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A Parametric Method for Proving Some Analytic Inequalities

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Abstract: In this paper, a parametric method for proving inequalities is described. The method is based on associating a considered inequality with the corresponding stratified family of functions. Many inequalities from the theory of analytic inequalities can be interpreted using families of functions that are stratified with respect to some parameter. By discussing the sign of the functions from the family by the parameter according to which the family is stratified, inequalities are obtained that contain the best possible constants, if they exist. The application of this method is demonstrated for four inequalities: the Cusa–Huygens inequality, the Wilker-type inequality and the two Mitrinović–Adamović-type inequalities. Significantly simpler proofs and improvements of all these inequalities are provided.

Keywords: analytic inequalities; a parametric method; stratified families of functions

MSC: 26D05; 26D07

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Citation: Malešević, B.; Mićović, M.; Mihailović, B. A Parametric Method for Proving Some Analytic Inequalities. *Axioms* **2024**, *13*, 520. https://doi.org/10.3390/axioms13080520

Academic Editor: Silvestru Sever Dragomir

Received: 30 June 2024
Revised: 27 July 2024
Accepted: 30 July 2024
Published: 1 August 2024

1. Introduction

In the theory of analytic inequalities, authors often give and prove improvements to some well-known inequalities. It has been shown in previous papers [1–6] that, in many cases, those inequalities can be considered through the concept of stratified families of functions [1].

In the following, we will specify the concept of the stratification of a family of functions over a real subset. We start with a family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ defined for values of the argument $x\in\mathbb{S}$ for some $\mathbb{S}\subseteq\mathbb{R}$ ($\mathbb{S}\neq\varnothing$) and parameter values $p\in\mathbb{P}$ for some $\mathbb{P}\subseteq\mathbb{R}$ ($\mathbb{P}\neq\varnothing$). The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *increasingly stratified at a point* $x_0\in\mathbb{S}$ if

$$(\forall p_1, p_2\in\mathbb{P})\ \ p_1 < p_2 \iff \varphi_{p_1}(x_0) < \varphi_{p_2}(x_0)\,,$$

and, conversely, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *decreasingly stratified at a point* $x_0\in\mathbb{S}$ if

$$(\forall p_1, p_2\in\mathbb{P})\ \ p_1 < p_2 \iff \varphi_{p_1}(x_0) > \varphi_{p_2}(x_0)\,.$$

The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is *increasingly* (*decreasingly*) *stratified on a set* $\mathbb{S}$ if it is increasingly (decreasingly) stratified at every point in the set $\mathbb{S}$. Note that stratified families of functions appear in some mathematical problems in engineering [7–10].

The paper is divided into an introduction, three sections and a conclusion. Some new results are given within each section. In Section 2, we provide a brief overview of one method for proving mixed trigonometric polynomial inequalities over some interval [11]. This method is used in Section 4. The parametric method, based on the concept of stratification, is the subject of Section 3. Applications of the method in the theory of analytic inequalities are given in Section 4. In Sections 3 and 4, specific choices of sets $\mathbb{S}$ and $\mathbb{P}$ for stratified families of functions are considered in accordance with the observed problems.

2. On a Sign of Mixed Trigonometric Polynomial Functions

In this section, we outline a method for proving mixed trigonometric polynomial (MTP) inequalities

$$f(x) > 0\,,$$

where $f(x)$ is an MTP function over a non-empty set $\mathbb{S} \subseteq \mathbb{R}$ given by

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x\,, \tag{1}$$

where $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $p_i, q_i, r_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Note that these MTP functions are continuous on $\mathbb{S}$. Additionally, if $\mathbb{S}$ is a set of the form (a,b), $[a,b)$, $(a,b]$, $[a,b]$, where $a < b$, then these MTP functions are differentiable any number of times on $\mathbb{S}$. Regarding MTP functions and inequalities, see [6,11–17].

In the following, we provide a brief description of the method for proving MTP inequalities according to [6,11]. As an application of this method, we will present methods for isolating zeros and extrema of MTP functions that have not been considered before.

2.1. A Positivity of MTP Functions

The MTP function (1), by applying substitutions from Table 1 [6,13] to each addend of the function $f(x)$, can be transformed into the equivalent form given by

$$f(x) = \sum_{i=1}^{m} \beta_i x^{s_i} \operatorname{trig}_i(k\,x)\,, \tag{2}$$

where $m \in \mathbb{N}$, $\beta_i \in \mathbb{R} \setminus \{0\}$, $s_i \in \mathbb{N}_0$, $\operatorname{trig}_i = \cos$ or $\operatorname{trig}_i = \sin$, and $k \in \mathbb{Z}$, for $x \in \mathbb{S}$. Notice that the expression of an MTP function in terms of multiple angles (2) does not contain any powers or products of trigonometric functions.

For a real function $\phi(x)$, $x \in \mathbb{S}$, a real polynomial $P(x)$ is *an upward polynomial approximation* on $\mathbb{A} \subseteq \mathbb{S}$, if it holds that

$$(\forall x \in \mathbb{A})\;\; \phi(x) \le P(x)\,,$$

and, conversely, a real polynomial $P(x)$ is *a downward polynomial approximation* on $\mathbb{A} \subseteq \mathbb{S}$, if it holds that

$$(\forall x \in \mathbb{A})\;\; \phi(x) \ge P(x)\,.$$

A method for proving an MTP inequality

$$f(x) > 0 \tag{3}$$

on $\mathbb{S}$ is described in [11] for $\mathbb{S} = (a,b)$, $a < b$. In that paper, the proof of the positivity of an MTP function is based on determining a downward polynomial approximation $P(x)$ with respect to the observed MTP function $f(x)$ such that

$$f(x) > P(x) > 0\,,$$

$x \in (a,b)$, where (a,b) is a bounded interval.

Note that if the MTP inequality (3) is considered on the interval (a,b), an examination of the positivity of the sign of the MTP function $f(x)$ can also be performed at the endpoints a and b. In this way, it is potentially possible to extend the inequality (3) to $x \in [a,b)$, $x \in (a,b]$ or $x \in [a,b]$.

In [11], upper and lower bounds of the MTP functions were considered using Maclaurin approximations for the sine and cosine functions. Generally, for analytic functions, the problem of determining upper and lower bounds and approximation errors can be approached using some other methods as well (see for example [18–22]).

To obtain a downward polynomial approximation $P(x)$ of the MTP function $f(x)$, according to [6,11], we approximate each addend of the function (2) using upward and downward Maclaurin approximations of the sine and cosine functions that are given in Lemmas 1.1 and 1.2 from [11]. Therefore, we obtain a real polynomial $P(x)$ such that

$$f(x) > P(x),$$

$x \in (a,b)$, holds. If for a such polynomial $P(x)$, it holds that

$$P(x) > 0,$$

$x \in (a,b)$, then

$$f(x) > 0,$$

$x \in (a,b)$.

Note that in Section 4, we denote the Taylor expansion of order n of some analytic function ϕ in the neighbourhood of some point a by $T_n^{\phi,a}(x)$.

It is well known that Sturm's theorem gives the number of zeros of a real polynomial function on a real segment, provided that the polynomial has no zeros at the endpoints of that segment ([23], Theorem 4.2 [24]). From a computational standpoint, the question arises regarding the effectiveness of the execution of Sturm's algorithm. In [24], it has been shown that for a polynomial with rational coefficients on a segment with rational endpoints such that the polynomial does not have a zero at the endpoints of the segment, Sturm's algorithm is executed effectively. In Section 4, the considered MTP functions $f(x)$ and the corresponding polynomials $P(x)$ have rational coefficients and are examined on a segment whose endpoints do not have to be rational numbers. If some endpoint is not a rational number, the proof is performed over an extended segment with rational endpoints.

2.2. Isolation Methods

In [25], a method for isolating intervals on which there are zeros of MTP functions is described and implemented. However, the computer implementation of this algorithm in Maple from [25] does not display all the steps that allow users to control and verify the proof. Therefore, in this subsection, we provide methods for isolating zeros and extrema of MTP functions based on the method for proving MTP inequalities from Section 2.1. The use of these methods allows the observation of all the steps in the proof process, i.e., verification.

