

On the Extremal Weighted Mostar Index of Bicyclic Graphs

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Abstract: Let G be a simple connected graph with edge set $E(G)$ and vertex set $V(G)$. The weighted Mostar index of a graph G is defined as $w^+ Mo(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v)) |n_u(e) - n_v(e)|$, where $n_u(e)$ denotes the number of vertices closer to u than to v for an edge uv in G . In this paper, we obtain the upper bound and lower bound of the weighted Mostar index among all bicyclic graphs and characterize the corresponding extremal graphs.

Keywords: bicyclic graph; the weighted Mostar index; extremal graph

MSC: 05C09; 05C35; 05C92; 92E10

1. Introduction

All graphs considered in this paper are finite, undirected, connected and simple. We refer the readers to [1] for the terminology and notations. Let G be a connected graph with edge set $E(G)$ and vertex set $V(G)$. Call u a *pendant vertex* in G , if $d_G(u) = 1$. Call uv a *pendant edge* in G , if $d_G(u) = 1$ or $d_G(v) = 1$. A connected graph is called a *unicyclic graph* if the number of edges equals the number of vertices. A connected graph is called a *bicyclic graph* if the number of edges is 1 more than the number of vertices. A cycle with k edges is called a k -cycle. For $u, v \in V(G)$, $d_G(u, v)$ denotes the distance between u and v in G , namely the length of the shortest path connecting u and v . For $e = uv \in E(G)$, the distance between the edge e and the vertex x , denoted by $d_G(e, x)$, is defined as $d_G(e, x) = \min\{d_G(u, x), d_G(v, x)\}$. For $e_1, e_2 \in E(G)$, the distance between e_1 and e_2 , denoted by $d_G(e_1, e_2)$, is defined as the least distance between one vertex in e_1 and the other vertex in e_2 . For $e = uv \in E(G)$, $n_u(e)$ denotes the number of vertices closer to u than to v for an edge uv in G .

The Mostar index [2] of a graph G is defined as

$$Mo(G) = \sum_{e=uv \in E(G)} |n_u(e) - n_v(e)|.$$

Some properties and applications of the Mostar index have been reported in [3–13]. The weighted Mostar index [14] of a graph G is defined as

$$w^+ Mo(G) = \sum_{e=uv \in E(G)} (d_G(u) + d_G(v)) |n_u(e) - n_v(e)|.$$

This topological index has been extensively studied in the mathematical literature. Kanddan [15] computed the weighted Mostar index for a conical and generalized gear graph. Asmat [16] obtained the upper bound of the weighted Mostar index for trees with a given diameter and the corresponding extremal graph. Zhen [17] obtained the extremal values of the weighted Mostar index among trees and unicyclic graphs with a given order.

In this paper, we obtain the upper bound and lower bound of the weighted Mostar index among all bicyclic graphs and characterize the corresponding extremal graphs.



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Theorem 1. Let G be a bicyclic graph with order $n \geq 12$, then

$$w^+ Mo(G) \geq \begin{cases} 4n + 6, & n \text{ is even,} \\ 4n + 10, & n \text{ is odd,} \end{cases}$$

with the equality if and only if $G \cong \Theta_{n-3,2,2}$ (See Figure 1).

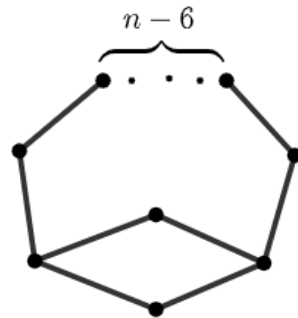


Figure 1. Graph $\Theta_{n-3,2,2}$.

Theorem 2. Let G be a bicyclic graph with order $n \geq 8$, then

$$w^+ Mo(G) \leq n^3 - 3n^2 + 2n - 4,$$

with the equality if and only if $G \cong B_2$ (See Figure 2).

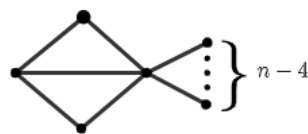


Figure 2. Graph B_2 .

Theorem 1 is proved in Section 3. The proof mainly refers to the proof of the lower bound of Mostar index for bicyclic graphs in [12]. We extend this method to the weighted Mostar index and improve this method. Theorem 2 is proved in Section 4. The method of proving the upper bound of Mostar index for bicyclic graphs by defining the deficit of edges in [12] is not applicable here. We use a method different from that in [12] to find the upper bound of weighted Mostar index among all bicyclic graphs.

2. Preliminaries

For convenience, we denote by $\phi_G(e) = |n_u(e) - n_v(e)|$ the contribution of the edge $e = uv$ to the Mostar index of graph G , and by $\Phi_G(e) = (d_G(u) + d_G(v))|n_u(e) - n_v(e)|$ the contribution of the edge $e = uv$ to the weighted Mostar index of graph G .

