New Improvements of the Jensen–Mercer Inequality for Strongly Convex Functions with Applications

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Abstract: In this paper, we use the generalized version of convex functions, known as strongly convex functions, to derive improvements to the Jensen–Mercer inequality. We achieve these improvements through the newly discovered characterizations of strongly convex functions, along with some previously known results about strongly convex functions. We are also focused on important applications of the derived results in information theory, deducing estimates for χ-divergence, Kullback–Leibler divergence, Hellinger distance, Bhattacharya distance, Jeffreys distance, and Jensen–Shannon divergence. Additionally, we prove some applications to Mercer-type power means at the end.

Keywords: convex and strongly convex functions; Jensen inequality; Jensen–Mercer inequality; strong f-divergences

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1. Introduction

The class of convex functions is widely used in numerous scientific fields. Several generalizations and variants of this class of functions have been introduced from different perspectives [1]. One important generalization of convex functions is the class of strongly convex functions. This class, initially developed by Polyak in 1966, has attracted the interest of numerous mathematicians due to its significant applications in various branches of mathematics, particularly in optimization theory [2].

Let I denote a real interval. It states that if

$$f(\kappa x + (1-\kappa)y) \leq \kappa f(x) + (1-\kappa)f(y) - c\kappa(1-\kappa)(y-x)^2$$  \hspace{1cm} (1)

holds for all $x,y \in I$, $\kappa \in [0,1]$ and for some real number $c > 0$, then $f$ is said to be a strongly convex function with modulus $c$. If $-f$ is strongly convex, then we say that a function $f$ is strongly concave. In other words, $f$ is strongly concave if

$$f(\kappa x + (1-\kappa)y) \geq \kappa f(x) + (1-\kappa)f(y) + c\kappa(1-\kappa)(y-x)^2$$  \hspace{1cm} (2)

holds for all $x,y \in I$ and $\kappa \in [0,1]$.

Throughout this manuscript, SCF is the abbreviation for strongly convex function. It is evident that $c\kappa(1-\kappa)(x-y)^2 \geq 0$; therefore, every SCF is also convex but the converse does not hold in general. Likewise, strong concavity implies ordinary concavity, but the reverse implication does not hold in general.
Example 1. Let \( a > 0 \) and the functions \( f, g, h, i: (0, a] \rightarrow \mathbb{R} \) be defined by \( f(t) = t \ln t, \) \( g(t) = (t - 1) \ln t, \) \( h(t) = -\ln t \) and \( i(t) = |t| \). Then, \( f, g, h, i \) are SCFs with modulus \( \frac{1}{e}, \frac{1}{2e} \) and \( \frac{1}{2e} \), respectively, and clearly, the function \( i \) is convex but not an SCF.

Remark 1. If \( z_1, z_2, z_3 \) are three points in \( I \) such that \( z_1 < z_2 < z_3 \), then (1) is equivalent to

\[
f(z_2) \leq \frac{z_3 - z_2}{z_3 - z_1} f(z_1) + \frac{z_2 - z_1}{z_3 - z_1} f(z_3) - c(z_2 - z_1)(z_3 - z_2). \tag{3}
\]

Compared with convex functions, the SCFs possess stronger versions of the analogous properties. Some of their useful characterizations are given in the following lemmas (see [3] (p. 268) as well as [4,5] and the references given therein).

Lemma 1. If \( f: I \rightarrow \mathbb{R} \) is a function, then the function \( T(x) = f(x) - cx^2 \) is convex for some \( c > 0 \) if and only if \( f \) is SCF with modulus \( c \).

The second lemma is a characterization for twice differentiable functions.

Lemma 2. Let \( f: I \rightarrow \mathbb{R} \) be a function such \( f'' \) exists. Then, \( f \) is SCF with modulus \( c > 0 \) if and only if \( f'' \geq 2c \).

While SCFs have been extensively researched and appear frequently in the literature, strongly concave functions are poorly represented, although their application can be widely used. We will give special emphasis to such a class of functions by stating our main results that also include strongly concave functions.

If we interpret the previous characterizations in terms of strongly concave functions, we can conclude the following. A function \( f \) is strongly concave with modulus \( c > 0 \) if the function \( T(x) = f(x) + cx^2 \) is concave. Also, a twice differentiable function \( f \) is strongly concave with modulus \( c > 0 \) if \( f'' \leq -2c \). Using this fact, considering specific strongly concave functions in the last section, we derive new estimates for Mercer-type means.

Below, we mention several variants of the well-known inequalities that are valid for SCFs. The first of them is Jensen’s inequality (see [5]).

Theorem 1. Let \( f: I \rightarrow \mathbb{R} \) be an SCF with modulus \( c > 0 \). Let \( z_i \in I \) and \( p_i \geq 0, i = 1, \ldots, n \), be such that \( \sum_{i=1}^{n} p_i = 1 \) and \( \bar{z}_p = \sum_{i=1}^{n} p_i z_i \). Then

\[
f(\bar{z}_p) \leq \sum_{i=1}^{n} p_i f(z_i) - c \sum_{i=1}^{n} p_i (z_i - \bar{z}_p)^2. \tag{4}
\]

Remark 2. Since \( c \sum_{i=1}^{n} p_i (z_i - \bar{z}_p)^2 \) is non-negative, from (4), we deduce Jensen’s inequality for convex functions as \( (f') \)

\[
f(\bar{z}_p) \leq \sum_{i=1}^{n} p_i f(z_i), \tag{5}
\]

and hence inequality (4) provides a better upper bound for \( f(\bar{z}_p) \) than (5).

By taking \( n = 2, p_1 = \frac{1}{2}, p_2 = \frac{1}{2} \) as the difference between right and left sides of (4), we deduce Jensen’s functional in the form

\[
\mathcal{J}_f(z_1, z_2) = f(z_1) + f(z_2) - 2f\left(\frac{z_1 + z_2}{2}\right) - c\left(z_1 - z_2\right)^2. \tag{6}
\]

Note that the non-negativity of Jensen’s functional (6) is a consequence of Jensen’s inequality (4).
If $f$ is strongly concave with modulus $c$, then
\[ f(\bar{z}_p) \geq \sum_{i=1}^{n} p_i f(z_i) + c \sum_{i=1}^{n} p_i (z_i - \bar{z}_p)^2. \tag{7} \]

In that case, Jensen’s functional has the form
\[ \mathcal{J}_c(-f; z_1, z_2) = f(z_1) + f(z_2) - 2f\left(\frac{z_1 + z_2}{2}\right) + 2c(z_2 - z_1)^2 \tag{8} \]
and it holds $\mathcal{J}_c(-f; z_1, z_2) \leq 0$.

The following variant of Jensen’s inequality, known as the Jensen-Mercer inequality, is presented in [7].

**Theorem 2.** Let $f : I \rightarrow \mathbb{R}$ be an SCF with modulus $c > 0$. Then
\[ f(\bar{z}_p) \leq f(q_1) + f(q_2) - \sum_{i=1}^{n} p_i f(z_i) - c \left(2(\bar{z}_p - q_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i (\bar{z}_p - z_i)^2\right), \tag{9} \]
where $q_1, q_2 \in I$, $q_1 < q_2$, $z_i \in [q_1, q_2]^n$, $\kappa_i \in [0, 1]$ such that $z_i = \kappa_i q_1 + (1 - \kappa_i) q_2$, $i = 1, 2, \ldots, n$, and $p = (p_1, \ldots, p_n)$ is a non-negative $n$-tuple with $\sum_{i=1}^{n} p_i = 1$ and $\bar{z}_p = \sum_{i=1}^{n} p_i z_i$.

**Remark 3.** Specifically, for $c = 0$, inequality (9) becomes
\[ f(q_1 + q_2 - \bar{z}_p) \leq f(q_1) + f(q_2) - \sum_{i=1}^{n} p_i f(z_i), \tag{10} \]
i.e., we get the Jensen-Mercer inequality for convex function proved in [8].

In [9], the authors applied the Jensen-Mercer inequality and derived several Hermite-Hadamard type inequalities for stochastic fractional integrals. Some properties of the Jensen-Mercer functional have been presented in [10]. This inequality has also been proved for the class of uniformly convex functions [11]. For more results about the Jensen-Mercer inequality, see for example [12–16].

Now, we quote recently obtained interpolating Jensen-type inequalities for SCFs [4] that present an improvement in results from [17], which has been pivotal in numerous recent investigations.

