

Article

Solving Fredholm Integral Equations Using Probabilistic F-Contractions

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Abstract: Fixed-point theory plays a pivotal role in addressing equations that model various real-life applications, offering robust methods to find solutions that remain stable under different conditions and dynamics. The main objective of this paper is to introduce and study the novel concept of probabilistic $\mathcal{F}$ -contraction within the framework of Menger spaces. This innovative approach extends classical fixed-point results to probabilistic settings, leveraging the probabilistic structure of Menger spaces to handle uncertainty and variability in the modeling process. By establishing the existence and uniqueness of fixed points in this versatile class of spaces, the study highlights the broader applicability and deeper significance of the probabilistic $\mathcal{F}$ -contraction. We explore the intricate interrelations among these fixed points, shedding light on their implications across different contexts and presenting insights into various theorem versions that enhance our understanding of their utility. Additionally, we propose a straightforward and effective approach for solving a system of Fredholm integral equations using fixed-point techniques specifically tailored for Menger spaces, illustrating their practical utility in tackling the complex mathematical models encountered in diverse fields.

Keywords: Menger spaces; fixed point; probabilistic $\mathcal{F}$ -contraction; integral equations

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1. Introduction

Recently, several authors have extensively studied the solution of the Fredholm integral equation via the fixed-point approach, recognizing its profound implications across various domains of mathematical analysis and applied mathematics [1–3]. The Fredholm integral equation, characterized by its integral operator acting on a function to produce another function, arises frequently in problems involving boundary value issues, potential theory, and quantum mechanics. Traditional methods of solving these integral equations often involve complex and computationally intensive techniques. The fixed-point approach offers an elegant and powerful alternative by transforming the integral equation into a form where fixed-point theorems can be applied. This methodology leverages the inherent properties of fixed points, where a function maps an element to itself within a certain space. Banach's contraction principle is a cornerstone in this area, ensuring the existence and uniqueness of fixed points under specific contraction conditions [4]. This principle has been extensively utilized to address the solvability of Fredholm integral equations, yielding significant results that simplify the solution process and enhance computational efficiency.

Several studies have expanded on this foundational work. For instance, Borkowski et al. explored generalized contractions in Banach spaces to solve nonlinear Fredholm integral equations, demonstrating the robustness of the fixed-point approach under relaxed conditions [5]. Additionally, recent advancements have integrated probabilistic elements into the fixed-point framework, introducing probabilistic metric spaces that account for uncertainty and variability in the problem's parameters. These developments, spearheaded by researchers like Younis, Karapinar, and others, have broadened the applicability of fixed-point methods to more complex and real-world scenarios [6–9]. Furthermore, hybrid fixed-point techniques combining deterministic and probabilistic methods have been developed to tackle systems of Fredholm integral equations, offering new avenues for research and application. These approaches not only enhance the theoretical understanding of fixed-point theorems but also provide practical tools for solving integral equations in diverse scientific and engineering fields.

The theory of metric fixed points finds its roots in the groundbreaking work of Banach on contractions. Known for its pivotal role, Banach's theorem has been instrumental in establishing the existence and uniqueness of solutions across diverse fields such as mathematical analysis, computer science, and engineering. Applications span from differential equations to machine learning theory and image processing, among others [1–3,10]. Expanding upon Banach's foundational contributions, mathematicians have refined the theory by introducing variations in contraction conditions and restructuring the underlying metric spaces. The concept of probabilistic metric spaces, introduced by Menger in 1942, represents an extension of metric spaces by replacing non-negative numbers with a random variable that assumes only non-negative real values [11]. This innovation was further developed by Schweizer and Sklar [12]. Fixed-point theory in probabilistic metric spaces constitutes a significant branch of probabilistic and stochastic analysis with diverse applications across various scientific disciplines [13–18]. The initial result in fixed-point theory within these spaces was obtained by Sehgal and Bharucha-Reid in 1972, with subsequent improvements by Sherwood and further developments in the field [19–21].

