

Article

Sufficient Efficiency Criteria for New Classes of Non-Convex Optimization Models

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Abstract: In this paper, we introduce and study a new class of minimization models driven by multiple integrals as cost functionals. Concretely, we formulate and establish some sufficient efficiency criteria for a feasible point in the considered optimization problem. To this end, we introduce and define the concepts of (Γ, ψ) -invexity and generalized (Γ, ψ) -invexity for the involved real-valued controlled multiple integral-type functionals. More precisely, we extend the notion of (generalized) (Γ, ψ) -invexity to the multiple objective control models driven by multiple integral functionals. In addition, innovative proofs are considered for the principal results derived in the paper.

Keywords: efficiency criteria; (Γ, ψ) -invexity; optimization models

MSC: 90C29; 65K10; 26B25



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1. Introduction

Applications of optimization and control theory in real-world problems are well known. Thus, finding of new techniques and methods to extremize (minimize, maximize) some functions or functionals has been a challenge for researchers in the last few years. In this regard, Weir and Mond [1], by considering the concept of weak minima, proposed different scalar duality results for multiobjective programming problems. Mond and Smart [2] studied duality and sufficiency in invex control problems. Optimality criteria and duality theorems were obtained by Chandra et al. [3] for a class of nondifferentiable control problems. Later, Bhatia and Kumar [4] investigated multiple-cost control problems governed by generalized invexity. Also, Bhatia and Mehra [5] formulated some optimality and duality results in generalized B -invex multiple objective optimization problems. Nahak and Nanda [6] investigated efficiency and duality for (F, ρ) -convex multiobjective variational control problems. On the other hand, Mishra and Mukherjee [7] studied vector control models under V -invexity hypotheses. Reddy and Mukherjee [8] generalized the study of efficiency and duality for vector fractional control models under (F, ρ) -convexity assumptions. A mixed duality associated with vector variational problems was established by Mukherjee and Rao [9]. Later, Zhian and Qingkai [10] provided some duality theorems for multiple-cost control models involving a generalized invexity. Also, Xiuhong [11] presented a duality theory for a class of multiobjective optimization problems. Under generalized (F, ρ) -convexity, Ahmad and Gulati established a mixed-type duality in vector variational models. Sufficiency efficiency conditions and dual models for some multiobjective optimization problems, governed by generalized type I functions, have been stated by Hachimi and Aghezzaf [12]. In this direction, Kim and Kim [13] defined a generalized

type I invexity for a family of multiobjective extremization problems. Mititelu [14], by using the notion of quasiinvexity, investigated efficiency criteria for some vector fractional variational problems. Khazafi et al. [15] extended the study to multi-cost control models determined by generalized (B, ρ) -type I functions. Recently, Treanță [16–18] analyzed well-posedness and optimality criteria in various constrained controlled optimization models. For a different approach, the reader can consult Bouregghda [19,20], and Joshi and Jha [21], where the authors developed a calcium dynamics model to investigate the interplay of calcium flux based on a reaction–diffusion equation.

In this paper, based on and motivated by the previously mentioned research works, we introduce and study a new class of minimization models driven by multiple integrals as cost functionals. Concretely, we formulate and establish some sufficient efficiency criteria for a feasible point in the considered optimization problem. To this end, we introduce and define the concepts of (Γ, ψ) -invexity and generalized (Γ, ψ) -invexity for the involved real-valued controlled multiple integral-type functionals. More precisely, we extend the notion of (generalized) (Γ, ψ) -invexity, presented by Caristi et al. [22] and Antczak [23], to the multiple objective control models driven by multiple integral functionals. Also, the novelty elements included in this study are provided by the innovative proofs associated with the main results derived in the paper.

The paper is structured as follows. Section 2 contains the preliminary results and notions that are used for establishing the principal outcomes of this study. Section 3 includes the main results of the paper. Concretely, the sufficient efficiency conditions are formulated and proved for the considered class of problems. The paper ends with Section 4, which states the conclusions and some future research directions of the present paper.

2. On Multi-Dimensional Variational Control Models

Multi-dimensional variational control models have applications in different branches of mathematical, engineering, and economical sciences, such as shape-optimization in medicine and fluid mechanics, structural optimization, material inversion in geophysics, or optimal control of processes (see Jayswal et al. [24] for a detailed description). Thus, partial differential equation-constrained multi-dimensional variational control models have been given considerable importance in recent years.

A general formulation of such an extremization model is given as below:

$$\begin{aligned} \text{(Problem)} \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Omega} (f_1(t, p(t), q(t)), \dots, f_k(t, p(t), q(t))) dt \\ & \text{subject to } g_i(t, p(t), q(t)) \leq 0, \\ & \frac{\partial p^j}{\partial t^\alpha} = h_j^\alpha(t, p(t), q(t)), \\ & p(t_0) = p_0, \quad p(t_1) = p_1, \end{aligned}$$

where $f = (f_l) : \Omega \times P \times Q \rightarrow \mathbb{R}^k, l = \overline{1, k}$ (objective or cost functional), the inequality constraint $g = (g_i) : \Omega \times P \times Q \rightarrow \mathbb{R}^u, i = \overline{1, u}$, and the equality constraint $h = (h_j^\alpha) : \Omega \times P \times Q \rightarrow \mathbb{R}^{rm}, j = \overline{1, m}, \alpha = \overline{1, r}$, are considered to be of C^∞ -class. In addition, we consider h^α satisfy $D_\beta h^\alpha = D_\alpha h^\beta, \alpha, \beta = \overline{1, r}, \alpha \neq \beta$, where $D_\alpha := \frac{\partial}{\partial t^\alpha}$.

A pair $(\bar{p}(t), \bar{q}(t)) \in P \times Q$ is said to be a feasible point or feasible solution to (Problem) if all the considered constraint functionals are satisfied. The feasible set of solutions can be written as follows

$$\begin{aligned} DOM = \{ (p(t), q(t)) \in P \times Q : & g_i(t, p(t), q(t)) \leq 0, h_j^\alpha(t, p(t), q(t)) = \frac{\partial p^j}{\partial t^\alpha}, \\ & p(t_0) = p_0, p(t_1) = p_1 \}. \end{aligned}$$

In multiobjective optimization, a unique feasible solution that optimizes all the objectives, in general, does not exist. Therefore, the concepts of a weak Pareto solution and a Pareto solution play a crucial role in solving such optimization problems.