**Theorem 3.** Let $f : I \rightarrow \mathbb{R}$ be an SCF with modulus $c > 0$. Let $z = (z_1, \ldots, z_n) \in I^n$ and $p = (p_1, \ldots, p_n)$ be a non-negative $n$-tuple, $q = (q_1, \ldots, q_n)$ be a positive $n$-tuple with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$ and $\bar{z}_p = \sum_{i=1}^{n} p_i z_i$, $\bar{z}_Q = \sum_{i=1}^{n} q_i z_i$. Then
\begin{align*}
0 & \leq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left( \sum_{i=1}^{n} q_i f(z_i) - f(\bar{z}_Q) - c \sum_{i=1}^{n} q_i (z_i - \bar{z}_Q)^2 \right) \tag{11} \\
& \leq \sum_{i=1}^{n} p_i f(z_i) - f(\bar{z}_p) - c \sum_{i=1}^{n} p_i (z_i - \bar{z}_p)^2 \\
& \leq \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left( \sum_{i=1}^{n} q_i f(z_i) - f(\bar{z}_Q) - c \sum_{i=1}^{n} q_i (z_i - \bar{z}_Q)^2 \right).
\end{align*}
If, in addition, \( q = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \) and \( \bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i \), then (11) becomes

\[
0 \leq \min_{1 \leq i \leq n} \{ p_i \} \left( \sum_{i=1}^{n} f(z_i) - n f(\bar{z}) - c \sum_{i=1}^{n} (z_i - \bar{z})^2 \right) \tag{12}
\]

\[
\leq \sum_{i=1}^{n} p_i f(z_i) - f(\bar{z} p) - c \sum_{i=1}^{n} p_i (z_i - \bar{z} p)^2
\]

\[
\leq \max_{1 \leq i \leq n} \{ p_i \} \left( \sum_{i=1}^{n} f(z_i) - n f(\bar{z}) - c \sum_{i=1}^{n} (z_i - \bar{z})^2 \right).
\]

Important and interesting applications of the class SCFs are evidenced by numerous papers published recently. In [18], the authors proved several inequalities for SCFs related to their first-order divided differences and as applications derived from related majorization-type inequalities, addressing majorized tuples and applying conditions for majorization inequalities, as described by Maligranda et al. Interesting examples illustrate and support the obtained results. Related integral results are presented in [19]. Some companion inequalities to Jensen’s inequality, specifically Slater-type inequalities for SCFs, are given in [20]. More advanced and enhanced forms of SCFs have been presented in the literature. The concept of coordinate SCFs has been introduced in [21], and several inequalities have been derived for this new class of functions. In [22], a thorough study conducted and presented some interesting background and history of the generalized forms of SCFs. The SCFs play a certain role in quantum calculus, and numerous results can be found for quantum integral using the class SCFs. In [23], several Trapezoidal-type inequalities are given through integral identities obtained by using Hölder’s and power mean inequalities. The main findings are verified through graphical illustrations. As an interesting reference, we mention the paper [24] that deals with applications of SCFs with problems for differential equations. Using the support line inequality a converse of the Jensen’s inequality has been obtained in [25]. Some generalized and advanced applications of convexity are given in [26,27]. Several integral inequalities for the class of extended SCFs are established in [28,29].

In this paper, we highlight more accurate characterizations for SCFs, which can be derived as easy consequences of recently obtained result (12). Those characterizations enable us to obtain new improvements to the Jensen–Mercer-type inequality. To specify, we obtain improvements of the Jensen–Mercer inequality (10) and, moreover, the improvement of (9). We use the obtained results to derive new estimates for strong \( f \)-divergences, the concept of Csiszár \( f \)-divergences for SCFs introduced in [30]. As outcomes, we deduce estimates for particular cases as \( \chi \)-divergence, Kullback–Leibler divergence, Hellinger distance, Bhattacharya distance, Jeffreys distance and Jensen–Shannon divergence. As applications of our main results, we also derive new estimates for Shannon entropy. In the last section, we consider Mercer-type power means. Using improved Mercer-type inequalities, we establish new Mercer-type means inequalities that improve Mercer’s result from [31]. The application of newly obtained inequalities for strongly concave functions is particularly interesting in our proof.

2. Main Results

At the beginning of this section, we extract an important consequence from Theorem 3, which will play a key role in proving our main results.
Remark 4. Note that the second inequality in (12) improves Jensen’s inequality (4) for SCFs. Precisely stated, for \( f : I \to \mathbb{R} \) an SCF with modulus \( c > 0 \), \( z = (z_1, \ldots, z_n) \in I^n \), a non-negative \( n \)-tuple \( p = (p_1, \ldots, p_n) \) with \( \sum_{i=1}^n p_i = 1 \), \( \bar{z}_p = \sum_{i=1}^n p_i z_i \) and \( \bar{z} = \frac{1}{n} \sum_{i=1}^n z_i \), by (12) we have

\[
\begin{align*}
f(\bar{z}_p) &\leq \sum_{i=1}^n p_i f(z_i) - c \sum_{i=1}^n p_i (z_i - \bar{z}_p)^2 \\
&\quad - \min_{1 \leq i \leq n} \{ p_i \left( \sum_{i=1}^n f(z_i) - nf(\bar{z}) - c \sum_{i=1}^n (z_i - \bar{z}_p)^2 \right) \}.
\end{align*}
\]

where

\[
\min_{1 \leq i \leq n} \{ p_i \left( \sum_{i=1}^n f(z_i) - nf(\bar{z}) - c \sum_{i=1}^n (z_i - \bar{z}_p)^2 \right) \} \geq 0.
\]

Choosing \( n = 2 \), for \( \kappa \in [0,1] \), \( p_1 = \kappa, p_2 = 1 - \kappa \), \( z_1 = x \) and \( z_2 = y \), from (13) we get

\[
\begin{align*}
f(\kappa x + (1 - \kappa)y) &\leq \kappa f(x) + (1 - \kappa)f(y) - c\kappa(1 - \kappa)(y - x)^2 \\
&\quad - \min\{\kappa, 1 - \kappa\} \left[ f(x) + f(y) - 2f\left( \frac{x+y}{2} \right) - \frac{c}{2}(y-x)^2 \right].
\end{align*}
\]

Further, since \( \min\{\kappa, 1 - \kappa\} = \frac{1}{2}(1 - |1 - 2\kappa|) \) and using notation (6), inequality (14) can be written in the form

\[
\begin{align*}
f(\kappa x + (1 - \kappa)y) &\leq \kappa f(x) + (1 - \kappa)f(y) - c\kappa(1 - \kappa)(y - x)^2 \\
&\quad - \frac{1}{2}(1 - |1 - 2\kappa|) J_c(f;x,y).
\end{align*}
\]

At the end, substituting \( \kappa x + (1 - \kappa)y = z_2, x = z_1, y = z_3, 1 - \kappa = \frac{z_3 - z_2}{z_3 - z_1} \) and \( \kappa = \frac{z_3 - z_2}{z_3 - z_1} \) in (15), and using notation (6), for \( z_1 < z_2 < z_3 \), we get

\[
\begin{align*}
f(z_2) &\leq \frac{z_3 - z_2}{z_3 - z_1} f(z_1) + \frac{z_2 - z_1}{z_3 - z_1} f(z_3) - c(z_2 - z_1)(z_3 - z_2) \\
&\quad - \left( \frac{1}{2} - \frac{1}{z_3 - z_1} \right) \left( z_2 - \frac{z_1 + z_3}{2} \right) J_c(f;z_1,z_3).
\end{align*}
\]

Note that inequalities (15) and (16) present more accurate versions of (1) and (3), respectively.

To demonstrate the main results, we need the following lemma.