2.2.1. A Method for Isolating Zeros of an MTP Function

Let us consider MTP functions $f : \mathbb{S} \longrightarrow \mathbb{R}$ on the segment $\mathbb{S} = [a,b]$, $a < b$. The following assertion evidently holds:

Theorem 1 (A method for isolating zeros)**.** *If there exist points $a_0, b_0 \in [a,b]$ such that $a < a_0 < b_0 < b$, satisfying the conditions*:

1. *f has a constant sign on $[a,a_0]$ and on $[b_0,b]$ under the condition that $f(a_0)f(b_0) < 0$;*
2. *f' has a constant sign on $[a_0,b_0]$;*

then the MTP function $f : [a,b] \longrightarrow \mathbb{R}$ has exactly one zero on the interval (a,b), more precisely, on the subinterval (a_0,b_0).

Remark 1. *The previous theorem holds for any differentiable function f over the segment $[a,b]$. By choosing f to be an MTP function, we can verify conditions 1 and 2 using the method for proving MTP inequalities from Section 2.1.*

Note that for the functions $f : [a,b] \longrightarrow \mathbb{R}$, we do not provide a selection procedure for a_0 and b_0. If concrete values for a_0 and b_0 are determined that satisfy conditions 1 and 2 using the method for proving MTP inequalities, then we have proof of the existence of exactly one zero and the isolation of the subinterval where the zero is located.

This method cannot be applied to isolate double zeros, i.e., zeros of the even multiplicity. However, these zeros also represent local extrema of the functions. Therefore, it is possible to isolate them using the method for isolating extrema (Section 2.2.2).

If an MTP function is a non-zero function, then on a bounded interval, it has finitely many zeros. Thus, it is possible to consider a method for isolating all zeros on some bounded interval on which there is more than one zero in a similar manner.

2.2.2. A Method for Isolating Extrema of an MTP Function

Let us consider MTP functions $f : \mathbb{S} \longrightarrow \mathbb{R}$ on the segment $\mathbb{S} = [a, b]$, $a < b$. The following assertion evidently holds:

Theorem 2 (A method for isolating extrema)**.** *If there exist points $a_0, b_0 \in [a, b]$ such that $a < a_0 < b_0 < b$, satisfying the conditions*:

1. *f' has a constant sign on $[a, a_0]$ and on $[b_0, b]$ under the condition that $f'(a_0)f'(b_0) < 0$;*
2. *f'' has a constant sign on $[a_0, b_0]$;*

then the MTP function $f : [a, b] \longrightarrow \mathbb{R}$ has exactly one extremum on the interval $[a, b]$, more precisely, on the subinterval (a_0, b_0).

Remark 2. *The previous theorem holds for any two times differentiable function f over the segment $[a, b]$. By choosing f to be an MTP function, we can verify conditions 1 and 2 using the method for proving MTP inequalities from Section 2.1.*

If f'' has a constant sign on $[a, b]$, then, to establish the existence of exactly one extremum and isolate the subinterval (a_0, b_0) where it lies, it is sufficient to prove that there exist points $a_0, b_0 \in [a, b]$ such that $a < a_0 < b_0 < b$ and that $f'(a_0)f'(b_0) < 0$.

If in the condition 2, it holds that $f'' > 0$ on $[a_0, b_0]$, then the function $f(x)$ has a minimum, whereas if $f'' < 0$ holds on $[a_0, b_0]$, the function $f(x)$ has a maximum. For the functions $f : [a, b] \longrightarrow \mathbb{R}$, we do not provide a selection procedure for a_0 and b_0. If concrete values for a_0 and b_0 are determined that satisfy conditions 1 and 2 using the method for proving MTP inequalities, then we have proof of the existence of exactly one extremum and the isolation of the subinterval where the extremum is located.

Note that the described method for isolating an extremum of an MTP function $f(x)$ is identical to the method for isolating a zero of the odd multiplicity of the MTP function $f'(x)$.

If an MTP function is not constant, then on a bounded interval, it has finitely many extrema. Thus, it is possible to consider a method for isolating all extrema on some bounded interval on which there is more than one extremum in a similar manner.

3. On the Parametric Method

In this section, the parametric method for proving inequalities will be described. Let $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ be a family of functions that takes real values for the argument $x \in \mathbb{S}$ ($\emptyset \neq \mathbb{S} \subseteq \mathbb{R}$) and the parameter $p \in \mathbb{P}$ ($\emptyset \neq \mathbb{P} \subseteq \mathbb{R}$). Let us observe for fixed $x \in \mathbb{S}$ the equation

$$\varphi_p(x) = 0$$

with respect to the parameter p. In the general case, this equation may have no solution with respect to $p \in \mathbb{P}$. For us, it is of particular interest to identify families for which there exists a solution with respect to $p \in \mathbb{P}$, for each $x \in \mathbb{S}$, especially those where the solution is unique.

With the following assertion, we provide some sufficient conditions such that for a family of functions $\{\varphi_p(x)\}_{p \in \mathbb{P}}$, the equation $\varphi_p(x) = 0$ has a solution with respect to p.

Theorem 3. *Let $\{\varphi_p(x)\}_{p\in[c,d]}$ be a family of functions with the argument $x\in[a,b]$, $a<b$, such that $\varphi_p(a)=0$ for $p\in[c,d]$, $c<d$. If*:

(a) *$\varphi_c(x)<0$ and $\varphi_d(x)>0$ for each $x\in(a,b)$, $\varphi_c(b)=0$, $\varphi_d(b)\in\mathbb{R}^+$;*
(b) *the functions $\varphi_p(x)$ are continuous with respect to $p\in[c,d]$ for each $x\in[a,b]$;*

then for each $x\in[a,b]$, the equation $\varphi_p(x)=0$ has a solution with respect to $p\in[c,d]$.

Proof. The conditions (a) and (b) ensure that for each $x\in(a,b)$, there will be at least one solution to the equation $\varphi_p(x)=0$ with respect to $p\in(c,d)$. If $x=a$, the assumption $\varphi_p(a)=0$ ensures that there is a solution at point a. If $x=b$, the assumption $\varphi_c(b)=0$ ensures that there is a solution at point b. □

A family of functions $\{\varphi_p(x)\}_{p\in[c,d]}$, $x\in[a,b]$, that satisfies the conditions of the previous theorem, we denote *compressed at the point a*. For such a family, the following assertion holds:

Theorem 4. *A family of functions $\{\varphi_p(x)\}_{p\in[c,d]}$ with the argument $x\in[a,b]$, $a<b$, that is compressed at the point a and stratified on $(a,b]$ has the property that the equation $\varphi_p(x)=0$ has exactly one solution with respect to $p\in[c,d]$, $c<d$, for each $x\in(a,b]$.*

Proof. Theorem 3 ensures the existence of a solution to the equation $\varphi_p(x)=0$ for each $x\in(a,b]$. If there were two solutions, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ would not be stratified. □

Note that the solvability of the equation $\varphi_p(x)=0$ with respect to the parameter p, under proper conditions for the family, is also considered with the well-known Implicit Function Theorem [26].

We further consider families of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ with the argument $x\in\mathbb{S}=(a,b)$, $a<b$, and the parameter $p\in\mathbb{P}$ ($\emptyset\neq\mathbb{P}\subseteq\mathbb{R}$) such that there exists a continuous function $g:(a,b)\longrightarrow\mathbb{P}$ for which

$$g(x)=p \quad\Longleftrightarrow\quad \varphi_p(x)=0\,. \tag{4}$$

In the theory of analytic inequalities, there exist numerous examples of trigonometric inequalities that can be connected to families of functions that are compressed at a point such that for them there exists a continuous function g such that (4) holds over the base interval $\mathbb{S}=(0,\pi/2)$, for example [1–3,5,6,27–41].

The function g itself, such that (4) holds, may or may not be given by some specific symbolic expression. In the applications discussed in this paper, cases where g can be determined symbolically are considered.

In the following, we examine cases of stratified families of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ when g is a monotonic function and when g is not a monotonic function. The function g determines the values of the parameter p for which the functions $\varphi_p(x)$ have zeros on the observed interval. The stratification of the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ provides the order of functions in that family which is used in the following.