Inspired by Wardowski's seminal work [22], where he introduced the concept of F-contraction as a generalization of Banach's contraction principle, this paper introduces a novel concept termed probabilistic F-contraction. We establish the existence and uniqueness of fixed points for this new notion within the framework of Menger spaces. Furthermore, we prove a novel common fixed-point theorem applicable to three self-mappings. To demonstrate the practical relevance of our theoretical developments, we apply these results to solve a system of Fredholm integral equations efficiently and effectively using the fixed-point technique based on our newly introduced contraction concept.

The structure of this article unfolds as follows: In Section 2, we revisit fundamental concepts and established results in probabilistic metric spaces. Section 3 introduces the concepts of probabilistic F-contraction, accompanied by the presentation of novel fixed-point theorems. Moreover, Section 4 is dedicated to studying the coincidence point theorem for three self-mappings, assuring its existence and uniqueness for this new contraction. Finally, in Section 5, we present a simple and efficient solution to a Fredholm integral equation by employing the fixed-point technique within the probabilistic metric space setting.

2. Preliminaries

To establish the foundation, we revisit some key Menger spaces. For a more comprehensive discussion, refer to [12].

Definition 1. A function $\phi : [0, +\infty] \rightarrow [0, 1]$ is a distance distribution function if it meets the following requirements:

1. ϕ is left continuous on $[0, +\infty)$;
2. ϕ is nondecreasing;
3. $\phi(0) = 0$ and $\phi(+\infty) = 1$.

Let Λ^+ denote the set of all distance distribution functions. The subset $E^+ \subset \Lambda^+$ is defined as $E^+ = \{\phi \in \Lambda^+ : \lim_{\mu \rightarrow +\infty} \phi(\mu) = 1\}$.

A particular element of E^+ is the Heaviside function $\mathcal{H}_0$, defined as follows:

$$\mathcal{H}_0(\mu) = \begin{cases} 0 & \text{if } \mu = 0, \\ 1 & \text{if } \mu > 0. \end{cases}$$

Definition 2. A triangular norm (t-norm) is a function $\Pi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the following requirements for any $\mu, \nu, \omega \in [0, 1]$:

1. $\Pi(\mu, \nu) = \Pi(\nu, \mu)$;
2. $\Pi(\mu, \Pi(\nu, \omega)) = \Pi(\Pi(\mu, \nu), \omega)$;
3. $\Pi(\mu, \nu) < \Pi(\mu, \omega)$ for $\nu < \omega$;
4. $\Pi(\mu, 1) = \Pi(1, \mu) = \mu$.

The most basic t-norms are $\Pi_M(\mu, \nu) = \min(\mu, \nu)$ and $\Pi_P(\mu, \nu) = \mu \cdot \nu$.

Definition 3. The triple (Ω, Ξ, Π) , where Ω is a nonempty set, Ξ is a function from $\Omega \times \Omega$ into Λ^+ and Π is a continuous t-norm, is called a Menger space if the following requirements are verified for any $\rho, \sigma, \varsigma \in \Omega$ and $x, y > 0$:

- (i) $\Xi_{\rho, \rho} = \mathcal{H}_0$;
- (ii) $\Xi_{\rho, \sigma} \neq \mathcal{H}_0$ if $\rho \neq \sigma$;
- (iii) $\Xi_{\rho, \sigma} = \Xi_{\sigma, \rho}$;
- (iv) $\Xi_{\rho, \sigma}(x + y) \geq \Pi(\Xi_{\rho, \varsigma}(x), \Xi_{\varsigma, \sigma}(y))$.