Lemma 3. Let \( I \subseteq \mathbb{R} \), \( e_1, e_2 \in I \), \( e_1 < e_2 \) and \( z_i \in [e_1, e_2], \kappa_i \in [0,1] \) such that \( z_i = \kappa_i e_1 + (1 - \kappa_i) e_2, i = 1, \ldots, n \). Then, for every \( f : I \to \mathbb{R} \), an SCF with modulus \( c > 0 \), we have

\[
\begin{align*}
f(e_1 + e_2 - z_i) &\leq f(e_1) + f(e_2) - f(z_i) - 2c(e_2 - e_1)^2 \kappa_i(1 - \kappa_i) \\
&\quad - (1 - |1 - 2\kappa_i|) J_c(f;e_1,e_2) \\
&\leq f(e_1) + f(e_2) - f(z_i) - 2c(e_2 - e_1)^2 \kappa_i(1 - \kappa_i) \\
&\leq f(e_1) + f(e_2) - f(z_i),
\end{align*}
\]

where

\[
J_c(f;e_1,e_2) = f(e_1) + f(e_2) - 2f\left( \frac{e_1 + e_2}{2} \right) - \frac{c}{2}(e_2 - e_1)^2.
\]
If $f$ is strongly concave, then
\[
f(e_1 + e_2 - z_i) \geq f(e_1) + f(e_2) - f(z_i) + 2c(e_2 - e_1)^2\kappa_i(1 - \kappa_i) - (1 - |1 - 2\kappa_i|)\mathcal{J}_c(-f; e_1, e_2) \\
\geq f(e_1) + f(e_2) - f(z_i) + 2c(e_2 - e_1)^2\kappa_i(1 - \kappa_i)
\]
where
\[
\mathcal{J}_c(-f; e_1, e_2) = f(e_1) + f(e_2) - 2f\left(\frac{e_1 + e_2}{2}\right) + \frac{c}{2}(e_2 - e_1)^2.
\]

**Proof.** For every $z_i \in [e_1, e_2]$, there exists a unique $\kappa_i \in [0, 1]$, $i \in \{1, \ldots, n\}$, such that
\[
z_i = (1 - \kappa_i)e_1 + \kappa_ie_2.
\]
Applying twice time (15), we have
\[
f(e_1 + e_2 - z_i) = f(e_1 + e_2 - (1 - \kappa_i)e_1 - \kappa_ie_2) \\
= f(\kappa_i e_1 + (1 - \kappa_i)e_2) \\
\leq \kappa_i f(e_1) + (1 - \kappa_i)f(e_2) - c\kappa_i(1 - \kappa_i)(e_2 - e_1)^2 \\
- \frac{1}{2}(1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2) \\
= f(e_1) + f(e_2) + \kappa_i f(e_1) - f(e_1) - \kappa_i f(e_2) - c\kappa_i(1 - \kappa_i)(e_2 - e_1)^2 \\
- \frac{1}{2}(1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2) \\
= f(e_1) + f(e_2) - [(1 - \kappa_i)f(e_1) + \kappa_i f(e_2)] - c\kappa_i(1 - \kappa_i)(e_2 - e_1)^2 \\
- \frac{1}{2}(1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2) \\
\leq f(e_1) + f(e_2) - f((1 - \kappa_i)e_1 + \kappa_ie_2) - 2c\kappa_i(1 - \kappa_i)(e_2 - e_1)^2 \\
- (1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2) \\
= f(e_1) + f(e_2) - f(z_i) - 2c\kappa_i(1 - \kappa_i)(e_2 - e_1)^2 \\
- (1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2)
\]
i.e., we get
\[
f(e_1 + e_2 - z_i) \leq f(e_1) + f(e_2) - f(z_i) - 2c(e_2 - e_1)^2\kappa_i(1 - \kappa_i) \\
- (1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2).
\]
Since
\[
2c(e_2 - e_1)^2\kappa_i(1 - \kappa_i) \geq 0 \text{ and } (1 - |1 - 2\kappa_i|)\mathcal{J}_c(f; e_1, e_2) \geq 0,
\]
then the series of inequalities (17) hold.

The last statement is a consequence of the fact that if $f$ is strongly concave, then $-f$ is an SCF. \qed

**Remark 5.** Lemma 3 presents an improvement of Theorem 2 for $n = 1$.

Now, we present generalization and improvement of the Jensen–Mercer inequality (10) as well as an improvement of inequality (9).

**Theorem 4.** Let $I \subseteq \mathbb{R}$, $e_1, e_2 \in I$, $e_1 < e_2$ and $z_i \in [e_1, e_2]$, $\kappa_i \in [0, 1]$ such that $z_i = \kappa_i e_1 + (1 - \kappa_i)e_2$, $i = 1, \ldots, n$. Let $p = (p_1, \ldots, p_n)$ be a non-negative $n$-tuple with
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\[ \sum_{i=1}^{n} p_i = 1, \tilde{z}_p = \sum_{i=1}^{n} p_i z_i \text{ and } \tilde{z} = \frac{1}{n} \sum_{i=1}^{n} z_i, \text{ and } \mathcal{J}_c(f; e_1, e_2), \mathcal{J}_c(-f; e_1, e_2) \text{ be defined by (18), (20), respectively. Then, for every } f : I \to \mathbb{R} \text{ an SCF with modulus } c > 0, \text{ we have} \]

\[ f(e_1 + e_2 - \tilde{z}_p) \leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) \quad (21) \]

\[ - c \left[ 2(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i (z_i - \tilde{z}_p)^2 \right] \]

\[ \mathcal{J}_c(f; e_1, e_2) \sum_{i=1}^{n} p_i (1 - |1 - 2 \kappa_i|) \]

\[ - \min_{1 \leq i \leq n} \{ p_i \} \left[ \sum_{i=1}^{n} f(e_1 + e_2 - z_i) - n f(e_1 + e_2 - \tilde{z}) - c \sum_{i=1}^{n} (\tilde{z} - z_i)^2 \right] \]

\[ \leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i). \]

If \( f \) is strongly concave, then

\[ f(e_1 + e_2 - \tilde{z}_p) \geq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) \quad (22) \]

\[ + c \left[ 2(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i (z_i - \tilde{z}_p)^2 \right] \]

\[ - \mathcal{J}_c(-f; e_1, e_2) \sum_{i=1}^{n} p_i (1 - |1 - 2 \kappa_i|) \]

\[ - \min_{1 \leq i \leq n} \{ p_i \} \left[ \sum_{i=1}^{n} f(e_1 + e_2 - z_i) - n f(e_1 + e_2 - \tilde{z}) + c \sum_{i=1}^{n} (\tilde{z} - z_i)^2 \right] \]

\[ \geq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) \]

\[ + c \left[ 2(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i (z_i - \tilde{z}_p)^2 \right] \]

\[ \geq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i). \]

**Proof.** Applying (13), we get

\[ f(e_1 + e_2 - \tilde{z}_p) \]

\[ = f \left( \sum_{i=1}^{n} p_i (e_1 + e_2 - z_i) \right) \]

\[ \leq \sum_{i=1}^{n} p_i f(e_1 + e_2 - z_i) - c \sum_{i=1}^{n} p_i (z_i - \tilde{z}_p)^2 \]

\[ - \min_{1 \leq i \leq n} \{ p_i \} \left[ \sum_{i=1}^{n} f(e_1 + e_2 - z_i) - n f(e_1 + e_2 - \tilde{z}) - c \sum_{i=1}^{n} (\tilde{z} - z_i)^2 \right]. \]
Using Lemma 3, we have
\[
\sum_{i=1}^{n} p_i f(e_i + e_2 - z_i) \leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) - 2c(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i)
\]
\[
- J_c(f; e_1, e_2) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|)
\]
\[
\leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) - 2c(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i)
\]
\[
\leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i).
\]

Now, combining (23) and (24), we get (21).

The last statement is a consequence of the fact that if \( f \) is strongly concave, then \(-f\) is an SCF. \(\square\)

**Theorem 5.** Let \( I \subseteq \mathbb{R}, e_1, e_2 \in I, e_1 < e_2, z = (z_1, \ldots, z_n) \in [e_1, e_2]^n \) and \( p = (p_1, \ldots, p_n) \) be a non-negative \( n \)-tuple with \( \sum_{i=1}^{n} p_i = 1 \) and \( \bar{z}_p = \sum_{i=1}^{n} p_i z_i \). Let \( J_c(f; e_1, e_2) \) and \( J_c(-f; e_1, e_2) \) be defined by (18) and (20), respectively. Then, for every \( f: I \to \mathbb{R} \) an SCF with modulus \( c > 0 \), we have
\[
f(e_1 + e_2 - \bar{z}_p) \leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) - 2c \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i)
\]
\[
- \left( \frac{1}{2} - \frac{1}{e_2 - e_1} \sum_{i=1}^{n} p_i \right) \left| z_i - \frac{e_1 + e_2}{2} \right| J_c(f; e_1, e_2)
\]
\[
\leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) - 2c \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i)
\]
\[
- 2 \left( \frac{1}{2} - \frac{1}{e_2 - e_1} \sum_{i=1}^{n} p_i \right) \left| z_i - \frac{e_1 + e_2}{2} \right| J_c(f; e_1, e_2)
\]
\[
\leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i) - 2c \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i)
\]
\[
\leq f(e_1) + f(e_2) - \sum_{i=1}^{n} p_i f(z_i). 
\]