We first consider the case when g is a monotonic function. The following assertions hold:

Theorem 5. *Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ be a family of functions for $x\in\mathbb{S}=(a,b)\subseteq\mathbb{R}$ and let $\mathbb{P}\subseteq\mathbb{R}$ ($\mathbb{P}\neq\emptyset$), satisfying the following conditions*:

1. *the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is increasingly (decreasingly) stratified on the interval (a,b);*
2. *there exists a continuous monotonically increasing function $g:(a,b)\longrightarrow\mathbb{P}$ that satisfies (4);*
3. *there exist limits $\lim\limits_{x\to a} g(x)=A$ and $\lim\limits_{x\to b} g(x)=B$ in $\overline{\mathbb{R}}$ such that $(A,B)\subseteq\mathbb{P}$.*

Then, it holds:

(i) *If* $p \leq A$, *then*

$$(\forall x \in (a,b))\ \ \varphi_p(x) \leq \varphi_A(x) < 0 \qquad \big((\forall x \in (a,b))\ \ \varphi_p(x) \geq \varphi_A(x) > 0\big).$$

(ii) *If* $p \in (A,B)$, *then the equality* $\varphi_p(x) = 0$ *has a unique solution* $x_0^{(p)} \in (a,b)$ *and it holds that*

$$\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\ \ \varphi_p(x) > 0 \qquad \Big(\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\ \ \varphi_p(x) < 0\Big)$$

and

$$\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\ \ \varphi_p(x) < 0 \qquad \Big(\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\ \ \varphi_p(x) > 0\Big).$$

(iii) *If* $p \geq B$, *then*

$$(\forall x \in (a,b))\ \ \varphi_p(x) \geq \varphi_B(x) > 0 \qquad \big((\forall x \in (a,b))\ \ \varphi_p(x) \leq \varphi_B(x) < 0\big).$$

Proof. The function g determines the values of the parameter p for which the functions from the family $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ have zeros on the observed interval. Based on the properties of stratified families of functions, the corresponding inequalities from (i) and (iii) follow. The assertion (ii) is a direct consequence of the stratification and the monotonicity of the function g. □

Theorem 6. *Let* $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ *be a family of functions for* $x \in \mathbb{S} = (a,b) \subseteq \mathbb{R}$ *and let* $\mathbb{P} \subseteq \mathbb{R}$ $(\mathbb{P} \neq \emptyset)$, *satisfying the following conditions*:

1. *the family of functions* $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ *is increasingly (decreasingly) stratified on the interval* (a,b);
2. *there exists a continuous monotonically decreasing function* $g : (a,b) \longrightarrow \mathbb{P}$ *that satisfies* (4);
3. *there exist limits* $\lim\limits_{x \to b} g(x) = A$ *and* $\lim\limits_{x \to a} g(x) = B$ *in* $\overline{\mathbb{R}}$ *such that* $(A,B) \subseteq \mathbb{P}$.

Then, it holds:

(i) *If* $p \leq A$, *then*

$$(\forall x \in (a,b))\ \ \varphi_p(x) \leq \varphi_A(x) < 0 \qquad \big((\forall x \in (a,b))\ \ \varphi_p(x) \geq \varphi_A(x) > 0\big).$$

(ii) *If* $p \in (A,B)$, *then the equality* $\varphi_p(x) = 0$ *has a unique solution* $x_0^{(p)} \in (a,b)$ *and it holds that*

$$\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\ \ \varphi_p(x) < 0 \qquad \Big(\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\ \ \varphi_p(x) > 0\Big)$$

and

$$\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\ \ \varphi_p(x) > 0 \qquad \Big(\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\ \ \varphi_p(x) < 0\Big).$$

(iii) *If* $p \geq B$, *then*

$$(\forall x \in (a,b))\ \ \varphi_p(x) \geq \varphi_B(x) > 0 \qquad \big((\forall x \in (a,b))\ \ \varphi_p(x) \leq \varphi_B(x) < 0\big).$$

Proof. It is analogous to the proof of Theorem 5. □

Remark 3. *In Theorems 5 and 6, if* $A = -\infty$, *the case* (i) *is not possible, and if* $B = +\infty$, *the case* (iii) *is not possible.*

The previous two theorems can also be considered, with minor modifications, in cases when the family of functions $\{\varphi_p(x)\}_{p \in \mathbb{P}}$ is defined at a or b.

Next, we consider the case when g is not a monotonic function. Let us consider a function that has exactly one local minimum on the observed interval $\mathbb{S} \subseteq \mathbb{R}$. For such a function, the following theorem holds:

Theorem 7. *Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ be a family of functions for $x \in \mathbb{S} = (a,b) \subseteq \mathbb{R}$ and let $\mathbb{P} \subseteq \mathbb{R}$ ($\mathbb{P} \neq \emptyset$), satisfying the following conditions:*

1. *the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is increasingly (decreasingly) stratified on the interval (a,b);*
2. *there exists a continuous function $g : (a,b) \longrightarrow \mathbb{P}$ that satisfies (4) and that is monotonically decreasing on (a, x_m) and monotonically increasing on (x_m, b) for some $x_m \in (a,b)$;*
3. *there exist limits $\lim\limits_{x\to a} g(x)$ and $\lim\limits_{x\to b} g(x)$ in $\overline{\mathbb{R}}$ such that for:*

$$A = g(x_m)\,,\;\; B = \min\left\{\lim_{x\to a} g(x), \lim_{x\to b} g(x)\right\} \;\; \textit{and} \;\; C = \max\left\{\lim_{x\to a} g(x), \lim_{x\to b} g(x)\right\},$$

it holds that $[A, C) \subseteq \mathbb{P}$.

Then, it holds:

(i) *If $p \le A$, then*

$$(\forall x \in (a,b))\;\; \varphi_p(x) \le \varphi_A(x) < 0 \qquad \big((\forall x \in (a,b))\;\; \varphi_p(x) \ge \varphi_A(x) > 0\big).$$

(ii) *If $p \in (A, B)$, then the equality $\varphi_p(x) = 0$ has exactly two solutions $x_0^{(p)}, x_1^{(p)} \in (a,b)$ such that $x_0^{(p)} < x_1^{(p)}$, and it holds that*

$$\Big(\forall x \in \Big(a, x_0^{(p)}\Big) \cup \Big(x_1^{(p)}, b\Big)\Big)\; \varphi_p(x) < 0 \quad \Big(\Big(\forall x \in \Big(a, x_0^{(p)}\Big) \cup \Big(x_1^{(p)}, b\Big)\Big)\; \varphi_p(x) > 0\Big)$$

and

$$\Big(\forall x \in \Big(x_0^{(p)}, x_1^{(p)}\Big)\Big)\;\; \varphi_p(x) > 0 \qquad \Big(\Big(\forall x \in \Big(x_0^{(p)}, x_1^{(p)}\Big)\Big)\;\; \varphi_p(x) < 0\Big).$$

(iii) *If $p \in [B, C)$, then the equality $\varphi_p(x) = 0$ has a unique solution $x_0^{(p)} \in (a,b)$.*
If $\lim\limits_{x\to a} g(x) < \lim\limits_{x\to b} g(x)$, then

$$\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\;\; \varphi_p(x) > 0 \qquad \Big(\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\;\; \varphi_p(x) < 0\Big)$$

and

$$\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\;\; \varphi_p(x) < 0 \qquad \Big(\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\;\; \varphi_p(x) > 0\Big).$$

If $\lim\limits_{x\to a} g(x) > \lim\limits_{x\to b} g(x)$, then

$$\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\;\; \varphi_p(x) < 0 \qquad \Big(\Big(\forall x \in \Big(a, x_0^{(p)}\Big)\Big)\;\; \varphi_p(x) > 0\Big)$$

and

$$\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\;\; \varphi_p(x) > 0 \qquad \Big(\Big(\forall x \in \Big(x_0^{(p)}, b\Big)\Big)\;\; \varphi_p(x) < 0\Big).$$

(iv) *If $p \ge C$, then*

$$(\forall x \in (a,b))\;\; \varphi_p(x) \ge \varphi_C(x) > 0 \qquad \big((\forall x \in (a,b))\;\; \varphi_p(x) \le \varphi_C(x) < 0\big).$$

Proof. It is analogous to the proof of Theorem 5. □

Remark 4. *In Theorem 7, if $C = +\infty$, the case (iv) is not possible, and if $B = +\infty$, the cases (iii) and (iv) are not possible.*

Theorem 7 considers the case when the continuous function g that satisfies (4) has exactly one local minimum on the observed interval. Analogously, it is possible to consider the case when this function has exactly one local maximum. Moreover, it is possible to analogously consider cases when this function has more than one local extremum.