(Ω, Ξ, Π) is a Hausdorff topological space in the topology induced by the family of (ϵ, λ) -neighborhoods:

$$\mathcal{Q} = \{\mathcal{Q}_\rho(\gamma, \kappa) : \rho \in \Omega, \gamma > 0 \text{ and } \kappa > 0\},$$

where

$$\mathcal{Q}_\rho(\gamma, \kappa) = \{\sigma \in \Omega : \Xi_{\rho, \sigma}(\gamma) > 1 - \kappa\}.$$

Definition 4. Consider a Menger space (Ω, Ξ, Π) . A sequence $\{v_n\}$ in Ω is defined as follows:

1. Converging to $v \in \Omega$ if, for any $\gamma > 0$ and $\kappa > 0$, there exists a positive integer $N(\gamma, \kappa)$ such that $\Xi_{v_n, v}(\kappa) > 1 - \gamma$ for all $n \geq N(\gamma, \kappa)$.
2. A Cauchy sequence if, for any $\gamma > 0$ and $\kappa > 0$, there exists a positive integer $N(\gamma, \kappa)$ such that $\Xi_{v_n, v_m}(\kappa) > 1 - \gamma$ for all $n, m \geq N(\gamma, \kappa)$.

A Menger space (Ω, Ξ, Π) is termed complete if every Cauchy sequence in Ω converges to a point within Ω .

Lemma 1. If $(\Omega, \mathcal{D})$ is a complete metric space, then (Ω, Ξ, Π_P) is a complete Menger space where $\Xi_{\rho, \sigma}(t) = \frac{t}{t + \mathcal{D}(\rho, \sigma)}$ for all $\rho, \sigma \in \Omega, t > 0$ and $\Xi_{\rho, \sigma}(0) = 0$.

Proof. To show that (Ω, Ξ, Π_P) is a complete Menger space, we verify the properties of a Menger space and then prove completeness. For the first three properties that are evident, we will verify just the triangle inequality. For $t = v + \omega$, we have

$$\Xi_{\rho,\sigma}(v + \omega) = \frac{v + \omega}{v + \omega + \mathcal{D}(\rho, \sigma)} \geq \frac{v}{v + \mathcal{D}(\rho, \zeta)} \cdot \frac{\omega}{\omega + \mathcal{D}(\zeta, \sigma)} = \Pi(\Xi_{\rho,\zeta}(v), \Xi_{\zeta,\sigma}(\omega)).$$

For the completeness, let $\{v_n\}$ be a Cauchy sequence in (Ω, Ξ, Π_P) . For any $\gamma > 0$ and $\kappa > 0$, there exists $N(\gamma, \kappa)$ such that

$$\Xi_{v_n, v_m}(\kappa) > 1 - \gamma \quad \text{whenever} \quad n, m \geq N(\gamma, \kappa).$$

This implies

$$\frac{\kappa}{\kappa + \mathcal{D}(v_n, v_m)} > 1 - \gamma \quad \text{then} \quad \mathcal{D}(v_n, v_m) < \frac{\kappa\gamma}{1 - \gamma}.$$

Hence, $\{v_n\}$ is a Cauchy sequence in the metric space $(\Omega, \mathcal{D})$.

Since $(\Omega, \mathcal{D})$ is complete, there exists $v \in \Omega$ such that $v_n \rightarrow v$.

We need to show that $v_n \rightarrow v$ in (Ω, Ξ, Π_P) . For any $\gamma > 0$ and $\kappa > 0$, there exists $N(\gamma, \kappa)$ such that, for all $n \geq N(\gamma, \kappa)$,

$$\mathcal{D}(v_n, v) < \frac{\kappa\gamma}{1 - \gamma}, \quad \text{which implies} \quad \Xi_{v_n, v}(\kappa) = \frac{\kappa}{\kappa + \mathcal{D}(v_n, v)} > 1 - \gamma.$$

Thus, $v_n \rightarrow v$ in the Menger space (Ω, Ξ, Π_P) .