If $f$ is strongly concave, then
\[
f(e_1 + e_2 - z_p)
\geq \frac{z_p - e_1}{e_2 - e_1} f(e_1) + \frac{e_2 - z_p}{e_2 - e_1} f(e_2) + \frac{1}{2} \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i)
\geq f(e_1) + f(e_2) - \frac{1}{2} \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i)
\geq f(e_1) + f(e_2) - \frac{1}{2} \sum_{i=1}^{n} p_i (z_i - e_1)(e_2 - z_i).
\]

**Proof.** Using (16), for every $x \in [e_1, e_2]$, we have
\[
f(x) \leq \frac{e_2 - x}{e_2 - e_1} f(e_1) + \frac{x - e_1}{e_2 - e_1} f(e_2) - c(x - e_1)(e_2 - x)
\leq \frac{1}{2} \left( x - \frac{e_1 + e_2}{2} \right) J_c(f; e_1, e_2).
\]

If we substitute $x$ with $e_1 + e_2 - z_i, i \in \{1, \ldots, n\}$, then (27) becomes
\[
f(e_1 + e_2 - z_i) \leq \frac{z_i - e_1}{e_2 - e_1} f(e_1) + \frac{e_2 - z_i}{e_2 - e_1} f(e_2) - c(z_i - e_1)(e_2 - z_i)
\leq \frac{1}{2} \left( z_i - \frac{e_1 + e_2}{2} \right) J_c(f; e_1, e_2).
\]

We denote
\[
f_{z_i, e_1, e_2} = \left( \frac{1}{2} \left( z_i - \frac{e_1 + e_2}{2} \right) J_c(f; e_1, e_2) \right).
\]

Further, we have
\[
\frac{z_i - e_1}{e_2 - e_1} f(e_1) + \frac{e_2 - z_i}{e_2 - e_1} f(e_2) - c(z_i - e_1)(e_2 - z_i) - f_{z_i, e_1, e_2}
\]
\[
= f(e_1) + f(e_2) - \left[ \frac{e_2 - z_i}{e_2 - e_1} f(e_1) + \frac{z_i - e_1}{e_2 - e_1} f(e_2) \right]
\leq c(z_i - e_1)(e_2 - z_i) - f_{z_i, e_1, e_2}
\leq f(e_1) + f(e_2) - f(z_i) - 2c(z_i - e_1)(e_2 - z_i) - 2f_{z_i, e_1, e_2}
\leq \frac{e_2 - z_i}{e_2 - e_1} f(e_1) + \frac{z_i - e_1}{e_2 - e_1} f(e_2) - f(z_i) - c(z_i - e_1)(e_2 - z_i) - f_{z_i, e_1, e_2}.
\]

Since $f$ is an SCF, then using (16), we have
\[
\frac{e_2 - z_i}{e_2 - e_1} f(e_1) + \frac{z_i - e_1}{e_2 - e_1} f(e_2) - f(z_i) - c(z_i - e_1)(e_2 - z_i) - f_{z_i, e_1, e_2} \geq 0.
\]
Now, combining (28) and (29), we get

\[ f(e_1 + e_2 - z_i) \leq z_i - \frac{e_1}{e_2 - e_1} f(e_1) + \frac{e_2 - z_i}{e_2 - e_1} f(e_2) - c(z_i - e_1)(e_2 - z_i) - f_{z_i,e_1,e_2} \]

Multiplying with \( p_i \) and summing over \( i, i = 1, \ldots, n \), the series of inequalities (30) and using the fact that

\[ f(e_1 + e_2 - z) = f \left( \sum_{i=1}^{n} p_i (e_1 + e_2 - z_i) \right) \leq \sum_{i=1}^{n} p_i f(e_1 + e_2 - z_i) \]

holds, we get (25).

The last statement is a consequence of the fact that if \( f \) is strongly concave, then \( -f \) is an SCF. \( \square \)

3. Applications to Strong \( f \)-Divergences and the Shannon Entropy

Let \( P_n = \{ p = (p_1, \ldots, p_n) : p_1, \ldots, p_n > 0, \sum_{i=1}^{n} p_i = 1 \} \) be the set of all complete finite discrete probability distributions. The restriction to positive distributions is only for convenience. If we take \( p_i = 0 \), for some \( i \in \{1, \ldots, n\} \), in the following results, we need to interpret undefined expressions as \( f(0) = \lim_{t \to 0^+} f(t) \), \( 0f(\frac{0}{0}) = 0 \) and \( 0f(\frac{\epsilon}{0}) = \lim_{\epsilon \to 0^+} f(\frac{\epsilon}{\epsilon}) = \epsilon \lim_{\epsilon \to 0^+} \frac{f(\epsilon)}{\epsilon}, \epsilon > 0 \).

I. Csiszár [32] introduced an important class of statistical divergences by means of convex functions.

**Definition 1.** Let \( f : (0, \infty) \to \mathbb{R} \) be a convex function and \( p, q \in P_n \). The Csiszár \( f \)-divergence is defined as

\[ D_f(q, p) = \sum_{i=1}^{n} p_i f \left( \frac{q_i}{p_i} \right). \] (31)

It has deep and fruitful applications in various branches of science (see, e.g., [33,34] with the references given therein) and is involved in the following Csiszár–Körner inequality (see [35]).

**Theorem 6.** Let \( p, q \in P_n \). If \( f : (0, \infty) \to \mathbb{R} \) is a convex function, then

\[ 0 \leq D_f(q, p) - f(1). \] (32)

**Remark 6.** If \( f \) is normalized, i.e., \( f(1) = 0 \), then it follows from (32) that

\[ 0 \leq D_f(q, p), \text{ with } D_f(q, p) = 0 \text{ if and only if } q = p. \] (33)

Two distributions \( q \) and \( p \) are very similar if \( D_f(q, p) \) is very close to zero.

Recently, in [30], a new concept of \( f \)-divergences was introduced.
**Definition 2.** Let \( f : (0, \infty) \to \mathbb{R} \) be an SCF with modulus \( c > 0 \) and \( p, q \in P_n \). Strong \( f \)-divergence is defined as

\[
\tilde{D}_f(q, p) = \sum_{i=1}^{n} p_i f\left( \frac{q_i}{p_i} \right).
\]

(34)

Accordingly, in [30], the following improvement of the Csiszár–Körner inequality for strong \( f \)-divergences was obtained, as follows.

**Theorem 7.** Let \( p, q \in P_n \). If \( f : (0, \infty) \to \mathbb{R} \) is an SCF with modulus \( c > 0 \), then

\[
0 \leq \tilde{D}_f(q, p) - f(1) - c \tilde{D}_{\chi^2}(q, p),
\]

(35)

where \( \tilde{D}_{\chi^2}(q, p) = \sum_{i=1}^{n} p_i \left( \frac{q_i}{p_i} \right)^2 - 1 \).

**Remark 7.** Here, \( \tilde{D}_{\chi^2}(q, p) = \sum_{i=1}^{n} p_i \left( \frac{q_i}{p_i} \right)^2 - 1 \) denotes strong chi-squared distance obtained for \( f(x) = (x - 1)^2 \) SCF with modulus \( c = 1 \).

Additionally, if \( f(1) = 0 \), then from (35), we have

\[
0 \leq c \tilde{D}_{\chi^2}(q, p) \leq \tilde{D}_f(q, p).
\]

(36)

Inequalities (35) and (36) improve (32) and (33).

In the sequel, we make use of the results from the previous sections in order to prove new estimates for strong \( f \)-divergences \( \tilde{D}_f(q, p) \).