Definition 1. A feasible solution $(\bar{p}(t), \bar{q}(t)) \in \text{DOM}$ is named an efficient solution to (Problem) if there does not exist $(p(t), q(t)) \in \text{DOM}$ satisfying

$$\int_{\Omega} f(t, p(t), q(t)) dt \leq \int_{\Omega} f(t, \bar{p}(t), \bar{q}(t)) dt,$$

with at least one strict inequality.

Definition 2. A feasible solution $(\bar{p}(t), \bar{q}(t)) \in \text{DOM}$ is named a weak efficient solution to (Problem) if there does not exist $(p(t), q(t)) \in \text{DOM}$ satisfying

$$\int_{\Omega} f(t, p(t), q(t)) dt < \int_{\Omega} f(t, \bar{p}(t), \bar{q}(t)) dt.$$

In general, the multi-dimensional functional is classified as follows:

I. Curvilinear integral cost functional

$$\begin{aligned} \text{(P1)} \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Gamma} f_{\alpha}(t, p(t), q(t)) dt^{\alpha} \\ & = \left(\int_{\Gamma} f_{\alpha}^1(t, p(t), q(t)) dt^{\alpha}, \dots, \int_{\Gamma} f_{\alpha}^k(t, p(t), q(t)) dt^{\alpha} \right), \end{aligned}$$

where $f_{\alpha}^l : \Omega \times P \times Q \rightarrow \mathbb{R}, \alpha = \overline{1, r}, l = \overline{1, k}$, are C^1 -class functionals and Γ is a curve (piecewise smooth), included in Ω , that joins t_0 and t_1 .

II. Multiple integral cost functional

$$\begin{aligned} \text{(P2)} \quad & \min_{(p(\cdot), q(\cdot))} \int_{\Omega} f(t, p(t), q(t)) dt \\ & = \left(\int_{\Omega} f_1(t, p(t), q(t)) dt, \dots, \int_{\Omega} f_k(t, p(t), q(t)) dt \right), \end{aligned}$$

where $f : \Omega \times P \times Q \rightarrow \mathbb{R}^k$ is a vector-valued C^1 -class functional.

Necessary and Sufficient Conditions Of Efficiency

Necessary and sufficient conditions of optimality are based on differential calculus that plays a crucial role in generating the optimal solutions in the considered optimization problems.

Theorem 1. (Fritz John type necessary conditions of efficiency.) If the pair $(\bar{p}(t), \bar{q}(t)) \in \text{DOM}$ is an efficient solution to (Problem), then there exist $\theta_l \in \mathbb{R}, l = \overline{1, k}, \mu_i(t) \in \mathbb{R}_+, i = \overline{1, u}$, and $\gamma_{\alpha}^j(t) \in \mathbb{R}, \alpha = \overline{1, r}, j = \overline{1, m}$, satisfying (for all $t \in \Omega$, except at discontinuities)

$$\begin{aligned} & \theta_l \frac{\partial f_l}{\partial p^{\varsigma}}(t, \bar{p}(t), \bar{q}(t)) + \mu_i(t) \frac{\partial g_i}{\partial p^{\varsigma}}(t, \bar{p}(t), \bar{q}(t)) \\ & + \gamma_{\alpha}^j(t) \frac{\partial h_j^{\alpha}}{\partial p^{\varsigma}}(t, \bar{p}(t), \bar{q}(t)) + \frac{\partial \gamma_{\alpha}^j}{\partial t^{\alpha}}(t) = 0, \quad \varsigma = \overline{1, m}, \end{aligned}$$

$$\begin{aligned} & \theta_l \frac{\partial f_l}{\partial q^{\tau}}(t, \bar{p}(t), \bar{q}(t)) + \mu_i(t) \frac{\partial g_i}{\partial q^{\tau}}(t, \bar{p}(t), \bar{q}(t)) \\ & + \gamma_{\alpha}^j(t) \frac{\partial h_j^{\alpha}}{\partial q^{\tau}}(t, \bar{p}(t), \bar{q}(t)) = 0, \quad \tau = \overline{1, n}, \end{aligned}$$

$$\mu_i(t)g_i(t, \bar{p}(t), \bar{q}(t)) = 0, \quad \theta \geq 0, \quad \mu(t) \geq 0.$$

To ensure that $\theta > 0$, some restrictions are imposed on the constraint functions, and these restrictions are known as constraint qualifications. In this regard, we formulate the following result.

Theorem 2. (Kuhn–Tucker type necessary conditions of efficiency.) *If the pair $(\bar{p}(t), \bar{q}(t)) \in \text{DOM}$ is an efficient solution to (Problem) and the constraint conditions hold, then there exist $\theta_l \in \mathbb{R}, l = \overline{1, k}, \mu_i(t) \in \mathbb{R}, i = \overline{1, u}, \gamma_\alpha^j(t) \in \mathbb{R}, \alpha = \overline{1, r}, j = \overline{1, m}$ satisfying (for all $t \in \Omega$, except at discontinuities)*

$$\theta_l \frac{\partial f_l}{\partial p^s}(t, \bar{p}(t), \bar{q}(t)) + \mu_i(t) \frac{\partial g_i}{\partial p^s}(t, \bar{p}(t), \bar{q}(t)) + \gamma_\alpha^j(t) \frac{\partial h_\alpha^j}{\partial p^s}(t, \bar{p}(t), \bar{q}(t)) + \frac{\partial \gamma_\alpha^j}{\partial t^s}(t) = 0,$$

$$\zeta = \overline{1, m},$$

$$\theta_l \frac{\partial f_l}{\partial q^\tau}(t, \bar{p}(t), \bar{q}(t)) + \mu_i(t) \frac{\partial g_i}{\partial q^\tau}(t, \bar{p}(t), \bar{q}(t)) + \gamma_\alpha^j(t) \frac{\partial h_\alpha^j}{\partial q^\tau}(t, \bar{p}(t), \bar{q}(t)) = 0,$$

$$\tau = \overline{1, n},$$

$$\mu_i(t)g_i(t, \bar{p}(t), \bar{q}(t)) = 0, \quad \theta_l > 0, \quad \mu(t) \geq 0.$$

The above-mentioned Kuhn–Tucker conditions are not sufficient for a feasible solution to be considered an optimal solution in an optimization problem. The sufficiency of these conditions is formulated in the following theorem.