4. Applications

In this section, applications of the parametric method in the theory of analytic inequalities will be demonstrated on the examples of the Cusa–Huygens inequality from [1], the Wilker-type inequality from [39,40], and the two Mitrinović–Adamović-type inequalities from [41].

All symbolic and numerical calculations in this section were performed in computer algebra system Maple 2019.

4.1. Application 1 (Cusa–Huygens Inequality)

The Cusa–Huygens inequality is given by:

Theorem 8 ([1]). *Let $x\in\left(0,\dfrac{\pi}{2}\right)$. Then, it holds that*

$$x > \frac{3\sin x}{2+\cos x}. \tag{5}$$

Many authors have studied the Cusa–Huygens inequality and generalized it [1,6,27,42–62]. In [1], this inequality is considered using the stratified family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}^+}$, where

$$\varphi_p(x) = x - \frac{(p+1)\sin x}{p+\cos x}, \tag{6}$$

for the argument $x\in(0,\pi/2)$.

It has been proven that the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}^+}$ is increasingly stratified on the interval $(0,\pi/2)$ [1]. Note that the family is defined for $x\in\mathbb{S}=(0,\pi/2)$ and for $p\in\mathbb{P}_1\cup\mathbb{P}_2$, where $\mathbb{P}_1=[0,+\infty)$ and $\mathbb{P}_2=(-\infty,-1]$. Moreover, the following assertion holds:

Lemma 1. *The families of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}_1}$ and $\{\varphi_p(x)\}_{p\in\mathbb{P}_2}$ are increasingly stratified on the interval $\mathbb{S}=(0,\pi/2)$.*

Proof. It holds that $\dfrac{\partial\varphi_p(x)}{\partial p}=\dfrac{\sin x(1-\cos x)}{(p+\cos x)^2}>0$ on the interval $(0,\pi/2)$ for $p\in(-\infty,-1]$ or $p\in[0,+\infty)$. □

Remark 5. *Note that the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P}=\mathbb{P}_1\cup\mathbb{P}_2$, is not stratified on the interval $\mathbb{S}=(0,\pi/2)$.*

We further consider the families $\{\varphi_p(x)\}_{p\in(-\infty,-1]}$ and $\{\varphi_p(x)\}_{p\in[0,+\infty)}$ for $x\in(0,\pi/2)$. By applying Theorem 6, as the first application of the parametric method, we provide the proof of the following statement:

Statement 1. *Let*:

$$A=\frac{2}{\pi-2}=1.75193\ldots \quad \textit{and} \quad B=2.$$

Then, it holds:

(i) If $p\in(0,A]$, then

$$\left(\forall x\in\left(0,\frac{\pi}{2}\right)\right)\; x<\frac{(A+1)\sin x}{A+\cos x}\le\frac{(p+1)\sin x}{p+\cos x}.$$

(ii) If $p\in(A,B)$, then the equality

$$\varphi_p(x)=x-\frac{(p+1)\sin x}{p+\cos x}=0$$

has a unique solution $x_0^{(p)}$ *and it holds that*

$$\left(\forall x \in \left(0, x_0^{(p)}\right)\right) \;\; x < \frac{(p+1)\sin x}{p+\cos x}$$

and

$$\left(\forall x \in \left(x_0^{(p)}, \frac{\pi}{2}\right)\right) \;\; x > \frac{(p+1)\sin x}{p+\cos x}\,.$$

(iii) If $p \in (-\infty, -1] \cup [B, +\infty)$, *then*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \;\; x > \frac{(B+1)\sin x}{B+\cos x} \geq \frac{(p+1)\sin x}{p+\cos x}\,.$$

Proof. The following equivalence holds:

$$\varphi_p(x) = x - \frac{(p+1)\sin x}{p+\cos x} = 0 \quad \Longleftrightarrow \quad p = g(x) = \frac{x\cos x - \sin x}{\sin x - x}\,, \tag{7}$$

through which we introduce the continuous function $g(x)$ on the interval $(0, \pi/2)$. Let us examine the monotonicity of the function $g(x)$ on that domain. The first derivative of the function $g(x)$ is

$$g'(x) = \frac{x\cos x + \cos x \sin x + x^2 \sin x - \sin x - x}{(\sin x - x)^2}\,.$$

For the MTP function

$$f(x) = x\cos x + \cos x \sin x + x^2 \sin x - \sin x - x\,,$$

we prove that $f(x) < 0$ on the interval $(0, \pi/2)$ by applying the method for proving MTP inequalities.

The MTP function $f(x)$ in terms of multiple angles is given by

$$f(x) = x\cos x + \tfrac{1}{2}\sin 2x + x^2 \sin x - \sin x - x\,.$$

By approximating the functions $\cos x$ and $\sin 2x$ with the Maclaurin polynomials of degrees 4 and 9, respectively, and the function $\sin x$ with the Maclaurin polynomial of degree 5 in the addend $x^2 \sin x$ and of degree 7 in the addend $-\sin x$, we obtain the upward polynomial approximation

$$P(x) = xT_4^{\cos,0}(x) + \tfrac{1}{2}T_9^{\sin,0}(2x) + x^2 T_5^{\sin,0}(x) - T_7^{\sin,0}(x) - x = \tfrac{2}{2835}x^9 - \tfrac{1}{240}x^7$$

of the function $f(x)$ on the interval $(0, \pi/2)$. It is evident that

$$f(x) < P(x) < 0$$

on the interval $(0, \pi/2)$. Hence, $g'(x) < 0$ on $(0, \pi/2)$; thus, the function $g(x)$ is decreasing on the observed interval.

Based on Lemma 1, the family of functions $\{\varphi_p(x)\}_{p \in [0, +\infty)}$ is increasingly stratified on the interval $(0, \pi/2)$.

Notice that

$$\lim_{x \to 0} g(x) = 2 = B \quad \text{and} \quad \lim_{x \to \pi/2} g(x) = \frac{2}{\pi - 2} = A\,.$$

Note that $A, B \in [0, +\infty)$. Hence, the family of functions $\{\varphi_p(x)\}_{p\in[0,+\infty)}$ satisfies all the conditions of Theorem 6. Therefore, for the considered family, everything stated in (i), (ii), and (iii) for $p \in [0, +\infty)$ of this statement holds. Let us note that

$$\lim_{p\to-\infty} \varphi_p(x) = \lim_{p\to+\infty} \varphi_p(x) = x - \sin x > 0$$

for each $x \in (0, \pi/2)$. Considering that the family of functions $\{\varphi_p(x)\}_{p\in(-\infty,-1]}$ is increasingly stratified on the interval $(0, \pi/2)$ based on Lemma 1, the assertion stated in (iii) is proven for parameter values $p \in (-\infty, -1]$ as well. □

Note that for the families of functions $\{\varphi_p(x)\}_{p\in(-\infty,-1]}$ and $\{\varphi_p(x)\}_{p\in[0,+\infty)}$, there exist values at the endpoints $\varphi_p(0) = 0$ and $\varphi_p(\pi/2) = \frac{\pi}{2} - \frac{p+1}{p}$. These families are compressed at the point 0. Figure 1 illustrates these families of functions and the corresponding g function for the family of functions $\{\varphi_p(x)\}_{p\in[0,+\infty)}$.