**Corollary 1.** Let \( e_2 > e_1 > 0 \), \( p, q \in P_n \) and \( \frac{q_i}{p_i} \in [e_1, e_2] \), \( \kappa_i \in [0, 1] \) such that

\[
\frac{q_i}{p_i} = \kappa_i e_1 + (1 - \kappa_i) e_2, \quad i = 1, \ldots, n.
\]

Let us denote \( \frac{q}{p} = \frac{1}{n} \sum_{i=1}^{n} \frac{q_i}{p_i} \), \( \tilde{D}_{\chi^2}(q, p) = \sum_{i=1}^{n} p_i \left( \frac{q_i}{p_i} \right)^2 - 1 \) and \( J_c(f; e_1, e_2) \) be defined by (18). Then, for every \( f : (0, \infty) \to \mathbb{R} \) an SCF with modulus \( c > 0 \), we have

\[
f(e_1 + e_2 - 1) - c \left[ 2(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \tilde{D}_{\chi^2}(q, p) \right]
\]

\[
- \frac{\min\{p_i\} \left[ \sum_{i=1}^{n} f\left( e_1 + e_2 - \frac{q_i}{p_i} \right) - nf\left( e_1 + e_2 - \frac{q}{p} \right) - c \sum_{i=1}^{n} \left( \frac{q_i}{p} - \frac{q_i}{p_i} \right)^2 \right]}{1 \leq i \leq n}
\]

\[
\leq f(e_1) + f(e_2) - \tilde{D}_f(q, p)
\]

\[
- c \left[ 2(e_2 - e_1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \tilde{D}_{\chi^2}(q, p) \right]
\]

\[
\leq f(e_1) + f(e_2) - \tilde{D}_f(q, p).
\]

(37)

**Proof.** Applying (21) to \( z_i = \frac{q_i}{p_i} \) with \( \tilde{z} = \frac{1}{n} \sum_{i=1}^{n} z_i = 1 \) and \( \hat{z} = \frac{1}{n} \sum_{i=1}^{n} \frac{q_i}{p_i} = \frac{q}{p} \), we get (37). \( \square \)
Corollary 2. Let $q_2 > q_1 > 0$, $p, q \in \mathcal{P}_n$ with $\frac{q_i}{p_i} \in [q_1, q_2]$, $i \in \{1, \ldots, n\}$ and $\mathcal{J}_c(f; q_1, q_2)$ be defined by (18). Then, for every $f : (0, \infty) \to \mathbb{R}$ an SCF with modulus $c > 0$, we have

$$ f(q_1 + q_2 - 1) \leq \frac{1 - q_1}{q_2 - q_1} f(q_1) + \frac{q_2 - 1}{q_2 - q_1} f(q_2) - c \sum_{i=1}^{n} p_i \left( \frac{q_i}{p_i} - q_1 \right) \left( q_2 - \frac{q_i}{p_i} \right) - \left( \frac{1}{2} - \frac{1}{q_2 - q_1} \sum_{i=1}^{n} p_i \left| \frac{q_i}{p_i} - q_1 \right| \right) \mathcal{J}_c(f; q_1, q_2) \leq f(q_1) + f(q_2) - D_f(q, p) - 2c \sum_{i=1}^{n} p_i \left( \frac{q_i}{p_i} - q_1 \right) \left( q_2 - \frac{q_i}{p_i} \right) \leq f(q_1) + f(q_2) - D_f(q, p). $$

Proof. Applying (25) to $z_i = \frac{q_i}{p_i}, i \in \{1, \ldots, n\}$, with $z_p = \sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} q_i = 1$, we get (38).

By utilizing earlier corollaries with the relevant generating SCF $f$, we can obtain new estimates for certain well-known divergences, which are specific cases of strong $f$-divergence (34). Here we analyze some of the most widely used divergences.

Example 2. Strong Kullback–Leibler divergence of $p, q \in \mathcal{P}_n$ is defined by

$$ D_{\text{KL}}(q, p) = \sum_{i=1}^{n} q_i \ln \left( \frac{q_i}{p_i} \right), $$

where the generating function is $f(t) = t \ln t$ for $t \in (0, \infty)$. Fix $q_2 > 0$. Since $f''(t) = \frac{1}{t^2}$, we have $f''(t) \geq \frac{1}{t^2}$ on $[q_1, q_2], 0 < q_1 < q_2$, and $f|_{[q_1, q_2]}$ is an SCF with modulus $c = \frac{1}{q_2^2}$.

Applying inequalities (37) and (38) to $f(t) = t \ln t$ with $c = \frac{1}{q_2^2}$, we may derive new estimates for strong Kullback–Leibler divergence $D_{\text{KL}}(q, p)$.

Example 3. Strong squared Hellinger divergence of $p, q \in \mathcal{P}_n$ is defined by

$$ D_{h^2}(q, p) = \sum_{i=1}^{n} \left( \sqrt{p_i} - \sqrt{q_i} \right)^2, $$

where the generating function is $f(t) = \left( \sqrt{t} - 1 \right)^2$ for $t \in (0, \infty)$. Fix $q_2 > 0$. Since $f''(t) = \frac{1}{2 \sqrt{t^3}}$, we have $f''(t) \geq \frac{1}{2 \sqrt{q_2^3}}$ on $[q_1, q_2], 0 < q_1 < q_2$, and $f|_{[q_1, q_2]}$ is an SCF with modulus $c = \frac{1}{4 \sqrt{q_2^3}}$.

Applying inequalities (37) and (38) to $f(t) = \left( \sqrt{t} - 1 \right)^2$ with $c = \frac{1}{4 \sqrt{q_2^3}}$, we may derive new estimates for strong squared Hellinger divergence $D_{h^2}(q, p)$.

Example 4. Strong Bhattacharyya distance of $p, q \in \mathcal{P}_n$ is defined by

$$ D_B(q, p) = -\sum_{i=1}^{n} \sqrt{p_i q_i}, $$
where the generating function is \( f(t) = -\sqrt{t} \) for \( t \in (0, \infty) \). Fix \( \epsilon_2 > 0 \). Since \( f''(t) = \frac{1}{4\epsilon_2^3} \), we have \( f'' \geq \frac{1}{4\epsilon_2^3} \) on \([\epsilon_1, \epsilon_2]\), \( 0 < \epsilon_1 < \epsilon_2 \), and \( f|_{[\epsilon_1, \epsilon_2]} \) is an SCF with modulus \( c = \frac{1}{8\epsilon_2^3} \).

Applying inequalities (37) and (38) to \( f(t) = -\sqrt{t} \) with \( c = \frac{1}{8\epsilon_2^3} \), we may derive new estimates for strong Bhattacharya distance \( \tilde{D}_B(q, p) \).

**Example 5.** Strong Jeffreys distance of \( p, q \in \mathcal{P}_n \) is defined by

\[
\tilde{D}_J(q, p) = \frac{n}{2} \sum_{i=1}^{n} (q_i - p_i) \ln \frac{q_i}{p_i} = \tilde{D}_{KL}(q, p) + \tilde{D}_{KL}(p, q).
\]

where the generating function is \( f(t) = (t - 1) \ln t \) for \( t \in (0, \infty) \). Fix \( \epsilon_2 > 0 \). Since \( f''(t) = \frac{t+1}{t^2}, \) we have \( f'' \geq \frac{t+1}{2\epsilon_2^3} \) on \([\epsilon_1, \epsilon_2]\), \( 0 < \epsilon_1 < \epsilon_2 \), and \( f|_{[\epsilon_1, \epsilon_2]} \) is an SCF with modulus \( c = \frac{2t+1}{2\epsilon_2^3} \).

Applying inequalities (37) and (38) to \( f(t) = (t - 1) \ln t \) with \( c = \frac{t+1}{2\epsilon_2^3} \), we may derive new estimates for strong Jeffreys distance \( \tilde{D}_J(q, p) \).

**Example 6.** Strong Jensen–Shannon divergence of \( p, q \in \mathcal{P}_n \) is defined by

\[
\tilde{D}_{JS}(q, p) = \frac{1}{2} \left[ \sum_{i=1}^{n} q_i \ln \frac{2q_i}{p_i + q_i} + \sum_{i=1}^{n} p_i \ln \frac{2p_i}{p_i + q_i} \right]
\]

where the generating function is \( f(t) = \frac{1}{2} (t \ln \frac{2t}{t+1} + \ln \frac{2}{t+1}) \), for \( t \in (0, \infty) \). Fix \( \epsilon_2 > 0 \). Since \( f''(t) = \frac{1}{2(1+t)}, \) we have \( f'' \geq \frac{1}{2\epsilon_2(1+\epsilon_2)} \) on \([\epsilon_1, \epsilon_2]\), \( 0 < \epsilon_1 < \epsilon_2 \), and \( f|_{[\epsilon_1, \epsilon_2]} \) is a SCF with modulus \( c = \frac{1}{4\epsilon_2(1+\epsilon_2)} \).

Applying inequalities (37) and (38) to \( f(t) = \frac{1}{2} (t \ln \frac{2t}{t+1} + \ln \frac{2}{t+1}) \) with \( c = \frac{1}{4\epsilon_2(1+\epsilon_2)} \), we may derive new estimates for strong Jensen–Shannon divergence \( \tilde{D}_{JS}(q, p) \).