Theorem 3. (Kuhn–Tucker sufficient conditions of efficiency.) *If the pair $(\bar{p}(t), \bar{q}(t)) \in \text{DOM}$ satisfies the necessary conditions of efficiency given above and the involved functionals are convex at $(\bar{p}(t), \bar{q}(t))$, then the pair $(\bar{p}(t), \bar{q}(t))$ is an efficient solution to (Problem).*

Definition 3. A functional $F(p, q) = \int_\Omega f(t, p(t), p_\sigma(t), q(t))dt$ is named convex at $(\bar{p}, \bar{q})$ if

$$\begin{aligned} F(p, q) - F(\bar{p}, \bar{q}) &\geq \int_\Omega (p(t) - \bar{p}(t))f_p(t, \bar{p}(t), \bar{p}_\sigma(t), \bar{q}(t))dt \\ &+ \int_\Omega (p_\sigma(t) - \bar{p}_\sigma(t))f_{p_\sigma}(t, \bar{p}(t), \bar{p}_\sigma(t), \bar{q}(t))dt \\ &+ \int_\Omega (q(t) - \bar{q}(t))f_q(t, \bar{p}(t), \bar{p}_\sigma(t), \bar{q}(t))dt \end{aligned}$$

is fulfilled for (p, q) on $P \times Q$.

3. Problem Formulation

For any $z = (z_1, z_2, \dots, z_n)^T, \gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$, where the symbol $()^T$ stands for the transpose, we consider:

- (i) $z = \gamma \iff z_h = \gamma_h, \text{ for } h = 1, \dots, n;$
- (ii) $z < \gamma \iff z_h < \gamma_h, \text{ for } h = 1, \dots, n;$
- (iii) $z \leq \gamma \iff z_h \leq \gamma_h, \text{ for } h = 1, \dots, n;$
- (iv) $z \leq \gamma \iff z \leq \gamma \text{ and } z \neq \gamma.$

By hypothesis, all vectors in the this study will be considered as column vectors. Let $L := [t_1, t_2] \subset \mathbb{R}^m$ be a multi-dimensional real interval, that is, a hyper-parallelepiped having the diagonally opposite corners $t_1 = (t_1^1, \dots, t_1^m)$ and $t_2 = (t_2^1, \dots, t_2^m)$. Also, let $\mathcal{P} = \{1, \dots, p\}, \mathcal{Q} = \{1, \dots, q\}, \mathcal{N} = \{1, \dots, n\}$, and $du = dt^1 \dots dt^m$ (volume element). Consider $\epsilon(t)$ as an n -dimensional (piecewise) differentiable function of t , and $\epsilon_\omega(t)$ as

the partial derivative of $\epsilon(t)$ related to t^ω , $\omega = \overline{1, m}$, in L . Also, we consider $\lambda(t)$ as an s -dimensional (piecewise) continuous function of t . Denote by $A \times B$ the space of all pairs $(\epsilon(t), \lambda(t))$ equipped with the uniform norm $\|\epsilon\| = \|\epsilon\|_\infty + \|\epsilon_\omega\|_\infty$, and $\|\lambda\| = \|\lambda\|_\infty$, respectively. In the following, we denote $\epsilon(t)$, $\lambda(t)$, and $\epsilon_\omega(t)$ as ϵ , λ , and ϵ_ω , respectively.

In [22], Caristi et al. defined the notion of (Γ, ψ) -invexity as an extension of invexity (previously defined by Hanson [25]). Now, we extend the notions of (generalized) (Γ, ψ) -invexity, presented by Caristi et al. [22] and Antczak [23], to the multiple objective control models driven by multiple integral functionals. In this regard, we define the concept of convexity associated with the following functional $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$.

Definition 4. The functional $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, $\Gamma = \Gamma(t, \epsilon, \lambda, y, v; (\cdot, \cdot, \cdot))$ is named convex on $\mathbb{R}^{n+s+1}$ if, for any $\epsilon, y \in \mathbb{R}^n, \lambda, v \in \mathbb{R}^s$, the following inequality

$$\begin{aligned} & \Gamma(t, \epsilon, \lambda, y, v; (\epsilon(\tau_1, \pi_1, \psi_1) + (1 - \epsilon)(\tau_2, \pi_2, \psi_2))) \\ & \leq \epsilon \Gamma(t, \epsilon, \lambda, y, v; (\tau_1, \pi_1, \psi_1)) + (1 - \epsilon) \Gamma(t, \epsilon, \lambda, y, v; (\tau_2, \pi_2, \psi_2)) \end{aligned}$$

holds, for all $\tau_1, \tau_2 \in \mathbb{R}^n, \pi_1, \pi_2 \in \mathbb{R}^s, \psi_1, \psi_2 \in \mathbb{R}$, and for any $\epsilon \in [0, 1]$.

Let $\Omega : A \times B \rightarrow \mathbb{R}$ be defined as $\Omega(\epsilon, \lambda) = \int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du$, where $\pi : L \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$ is a C^1 -class functional. The following definitions, in accordance with Hanson [25], Caristi et al. [22], and Antczak [23], and following Treanță [17,26], state the notion of generalized (Γ, ψ) -invexity for the above-mentioned real-valued controlled multiple integral-type functional Ω .

Definition 5. For an arbitrary fixed $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, if there exist $\psi \in \mathbb{R}$ and $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, with $\Gamma = \Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{n+s+1}$, $\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$ for every $(\epsilon, \lambda) \in A \times B$ and $\rho \in \mathbb{R}_+$, such that

$$\begin{aligned} & \int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du - \int_L \pi(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq [>] \int_L \Gamma \left(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\pi_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\pi_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \pi_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi) \right) du \end{aligned}$$

holds, for all $(\epsilon, \lambda) \in A \times B, [(\epsilon, \lambda) \neq (\bar{\epsilon}, \bar{\lambda})]$, then the multiple integral functional Ω is named [strictly] (Γ, ψ) -invex at $(\bar{\epsilon}, \bar{\lambda})$ on $A \times B$. If the above-mentioned relation is fulfilled for every $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, then Ω is named [strictly] (Γ, ψ) -invex on $A \times B$.

Definition 6. For an arbitrary fixed $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, if there exist $\psi \in \mathbb{R}$ and $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, with $\Gamma = \Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{n+s+1}$, $\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$ for every $(\epsilon, \lambda) \in A \times B$ and $\rho \in \mathbb{R}_+$, such that

$$\begin{aligned} & \int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du - \int_L \pi(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \leq [<] \int_L \Gamma \left(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\pi_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\pi_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \pi_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi) \right) du \end{aligned}$$

holds, for all $(\epsilon, \lambda) \in A \times B, [(\epsilon, \lambda) \neq (\bar{\epsilon}, \bar{\lambda})]$, then the multiple integral functional Ω is named [strictly] (Γ, ψ) -incave at $(\bar{\epsilon}, \bar{\lambda})$ on $A \times B$. If the above-mentioned relation is fulfilled for every $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, then Ω is named [strictly] (Γ, ψ) -incave on $A \times B$.