Let G be a unicyclic graph with order n . Let C be the unique cycle in G and $|C| = k$. For $w \in V(C)$, denote by T_w the component of $(G - \{e_1, e_2\})$ which contains w , where e_1 and e_2 are the two edges incident to w in C . Let e be an edge in C . If k is even, there exists an edge $\bar{e}$ in C such that $d(e, \bar{e}) = \frac{k}{2} - 1$, then $\bar{e}$ is called the *parallel edge* of e . If k is odd, there exists a vertex w_e in C such that $d(e, w_e) = \frac{k-1}{2}$, then w_e is called the *symmetric vertex* of e . Conversely, if $w_e = v$, define $e = e_v$, then say that e is symmetric about v .

Theta graph $\Theta_{a,b,c}$ consists of two vertices, v_1 and v_2 , which are connected by three internally disjoint paths, P_a , P_b , and P_c of lengths a , b , and c , respectively. We assume that $a \geq b \geq c$.

Lemma 1. ([2]) Let G be a connected graph with order n , and $e = uv \in E(G)$, then

$$\phi_G(e) = |n_u(e) - n_v(e)| \leq n - 2,$$

with the equality if and only if e is a pendant edge in G .

Lemma 2. Let G be a connected graph, and $e = uv$ be a cut edge but not a pendant edge in G . Let G' be the graph obtained from G by contracting the edge $e = uv$ into a vertex u , and adding a pendant edge uv to this vertex (see Figure 3). Then $w^+Mo(G') > w^+Mo(G)$.

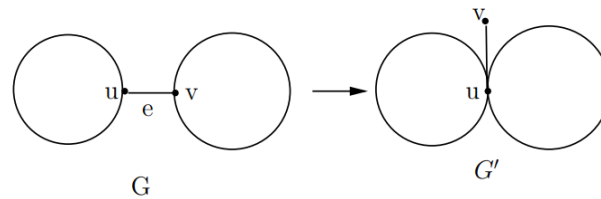


Figure 3. Transformation of cut edge.

Proof. It is clear that $d_{G'}(u) + d_{G'}(v) = d_G(u) + d_G(v)$. By Lemma 1, we have $\phi_{G'}(uv) > \phi_G(uv)$, then $\Phi_{G'}(uv) > \Phi_G(uv)$. The sum of the degrees of the two end vertices of each edge (except $e = uv$) incident to u or v becomes larger, but the contribution to the Mostar index stays unchanged, so the contribution to the weighted Mostar index becomes larger. Furthermore, for the remaining edges, the contributions to the weighted Mostar index stay unchanged. Thus, $w^+Mo(G') > w^+Mo(G)$. $\square$

Lemma 3. ([11]) Let G be a unicyclic graph with order n , and C be the unique cycle in G .

(1) If $|C|$ is even and $e \in E(C)$, then

$$\phi_G(e) = \phi_G(\bar{e}) = ||G_1| - |G_2|| = n - 2 \min\{|G_1|, |G_2|\},$$

where G_1 and G_2 are the two components of $G - \{e, \bar{e}\}$.

(2) If $|C|$ is odd and $e \in E(C)$, then

$$\phi_G(e) = ||G_1| - |G_2|| = n - |T_{w_e}| - 2 \min\{|G_1|, |G_2|\},$$

where G_1 and G_2 are the two components of $G - \{e\} - T_{w_e}$.

3. Proof of Theorem 1

In this section, we start by calculating the weighted Mostar index of graph $\Theta_{n-3,2,2}$ to find the bicyclic graph with the minimum weighted Mostar index.

Lemma 4. If n is odd, then $w^+Mo(\Theta_{n-3,2,2}) = 4n + 10$; if n is even, then $w^+Mo(\Theta_{n-3,2,2}) = 4n + 6$.

Proof. If n is odd, then $a = n - 3$ is even. For each edge e of $\Theta_{n-3,2,2}$, we have $\phi(e) = 1$. Thus, $w^+Mo(\Theta_{n-3,2,2}) = 30 + 4(n - 5) = 4n + 10$. If n is even, then $a = n - 3$ is odd. The central edge on P_a has a contribution of 0 and each remaining edge of $\Theta_{n-3,2,2}$ has a contribution of 1 to $Mo(\Theta_{n-3,2,2})$. Thus, $w^+Mo(\Theta_{n-3,2,2}) = 30 + 4(n - 6) = 4n + 6$. $\square$

A path $P := v_0v_1, \dots, v_r$ in a graph G is called a *pendant path* if one of the two vertices of v_0 and v_r is a pendant vertex and the other one has degree at least 3, and each remaining vertex of P has degree 2.

Lemma 5. Let G be a bicyclic graph with order $n \geq 12$. If G has at least two pendant paths or one pendant path with order at least 3, then $w^+Mo(G) > w^+Mo(\Theta_{n-3,2,2})$.