In the sequel, we consider Shannon’s entropy [36], defined in terms of its probability distribution \( p \) for a random variable \( X \) as

\[
S(p) = \sum_{i=1}^{n} p_i \ln \frac{1}{p_i} = -\sum_{i=1}^{n} p_i \ln p_i.
\]

It quantifies the unevenness in \( p \) and satisfies the inequality

\[
0 \leq S(p) \leq \ln n.
\]

We derive new estimates for Shannon’s entropy using results from the previous sections.
**Corollary 3.** Let \( q_2 > q_1 > 0, \ p \in \mathcal{P}_n, \) and \( \frac{1}{p_i} \in [q_1, q_2], \ k_i \in [0, 1] \) such that \( \frac{1}{p_i} = k_i q_1 + (1 - k_i) q_2, \ i = 1, \ldots, n. \) Then

\[
\ln \frac{1}{q_1 + q_2 - n} \leq S(p) - \ln q_1 q_2 \leq \frac{1}{2c_2^2} \left[ 2(q_2 - q_1)^2 \sum_{i=1}^{n} p_i k_i (1 - k_i) + \sum_{i=1}^{n} p_i \left( \frac{1}{p_i} - n \right)^2 \right] - J(-\ln; q_1, q_2) \left( \sum_{i=1}^{n} p_i (1 - |1 - 2k_i|) \right) - \min_{1 \leq i \leq n} \{ p_i \} \left[ \ln \frac{(q_1 + q_2 - \hat{p})^n}{\prod_{i=1}^{n} (q_1 + q_2 - \frac{1}{p_i})} - c \sum_{i=1}^{n} \left( \hat{p} - \frac{1}{p_i} \right)^2 \right] \]

where we denote \( \hat{p} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \) and

\[
J(-\ln; q_1, q_2) = \ln \frac{q_1 + q_2}{q_1 q_2} - \frac{1}{2c_2^2} (q_2 - q_1)^2.
\]

**Proof.** Applying (21) to the function \( f(t) = -\ln t, t \in [q_1, q_2], \) SCF with modulus \( c = \frac{1}{2c_2^2} \) and \( z_i = \frac{1}{p_i} \) with \( z_p = \sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} p_i \frac{1}{p_i} = n \) and \( z = \frac{1}{n} \sum_{i=1}^{n} z_i = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} = \hat{p} \), we get (41).

**Corollary 4.** Let \( q_2 > q_1 > 0 \) and \( p \in \mathcal{P}_n \) be such that \( \frac{1}{p_i}, \ldots, \frac{1}{p_n} \in [q_1, q_2]. \) Let \( J(-\ln; q_1, q_2) \) be defined as in (42). Then

\[
\ln \frac{1}{q_1 + q_2 - n} \leq \frac{n - q_1}{q_2 - q_1} \ln \frac{1}{q_2 - q_1} \left( \frac{1}{q_2 - q_1} - \frac{1}{q_2} - \frac{1}{q_2} \right) \leq S(p) - \ln q_1 q_2 - \frac{1}{c_2^2} \sum_{i=1}^{n} p_i (z_i - q_1)(q_2 - z_i) - \left( 1 - \frac{2}{q_2 - q_1} - \frac{1}{p_i} - \frac{1}{q_2} \right) J(-\ln; q_1, q_2) \leq S(p) - \ln q_1 q_2 - \frac{1}{c_2^2} \sum_{i=1}^{n} p_i (z_i - q_1)(q_2 - z_i) \leq S(p) - \ln q_1 q_2.
\]

**Proof.** Applying (25) to the function \( f(t) = -\ln t, t \in [q_1, q_2], \) SCF with modulus \( c = \frac{1}{2c_2^2} \) and \( z_i = \frac{1}{p_i} \) with \( z_p = \sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} p_i \frac{1}{p_i} = n \) and \( z = \frac{1}{n} \sum_{i=1}^{n} z_i = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} = \hat{p} \), we get (43).
4. Estimates for Mercer-Type Means

Let us recall that for $0 < e_1 < e_2$, $z = (z_1, \ldots, z_n) \in [e_1, e_2]^n$ and $p = (p_1, \ldots, p_n)$ a non-negative $n$-tuple such that $\sum_{i=1}^n p_i = 1$, the weighted power mean of order $r \in \mathbb{R}$ is defined by

$$
\eta_r(z, p) = \begin{cases}
(\sum_{i=1}^n p_i z_i^r)^{1/r}, & r \neq 0 \\
\prod_{i=1}^n z_i^{p_i}, & r = 0
\end{cases}
$$

(44)

By slightly modifying (44), we get Mercer type power mean of order $r \in \mathbb{R}$ as follows

$$
Q_r(e_1, e_2, z, p) = \left\{ \frac{e_1^r + e_2^r - \eta_r(z, p)}{\eta_r(z, p)} \right\}^{1/r}, \quad r \neq 0,
$$

$$
Q_r(e_1, e_2, z, p) = r, \quad r = 0
$$

(see [31]). In the same paper, Mercer proved that for $r, s \in \mathbb{R}, r < s$, it holds:

$$
Q_r(e_1, e_2, z, p) \leq Q_s(e_1, e_2, z, p).
$$

(45)

Applying Theorem 4 to some specified strongly convex and strongly concave functions, we get an improvement of the previous Mercer’s result. More precisely, we derive a series of inequalities that refine (45).

**Theorem 8.** Let $0 < e_1 < e_2$ and $z_i \in [e_1, e_2], \kappa_i \in [0, 1]$ such that $z_i = \kappa_i e_1 + (1 - \kappa_i) e_2$, $i = 1, \ldots, n$. Let $p = (p_1, \ldots, p_n)$ be non-negative $n$-tuple with $\sum_{i=1}^n p_i = 1, z_p = \sum_{i=1}^n p_i z_i$ and $z = \frac{1}{n} \sum_{i=1}^n z_i$.

Let $r, s \in \mathbb{R}$ such that $r < s$. Then, the following cases hold.

**CASE 1:** If $0 < r < s$, then

$$
Q_r(e_1, e_2, z, p) \leq [Q_s^2(e_1, e_2, z, p) - c_1 \Delta_r(e_1, e_2, \kappa, z, p)]^{1/2} - J_1(z, e_1, e_2) + \sum_{i=1}^n p_i (1 - [1 - 2\kappa_i]) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r,s} \Omega_{r,s}(e_1, e_2, z, p)
$$

(46)

where

$$
c_1 = \frac{s}{2r} \left( \frac{s}{r} - 1 \right) e_2^{s-2}, \quad \text{for} \quad \frac{s}{r} \in (1, 2),
$$

$$
c_1 = \frac{s}{2r} \left( \frac{s}{r} - 1 \right) e_2^{s-2}, \quad \text{for} \quad \frac{s}{r} \in [2, \infty),
$$

and

$$
J_1(z, e_1, e_2) = e_1^r + e_2^r - 2 \left( \frac{e_1^r + e_2^r}{2} \right)^{1/2} - c_1 (e_2^r - e_1^r)^2,
$$

$$
\Delta_r(e_1, e_2, \kappa, z, p) = 2(e_2^r - e_1^r)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i z_i^2 - \sum_{i=1}^n p_i z_i^2
$$

$$
\Omega_{r,s} \Omega_{r,s}(e_1, e_2, z, p) = \sum_{i=1}^n e_i^r (e_1^r + e_2^r - z_i^r)^{1/2} - n \left( e_1^r + e_2^r - \frac{n}{n} \sum_{i=1}^n z_i^r \right)^{1/2} - c_1 \sum_{i=1}^n \left( \frac{z_i^r - 1}{n} \sum_{i=1}^n z_i^r \right)^{1/2}.
$$
CASE 2: If \( r < s < 0 \), then inequalities (46) hold with \( c_1 \) replaced by
\[
c_2 = -\frac{s}{2r} \left( \frac{s}{r} - 1 \right) e_2^s - 2. \tag{47}
\]

CASE 3: If \( r < 0 < s \), then inequalities (46) hold with \( c_3 = -c_1 \).