Definition 7. For an arbitrary fixed $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, if there exist $\psi \in \mathbb{R}$ and $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, with $\Gamma = \Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{n+s+1}$, $\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$ for every $(\epsilon, \lambda) \in A \times B$ and $\rho \in \mathbb{R}_+$, such that

$$\int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du < \int_L \pi(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du$$

implies

$$\int_L \Gamma \left(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; \left(\pi_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\pi_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \pi_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi \right) \right) du < 0$$

holds, for all $(\epsilon, \lambda) \in A \times B$, then the multiple integral functional Ω is named (Γ, ψ) -pseudoinvex at $(\bar{\epsilon}, \bar{\lambda})$ on $A \times B$. If the relation given above is satisfied for every $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, then Ω is named (Γ, ψ) -pseudoinvex on $A \times B$.

Definition 8. For an arbitrary fixed $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, if there exist $\psi \in \mathbb{R}$ and $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, with $\Gamma = \Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{n+s+1}$, $\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$ for every $(\epsilon, \lambda) \in A \times B$ and $\rho \in \mathbb{R}_+$, such that

$$\int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du \leq \int_L \pi(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du$$

implies

$$\int_L \Gamma \left(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; \left(\pi_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\pi_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \pi_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi \right) \right) du < 0$$

holds, for all $(\epsilon, \lambda) \in A \times B$, $(\epsilon, \lambda) \neq (\bar{\epsilon}, \bar{\lambda})$, then the multiple integral functional Ω is named strictly (Γ, ψ) -pseudoinvex at $(\bar{\epsilon}, \bar{\lambda})$ on $A \times B$. If the relation given above is satisfied for every $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, then Ω is named strictly (Γ, ψ) -pseudoinvex on $A \times B$.

Definition 9. For an arbitrary fixed $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, if there exist $\psi \in \mathbb{R}$ and $\Gamma : L \times (\mathbb{R}^n \times \mathbb{R}^s)^2 \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}$, with $\Gamma = \Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ convex on $\mathbb{R}^{n+s+1}$, $\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$ for every $(\epsilon, \lambda) \in A \times B$ and $\rho \in \mathbb{R}_+$, such that

$$\int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du \leq \int_L \pi(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du$$

implies

$$\int_L \Gamma \left(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; \left(\pi_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\pi_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \pi_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi \right) \right) du \leq 0$$

holds, for all $(\epsilon, \lambda) \in A \times B$, then the multiple integral functional Ω is named (Γ, ψ) -quasiinvex at $(\bar{\epsilon}, \bar{\lambda})$ on $A \times B$. If the relation given above is satisfied for every $(\bar{\epsilon}, \bar{\lambda}) \in A \times B$, then Ω is named (Γ, ψ) -quasiinvex on $A \times B$.

The concept of (Γ, ψ) -invexity extends many generalized convexity concepts, previously defined in the specialized literature. To highlight this fact, we formulate an illustrative example of a controlled multiple integral-type functional Ω , which is (Γ, ψ) -invex but not invex.

Example 1. Consider the functional $\pi : [0, 1]^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\pi(t, \epsilon, \epsilon_\omega, \lambda) = \lambda_1 \lambda_2$, where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, generating the controlled multiple integral-type functional Ω , defined by $\Omega(\epsilon, \lambda) = \int_L \pi(t, \epsilon, \epsilon_\omega, \lambda) du^1 du^2$. For $\psi = -1$ and

$$\Gamma(t, \epsilon, \lambda, \bar{\epsilon}, \bar{\lambda}; (0, \beta, \psi)) = -\frac{1}{2}(\bar{\lambda}_1 \beta_1 + \bar{\lambda}_2 \beta_2) - 2(1 - 2^\psi)|\lambda_1 \lambda_2|,$$

by Definition 5, it can be easily shown that the functional Ω is (Γ, ψ) -invex on $\mathbb{R}^4$. Note, moreover, that the functional Ω is not invex on $\mathbb{R}^4$ related to any function η (see, Definition 7, Nahak and Nanda [27]).

In this paper, we focus on efficiency criteria and dual models for the following non-convex multiple-cost minimization problem:

$$\begin{aligned} \text{(Problem)} \quad & \min_{(\epsilon, \lambda)} \int_L \mathcal{F}(t, \epsilon(t), \lambda(t)) du \\ & = \left(\int_L \mathcal{F}^1(t, \epsilon(t), \lambda(t)) du, \dots, \int_L \mathcal{F}^p(t, \epsilon(t), \lambda(t)) du \right) \\ \text{subject to} \quad & \mathcal{G}(t, \epsilon(t), \lambda(t)) \leq 0, \quad t \in L, \\ & M(t, \epsilon(t), \lambda(t)) = \epsilon_\omega(t), \quad t \in L, \\ & \epsilon(t_1) = \epsilon_0 = \text{given}, \quad \epsilon(t_2) = \epsilon_1 = \text{given}, \end{aligned}$$

with $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^p) : L \times \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^p$ as a p -dimensional C^1 -class functional, and the constraint functionals $\mathcal{G} : L \times \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^q$ and $M : L \times \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^{nm}$ are assumed C^1 -class q and nm -dimensional functionals, respectively.

Let SOL denote the feasible point set associated with (Problem), that is

$$SOL = \{(\epsilon, \lambda) \in A \times B \text{ verifying the constraints of (Problem)}\}.$$

Definition 10. A feasible solution $(\bar{\epsilon}, \bar{\lambda})$ of the considered multi-cost variational model (Problem) is named an efficient point of (Problem) if there exists no other $(\epsilon, \lambda) \in SOL$, such that

$$\int_L \mathcal{F}(t, \epsilon, \lambda) du \leq \int_L \mathcal{F}(t, \bar{\epsilon}, \bar{\lambda}) du$$

that is, there exists no other $(\epsilon, \lambda) \in SOL$, such that

$$\begin{aligned} \int_L \mathcal{F}^h(t, \epsilon, \lambda) du &\leq \int_L \mathcal{F}^h(t, \bar{\epsilon}, \bar{\lambda}) du, \quad \forall h \in \mathcal{P}, \\ \int_L \mathcal{F}^r(t, \epsilon, \lambda) du &< \int_L \mathcal{F}^r(t, \bar{\epsilon}, \bar{\lambda}) du, \quad \text{for some } r \in \mathcal{P}. \end{aligned}$$

4. Sufficient Efficiency Criteria for (Problem)

In order to establish sufficient efficiency criteria for the considered multiple-cost control model (Problem), we formulate the Karush–Kuhn–Tucker (KKT) necessary efficiency conditions for such a vector extremization problem (see Treanță [26]).

