}{n(n - 1)} \right) [ g_{jl} \nabla_k R - g_{jk} \nabla_l R ] \] (49)
The divergence of the Riemann tensor of the base manifold is given by

\[ 0 = (A + B) \left[ \nabla_b R^b_{\alpha \beta \gamma} - \nabla_{\alpha \beta \gamma} R \right]. \]

This means that the Ricci tensor is of Codazzi type given that \( A + B \neq 0 \).

We now study these results in warped product manifolds. Let us find the divergence of the quasi-conformal curvature tensor on factor manifolds using the \( \nabla_c Q_{abc} = 0 \).

**Theorem 10.** Assume that the quasi-conformal curvature tensor of a warped product manifold \( N \) is divergence-free. The divergence of the Riemann tensor of the base manifold is given by

\[ (A + B) \nabla_b R^b_{\alpha \beta \gamma} = F \left( \frac{A}{n(n-1)} - \frac{n-4}{2n} B \right) \left[ g_{\alpha \beta} \nabla_b R - g_{\alpha \beta} \nabla \right]. \]

The divergence of the Riemann tensor of the base manifold is given by

\[ (A + B) \nabla_b R^b_{\alpha \beta \gamma} = F \left( \frac{A}{n(n-1)} - \frac{n-4}{2n} B \right) \left[ g_{\alpha \beta} \nabla R - g_{\alpha \beta} \nabla \right]. \]

**Proof.** The divergence of the quasi-conformal curvature tensor implies

\[ (A + B) \left[ \nabla_b R^b_{\alpha \beta \gamma} - \nabla_{\alpha \beta \gamma} R \right] = - \left( \frac{n-4}{2n} B - \frac{A}{n(n-1)} \right) \left[ g_{\alpha \beta} \nabla R - g_{\alpha \beta} \nabla \right]. \]

Firstly, on the base manifold, it leads to

\[ (A + B)(\nabla_b R^b_{\alpha \beta \gamma} - \nabla_{\alpha \beta \gamma} R) = \frac{A}{n(n-1)} - \frac{n-4}{2n} B \left[ g_{\alpha \beta} \nabla R - g_{\alpha \beta} \nabla \right]. \]

Using the geometry of the warped product manifolds, it is

\[ (A + B)(\nabla_b R^b_{\alpha \beta \gamma} - \nabla_{\alpha \beta \gamma} R) = \frac{A}{n(n-1)} - \frac{n-4}{2n} B \left[ g_{\alpha \beta} \nabla R - g_{\alpha \beta} \nabla \right] + \frac{\bar{n}}{2} (A + B) \left[ \nabla_{\alpha \beta} \left( \frac{1}{F} T_{\alpha \beta} \right) + \nabla_\gamma \left( \frac{1}{F} T_{\alpha \beta} \right) \right]. \]

The second Bianchi identity yields

\[ (A + B)\nabla_b R^b_{\alpha \beta \gamma} = \frac{A}{n(n-1)} - \frac{n-4}{2n} B \left[ g_{\alpha \beta} \nabla R - g_{\alpha \beta} \nabla \right] + \frac{\bar{n}}{2} (A + B) \left[ \nabla_{\alpha \beta} \left( \frac{1}{F} T_{\alpha \beta} \right) + \nabla_\gamma \left( \frac{1}{F} T_{\alpha \beta} \right) \right]. \]

However, on the fiber manifold, we have

\[ (A + B)(\nabla_\beta R^\beta_{\alpha \gamma} - \nabla_{\alpha \gamma} R) = \left( \frac{n-4}{2n} B - \frac{A}{n(n-1)} \right) \left[ g_{\alpha \gamma} \nabla_\beta R - g_{\alpha \beta} \nabla_\gamma R \right]. \]
Again, with the use of the geometry of a warped product manifold, one obtains

\[(A + B) \left( \nabla_{\beta} R_{\alpha \gamma} - \nabla_{\gamma} R_{\alpha \beta} \right) = F \left( \frac{A}{n(n - 1)} - \frac{n - 4}{2n} B \right) \left[ \theta_{a \beta} \nabla_{\gamma} R - \theta_{a \beta} \nabla_{\gamma} R \right].\]

The second Bianchi identity on the fiber manifold infers

\[(A + B) \nabla_{\lambda} R_{\alpha \beta \gamma} = F \left( \frac{A}{n(n - 1)} - \frac{n - 4}{2n} B \right) \left[ \theta_{a \beta} \nabla_{\gamma} R - \theta_{a \beta} \nabla_{\gamma} R \right].\]

Thus, we have the result. \(\square\)

This result may lead to a set of interesting corollaries.

**Corollary 6.** Assume that the quasi-conformal curvature tensor of a warped product manifold \(N\) is divergence-free where \(A + B \neq 0\). The Riemann curvature tensor of the base manifold is harmonic if \(\nabla_{\gamma} R = 0\) and \(\frac{1}{2} T_{\alpha c}^{\lambda} \) is of Codazzi type. Also, the Riemann tensor of the fiber manifold is harmonic if \(\nabla_{\beta} R = 0\).


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**References**


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