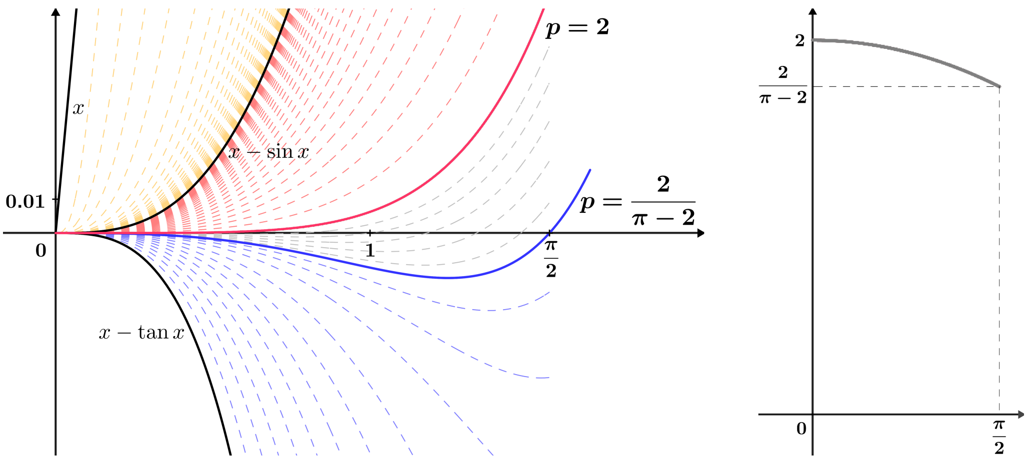

Figure 1. Stratified families of functions from Lemma 1 with the corresponding g function; see (7).

Let us emphasize that the value $A = \frac{2}{\pi-2}$ was determined for the first time in [62], while the value $B = 2$ is given by the Cusa–Huygens inequality (5). By utilizing the stratification, cases for other values of the parameter $p \in (-\infty, -1] \cup [0, +\infty)$ were examined and are visually depicted in Figure 1.

4.2. Application 2 (Wilker-Type Inequality)

In this paper, by the Wilker inequality, we consider the inequality from the following theorem:

Theorem 9 ([63])**.** *Let $x \in \left(0, \frac{\pi}{2}\right)$. Then, it holds that*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2\,. \tag{8}$$

Extensions and refinements of the Wilker inequalities have been considered in many papers [6,27,39,40,52–57,64–84]. In [39], the author L. Zhu proved the following statement:

Theorem 10. *Let $x \in \left(0, \frac{\pi}{2}\right)$. Then, it holds that*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{8}{45}x^4\left(\frac{\tan x}{x}\right)^{6/7}. \tag{9}$$

A new proof of Theorem 10 was given by R. Shinde, C. Chesneau, N. Darkunde, S. Ghodechor and A. Lagad in [40]. In this paper, we provide a significantly simpler proof of the previous theorem using the parametric method.

In order to provide a new proof and refine the previous theorem, we will introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P}=\mathbb{R}$, where

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2-px^4\left(\frac{\tan x}{x}\right)^{6/7}, \tag{10}$$

for the argument $x\in\mathbb{S}=(0,\pi/2)$.

The following assertion holds:

Lemma 2. *The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval $\mathbb{S}=(0,\pi/2)$.*

Proof. It holds that $\dfrac{\partial\varphi_p(x)}{\partial p}=-x^4\left(\dfrac{\tan x}{x}\right)^{6/7}<0$ on the interval $(0,\pi/2)$ for $p\in\mathbb{R}$. □

By applying Theorem 5, we provide the proof of the following statement:

Statement 2. *Let*:

$$A=\frac{8}{45}=0.1\overline{7}\,.$$

Then, it holds:

(i) If $p\in(-\infty,A]$, then

$$\left(\forall x\in\left(0,\frac{\pi}{2}\right)\right)\;\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2>Ax^4\left(\frac{\tan x}{x}\right)^{6/7}\geq px^4\left(\frac{\tan x}{x}\right)^{6/7}.$$

(ii) If $p\in(A,+\infty)$, then the equality

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2-px^4\left(\frac{\tan x}{x}\right)^{6/7}=0$$

has a unique solution $x_0^{(p)}$ and it holds that

$$\left(\forall x\in\left(0,x_0^{(p)}\right)\right)\;\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2<px^4\left(\frac{\tan x}{x}\right)^{6/7}$$

and

$$\left(\forall x\in\left(x_0^{(p)},\frac{\pi}{2}\right)\right)\;\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2>px^4\left(\frac{\tan x}{x}\right)^{6/7}.$$

Proof. The following equivalence holds:

$$\begin{aligned}&\varphi_p(x)=\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2-px^4\left(\frac{\tan x}{x}\right)^{6/7}=0\\ \iff\quad & p=g(x)=\frac{\sin^2 x\cos x-2x^2\cos x+x\sin x}{x^6\left(\dfrac{\tan x}{x}\right)^{6/7}\cos x},\end{aligned} \tag{11}$$

through which we introduce the continuous function $g(x)$ on the interval $(0, \pi/2)$. Let us examine the monotonicity of the function $g(x)$ on that domain. The first derivative of the function $g(x)$ is

$$g'(x) = \frac{f(x)}{7\left(\dfrac{\tan x}{x}\right)^{6/7} x^7 \cos^2 x \sin x},$$

where $f(x)$ is the MTP function given by

$$\begin{aligned} f(x) &= -14x\cos^5 x + 36\cos^4 x \sin x + 49x\cos^3 x + 44x^2\cos^2 x \sin x \\ &\quad -36\cos^2 x \sin x + 12x^3 \cos x - 35x\cos x + x^2 \sin x\,. \end{aligned}$$

Let us prove that $f(x) > 0$ on the interval $(0, \pi/2)$ by applying the method for proving MTP inequalities.

The MTP function $f(x)$ in terms of multiple angles is given by

$$\begin{aligned} f(x) &= -\tfrac{7}{8}x\cos 5x + \tfrac{63}{8}x\cos 3x + 12x^3\cos x - 7x\cos x \\ &\quad +\tfrac{9}{4}\sin 5x + 11x^2\sin 3x - \tfrac{9}{4}\sin 3x + 12x^2 \sin x - \tfrac{9}{2}\sin x\,. \end{aligned}$$

By approximating cosine functions with the Maclaurin polynomials of degree 20 in negative addends and degree 18 in positive addends, and sine functions with the Maclaurin polynomials of degree 21 in negative addends and degree 19 in positive addends, we obtain the downward polynomial approximation

$$\begin{aligned} P(x) &= -\tfrac{7}{8}xT_{20}^{\cos,0}(5x) + \tfrac{63}{8}xT_{18}^{\cos,0}(3x) + 12x^3T_{18}^{\cos,0}(x) - 7xT_{20}^{\cos,0}(x) \\ &\quad +\tfrac{9}{4}T_{19}^{\sin,0}(5x) + 11x^2T_{19}^{\sin,0}(3x) - \tfrac{9}{4}T_{21}^{\sin,0}(3x) + 12x^2T_{19}^{\sin,0}(x) - \tfrac{9}{2}T_{21}^{\sin,0}(x) \\ &= -\tfrac{1562461991350829}{45414170819297280000}x^{21} + \tfrac{1868039}{10854718875}x^{19} \\ &\quad -\tfrac{4712}{2837835}x^{17} + \tfrac{791792}{70945875}x^{15} - \tfrac{2368}{51975}x^{13} + \tfrac{128}{1575}x^{11} \end{aligned}$$

of the function $f(x)$ on the interval $(0, \pi/2)$. By applying Sturm's theorem to the polynomial $P(x)$ over the extended segment with rational endpoints $[-0.001, 1.571]$, it can be concluded that the polynomial $P(x)$ has exactly one zero on that segment, which is evidently attained at the point $x = 0$. On the interval $(0, \pi/2)$, the polynomial $P(x)$ is positive since $P(1.571) = 2.16927\ldots > 0$. Thus, it holds that

$$f(x) > P(x) > 0$$

on the interval $(0, \pi/2)$. Hence, $g'(x) > 0$ on $(0, \pi/2)$; thus, the function $g(x)$ is increasing on the observed interval.

Based on Lemma 2, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ is decreasingly stratified on the interval $(0, \pi/2)$.

Notice that

$$\lim_{x\to 0+} g(x) = \frac{8}{45} = A \quad \text{and} \quad \lim_{x\to\pi/2-} g(x) = +\infty = B\,.$$

Note that $A \in \mathbb{R}$. Hence, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ satisfies all the conditions for the application of Theorem 5, which concludes the proof. □

Figure 2 illustrates the stratified family of functions from Lemma 2. Cases for some values of the real parameter p are shown, and the known constant $(A = 8/45)$, obtained in Theorem 10, is highlighted.