Theorem 4. If $(\bar{\epsilon}, \bar{\lambda})$ is a normal efficient point of (Problem) and the KKT constraint qualification is fulfilled, then there exist $\bar{\delta} \in \mathbb{R}^p$ and the differentiable functions (piecewise) $\bar{\mu}(\cdot) : L \rightarrow \mathbb{R}^q$ and $\bar{\theta}(\cdot) : L \rightarrow \mathbb{R}^{nm}$ satisfying

$$\begin{aligned} & \bar{\delta}^T \mathcal{F}_\epsilon(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\mu}(t)^T \mathcal{G}_\epsilon(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\theta}(t)^T \mathcal{H}_\epsilon(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) \\ & = \frac{\partial}{\partial t^\omega} \left[\bar{\delta}^T \mathcal{F}_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\mu}(t)^T \mathcal{G}_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\theta}(t)^T \mathcal{H}_{\epsilon_\omega}(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) \right], \quad t \in L, \end{aligned} \quad (1)$$

$$\bar{\delta}^T \mathcal{F}_\lambda(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\mu}(t)^T \mathcal{G}_\lambda(t, \bar{\epsilon}, \bar{\lambda}) + \bar{\theta}(t)^T \mathcal{H}_\lambda(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) = 0, \quad t \in L, \quad (2)$$

$$\int_L \bar{\mu}(t)^T \mathcal{G}(t, \bar{\epsilon}, \bar{\lambda}) du = 0, \quad \bar{\delta} \geq 0, \quad \bar{\delta}^T e = 1, \quad \bar{\mu}(t) \geq 0, \quad t \in L, \quad (3)$$

except at discontinuity points, with $\mathcal{H} := M - \epsilon_\omega$.

Further, let us consider μ for $\bar{\mu}(t)$ and θ for $\bar{\theta}(t)$.

Theorem 5. If $(\bar{\epsilon}, \bar{\lambda}) \in \text{SOL}$ and the necessary efficiency conditions in (1)–(3) are satisfied, with $\bar{\delta} \in \mathbb{R}^p$ and the differentiable functions (piecewise) $\bar{\mu}(\cdot) : L \rightarrow \mathbb{R}^q$ and $\bar{\theta}(\cdot) : L \rightarrow \mathbb{R}^{nm}$, and, in addition, the following assumptions are considered:

- (a) $\int_L \mathcal{F}^h(t, \epsilon, \lambda) du, h \in \mathcal{P}$, is strictly $(\Gamma, \psi_{\mathcal{F}^h})$ -invex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL;
- (b) $\int_L \mathcal{G}^f(t, \epsilon, \lambda) du, f \in \mathcal{Q}$, is $(\Gamma, \psi_{\mathcal{G}^f})$ -invex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL;
- (c) $\int_L \mathcal{H}^i(t, \epsilon, \epsilon_\omega, \lambda) du, i \in \mathcal{N}^+(t) = \{i \in \mathcal{N} : \bar{\theta}_i(t) > 0\}$, is $(\Gamma, \psi_{\mathcal{H}^i})$ -invex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL;
- (d) $-\int_L \mathcal{H}^i(t, \epsilon, \epsilon_\omega, \lambda) du, i \in \mathcal{N}^-(t) = \{i \in \mathcal{N} : \bar{\theta}_i(t) < 0\}$, is $(\Gamma, \psi_{\mathcal{H}^i})$ -invex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL;
- (e) $\sum_{h=1}^p \bar{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \bar{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \psi_{\mathcal{H}^i} - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i \psi_{\mathcal{H}^i} \geq 0$;

then the pair $(\bar{\epsilon}, \bar{\lambda})$ is an efficient point of (Problem).

Proof. Contrary to the result, assume that $(\bar{\epsilon}, \bar{\lambda})$ is not an efficient pair of (Problem). Consequently, there exists $(\tilde{\epsilon}, \tilde{\lambda}) \in \text{SOL}$ satisfying

$$\int_L \mathcal{F}^h(t, \tilde{\epsilon}, \tilde{\lambda}) du \leq \int_L \mathcal{F}^h(t, \bar{\epsilon}, \bar{\lambda}) du, \quad \forall h \in \mathcal{P} \quad (4)$$

and

$$\int_L \mathcal{F}^r(t, \tilde{\epsilon}, \tilde{\lambda}) du < \int_L \mathcal{F}^r(t, \bar{\epsilon}, \bar{\lambda}) du, \quad \text{for some } r \in \mathcal{P}. \quad (5)$$

Since the hypotheses (a)–(d) are assumed, the following inequalities are fulfilled

$$\begin{aligned} & \int_L \mathcal{F}^h(t, \tilde{\epsilon}, \tilde{\lambda}) du - \int_L \mathcal{F}^h(t, \bar{\epsilon}, \bar{\lambda}) du \\ & > \int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^h(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_\lambda^h(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du, \quad h \in \mathcal{P}, \end{aligned} \quad (6)$$

$$\begin{aligned} & \int_L \mathcal{G}^f(t, \tilde{\epsilon}, \tilde{\lambda}) du - \int_L \mathcal{G}^f(t, \bar{\epsilon}, \bar{\lambda}) du \\ & \geq \int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{G}_{\epsilon_\omega}^f(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{G}_\lambda^f(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du, \quad f \in \mathcal{Q}, \end{aligned} \quad (7)$$

$$\begin{aligned} & \int_L \mathcal{H}^i(t, \tilde{\epsilon}, \tilde{\epsilon}_\omega, \tilde{\lambda}) du - \int_L \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq \int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du, \\ & i \in \mathcal{N}^+(t), \end{aligned} \quad (8)$$

$$\begin{aligned} & - \int_L \mathcal{H}^i(t, \tilde{\epsilon}, \tilde{\epsilon}_\omega, \tilde{\lambda}) du + \int_L \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq \int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. -\mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du, \quad i \in \mathcal{N}^-(t). \end{aligned} \quad (9)$$