Proof. Let G be a bicyclic graph with order $n \geq 12$, if G has at least two pendant paths, then $w^+Mo(G) \geq 2 \times 3(n-2) > 4n+10$. If G has one pendant path with order at least 3, then $w^+Mo(G) \geq 3(n-2) + 4(n-4) > 4n+10$. $\square$

Lemma 6. Let G be a bicyclic graph with order $n \geq 12$. If G has no pendant path or only one pendant path with order 2, then $w^+Mo(G) \geq w^+Mo(\Theta_{n-3,2,2})$, with the equality if and only if $G \cong \Theta_{n-3,2,2}$.

Proof. Let G be a bicyclic graph with order $n \geq 12$. If G has no pendant path or only one pendant path with order 2, then G consists of a graph G' which has no pendant edge, and possibly an extra vertex attached to G' by a single edge. Hence, $|V(G)| = |V(G')|$ or $|V(G)| = |V(G')| + 1$.

Suppose that G' has exactly two cycles, C_a and C_b , with lengths a and b , respectively. Let u_1 and u_2 be the two terminal vertices on the path of length t connecting the two cycles, where u_1 is in C_a and u_2 is in C_b . Clearly, $|V(G')| = a + b + t - 1$. If $t = 0$, then $u_1 = u_2$. Let e_1, e_2 be the edges incident with u_1 in C_a , and let f_1, f_2 be the edges incident with u_2 in C_b . We distinguish three cases.

Case 1: If $G \cong G'$, then $n = a + b + t - 1$, $\phi_G(e_1) = \phi_G(e_2) = n - a$, $\phi_G(f_1) = \phi_G(f_2) = n - b$. For each of these four edges, the sum of degrees of the end vertices is at least 5, so we have

$$w^+Mo(G) \geq 10(n-a) + 10(n-b) = 10n + 10t - 10 \geq 10n - 10 > 4n + 10.$$

Case 2: If $G \not\cong G'$, and the pendant edge is attached to any vertex on the path, then $n = a + b + t$, $\phi_G(e_1) = \phi_G(e_2) = n - a$, $\phi_G(f_1) = \phi_G(f_2) = n - b$. We have

$$w^+Mo(G) \geq 10(n-a) + 10(n-b) = 10n + 10t \geq 10n > 4n + 10.$$

Case 3: If $G \not\cong G'$, and the pendant edge is attached to any vertex on one cycle, say C_a , then $n = a + b + t$, $\phi_G(e_1), \phi_G(e_2) \geq n - a - 2$, $\phi_G(f_1) = \phi_G(f_2) = n - b$. We have

$$w^+Mo(G) \geq 10(n-a-2) + 10(n-b) = 10n + 10t - 20 \geq 10n - 20 > 4n + 10.$$

Suppose that $G' \cong \Theta_{a,b,c}$, let P_a, P_b, P_c be the three paths of $\Theta_{a,b,c}$, and v_1, v_2 be the two vertices of degree 3 of $\Theta_{a,b,c}$. Let e_1, e_2, e_3 be the edges incident with v_1 , where e_1 is in P_a , e_2 is in P_b , and e_3 is in P_c . $\phi[\phi(e_1), \phi(e_2), \phi(e_3)]$ denotes the contribution of the three edges to $Mo(G')$. By symmetry, the edges incident with v_2 have the same contribution.

In $\Theta_{a,b,c}$, for $e_1 = v_1v_3$, the number of vertices closer to v_1 than to v_3 is equal to the number of vertices closer to v_3 than to v_1 on P_a and P_c . The remaining $b-1$ vertices on P_b are all closer to v_1 than to v_3 , then $\phi(e_1) = b-1$.

For $e_3 = v_1v_4$, the number of vertices closer to v_1 than to v_4 is equal to the number of vertices closer to v_4 than to v_1 on P_a and P_c . If $b+c$ is even, among the remaining $b-1$ vertices on P_b , $\frac{b+c}{2}-1$ vertices are closer to v_1 than to v_4 , and $\frac{b-c}{2}$ vertices are closer to v_4 than to v_1 , then $\phi(e_3) = c-1$. Similarly, if $b+c$ is odd, we can also obtain $\phi(e_3) = c-1$.

For $e_2 = v_1v_5$, the number of vertices closer to v_1 than to v_5 is equal to the number of vertices closer to v_5 than to v_1 on P_b and P_c . If $b \geq c+2$, then $a-1$ vertices on P_a are all closer to v_1 than to v_5 , thus $\phi(e_2) = a-1$. If $b = c+1$, let u_i be any vertex on P_a , where i denotes the length of the path from v_2 to u_i on P_a , and $1 \leq i \leq a-1$. Note that if $a-i < c+i$, then $d(u_i, v_1) < d(u_i, v_5)$ and if $a-i \geq c+i$, then $d(u_i, v_1) = d(u_i, v_5)$, so among the $a-1$ vertices on P_a , $a-1 - \lfloor \frac{a-c}{2} \rfloor$ vertices are closer to v_1 than to v_5 , and no vertex is closer to v_5 than to v_1 , thus $\phi(e_2) = a-1 - \lfloor \frac{a-c}{2} \rfloor = \lfloor \frac{a+c-1}{2} \rfloor$. If $b = c$, then $\phi(e_2) = \phi(e_3) = c-1$.