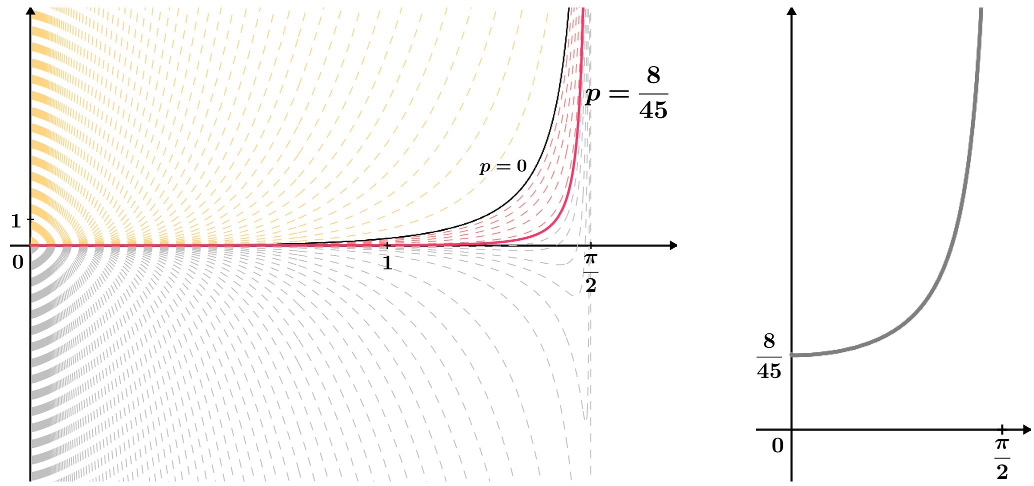

Figure 2. Stratified family of functions from Lemma 2 with the corresponding g function; see (11).

4.3. Applications 3 and 4 (Mitrinović–Adamović-Type Inequalities)

The Mitrinović–Adamović inequality is given by:

Theorem 11 ([85])**.** *Let* $x \in \left(0, \dfrac{\pi}{2}\right)$*. Then, it holds that*

$$\left(\frac{\sin x}{x}\right)^3 > \cos x\,. \tag{12}$$

Extensions and refinements of the Mitrinović–Adamović inequality have been considered in many papers [35,41,56–61,82–84,86–91]. In [41], the authors L. Zhu and R. Zhang gave the two following extensions:

Theorem 12. *Let* $x \in \left(0, \dfrac{\pi}{2}\right)$*. Then, it holds that*

$$0.06593\ldots x^3 \sin x < \left(\frac{\sin x}{x}\right)^3 - \cos x < \frac{1}{15}x^3 \sin x \tag{13}$$

and the constants $0.06593\ldots$ *and* $\frac{1}{15}$ *are the best possible.*

Theorem 13. *Let* $x \in \left(0, \dfrac{\pi}{2}\right)$*. Then, it holds that*

$$\frac{1}{15}x^4\left(\frac{\sin x}{x}\right)^{23/21} < \left(\frac{\sin x}{x}\right)^3 - \cos x < \left(\frac{2}{\pi}\right)^{124/21} x^4\left(\frac{\sin x}{x}\right)^{23/21} \tag{14}$$

and the constants $\frac{1}{15}$ *and* $\left(\frac{2}{\pi}\right)^{124/21}$ *are the best possible.*

Theorems 12 and 13 were proved in [41] in a manner similar to the parametric method described in this paper. However, the authors did not introduce stratified families of functions, nor did they use the method for proving MTP inequalities. In the following, we will prove Theorems 12 and 13 using the parametric method and show that this proof is simpler than the original proof from [41]. Additionally, by applying this method, we will obtain further refinements of Theorems 12 and 13 for parameter values for which these inequalities have not been previously considered.

4.3.1. Application 3

In this part, we provide new proof and refinement of Theorem 12. For this purpose, we will introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P}=\mathbb{R}$, where

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^3-\cos x-px^3\sin x\,, \tag{15}$$

for the argument $x\in\mathbb{S}=(0,\pi/2)$.

The following assertion holds:

Lemma 3. *The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval $\mathbb{S}=(0,\pi/2)$.*

Proof. It holds that $\dfrac{\partial\varphi_p(x)}{\partial p}=-x^3\sin x<0$ on the interval $(0,\pi/2)$ for $p\in\mathbb{R}$. □

By applying Theorem 7, we provide the proof of the following statement:

Statement 3. *Let*:

$$A=0.065931\ldots\,,\quad B=\frac{64}{\pi^6}=0.066570\ldots\quad \textit{and}\quad C=\frac{1}{15}=0.0\overline{6}\,.$$

The value $0.06593\ldots$ *is the unique minimum of the function* $g(x)=(\sin^3 x-x^3\cos x)/(x^6\sin x)$ *on the interval* $(0,\pi/2)$.
Then, it holds:

(i) *If* $p\in(-\infty,A]$, *then*

$$\left(\forall x\in\left(0,\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x>Ax^3\sin x\geq px^3\sin x\,.$$

(ii) *If* $p\in(A,B)$, *then the equality*

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^3-\cos x-px^3\sin x=0$$

has exactly two solutions $x_0^{(p)}$ *and* $x_1^{(p)}$, *and it holds that*

$$\left(\forall x\in\left(0,x_0^{(p)}\right)\cup\left(x_1^{(p)},\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x>px^3\sin x$$

and

$$\left(\forall x\in\left(x_0^{(p)},x_1^{(p)}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x<px^3\sin x\,.$$

(iii) *If* $p\in[B,C)$, *then the equality*

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^3-\cos x-px^3\sin x=0$$

has a unique solution $x_0^{(p)}$ *and it holds that*

$$\left(\forall x\in\left(0,x_0^{(p)}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x>px^3\sin x$$

and

$$\left(\forall x\in\left(x_0^{(p)},\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x<px^3\sin x\,.$$

(iv) If $p\in[C,+\infty)$, *then*

$$\left(\forall x\in\left(0,\frac{\pi}{2}\right)\right)\ \left(\frac{\sin x}{x}\right)^3-\cos x<Cx^3\sin x\le px^3\sin x\,.$$

Proof. The following equivalence holds:

$$\varphi_p(x)=\left(\frac{\sin x}{x}\right)^3-\cos x-px^3\sin x=0\quad\Longleftrightarrow\quad p=g(x)=\frac{\sin^3 x-x^3\cos x}{x^6\sin x},\quad(16)$$

through which we introduce the continuous function $g(x)$ on the interval $(0,\pi/2)$. We will show that the function g has exactly one minimum on that domain. For this purpose, let us consider the first derivative of the function $g(x)$

$$g'(x)=\frac{f(x)}{x^7\sin^2 x},\qquad(17)$$

where $f(x)$ is the MTP function given by

$$f(x)=-6\cos^4 x-2x\cos^3 x\sin x+12\cos^2 x+3x^3\cos x\sin x+2x\cos x\sin x+x^4-6\,.$$

The MTP function $f(x)$ in terms of multiple angles is given by

$$f(x)=-\tfrac{3}{4}\cos 4x+3\cos 2x-\tfrac{1}{4}x\sin 4x+\left(\tfrac{3}{2}x^3+\tfrac{1}{2}x\right)\sin 2x+x^4-\tfrac{9}{4}\,.$$

Let us apply the method for isolating the zeros of the MTP function $f(x)$ by selecting the points $a_0=1.1$ and $b_0=1.2$ on the interval $(a,b)=(0,\pi/2)$ such that Theorem 1 can be applied.

1. We prove that $f(x)<0$ for $x\in(a,a_0]=(0,1.1]$ and that $f(x)>0$ for $x\in[b_0,b)=[1.2,\pi/2)$ by applying the method for proving MTP inequalities.