By considering (4)–(6) and using the assumption $\bar{\delta} \geq 0$, it follows

$$\int_L \bar{\delta}_h \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^h(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_\lambda^h(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du \leq 0, \quad (10)$$

for $h \in \mathcal{P}$, and

$$\int_L \bar{\delta}_r \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^r(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^r(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_\lambda^r(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^r} \right) \right) du < 0, \quad (11)$$

for at least one $r \in \mathcal{P}$. By adding (10) and (11), it results

$$\int_L \sum_{h=1}^p \bar{\delta}_h \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^h(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_\lambda^h(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du < 0. \quad (12)$$

Since $\bar{\mu}_f(t) \geq 0, f \in \mathcal{Q}$, then (7) gives

$$\begin{aligned} & \int_L \sum_{f=1}^q \bar{\mu}_f \mathcal{G}^f(t, \bar{\epsilon}, \bar{\lambda}) du - \int_L \sum_{f=1}^q \bar{\mu}_f \mathcal{G}^f(t, \bar{\epsilon}, \bar{\lambda}) du \\ & \geq \int_L \sum_{f=1}^q \bar{\mu}_f \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{G}_{\epsilon_\omega}^f(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{G}_\lambda^f(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du. \end{aligned} \quad (13)$$

Taking into account the feasibility property of $(\bar{\epsilon}, \bar{\lambda})$ in (Problem), together with the KKT necessary efficiency condition given in (3), we obtain

$$\int_L \sum_{f=1}^q \bar{\mu}_f \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{G}_{\epsilon_\omega}^f(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{G}_\lambda^f(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du \leq 0. \quad (14)$$

The inequalities (8) and (9) yield, respectively,

$$\begin{aligned} & \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}, \bar{\lambda}) du - \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. \mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du, \end{aligned} \quad (15)$$

$$\begin{aligned} & \int_L \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}, \bar{\lambda}) du - \int_L \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq \int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. -\mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du. \end{aligned} \quad (16)$$

Adding the inequalities (15) and (16), it follows

$$\begin{aligned} & \int_L \sum_{i \in \mathcal{N}} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}, \bar{\lambda}) du - \int_L \sum_{i \in \mathcal{N}} \bar{\theta}_i \mathcal{H}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) du \\ & \geq \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. \mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du \\ & + \int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma \left(t, \bar{\epsilon}, \bar{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. -\mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du. \end{aligned}$$

Using the feasibility of $(\tilde{\epsilon}, \tilde{\lambda})$ in (Problem), together with (2) and (3), we have

$$\begin{aligned} & \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. \mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du \\ & + \int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. -\mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du \leq 0. \end{aligned} \quad (17)$$

Combining (12), (14), and (17), we obtain

$$\begin{aligned} & \int_L \sum_{h=1}^p \bar{\delta}_h \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \bar{\epsilon}, \bar{\epsilon}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^h(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_\lambda^h(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du \\ & + \int_L \sum_{f=1}^q \bar{\mu}_f \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{G}_{\epsilon_\omega}^f(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{G}_\lambda^f(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du \\ & + \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. \mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du \\ & + \int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_\epsilon^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\ & \left. \left. -\mathcal{H}_\lambda^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du < 0. \end{aligned} \quad (18)$$

We denote

$$\hat{\delta}_h = \frac{\bar{\delta}_h}{\sum_{h=1}^p \bar{\delta}_h + \sum_{f=1}^q \bar{\mu}_f(t) + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i(t) - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i(t)}, \quad h \in \mathcal{P}, \quad (19)$$

$$\hat{\mu}_f(t) = \frac{\bar{\mu}_f(t)}{\sum_{h=1}^p \bar{\delta}_h + \sum_{f=1}^q \bar{\mu}_f(t) + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i(t) - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i(t)}, \quad f \in \mathcal{Q}, \quad (20)$$

$$\hat{\theta}_i(t) = \frac{\bar{\theta}_i(t)}{\sum_{h=1}^p \bar{\delta}_h + \sum_{f=1}^q \bar{\mu}_f(t) + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i(t) - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i(t)}, \quad i \in \mathcal{N}^+(t), \quad (21)$$

$$\hat{\theta}_i(t) = \frac{-\bar{\theta}_i(t)}{\sum_{h=1}^p \bar{\delta}_h + \sum_{f=1}^q \bar{\mu}_f(t) + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i(t) - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i(t)}, \quad i \in \mathcal{N}^-(t). \quad (22)$$

By (19)–(22), it follows that $0 \leq \hat{\delta}_h \leq 1, h \in \mathcal{P}$, but $\hat{\delta}_h > 0$ for at least one $h \in \mathcal{P}$, $0 \leq \hat{\mu}_f(t) \leq 1, f \in \mathcal{Q}$, $0 \leq \hat{\theta}_i(t) \leq 1, i \in \mathcal{N}$, and, moreover,

$$\sum_{h=1}^p \hat{\delta}_h + \sum_{f=1}^q \hat{\mu}_f(t) + \sum_{i \in \mathcal{N}^+(t)} \hat{\theta}_i(t) + \sum_{i \in \mathcal{N}^-(t)} \hat{\theta}_i(t) = 1. \quad (23)$$

Combining (18)–(22), we obtain

$$\begin{aligned}
& \int_L \sum_{h=1}^p \hat{\delta}_h \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_{\tilde{\epsilon}}^h(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t \omega} [\mathcal{F}_{\tilde{\epsilon}\omega}^h(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{F}_{\tilde{\lambda}}^h(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du \\
& + \int_L \sum_{f=1}^q \hat{\mu}_f \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_{\tilde{\epsilon}}^f(t, \bar{\epsilon}, \bar{\lambda}) - \frac{\partial}{\partial t \omega} [\mathcal{G}_{\tilde{\epsilon}\omega}^f(t, \bar{\epsilon}, \bar{\lambda})], \mathcal{G}_{\tilde{\lambda}}^f(t, \bar{\epsilon}, \bar{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du \\
& + \int_L \sum_{i \in \mathcal{N}^+(t)} \hat{\theta}_i \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{H}_{\tilde{\epsilon}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t \omega} [\mathcal{H}_{\tilde{\epsilon}\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\
& \left. \left. \mathcal{H}_{\tilde{\lambda}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du \\
& + \int_L \sum_{i \in \mathcal{N}^-(t)} \hat{\theta}_i \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(-\mathcal{H}_{\tilde{\epsilon}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t \omega} [-\mathcal{H}_{\tilde{\epsilon}\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda})], \right. \right. \\
& \left. \left. -\mathcal{H}_{\tilde{\lambda}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i} \right) \right) du < 0. \tag{24}
\end{aligned}$$