Case 1: $a \geq b \geq c + 2$. In $\Theta_{a,b,c}$, we have $\phi[b - 1, a - 1, c - 1]$. If $G \cong G'$, then $n = a + b + c - 1$. Considering the contributions of the six edges incident to v_1 and v_2 in $\Theta_{a,b,c}$ to $w^+Mo(G)$, we obtain

$$w^+Mo(G) \geq 10(b - 1) + 10(a - 1) + 10(c - 1) = 10n - 20 > 4n + 10.$$

If $G \not\cong G'$, then $n = a + b + c$, $\phi_G(e_1) \geq b - 2$, $\phi_G(e_2) \geq a - 2$, $\phi_G(e_3) \geq c - 2$. Considering the contributions of the pendant edge and the six edges incident to v_1 and v_2 in $\Theta_{a,b,c}$ to $w^+Mo(G)$, we know that

$$w^+Mo(G) \geq 10(b - 2) + 10(a - 2) + 10(c - 2) + 4(n - 2) = 14n - 68 > 4n + 10.$$

Case 2: $a \geq b = c + 1$. In $\Theta_{a,c+1,c}$, we have $\phi\left[c, \left\lfloor \frac{a+c-1}{2} \right\rfloor, c - 1\right]$. If $G \cong G'$, then $n = a + 2c$, we obtain

$$\begin{aligned} w^+Mo(G) &\geq 10c + 10\left\lfloor \frac{a+c-1}{2} \right\rfloor + 10(c - 1) \\ &\geq 10c + 10\left(\frac{a+c-2}{2}\right) + 10(c - 1) \\ &= 5n + 15c - 20. \end{aligned}$$

If $c \geq 2$, then $w^+Mo(G) \geq 5n + 10 > 4n + 10$. If $c = 1$, then $b = 2$, thus $G \cong \Theta_{a,2,1}$. When n is odd, $a = n - 2$ is odd, then we have

$$w^+Mo(\Theta_{a,2,1}) = 10 + 10 \times \frac{a-1}{2} + 4(a-3) = 9n - 25 > 4n + 10.$$

When n is even, $a = n - 2$ is even, then we find

$$w^+Mo(\Theta_{a,2,1}) = 10 + 10 \times \frac{a}{2} + 4(a-2) = 9n - 16 > 4n + 10.$$

If $G \not\cong G'$, then $n = a + 2c + 1$, $\phi_G(e_1) \geq c - 1$, $\phi_G(e_2) \geq \left\lfloor \frac{a+c-1}{2} \right\rfloor - 1$, $\phi_G(e_3) \geq c - 2$. We obtain

$$\begin{aligned} w^+Mo(G) &\geq 10(c - 1) + 10\left(\left\lfloor \frac{a+c-1}{2} \right\rfloor - 1\right) + 10(c - 2) + 4(n - 2) \\ &\geq 9n + 15c - 63 \\ &\geq 9n - 48 \\ &> 4n + 10. \end{aligned}$$

Case 3: $a \geq b = c \geq 2$. In $\Theta_{a,c,c}$, we have $\phi[c - 1, c - 1, c - 1]$. If $G \cong G'$, it is clear that the contribution of each edge incident to v_1 or v_2 to $Mo(G)$ is $c - 1$. Each central edge on the path with an odd length has a contribution of 0 and each remaining edge has a contribution of at least 1 to $Mo(G)$. Thus, if $c \geq 3$, then

$$w^+Mo(G) \geq 30(c - 1) + 4[(n + 1) - 9] = 30c + 4n - 62 > 4n + 10.$$

If $c = 2$, then $G \cong \Theta_{n-3,2,2}$.

If $G \not\cong G'$, then $n = a + 2c$, $\phi_G(e_1), \phi_G(e_2), \phi_G(e_3) \geq c - 2$. We obtain

$$w^+Mo(G) \geq 30(c - 2) + 4(n - 2) = 4n + 30c - 68.$$

If $c \geq 3$, then $w^+Mo(G) \geq 4n + 22 > 4n + 10$. If $c = 2$ and the pendant edge is incident to v_1 or v_2 , then $\phi_G(e_1) = \phi_G(e_2) = \phi_G(e_3) = 2$, thus

$$w^+Mo(G) \geq 5(n - 2) + 12 \times 3 = 5n + 26 > 4n + 10.$$

If $c = 2$ and the pendant edge is incident to the center vertex of P_c , then we have $\phi_G(e_1) = \phi_G(e_2) = 2, \phi_G(e_3) = 0$, thus

$$w^+ Mo(G) \geq 4(n-2) + 10 + 10 = 4n + 12 > 4n + 10.$$

If $c = 2$ and the pendant edge is incident to a vertex x in $V(P_a) \setminus \{v_1, v_2\}$, where $d(x, v_1) \leq d(x, v_2)$, then $\phi_G(e_1) = 0, \phi_G(e_2) = \phi_G(e_3) = 2$, thus

$$w^+ Mo(G) \geq 4(n-2) + 10 + 10 = 4n + 12 > 4n + 10.$$

□

Finally, Theorem 1 can be proved by Lemmas 4–6.