1.1. $x\in(0,1.1]$

By approximating the functions $\cos 4x$, $\cos 2x$, $\sin 4x$ and $\sin 2x$ with the Maclaurin polynomials of degrees 18, 16, 15 and 13, respectively, we obtain the downward polynomial approximation

$$\begin{aligned}P_1(x)&=-\tfrac{3}{4}T_{18}^{\cos,0}(4x)+3T_{16}^{\cos,0}(2x)-\tfrac{1}{4}xT_{15}^{\sin,0}(4x)+\left(\tfrac{3}{2}x^3+\tfrac{1}{2}x\right)T_{13}^{\sin,0}(2x)+x^4-\tfrac{9}{4}\\&=\tfrac{262144}{32564156625}x^{18}+\tfrac{34}{637875}x^{16}-\tfrac{8}{17325}x^{14}+\tfrac{2}{945}x^{12}-\tfrac{2}{945}x^{10}\end{aligned}$$

of the function $f(x)$ on the interval $(0,1.1]$. By applying Sturm's theorem to the polynomial $P_1(x)$ over the extended segment with rational endpoints $[-0.001,1.1]$, it can be concluded that the polynomial $P_1(x)$ has exactly one zero on that segment, which is evidently attained at the point $x=0$. On the interval $(0,1.1]$, the polynomial $P_1(x)$ is negative since $P_1(1.1)=-0.00031108\ldots<0$. Thus, it holds that

$$f(x)<P_1(x)<0$$

on the interval $(0,1.1]$.

1.2. $x \in [1.2, \pi/2)$
By approximating the functions $\cos 4x$, $\cos 2x$, $\sin 4x$ and $\sin 2x$ with the Maclaurin polynomials of degrees 16, 14, 17 and 15, respectively, we obtain the downward polynomial approximation

$$\begin{aligned} P_2(x) &= -\tfrac{3}{4}T_{16}^{\cos,0}(4x) + 3T_{14}^{\cos,0}(2x) - \tfrac{1}{4}xT_{17}^{\sin,0}(4x) + \left(\tfrac{3}{2}x^3 + \tfrac{1}{2}x\right)T_{15}^{\sin,0}(2x) + x^4 - \tfrac{9}{4} \\ &= -\tfrac{26296}{2170943775}x^{18} + \tfrac{4}{75075}x^{16} - \tfrac{8}{17325}x^{14} + \tfrac{2}{945}x^{12} - \tfrac{2}{945}x^{10} \end{aligned}$$

of the function $f(x)$ on the interval $[1.2, \pi/2)$. By applying Sturm's theorem to the polynomial $P_2(x)$ over the extended segment with rational endpoints $[1.2, 1.571]$, it can be concluded that the polynomial $P_2(x)$ has no zeros on that segment. On the interval $[1.2, \pi/2)$, the polynomial $P_2(x)$ is positive since $P_2(1.571) = 0.059140\ldots > 0$. Thus, it holds that

$$f(x) > P_2(x) > 0$$

on the interval $[1.2, \pi/2)$.
2. We prove that $f'(x) > 0$ for $x \in [1.1, 1.2]$.
It holds that

$$\begin{aligned} f'(x) &= -8x\cos^4 x + 22\cos^3 x \sin x + 6x^3\cos^2 x + 10x\cos^2 \\ &\quad +9x^2\cos x \sin x - 22\cos x \sin x + x^3 - 2x\,. \end{aligned}$$

Let us apply the method for proving MTP inequalities. The MTP function $f'(x)$ in terms of multiple angles is given by

$$f'(x) = -x\cos 4x + \left(3x^3 + x\right)\cos 2x + \tfrac{11}{4}\sin 4x + \tfrac{9}{2}x^2\sin 2x - \tfrac{11}{2}\sin 2x + 4x^3\,.$$

By approximating the functions $\cos 4x$, $\cos 2x$ and $\sin 4x$ with the Maclaurin polynomials of degrees 16, 14 and 15, respectively, and the function $\sin 2x$ with the Maclaurin polynomial of degree 15 in the addend $\frac{9}{2}x^2\sin 2x$ and with the Maclaurin polynomial of degree 17 in the addend $-\frac{11}{2}\sin 2x$, we obtain the downward polynomial approximation

$$\begin{aligned} P_3(x) &= -xT_{16}^{\cos,0}(4x) + (3x^3 + x)T_{14}^{\cos,0}(2x) + \tfrac{11}{4}T_{15}^{\sin,0}(4x) \\ &\quad +\tfrac{9}{2}x^2T_{15}^{\sin,0}(2x) - \tfrac{11}{2}T_{17}^{\sin,0}(2x) + 4x^3 \\ &= -\tfrac{63874}{310134825}x^{17} + \tfrac{181472}{212837625}x^{15} - \tfrac{16}{2475}x^{13} + \tfrac{8}{315}x^{11} - \tfrac{4}{189}x^9 \end{aligned}$$

of the function $f'(x)$ on the interval $[1.1, 1.2]$. By applying Sturm's theorem to the polynomial $P_3(x)$ over the segment with rational endpoints $[1.1, 1.2]$, it can be concluded that the polynomial $P_3(x)$ has no zeros on that segment. On the interval $[1.1, 1.2]$, the polynomial $P_3(x)$ is positive since $P_3(1.2) = 0.018898\ldots > 0$. Thus, it holds that

$$f'(x) > P_3(x) > 0$$

on the interval $[1.1, 1.2]$.

According to Theorem 1, there exists exactly one zero of the function $f(x)$. Given that $f(1.15510) = -0.67427\ldots 10^{-7} < 0$ and $f(1.15511) = 0.43311\ldots 10^{-7} > 0$, the zero of the function $f(x)$ is numerically determined as

$$x^* = 1.15510\ldots \in [1.1, 1.2]\,.$$

Considering that $f(x) < 0$ on the interval $(0, 1.1]$ and that $f(x) > 0$ on the interval $[1.2, \pi/2)$, it also holds that $g'(x) = f(x)/x^7\sin^2 x < 0$ on the interval $(0, 1.1]$ and $g'(x) > 0$

on the interval $[1.2, \pi/2)$. Based on this, we conclude that at the point $x^* = 1.15510\ldots$, the function g has a minimum
$$g(x^*) = 0.065931\ldots = A\,.$$

Based on Lemma 3, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ is decreasingly stratified on the interval $(0, \pi/2)$.

Notice that
$$\lim_{x\to 0} g(x) = \frac{1}{15} = C \quad \text{and} \quad \lim_{x\to\pi/2} g(x) = \frac{64}{\pi^6} = B\,.$$

Note that $B, C \in \mathbb{R}$. Hence, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ satisfies all the conditions for the application of Theorem 7, which concludes the proof. □

Remark 6. *It was also possible to localize the zero x^* using the method from* [25]*. By using the Maple library implemented within the scope of* [25]*, we obtain $x^* \in [1.15437\ldots, 1.15669\ldots]$ when choosing $\delta = 0.1$, where δ represents the maximal length of isolating intervals, without displaying all steps in the proof.*

Figure 3 illustrates the stratified family of functions from Lemma 3. Cases for some values of the real parameter p are shown, and the known constants ($A = 0.065931\ldots$, $B = 64/\pi^6$ and $C = 1/15$), obtained in Theorem 12 and Statement 3, are highlighted.

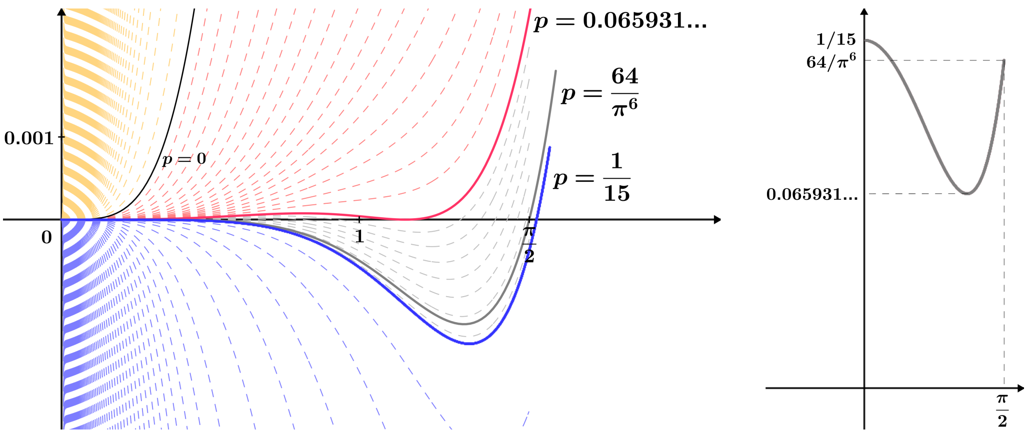

Figure 3. Stratified family of functions from Lemma 3 with the corresponding g function; see (16).