CASE 4: If \( r < s = 0 \), then
\[
Q_r(e_1, e_2, z, p) \leq \exp \left[ \ln Q_0(e_1, e_2, z, p) - c_4 \tilde{\Delta}_r(e_1, e_2, \kappa_i, z, p) \right. \\
- \mathcal{J}_4 \left( \frac{1}{r} \ln(\cdot); e'_1, e'_2 \right) \sum_{i=1}^n p_i(1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \tilde{\Omega}_{r,s,c_4}(e_1, e_2, z, p) \\
\leq \exp \left[ \ln Q_0(e_1, e_2, z, p) - c_4 \tilde{\Delta}_r(e_1, e_2, \kappa_i, z, p) \right] \\
\leq Q_0(e_1, e_2, z, p),
\]

where \( c_4 = \frac{1}{2\sqrt{e}} \) and
\[
\mathcal{J}_4 \left( \frac{1}{r} \ln(\cdot); e'_1, e'_2 \right) = \ln e_1 e_2 - 2 \ln \left( \frac{\tilde{e}_1' + \tilde{e}_2'}{2} \right) - \frac{c_4}{2} (e_2' - e_1')^2,
\]
\[
\tilde{\Delta}_r(e_1, e_2, \kappa_i, z, p) = 2(e_2' - e_1')^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( z_i' - \sum_{i=1}^n p_i z_i' \right)^2,
\]
\[
\tilde{\Omega}_{r,s,c_4}(e_1, e_2, z, p) = \frac{1}{r} \sum_{i=1}^n \ln(e_1' + e_2' - z_i') - \frac{n}{r} \ln \left( e_1' + e_2' - \frac{1}{n} \sum_{i=1}^n z_i' \right) \\
- c_4 \sum_{i=1}^n \left( z_i' - \frac{1}{n} \sum_{i=1}^n z_i' \right)^2.
\]

CASE 5: If \( r = 0 < s \), then
\[
Q_0(e_1, e_2, z, p) \leq \left| Q_0^s(e_1, e_2, z, p) - c_5 \Delta(e_1, e_2, \kappa_i, z, p) \right. \\
- \mathcal{J}_5 \left( s \ln(\cdot); \ln e_1, \ln e_2 \right) \sum_{i=1}^n p_i(1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r,s,c_5}(e_1, e_2, z, p) \right)^{\frac{1}{2}} \\
\leq \left| Q_0^s(e_1, e_2, z, p) - c_5 \Delta(e_1, e_2, \kappa_i, z, p) \right|^{\frac{1}{2}} \\
\leq Q_0(e_1, e_2, z, p),
\]

where \( c_5 = \frac{1}{2s^2} e^{s \ln e_1} \) and
\[
\mathcal{J}_5 \left( s \ln(\cdot); \ln e_1, \ln e_2 \right) = e_1^s + e_2^s - 2(e_1 e_2) ^{\frac{s}{2}} - \frac{c_5}{2} \left( \ln \frac{e_2}{e_1} \right)^2,
\]
\[
\Delta(e_1, e_2, \kappa_i, z, p) = 2 \left( \ln \frac{e_2}{e_1} \right)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( \ln \frac{z_i}{\prod_{i=1}^n z_i} \right)^2,
\]
\[
\Omega_{r,s,c_5}(e_1, e_2, z, p) = \sum_{i=1}^n \left( \frac{e_1 e_2}{z_i} \right)^s - n \left( \frac{e_1 e_2}{\prod_{i=1}^n z_i} \right)^s c_5 \sum_{i=1}^n \left( \ln \frac{z_i}{\prod_{i=1}^n z_i} \right)^2.
\]

**Proof.** CASE 1: Let \( 0 < r < s \). Then \( \frac{s}{r} > 1 \).
By definition of $z_i, i \in \{1, \ldots, n\}$, we have

$$0 < e_1 \leq z_i \leq e_2 < \infty.$$  

Then also

$$0 < e_1^x \leq z_i^x \leq e_2 < \infty.$$  

We apply inequality (21), to $f(t) = t^\frac{x}{r}, t \in [e_1, e_2]$, an SCF with modulus

$$c = \frac{1}{2} \inf_{t \in [e_1, e_2]} t^\theta (t - 1)^{\frac{x}{r} - 2}$$

and replacing $e_1, e_2, z_i$ with $e_1^x, e_2^x, z_i^x$, respectively, we get

$$\left( e_1^x + e_2^x - \sum_{i=1}^n p_i z_i^x \right)^{\frac{\theta}{\theta}} \leq e_1^x + e_2^x - \sum_{i=1}^n p_i z_i^x$$

$$- c_1 \left[ 2(e_2^x - e_1^x)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( z_i^x - \sum_{i=1}^n p_i z_i^x \right)^2 \right]$$

$$- \mathcal{J}_{c_1} \left( (\cdot)^{\frac{\theta}{r}}; e_1^x, e_2^x \right) \sum_{i=1}^n p_i (1 - |1 - 2 \kappa_i|)$$

$$- \min_{1 \leq i \leq n} \{ p_i \} \left[ \sum_{i=1}^n (e_1^x + e_2^x - z_i^x)^{\frac{\theta}{r}} - n \left( e_1^x + e_2^x - \frac{1}{n} \sum_{i=1}^n z_i^x \right)^{\frac{\theta}{r}} \right]$$

$$+ \min_{1 \leq i \leq n} \{ p_i \} c_1 \sum_{i=1}^n \left( z_i^x - \frac{1}{n} \sum_{i=1}^n z_i^x \right)^2$$

$$\leq e_1^x + e_2^x - \sum_{i=1}^n p_i z_i^x$$

$$- c_1 \left[ 2(e_2^x - e_1^x)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( z_i^x - \sum_{i=1}^n p_i z_i^x \right)^2 \right]$$

$$\leq e_1^x + e_2^x - \sum_{i=1}^n p_i z_i^x$$

where for $\frac{\theta}{r} \in (1, 2)$, we have $c_1 = \frac{\theta}{r} (\frac{\theta}{r} - 1) e_2^{\theta - 2}$ and for $\frac{\theta}{r} \in [2, \infty), c_1 = \frac{\theta}{r} (\frac{\theta}{r} - 1) e_2^{\theta - 2}$.

We can write equivalently

$$Q^{\frac{\theta}{r}}_{c_1}(e_1, e_2, z, p)$$

$$\leq Q^\theta_{c_1}(e_1, e_2, z, p) - c_1 \mathcal{J}_{c_1}(e_1, e_2, \kappa_i, z, p)$$

$$- \mathcal{J}_{c_1} \left( (\cdot)^{\frac{\theta}{r}}; e_1^x, e_2^x \right) \sum_{i=1}^n p_i (1 - |1 - 2 \kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r, \kappa_i} \left( e_1, e_2, \kappa_i, z, p \right)$$

$$\leq Q^\theta_{c_1}(e_1, e_2, z, p) - c_1 \mathcal{J}_{c_1}(e_1, e_2, \kappa_i, z, p)$$

$$\leq Q^\theta_{c_1}(e_1, e_2, z, p).$$
Since $0 < r < s$, rising to the power $\frac{1}{s} > 0$, we get

\[
Q_r(e_1, e_2, z, p) \\
\leq |Q^\alpha_r(e_1, e_2, z, p) - c_1A_r(e_1, e_2, \kappa, z, p)| \\
- J_{c_1} \left( \left( c^\alpha \right) ; e_1^\alpha, e_2^\alpha \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r, s, c_1} (e_1, e_2, \kappa, z, p) \right]^{\frac{1}{s}} \\
\leq |Q^\alpha_r(e_1, e_2, z, p) - c_1A_r(e_1, e_2, \kappa, z, p)|^{\frac{1}{s}} \\
\leq Q_s(e_1, e_2, z, p),
\]

which we need to prove.

**CASE 2:** Let $r < s < 0$. Then $0 < \frac{r}{s} < 1$.

In this case, we have

\[
0 < e^r_2 \leq z^r_1 \leq e^r_1 < \infty.
\]

Applying (22) to the function $f(t) = t^r$, $t \in [e_1, e_2]$, strongly concave with modulus $c = -\frac{1}{s}\sup_{t \in [e_1, e_2]} \left( \left( \frac{r}{s} - 1 \right) t^{r-2} \right)$ and replacing $e_1, e_2, z_i$ with $e^r_2, e^{-r}_1, z^r_i$, respectively, we get

\[
Q^r_s(e_1, e_2, z, p) \\
\geq Q^r_s(e_1, e_2, z, p) - c_2A_r(e_1, e_2, \kappa, z, p) \\
- J_{c_2} \left( \left( \frac{r}{s} \right) ; e^{-r}_1, e^r_2 \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r, s, c_2} (e_1, e_2, \kappa, z, p) \right) \\
\geq Q^r_s(e_1, e_2, z, p) - c_2A_r(e_1, e_2, \kappa, z, p) \\
\geq Q^r_s(e_1, e_2, z, p),
\]

where $c_2 = -\frac{1}{s}\left( \frac{r}{s} - 1 \right) e^{-2}_2$.