By Definition 4, it follows that the functional $\Gamma(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; (\cdot, \cdot, \cdot))$ is convex on $\mathbb{R}^{n+s+1}$. Considering (23) is valid, by using (24), we obtain

$$\begin{aligned}
& \int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\left[\sum_{h=1}^p \hat{\delta}_h \mathcal{F}_{\tilde{\epsilon}}^h(t, \bar{\epsilon}, \bar{\lambda}) + \sum_{f=1}^q \hat{\mu}_f \mathcal{G}_{\tilde{\epsilon}}^f(t, \bar{\epsilon}, \bar{\lambda}) \right. \right. \right. \\
& \left. \left. + \sum_{i \in \mathcal{N}^+(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\epsilon}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) + \sum_{i \in \mathcal{N}^-(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\epsilon}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) \right] \right. \\
& \left. - \frac{\partial}{\partial t \omega} \left[\sum_{h=1}^p \hat{\delta}_h \mathcal{F}_{\tilde{\epsilon}\omega}^h(t, \bar{\epsilon}, \bar{\lambda}) + \sum_{f=1}^q \hat{\mu}_f \mathcal{G}_{\tilde{\epsilon}\omega}^f(t, \bar{\epsilon}, \bar{\lambda}) \right. \right. \\
& \left. \left. + \sum_{i \in \mathcal{N}^+(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\epsilon}\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) + \sum_{i \in \mathcal{N}^-(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\epsilon}\omega}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) \right] \right. \\
& \left. \left. \sum_{h=1}^p \hat{\delta}_h \mathcal{F}_{\tilde{\lambda}}^h(t, \bar{\epsilon}, \bar{\lambda}) + \sum_{f=1}^q \hat{\mu}_f \mathcal{G}_{\tilde{\lambda}}^f(t, \bar{\epsilon}, \bar{\lambda}) \right. \right. \\
& \left. \left. + \sum_{i \in \mathcal{N}^+(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\lambda}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}) + \sum_{i \in \mathcal{N}^-(t)} \hat{\theta}_i \mathcal{H}_{\tilde{\lambda}}^i(t, \bar{\epsilon}, \bar{\epsilon}_\omega, \bar{\lambda}), \right. \right. \\
& \left. \left. \sum_{h=1}^p \hat{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \hat{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t) \cup \mathcal{N}^-(t)} \hat{\theta}_i \psi_{\mathcal{H}^i} \right) \right) du < 0.
\end{aligned}$$

Hence, the KKT necessary efficiency conditions yield

$$\int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(0, 0, \sum_{h=1}^p \hat{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \hat{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t) \cup \mathcal{N}^-(t)} \hat{\theta}_i \psi_{\mathcal{H}^i} \right) \right) du < 0. \tag{25}$$

From assumption (e), we obtain

$$\sum_{h=1}^p \hat{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \hat{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t) \cup \mathcal{N}^-(t)} \hat{\theta}_i \psi_{\mathcal{H}^i} \geq 0. \tag{26}$$

By Definition 5, it follows that $\Gamma(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; (0, 0, \rho)) \geq 0$, for $\rho \in \mathbb{R}_+$. Therefore, relation (26) involves

$$\int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(0, 0, \sum_{h=1}^p \hat{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \hat{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t) \cup \mathcal{N}^-(t)} \hat{\theta}_i \psi_{\mathcal{H}^i} \right) \right) du \geq 0$$

is true, which is a contradiction to (25). Consequently, the proof is complete. $\square$

Theorem 6. If $(\bar{\epsilon}, \bar{\lambda}) \in SOL$ and the KKT necessary efficiency criteria given in (1)–(3) are fulfilled at this point, with $\bar{\delta} \in \mathbb{R}^p$ and the differentiable functions (piecewise) $\bar{\mu}(\cdot) : L \rightarrow \mathbb{R}^q$ and $\bar{\theta}(\cdot) : L \rightarrow \mathbb{R}^{nm}$, and moreover, suppose the statements are fulfilled:

- (a) $\int_L \mathcal{F}^h(t, \epsilon, \lambda) du, h \in \mathcal{P}$, is strictly $(\Gamma, \psi_{\mathcal{F}^h})$ -pseudoinvex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL ;
- (b) $\int_L \mathcal{G}^f(t, \epsilon, \lambda) du, f \in \mathcal{Q}$, is $(\Gamma, \psi_{\mathcal{G}^f})$ -quasiinvex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL ;
- (c) $\int_L \mathcal{H}^i(t, \epsilon, \epsilon_\omega, \lambda) du, i \in \mathcal{N}^+(t)$, is $(\Gamma, \psi_{\mathcal{H}^i})$ -quasiinvex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL ;
- (d) $-\int_L \mathcal{H}^i(t, \epsilon, \epsilon_\omega, \lambda) du, i \in \mathcal{N}^-(t)$, is $(\Gamma, \psi_{\mathcal{H}^i})$ -quasiinvex at $(\bar{\epsilon}, \bar{\lambda})$ on SOL ;
- (e) $\sum_{h=1}^p \bar{\delta}_h \psi_{\mathcal{F}^h} + \sum_{f=1}^q \bar{\mu}_f \psi_{\mathcal{G}^f} + \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \psi_{\mathcal{H}^i} - \sum_{i \in \mathcal{N}^-(t)} \bar{\theta}_i \psi_{\mathcal{H}^i} \geq 0$;

then the pair $(\bar{\epsilon}, \bar{\lambda})$ is an efficient point of (Problem).

Proof. By contradiction, suppose that $(\bar{\epsilon}, \bar{\lambda})$ is not an efficient pair of (Problem). Therefore, there exists $(\tilde{\epsilon}, \tilde{\lambda}) \in SOL$ satisfying

$$\int_L \mathcal{F}^h(t, \tilde{\epsilon}, \tilde{\lambda}) du \leq \int_L \mathcal{F}^h(t, \bar{\epsilon}, \bar{\lambda}) du, \quad h \in \mathcal{P} \quad (27)$$

and

$$\int_L \mathcal{F}^r(t, \tilde{\epsilon}, \tilde{\lambda}) du < \int_L \mathcal{F}^r(t, \bar{\epsilon}, \bar{\lambda}) du, \quad \text{for some } r \in \mathcal{P}. \quad (28)$$