4.3.2. Application 4

In this part, we provide new proof and refinement of Theorem 13. For this purpose, we will introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P} = \mathbb{R}$, where
$$\varphi_p(x) = \left(\frac{\sin x}{x}\right)^3 - \cos x - px^4\left(\frac{\sin x}{x}\right)^{23/21}, \tag{18}$$
for the argument $x \in \mathbb{S} = (0, \pi/2)$.

The following assertion holds:

Lemma 4. *The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval $\mathbb{S} = (0, \pi/2)$.*

Proof. It holds that $\dfrac{\partial\varphi_p(x)}{\partial p} = -x^4\left(\dfrac{\sin x}{x}\right)^{23/21} < 0$ on the interval $(0, \pi/2)$ for $p \in \mathbb{R}$. □

By applying Theorem 5, we provide the proof of the following statement:

Statement 4. *Let*:

$$A = \frac{1}{15} = 0.0\overline{6} \quad \textit{and} \quad B = \left(\frac{2}{\pi}\right)^{124/21} = 0.069495\ldots.$$

Then, it holds:

(*i*) *If* $p \in (-\infty, A]$, *then*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \left(\frac{\sin x}{x}\right)^3 - \cos x > Ax^4 \left(\frac{\sin x}{x}\right)^{23/21} \geq px^4 \left(\frac{\sin x}{x}\right)^{23/21}.$$

(*ii*) *If* $p \in (A, B)$, *then the equality*

$$\varphi_p(x) = \left(\frac{\sin x}{x}\right)^3 - \cos x - px^4 \left(\frac{\sin x}{x}\right)^{23/21} = 0$$

has a unique solution $x_0^{(p)}$ *and it holds that*

$$\left(\forall x \in \left(0, x_0^{(p)}\right)\right) \ \left(\frac{\sin x}{x}\right)^3 - \cos x < px^4 \left(\frac{\sin x}{x}\right)^{23/21}$$

and

$$\left(\forall x \in \left(x_0^{(p)}, \frac{\pi}{2}\right)\right) \ \left(\frac{\sin x}{x}\right)^3 - \cos x > px^4 \left(\frac{\sin x}{x}\right)^{23/21}.$$

(*iii*) *If* $p \in [B, +\infty)$, *then*

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \left(\frac{\sin x}{x}\right)^3 - \cos x < Bx^4 \left(\frac{\sin x}{x}\right)^{23/21} \leq px^4 \left(\frac{\sin x}{x}\right)^{23/21}.$$

Proof. The following equivalence holds:

$$\begin{aligned} & \varphi_p(x) = \left(\frac{\sin x}{x}\right)^3 - \cos x - px^4 \left(\frac{\sin x}{x}\right)^{23/21} = 0 \\ \iff \quad & p = g(x) = \frac{\sin^3 x - x^3 \cos x}{x^7 \left(\frac{\sin x}{x}\right)^{23/21}}, \end{aligned} \tag{19}$$

through which we introduce the continuous function $g(x)$ on the interval $(0, \pi/2)$. Let us examine the monotonicity of the function $g(x)$ on that domain. The first derivative of the function $g(x)$ is

$$g'(x) = \frac{f(x)}{21 \left(\frac{\sin x}{x}\right)^{2/21} x^7 \sin^2 x},$$

where $f(x)$ is the MTP function given by

$$\begin{aligned} f(x) \ = \ & -124 \cos^4 x - 40x \cos^3 x \sin x + 2x^4 \cos^2 x + 248 \cos^2 x \\ & +61x^3 \cos x \sin x + 40x \cos x \sin x + 21x^4. \end{aligned}$$

Let us prove that $f(x) > 0$ on the interval $(0, \pi/2)$. The MTP function $f(x)$ in terms of multiple angles is given by

$$f(x) = -\tfrac{31}{2} \cos 4x + (x^4 + 62) \cos 2x - 5x \sin 4x + (\tfrac{61}{2} x^3 + 10x) \sin 2x + 22x^4 + \tfrac{155}{2}.$$

Based on the evaluation of individual addends, it is evident that

$$f(x) > 0$$

on the interval $(0, \pi/2)$. Hence, $g'(x) > 0$ on $(0, \pi/2)$; thus, the function $g(x)$ is increasing on the observed interval.

Based on Lemma 4, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ is decreasingly stratified on the interval $(0, \pi/2)$.

Notice that

$$\lim_{x\to 0+} g(x) = \frac{1}{15} = A \quad \text{and} \quad \lim_{x\to \pi/2-} g(x) = \left(\frac{2}{\pi}\right)^{124/21} = B\,.$$

Note that $A, B \in \mathbb{R}$. Hence, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{R}}$ satisfies all the conditions for the application of Theorem 5, which concludes the proof. □

Figure 4 illustrates the stratified family of functions from Lemma 4. Cases for some values of the real parameter p are shown, and the known constants ($A = 1/15$ and $B = (2/\pi)^{124/21}$), obtained in Theorem 13, are highlighted.

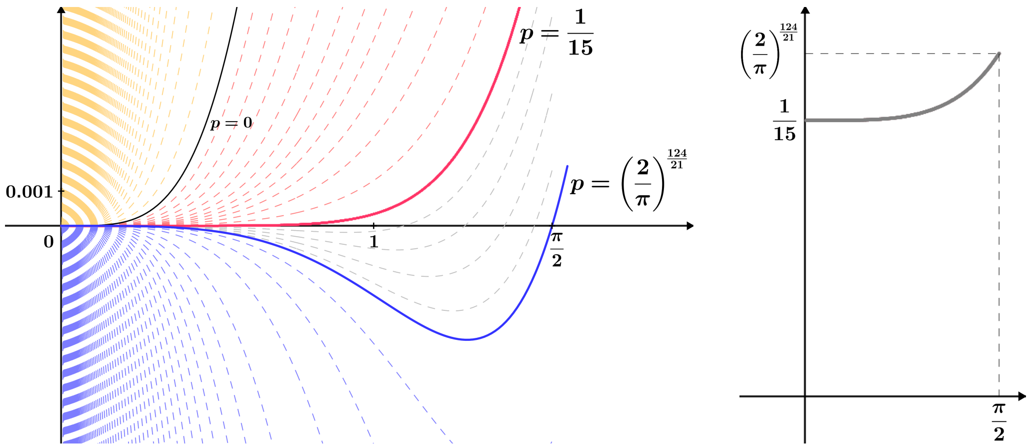

Figure 4. Stratified family of functions from Lemma 4 with the corresponding g function; see (19).

5. Conclusions

In this paper, a method for proving inequalities via a function g, when the equivalence (4) holds and when the function g is continuous, is described. The cases when the function g is not explicitly given or when g is discontinuous will be discussed in following papers.

Let us especially emphasize that, in this paper, the concept of stratification is specified compared to [1], where it was originally introduced. With the aim of examining the monotonicity of the function g, methods for isolating zeros and extrema of MTP functions, which are based on a method for proving MTP inequalities from [11], have been described. These methods allow verification of steps as given in the proofs of the statements in this paper. The described method from [11] is also computer-implemented [92]. Therefore, it can be said that proof of inequality, in cases where examining the monotonicity of the function g is reduced to examining the positivity of MTP functions, is an algorithmically solvable problem using the parametric method as described in this paper.

Connecting inequalities with the corresponding stratified family of functions and then obtaining the corresponding g function is applicable to an exceptionally large number of inequalities in the theory of analytic inequalities [93–100]. It is particularly noteworthy to emphasize that by applying this method, additional refinements of inequalities for various parameter values can be obtained.

Author Contributions: Conceptualization, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); methodology, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); software, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); validation, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); formal analysis, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); investigation, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); data curation, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); writing—original draft preparation, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); writing—review and editing, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović); visualization, B.M. (Branko Malešević), M.M. and B.M. (Bojana Mihailović). All authors have read and agreed to the published version of the manuscript.

Funding: This work was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract numbers: 451-03-65/2024-03/200103 (for the first and third authors) and 451-03-66/2024-03/200103 (for the second author).

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors wish to express their gratitude to the referees for their thorough reading of the paper and their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviation

The following abbreviation is used in this manuscript:

MTP	Mixed Trigonometric Polynomial

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