4. Proof of Theorem 2

Based on the definition of bicyclic graphs, bicyclic graphs can be divided into two types: $\mathcal{B}_n^1$ is the set of bicyclic graphs of order n with exactly two cycles. $\mathcal{B}_n^2$ is the set of bicyclic graphs of order n with three cycles. In this section, for every type of bicyclic graph, we obtain the bicyclic graph with the maximum weighted Mostar index.

$G_1 \cdot G_2$ denotes the graph obtained from G_1 and G_2 by identifying one vertex. $S_{n,k}$ denotes the unicyclic graph on n vertices with a cycle of length k , where $n - k$ pendant vertices are incident to a fixed vertex on the cycle, and we call the fixed vertex the center vertex of $S_{n,k}$.

Theorem 3. Let $G \in \mathcal{B}_n^1$ and $n \geq 5$, then $w^+ Mo(G) \leq n^3 - 3n^2 + 2n - 12$, with the equality if and only if $G \cong B_1$ (see Figure 4).

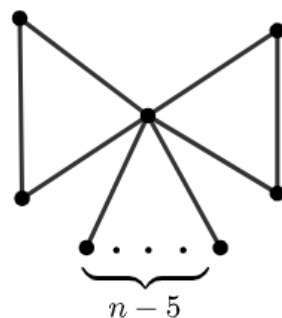


Figure 4. Graph B_1 .

Proof. Let $G \cong H_1 \cdot H_2$, where H_1 is a unicyclic graph of order n_1 with k_1 -cycle, and H_2 is a unicyclic graph of order n_2 with k_2 -cycle. By Lemma 2, we can easily obtain $w^+ Mo(H_1 \cdot H_2) \leq w^+ Mo(H'_1 \cdot H_2)$, where H'_1 is a unicyclic graph of order n_1 with k_1 -cycle, and all cut edges are pendant edges. Let C be the unique cycle of H'_1 . Note that

$$\sum_{e \in E(H'_1 \cdot H_2) \setminus E(C)} \Phi_{H'_1 \cdot H_2}(e) \leq \sum_{e \in E(S_{n_1, k_1} \cdot H_2) \setminus E(C)} \Phi_{S_{n_1, k_1} \cdot H_2}(e).$$

Let u be the common vertex of H'_1 and H_2 , and let p be the number of pendant edges attached at u in H'_1 . D denotes the sum of the degrees of all vertices on C , and δ_e denotes the sum of the degrees of the end vertices of the edge e .

If k_1 is odd, e_u is the symmetric edge of u on C , and let $F = E(C) \setminus \{e_u\}$. It is clear that $n = n_2 - 1 + p + k_1 + \sum_{e \in F} (|T_{w_e}| - 1)$. By Lemma 3, we obtain

$$\phi(e_u) \leq n - (n_2 + p) - 2 \times \frac{k_1 - 1}{2} = n - k_1 - (n_2 + p - 1).$$

For each $e \in F$, we obtain $\phi(e) \leq n - |T_{w_e}| - 2 \times \frac{k_1-1}{2} = n - k_1 - (|T_{w_e}| - 1)$. By $\delta_{e_u} \geq 4$ and $|T_{w_e}| - 1 \geq 0$, then

$$\begin{aligned} \sum_{e \in E(C)} \Phi_{H'_1 \cdot H_2}(e) &\leq \delta_{e_u}[n - k_1 - (n_2 + p - 1)] + \sum_{e \in F} \delta_e[n - k_1 - (|T_{w_e}| - 1)] \\ &= 2D(n - k_1) - \delta_{e_u}(n_2 + p - 1) - \sum_{e \in F} \delta_e(|T_{w_e}| - 1) \\ &\leq 2D(n - k_1) - 4 \left[n - k_1 - \sum_{e \in F} (|T_{w_e}| - 1) \right] - \sum_{e \in F} \delta_e(|T_{w_e}| - 1) \\ &= (2D - 4)(n - k_1) + \sum_{e \in F} (4 - \delta_e)(|T_{w_e}| - 1) \\ &\leq (2D - 4)(n - k_1), \end{aligned}$$

with the equality if and only if $H'_1 \cong S_{n_1, k_1}$.

If k_1 is even, by Lemma 3, for each $e \in E(C)$, we find $\phi(e) \leq n - 2 \times \frac{k_1}{2} = n - k_1$. Then

$$\sum_{e \in E(C)} \Phi_{H'_1 \cdot H_2}(e) \leq 2D(n - k_1),$$

with the equality if and only if $H'_1 \cong S_{n_1, k_1}$.