Therefore, (Ω, Ξ, Π_P) is a complete Menger space. $\square$

3. The Probabilistic $\mathcal{F}$ -Contraction

The contraction we introduce draws inspiration from the widely renowned contractive condition formulated by D. Wardowski in 2012, which extends the seminal Banach result. The condition is defined as follows:

For a given $\pi > 0$, if $d(T\omega, Ty) > 0$, then the inequality $\pi + \mathcal{F}(d(T\omega, Ty)) \leq \mathcal{F}(d(\omega, y))$ must hold, where $\mathcal{F}$ is a function mapping on $(0, +\infty)$ to $\mathbb{R}$ and satisfies the following three conditions:

$\mathcal{F}_1$) $\mathcal{F}$ exhibits strict monotonicity, meaning, for any $v, \tau \in \mathbb{R}_+$ with $v < \tau$, it holds that $\mathcal{F}(v) < \mathcal{F}(\tau)$.

$\mathcal{F}_2$) For any sequence $\{\delta_n\}_{n \in \mathbb{N}}$ of positive numbers, we have $\lim_{n \rightarrow +\infty} \delta_n = 0$ if and only if $\lim_{n \rightarrow +\infty} \mathcal{F}(\delta_n) = -\infty$.

$\mathcal{F}_3$) There exists a constant k belonging to the interval $(0, 1)$ such that $\lim_{v \rightarrow 0^+} v^k \mathcal{F}(v) = 0$.

We represent by Ψ the class of mapping which satisfied these three properties. Many works dealing with this type of contraction have emerged recently [18,23–26]. For instance, Fabiano et al. [23] successfully proved the fixed-point theorem for $\mathcal{F}$ -contractions by considering only the condition $\mathcal{F}_1$. They achieved this by leveraging the results established by Radenović in [27].

We present the probabilistic version of $\mathcal{F}$ -contraction within the framework of Menger space as follows:

Definition 5. Consider (Ω, Ξ, Π) as a Menger space and $\mathcal{F}$ a strictly increasing mapping on $[0, 1)$ into $(0, +\infty)$. A mapping $\mathcal{T} : \Omega \rightarrow \Omega$ is termed a probabilistic $\mathcal{F}$ -contraction if there exists $\omega \in (0, 1)$ such that for all $v, \tau \in \Omega$ and $t > 0$, the inequality

$$\omega \cdot \mathcal{F}(\Xi_{\mathcal{T}v, \mathcal{T}\tau}(t)) \geq \mathcal{F}(\Xi_{v, \tau}(t)) \quad (1)$$

holds.

Theorem 1. Consider (Ω, Ξ, Π) as a complete Menger space and let $\mathcal{T} : \Omega \rightarrow \Omega$ be a probabilistic $\mathcal{F}$ -contraction where $\mathcal{F}$ is continuous. Then, $\mathcal{T}$ possesses a unique fixed point in Ω .

Proof. Choose $\omega_0 \in \Omega$ and $\omega_{n+1} = \mathcal{T}(\omega_n)$ for all $n \in \mathbb{N}$. By the contractivity condition, we have for every $n \in \mathbb{N}$ and $t > 0$,

$$\mathcal{F}(\Xi_{\omega_{n+1}, \omega_n}(t)) > \omega \cdot \mathcal{F}(\Xi_{\omega_{n+1}, \omega_n}(t)) \geq \mathcal{F}(\Xi_{\omega_n, \omega_{n-1}}(t)).$$

By the monotonicity of $\mathcal{F}$, we obtain

$$\Xi_{\omega_{n+1}, \omega_n}(t) > \Xi_{\omega_n, \omega_{n-1}}(t).$$

Therefore, the sequence $\Xi_{\omega_{n+1}, \omega_n}(t)$ exhibits strict monotonicity and is bounded above, implying its convergence. Hence, there exists $r(t) \in [0, 1]$ such that, for any $t > 0$, we obtain

$$\lim_{n \rightarrow +\infty} \Xi_{\omega_{n+1}, \omega_n}(t) = r(t). \quad (2)$$