By using Definition 8, the relations (27) and (28) yield

$$\int_L \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \tilde{\epsilon}, \tilde{\lambda}) - \frac{\partial}{\partial t^\omega} \left[\mathcal{F}_{\epsilon_\omega}^h(t, \tilde{\epsilon}, \tilde{\lambda}) \right], \mathcal{F}_\lambda^h(t, \tilde{\epsilon}, \tilde{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du < 0, \quad h \in \mathcal{P},$$

and, since $\bar{\delta} \geq 0$, then the inequality given above gives

$$\int_L \sum_{h=1}^p \bar{\delta}_h \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{F}_\epsilon^h(t, \tilde{\epsilon}, \tilde{\lambda}) - \frac{\partial}{\partial t^\omega} \left[\mathcal{F}_{\epsilon_\omega}^h(t, \tilde{\epsilon}, \tilde{\lambda}) \right], \mathcal{F}_\lambda^h(t, \tilde{\epsilon}, \tilde{\lambda}), \psi_{\mathcal{F}^h} \right) \right) du < 0. \quad (29)$$

Considering the feasibility property of $(\bar{\epsilon}, \bar{\lambda})$ and $(\tilde{\epsilon}, \tilde{\lambda})$ in problem (Problem), together with the KKT necessary efficiency criteria, we obtain

$$\int_L \bar{\mu}_f \mathcal{G}^f(t, \tilde{\epsilon}, \tilde{\lambda}) du \leq \int_L \bar{\mu}_f \mathcal{G}^f(t, \bar{\epsilon}, \bar{\lambda}) du, \quad f \in \mathcal{Q}.$$

Thus, by Definition 9, the hypothesis (b) yields

$$\int_L \bar{\mu}_f \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \tilde{\epsilon}, \tilde{\lambda}) - \frac{\partial}{\partial t^\omega} \left[\mathcal{G}_{\epsilon_\omega}^f(t, \tilde{\epsilon}, \tilde{\lambda}) \right], \mathcal{G}_\lambda^f(t, \tilde{\epsilon}, \tilde{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du \leq 0, \quad f \in \mathcal{Q}.$$

Adding the inequalities given above, it results

$$\int_L \sum_{f=1}^q \bar{\mu}_f \Gamma \left(t, \tilde{\epsilon}, \tilde{\lambda}, \bar{\epsilon}, \bar{\lambda}; \left(\mathcal{G}_\epsilon^f(t, \tilde{\epsilon}, \tilde{\lambda}) - \frac{\partial}{\partial t^\omega} \left[\mathcal{G}_{\epsilon_\omega}^f(t, \tilde{\epsilon}, \tilde{\lambda}) \right], \mathcal{G}_\lambda^f(t, \tilde{\epsilon}, \tilde{\lambda}), \psi_{\mathcal{G}^f} \right) \right) du \leq 0. \quad (30)$$

Further, by using the feasibility property of $(\bar{\epsilon}, \bar{\lambda})$ and $(\tilde{\epsilon}, \tilde{\lambda})$ in (Problem), it follows

$$\int_L \mathcal{H}^i(t, \tilde{e}, \tilde{e}, \tilde{\lambda}) du = \int_L \mathcal{H}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) du, \quad i \in \mathcal{N}^+(t), \quad (31)$$

$$\int_L -\mathcal{H}^i(t, \tilde{e}, \tilde{e}, \tilde{\lambda}) du = \int_L -\mathcal{H}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) du, \quad i \in \mathcal{N}^-(t). \quad (32)$$

Hence, by assumptions given in (c) and (d), the above-mentioned inequalities in (31) and (32) involve, respectively,

$$\int_L \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], \mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du \leq 0, \quad i \in \mathcal{N}^+(t), \quad (33)$$

$$\int_L \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (-\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], -\mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du \leq 0, \quad i \in \mathcal{N}^-(t). \quad (34)$$

Thus,

$$\int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], \mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du \leq 0, \quad (35)$$

$$\int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (-\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], -\mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du \leq 0. \quad (36)$$

Combining (29), (30), (35), and (36), we obtain

$$\begin{aligned} & \int_L \sum_{h=1}^p \bar{\delta}_h \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (\mathcal{F}_\epsilon^h(t, \bar{e}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{F}_{\epsilon_\omega}^h(t, \bar{e}, \bar{\lambda})], \mathcal{F}_\lambda^h(t, \bar{e}, \bar{\lambda}), \psi_{\mathcal{F}^h})) du \\ & + \int_L \sum_{f=1}^q \bar{\mu}_f \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (\mathcal{G}_\epsilon^f(t, \bar{e}, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{G}_{\epsilon_\omega}^f(t, \bar{e}, \bar{\lambda})], \mathcal{G}_\lambda^f(t, \bar{e}, \bar{\lambda}), \psi_{\mathcal{G}^f})) du \\ & + \int_L \sum_{i \in \mathcal{N}^+(t)} \bar{\theta}_i \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], \mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du \\ & + \int_L \sum_{i \in \mathcal{N}^-(t)} (-\bar{\theta}_i) \Gamma(t, \tilde{e}, \tilde{\lambda}, \bar{e}, \bar{\lambda}; (-\mathcal{H}_\epsilon^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}) - \frac{\partial}{\partial t^\omega} [-\mathcal{H}_{\epsilon_\omega}^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda})], -\mathcal{H}_\lambda^i(t, \bar{e}, \bar{e}_\omega, \bar{\lambda}), \psi_{\mathcal{H}^i})) du < 0. \end{aligned}$$

The second part of the proof is similar to the proof provided for Theorem 5. $\square$

Remark 1. As potential extensions of the proposed technique for various types of optimization problems or domains, we could mention the study of well-posedness and efficiency criteria associated with similar classes of extremization problems governed by path-independent curvilinear integral functionals (very important in applications due to their physical meaning (mechanical work)). This is a specific research question or unresolved issue that could be addressed in future studies to build upon the current findings. Consequently, we note the applicability of the proposed technique to larger or more complex optimization problems.

5. Conclusions and Future Research Directions

In this paper, we have introduced and studied new minimization problems determined by multiple integrals as objective functionals. More precisely, we have formulated and established new sufficient efficiency criteria for a feasible point in the studied optimization model. To this end, we have introduced and defined the notions of (Γ, ψ) -invexity and generalized (Γ, ψ) -invexity for the implied real-valued controlled multiple integral-type functionals. More precisely, we extended the notion of (generalized) (Γ, ψ) -invexity, presented by Caristi et al. [22] and Antczak [23], to the multiple objective control models driven by multiple integral functionals. In addition, innovative proofs have been considered for the principal results derived in the paper. An immediate future research direction associated with this paper could be the study of duality theory (of Wolfe, Mond-Weir, Lagrange, or mixed type).

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