Thus, we have $w^+ Mo(H'_1 \cdot H_2) \leq w^+ Mo(S_{n_1, k_1} \cdot H_2)$, with the equality if and only if $H'_1 \cong S_{n_1, k_1}$, where the common vertex of S_{n_1, k_1} and H_2 is the center vertex of S_{n_1, k_1} .

Let C' denote the unique cycle of S_{n_1, k_1-1} and f be a pendant edge of S_{n_1, k_1-1} , $E_1 = E(C)$, $E_2 = E(C') \cup \{f\}$. Note that

$$\sum_{e \in E(S_{n_1, k_1} \cdot H_2) \setminus E_1} \Phi_{S_{n_1, k_1} \cdot H_2}(e) \leq \sum_{e \in E(S_{n_1, k_1-1} \cdot H_2) \setminus E_2} \Phi_{S_{n_1, k_1-1} \cdot H_2}(e).$$

If k_1 is odd, then $k_1 - 1$ is even, we have

$$\begin{aligned} \sum_{e \in E_1} \Phi_{S_{n_1, k_1} \cdot H_2}(e) &= [2d(u) + 2 \times 2(k_1 - 1) - 4](n - k_1) \\ &= [2d(u) + 4k_1 - 8](n - k_1), \end{aligned}$$

$$\begin{aligned} \sum_{e \in E_2} \Phi_{S_{n_1, k_1-1} \cdot H_2}(e) &= [2d(u) + 2 + 2 \times 2(k_1 - 2)](n - k_1 + 1) + [d(u) + 2](n - 2) \\ &= [2d(u) + 4k_1 - 6](n - k_1 + 1) + [d(u) + 2](n - 2). \end{aligned}$$

Then,

$$\sum_{e \in E_2} \Phi_{S_{n_1, k_1-1} \cdot H_2}(e) - \sum_{e \in E_1} \Phi_{S_{n_1, k_1} \cdot H_2}(e) = 2(n - k_1) + 2d(u) + 4k_1 - 6 + [d(u) + 2](n - 2) > 0.$$

If k_1 is even, then $k_1 - 1$ is odd, we have

$$\begin{aligned} \sum_{e \in E_1} \Phi_{S_{n_1, k_1} \cdot H_2}(e) &= [2d(u) + 2 \times 2(k_1 - 1)](n - k_1) \\ &= [2d(u) + 4k_1 - 4](n - k_1), \end{aligned}$$

$$\begin{aligned} \sum_{e \in E_2} \Phi_{S_{n_1, k_1-1} \cdot H_2}(e) &= [2d(u) + 2 + 2 \times 2(k_1 - 2) - 4](n - k_1 + 1) + [d(u) + 2](n - 2) \\ &= [2d(u) + 4k_1 - 10](n - k_1 + 1) + [d(u) + 2](n - 2). \end{aligned}$$

By $d(u) \geq 4$, then

$$\sum_{e \in E_2} \Phi_{S_{n_1, k_1-1} \cdot H_2}(e) - \sum_{e \in E_1} \Phi_{S_{n_1, k_1} \cdot H_2}(e) = -6(n - k_1) + 2d(u) + 4k_1 - 10 + [d(u) + 2](n - 2) > 0.$$

So $w^+ Mo(S_{n_1, k_1} \cdot H_2) < w^+ Mo(S_{n_1, k_1-1} \cdot H_2) < \dots < w^+ Mo(S_{n_1, 3} \cdot H_2)$.

Thus, $w^+ Mo(H_1 \cdot H_2) \leq w^+ Mo(S_{n_1, 3} \cdot H_2)$, with the equality if and only if $H_1 \cong S_{n_1, 3}$, where the common vertex of $S_{n_1, 3}$ and H_2 is the center vertex of $S_{n_1, 3}$. By symmetry, we have $w^+ Mo(H_1 \cdot H_2) \leq w^+ Mo(S_{n_1, 3} \cdot S_{n_2, 3})$, with the equality if and only if $H_1 \cong S_{n_1, 3}$ and $H_2 \cong S_{n_2, 3}$, where the common vertex of $S_{n_1, 3}$ and $S_{n_2, 3}$ is the center vertex of $S_{n_1, 3}$ and $S_{n_2, 3}$.

Hence, B_1 is the bicyclic graph with the maximum weighted Mostar index in $\mathcal{B}_n^1$, and it is clear that

$$w^+ Mo(B_1) = 4(n + 1)(n - 3) + n(n - 2)(n - 5) = n^3 - 3n^2 + 2n - 12.$$

This completes the proof of Theorem 3. $\square$

Theorem 4. Let $G \in \mathcal{B}_n^2$ and $n \geq 8$, then $w^+ Mo(G) \leq n^3 - 3n^2 + 2n - 4$, with the equality if and only if $G \cong B_2$.