Since $r < s < 0$, raising to the power $\frac{1}{s}$, we get

\[
Q_r(e_1, e_2, z, p) \\
\leq |Q^r_s(e_1, e_2, z, p) - c_2A_r(e_1, e_2, \kappa, z, p)|^{\frac{1}{s}} \\
- J_{c_2} \left( \left( \frac{r}{s} \right) ; e^{-r}_1, e^r_2 \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r, s, c_2} (e_1, e_2, \kappa, z, p) \right]^{\frac{1}{s}} \\
\leq |Q^r_s(e_1, e_2, z, p) - c_2A_r(e_1, e_2, \kappa, z, p)|^{\frac{1}{s}} \\
\leq Q_s(e_1, e_2, z, p),
\]

**CASE 3:** Let $r < 0 < s$. Then $\frac{r}{s} < 0$.

In this case, we have

\[
0 < e^r_2 \leq z^r_1 \leq e^r_1 < \infty.
\]

Applying (21) to $f(t) = t^r$, $t \in [e_1, e_2]$, an SCF with $c = \frac{1}{s}\inf_{t \in [e_1, e_2]} \left( \left( \frac{r}{s} - 1 \right) t^{r-2} \right)$ and replacing $e_1, e_2, z_i$ with $e^r_2, e^{-r}_1, z^r_i$, respectively, we get

\[
Q^r_s(e_1, e_2, z, p) \\
\leq Q^r_s(e_1, e_2, z, p) - c_3A_r(e_1, e_2, \kappa, z, p) \\
- J_{c_3} \left( \left( \frac{r}{s} \right) ; e^{-r}_1, e^r_2 \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) - \min_{1 \leq i \leq n} \{ p_i \} \Omega_{r, s, c_3} (e_1, e_2, \kappa, z, p) \right) \\
\leq Q^r_s(e_1, e_2, z, p) - c_3A_r(e_1, e_2, \kappa, z, p) \\
\leq Q^r_s(e_1, e_2, z, p),
\]

where $c_3 = \frac{1}{s}\left( \frac{r}{s} - 1 \right) e^{-2}_2 = -c_2$. 
Since \( r < 0 < s \), raising to the power \( \frac{1}{2} \), we again get (50).

CASE 4: Let \( r < s = 0 \).

In this case

\[
0 < e_1^2 \leq z'_j \leq e_1^1 < \infty.
\]

Applying (21) to the SCF \( f(t) = \frac{1}{r} \ln t, t \in [e_1, e_2] \), with \( c = \frac{1}{2} \inf_{t \in [e_1, e_2]} \left( -\frac{1}{r'} \right) \), and replacing \( e_1, e_2, z_i \) with \( e_1^2, e_1^1, z'_j \), respectively, we get

\[
\frac{1}{r} \ln \left( e_1^1 + e_1^2 - \sum_{i=1}^{n} p_i z'_i \right) \\
\leq \frac{1}{r} \ln e_1^1 + \frac{1}{r} \ln e_1^2 - \sum_{i=1}^{n} p_i - \ln z'_i \\
c_4 \left[ 2(e_1^2 - e_1^1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i \left( z'_i - \sum_{i=1}^{n} p_i z'_i \right)^2 \right] \\
- J_{c_4} \left( \frac{1}{r} \ln (\cdot); e_1^1, e_1^2 \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) \\
- \min \{p_i\} \frac{1}{r} \left[ n \sum_{i=1}^{n} \ln (e_1^1 + e_1^2 - z'_i) - n \frac{1}{r} \ln \left( e_1^1 + e_1^2 - \frac{1}{n} \sum_{i=1}^{n} z'_i \right) \right] \\
+ \min \{p_i\} c_4 n \sum_{i=1}^{n} \left( z'_i - \frac{1}{n} \sum_{i=1}^{n} z'_i \right)^2 \\
\leq \frac{1}{r} \ln e_1^1 + \frac{1}{r} \ln e_1^2 - \sum_{i=1}^{n} p_i - \ln z'_i \\
c_4 \left[ 2(e_1^2 - e_1^1)^2 \sum_{i=1}^{n} p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^{n} p_i \left( z'_i - \sum_{i=1}^{n} p_i z'_i \right)^2 \right] \\
\leq \frac{1}{r} \ln e_1^1 + \frac{1}{r} \ln e_1^2 - \sum_{i=1}^{n} p_i - \ln z'_i,
\]

where \( c_4 = -\frac{1}{2e_1^2} \), i.e., we get

\[
\ln Q_r(e_1, e_2, z, p) \\
\leq \ln Q_0(e_1, e_2, z, p) - c_4 \Delta_r(e_1, e_2, \kappa_i, z, p) \\
- J_{c_4} \left( \frac{1}{r} \ln (\cdot); e_1^1, e_1^2 \right) \sum_{i=1}^{n} p_i (1 - |1 - 2\kappa_i|) - \min \{p_i\} c_4 \sum_{i=1}^{n} \Omega_r(e_1, e_2, \kappa_i, z, p) \\
\leq \ln Q_0(e_1, e_2, z, p) - c_4 \Delta_r(e_1, e_2, \kappa_i, z, p) \\
\leq \ln Q_0(e_1, e_2, z, p),
\]

which is equivalent to (48).

CASE 5: Let \( r = 0 < s \).

In this case

\[-\infty < \ln e_1 \leq \ln z_i \leq \ln e_2 < \infty.\]
Applying (21) to the SCF \( f(t) = \exp(st), t \in [e_1, e_2] \), with \( c = \frac{1}{2} \inf_{t \in [e_1, e_2]} s^2 e^{st} \), and replacing \( e_1, e_2, z_i \) with \( \ln e_1, \ln e_2, \ln z_i \), respectively, we get

\[
\exp s \left( \ln e_1 + \ln e_2 - \sum_{i=1}^n p_i \ln z_i \right)
\leq \exp(s \ln e_1) + \exp(s \ln e_2) - \sum_{i=1}^n p_i \exp(s \ln z_i)
\]

\[
- c_5 \left[ 2(\ln e_2 - \ln e_1)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( \ln z_i - \frac{n}{n} \sum_{i=1}^n p_i \ln z_i \right) \right]
\]

\[
- J_{c_5} \left( \exp s(\cdot); \ln e_1, \ln e_2 \right) \sum_{i=1}^n p_i (1 - |1 - 2\kappa_i|)
\]

\[
- \min_{1 \leq i \leq n} \left\{ p_i \right\} \sum_{i=1}^n \left( \ln z_i - \frac{n}{n} \sum_{i=1}^n \ln z_i \right)^2
\]

\[
\leq \exp(s \ln e_1) + \exp(s \ln e_2) - \sum_{i=1}^n p_i \exp(s \ln z_i)
\]

\[
- c_5 \left[ 2(\ln e_2 - \ln e_1)^2 \sum_{i=1}^n p_i \kappa_i (1 - \kappa_i) + \sum_{i=1}^n p_i \left( \ln z_i - \frac{n}{n} \sum_{i=1}^n p_i \ln z_i \right) \right]
\]

\[
\leq \exp(s \ln e_1) + \exp(s \ln e_2) - \sum_{i=1}^n p_i \exp(s \ln z_i),
\]

where \( c_5 = \frac{1}{2} s^2 e^{s \ln e_1} \), i.e., we get

\[
Q_s^0(e_1, e_2, z, p) \leq Q_s^c(e_1, e_2, z, p) - c_5 \Delta(e_1, e_2, \kappa_i, z, p)
\]

\[
- J_{c_5} \left( \exp s(\cdot); \ln e_1, \ln e_2 \right) \sum_{i=1}^n p_i (1 - |1 - 2\kappa_i|)
\]

\[
- \min_{1 \leq i \leq n} \left\{ p_i \right\} \Omega_{r,s,c_5}(e_1, e_2, \kappa_i, z, p)
\]

\[
\leq Q_s^c(e_1, e_2, z, p) - c_5 \Delta(e_1, e_2, z, p)
\]

\[
\leq Q_s^c(e_1, e_2, z, p).
\]

Since \( s > 0 \), raising to the power \( \frac{1}{s} \), we get (49).

This ends the proof. \( \Box \)

**Remark 8.** Analogously, we can apply Theorem 5 for new estimates, but we leave those results to the reader to derive them.

**5. Conclusions**

In the literature, numerous results are presented for Jensen’s and the Jensen–Mercer– type inequalities pertaining convex functions, focusing on their generalizations, refinements, and improvements using different approaches and tools. After the appearance of Jensen’s and Jensen–Mercer inequalities for the class of SCFs, a natural question arises: is it possible to provide related results for these inequalities for the class of SCFs? Most of the results are not straightforward to obtain such related results. In this manuscript, we derived improvements of the Jensen–Mercer inequality using some earlier results of the class of SCFs. Also, for the desired inequalities, we elaborated interesting properties of the SCFs. Our main results also hold for strongly concave functions, which are poorly represented in
the literature, although their application can be very useful. By observing such functions and using their characterizations, we derived interesting results involving Mercer-type means. We also gave applications of the main inequalities in information theory.

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