Proof. Let $G \in \mathcal{B}_n^2$ and $n \geq 8$, then G contains a subgraph isomorphic to $\Theta_{a, b, c}$, and we assume that $a \geq b \geq c$. If G is the bicyclic graph with the maximum weighted Mostar index, by Lemma 2.2, we know that all cut edges are pendant edges. We distinguish five cases.

Case 1: $a + c \geq 5$.

Let p denote the number of pendant edges in G . We have $n = a + b + c - 1 + p \geq p + 6$ by $b \geq 2$. For each edge e in $P_a \cup P_c$, we have $\Phi(e) \leq (p + 6)[n - (a + c)] \leq n(n - 5)$; for each edge e in P_b , we have $\Phi(e) \leq (p + 5)[n - (b + c)] \leq (n - 1)(n - 3)$; for each pendant edge e , we have $\Phi(e) \leq (p + 4)(n - 2) \leq (n - 2)^2$. Since these equations cannot hold at the same time, we know that

$$\begin{aligned} w^+ Mo(G) &< (a + c)n(n - 5) + b(n - 1)(n - 3) + [n - (a + b + c - 1)](n - 2)^2 \\ &= n^3 - 3n^2 - (a + c)n - b - 4(a + c) + 4 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

Case 2: $a = 3, b = 3, c = 1$.

Let v_1 and v_2 be two vertices of degree 3 of $\Theta_{3, 3, 1}$. Clearly, $\Phi(v_1 v_2) \leq n(n - 6)$; for each edge e (except $v_1 v_2$) incident to v_1 or v_2 in $\Theta_{3, 3, 1}$, we have $\Phi(e) \leq (n - 1)(n - 4)$; for each remaining edge e in $\Theta_{3, 3, 1}$, we have $\Phi(e) \leq (n - 2)(n - 6)$; for each pendant edge e , we have $\Phi(e) \leq (n - 2)^2$. We reveal

$$\begin{aligned} w^+ Mo(G) &< n(n - 6) + 4(n - 1)(n - 4) + 2(n - 2)(n - 6) + (n - 6)(n - 2)^2 \\ &= n^3 - 3n^2 - 14n + 16 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

Case 3: $a = 3, b = 2, c = 1$.

Let v_1 and v_2 be two vertices of degree 3 of $\Theta_{3, 2, 1}$. v_3 is the vertex of degree 2 on P_b , v_4 is the neighbor of v_1 on P_a , and v_5 is the neighbor of v_2 on P_a . If there is at least one pendant edge attached at the vertices of degree 2 of $\Theta_{3, 2, 1}$, then $\Phi(v_1 v_2) \leq n(n - 5)$, $\Phi(v_1 v_3), \Phi(v_2 v_3) < n(n - 4)$, $\Phi(v_1 v_4), \Phi(v_2 v_5) < n(n - 4)$, $\Phi(v_4 v_5) \leq (n - 1)(n - 5)$; for each pendant edge e , we have $\Phi(e) \leq (n - 2)^2$. We find

$$\begin{aligned} w^+ Mo(G) &< n(n - 5) + 4n(n - 4) + (n - 1)(n - 5) + (n - 5)(n - 2)^2 \\ &= n^3 - 3n^2 - 3n - 15 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

If there is no pendant edge attached at vertices of degree 2 of $\Theta_{3,2,1}$, then we have $\Phi(v_1v_2) \leq (n+1)(n-5)$, $\Phi(v_1v_3), \Phi(v_2v_3) \leq n(n-4)$, $\Phi(v_1v_4), \Phi(v_2v_5) \leq n(n-4)$, $\Phi(v_4v_5) \leq 4(n-5)$; for each pendant edge e , we have $\Phi(e) \leq (n-1)(n-2)$. We obtain

$$\begin{aligned} w^+Mo(G) &< (n+1)(n-5) + 4n(n-4) + 4(n-5) + (n-5)(n-1)(n-2) \\ &= n^3 - 3n^2 + n - 35 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

Case 4: $a = 2, b = 2, c = 2$.

If there is at least one pendant edge attached at the vertices of degree 2 of $\Theta_{2,2,2}$, then for each edge e in $\Theta_{2,2,2}$, we have $\Phi(e) < n(n-4)$; for each pendant edge e , we have $\Phi(e) \leq (n-2)^2$. We obtain

$$\begin{aligned} w^+Mo(G) &< 6n(n-4) + (n-5)(n-2)^2 \\ &= n^3 - 3n^2 - 20 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

If all pendant edges are attached at v_1 , where v_1 is a vertex of degree 3 of $\Theta_{2,2,2}$, then for each edge e incident to v_1 in $\Theta_{2,2,2}$, we have $\Phi(e) = n(n-4)$; for each remaining edge e in $\Theta_{2,2,2}$, we have $\Phi(e) = 5(n-6)$; for each pendant edge e , we have $\Phi(e) = (n-1)(n-2)$. We find

$$\begin{aligned} w^+Mo(G) &= 3n(n-4) + 3 \times 5(n-6) + (n-5)(n-1)(n-2) \\ &= n^3 - 5n^2 + 20n - 100 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

If all pendant edges are attached at v_1 and v_2 , where v_1 and v_2 are the vertices of degree 3 of $\Theta_{2,2,2}$, then for each edge e in $\Theta_{2,2,2}$, we have $\Phi(e) \leq (n-1)(n-6)$; for each pendant edge e , we have $\Phi(e) \leq (n-2)^2$. We have

$$\begin{aligned} w^+Mo(G) &< 6(n-1)(n-6) + (n-5)(n-2)^2 \\ &= n^3 - 3n^2 - 18n + 16 \\ &< n^3 - 3n^2 + 2n - 4. \end{aligned}$$

Case 5: $a = 2, b = 2, c = 1$.

Let v_1 and v_2 be two vertices of degree 3 of $\Theta_{2,2,1}$, and let v_3 and v_4 be two vertices of degree 2 of $\Theta_{2,2,1}$. p_j denotes the number of pendant edges attached at v_j , where $j = 1, 2, 3, 4$. It is clear that $n = \sum_{j=1}^4 p_j + 4$. Without a loss of generality, we assume that $p_1 \geq p_2, p_3 \geq p_4$, then

$$\begin{aligned} \Phi(v_1v_3) &= (p_1 + p_3 + 5)|p_1 + p_4 + 1 - p_3| \leq (p_1 + p_3 + 5)(n - p_2 - 3), \\ \Phi(v_1v_4) &= (p_1 + p_4 + 5)(p_1 + p_3 + 1 - p_4) \leq (p_1 + p_4 + 5)(n - p_2 - 3), \\ \Phi(v_2v_3) &= (p_2 + p_3 + 5)|p_2 + p_4 + 1 - p_3| \leq (p_2 + p_3 + 5)(n - p_1 - 3), \\ \Phi(v_2v_4) &= (p_2 + p_4 + 5)(p_2 + p_3 + 1 - p_4) \leq (p_2 + p_4 + 5)(n - p_1 - 3), \\ \Phi(v_1v_2) &= (p_1 + p_2 + 6)(p_1 - p_2), \end{aligned}$$

for each pendant edge e , we have

$$\Phi(e) \leq n(n-2).$$

By $n = \sum_{j=1}^4 p_j + 4 \geq 8$, $p_2, p_3, p_4 \geq 0$, and $0 \leq p_1 \leq n - 4$, we obtain

$$\begin{aligned} w^+ Mo(G) &\leq (p_1 + p_3 + 5)(n - p_2 - 3) + (p_1 + p_4 + 5)(n - p_2 - 3) + (p_2 + p_3 + 5)(n - p_1 - 3) \\ &\quad + (p_2 + p_4 + 5)(n - p_1 - 3) + (p_1 + p_2 + 6)(p_1 - p_2) + n(n - 2)(n - 4) \\ &= 2(p_1 + p_2 + p_3 + p_4 + 10)(n - 3) - p_2(2p_1 + p_3 + p_4 + 10) \\ &\quad - p_1(2p_2 + p_3 + p_4 + 10) + p_1(p_1 + p_2 + 6) - p_2(p_1 + p_2 + 6) + n^3 - 6n^2 + 8n \\ &= n^3 - 6n^2 + 8n + 2(n + 6)(n - 3) + p_1(p_1 - p_2 - p_3 - p_4 - 4) \\ &\quad - p_2(3p_1 + p_2 + p_3 + p_4 + 16) \\ &\leq n^3 - 4n^2 + 14n - 36 + p_1(p_1 - 4) \\ &\leq n^3 - 4n^2 + 14n - 36 + (n - 4)(n - 8) \\ &= n^3 - 3n^2 + 2n - 4, \end{aligned}$$

with the equality if and only if $p_2 = p_3 = p_4 = 0$, and $p_1 = n - 4$, i.e., $G \cong B_2$.

This completes the proof of Theorem 4. $\square$

Finally, Theorem 2 can be proved by Theorems 3 and 4.

5. Conclusions and a Further Research Problem

In this paper, firstly, by dividing the bicyclic graphs into two types according to the number of pendant paths, we determine the lower bound of the weighted Mostar index among all bicyclic graphs with order $n \geq 12$ and characterize the corresponding extremal graphs. Secondly, we divide the bicyclic graphs into two types based on the definition of bicyclic graphs and Theta graph, and find its upper bound for every type of bicyclic graph, then we obtain the upper bound of the weighted Mostar index for bicyclic graphs with order $n \geq 8$ and the graph that achieves the upper bound. Based on the bicyclic graphs, we can consider the related research of tricyclic graphs. The following is a further research problem:

Problem 1. Among all tricyclic graphs with order n , identify the graphs with the largest and smallest weighted Mostar index.

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