Evidently, for every $t > 0$ and $n \in \mathbb{N}$, it follows that

$$\Xi_{\omega_{n+1}, \omega_n}(t) < r(t). \quad (3)$$

Then, by (2) and (3), we have for any $t > 0$,

$$\lim_{n \rightarrow +\infty} \mathcal{F}(\Xi_{\omega_{n+1}, \omega_n}(t)) = \mathcal{F}(r(t)). \quad (4)$$

We suppose that $r(t) < 1$, for some $t > 0$, and by the contractive condition, we obtain

$$\mathcal{F}(\Xi_{\omega_{n+1}, \omega_n}(t)) > \omega \cdot \mathcal{F}(\Xi_{\omega_{n+1}, \omega_n}(t)) \geq \mathcal{F}(\Xi_{\omega_n, \omega_{n-1}}(t)). \quad (5)$$

By taking the limit from both sides of (5) alongside (4), we derive

$$\mathcal{F}(r(t)) > \omega \cdot \mathcal{F}(r(t)) \geq \mathcal{F}(r(t)).$$

This implies that $\mathcal{F}(r(t)) = 0$. However, this contradicts the fact that $\mathcal{F}(r(t)) > 0$. Therefore,

$$\lim_{n \rightarrow +\infty} \Xi_{\omega_{n+1}, \omega_n}(t) = 1 \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \quad (6)$$

Now we have to prove that $\{\omega_n\}$ is a Cauchy sequence. For this, we suppose that this claim is not true, which means that there exists $\kappa \in (0, 1)$, $\gamma > 0$ and $\delta \in (0, \gamma)$ with the sequences $\{\omega_{m_j}\}$ and $\{\omega_{n_j}\}$ satisfying

$$\Xi_{\omega_{m_j}, \omega_{n_j}}(\gamma) \leq 1 - \kappa, \quad \Xi_{\omega_{m_{j-1}}, \omega_{n_j}}(\gamma) > 1 - \kappa \text{ and } \Xi_{\omega_{m_{j-1}}, \omega_{n_j}}(\gamma - \delta) > 1 - \kappa. \quad (7)$$

Hence, from the triangular inequality, we obtain

$$1 - \kappa \geq \Xi_{\omega_{m_j}, \omega_{n_j}}(\gamma) \geq \Pi(\Xi_{\omega_{m_j}, \omega_{m_{j-1}}}(2\delta), \Xi_{\omega_{m_{j-1}}, \omega_{n_j}}(\gamma - 2\delta)). \quad (8)$$

Letting $j \rightarrow \infty$ in (7) and (8), we have

$$\lim_{j \rightarrow \infty} \Xi_{\omega_{m_j}, \omega_{n_j}}(\gamma) = \lim_{j \rightarrow \infty} \Xi_{\omega_{m_{j-1}}, \omega_{n_j}}(\gamma - \delta) = 1 - \kappa.$$

The fact that $\Xi_{\omega_{m_j-1}, \omega_{n_j-1}}(\gamma) \geq \Pi(\Xi_{\omega_{m_j-1}, \omega_{n_j}}(\gamma - \delta), \Xi_{\omega_{n_j}, \omega_{n_j-1}}(\delta))$, implies that

$$\lim_{j \rightarrow \infty} \Xi_{\omega_{m_j-1}, \omega_{n_j-1}}(\gamma) \geq 1 - \kappa.$$

It follows by the contractive condition that

$$\mathcal{F}(\Xi_{\omega_{m_j}, \omega_{n_j}}(\gamma)) > \omega \cdot \mathcal{F}(\Xi_{\omega_{m_j}, \omega_{n_j}}(\gamma)) \geq \mathcal{F}(\Xi_{\omega_{m_j-1}, \omega_{n_j-1}}(\gamma)).$$

Letting again $j \rightarrow +\infty$, we obtain

$$\mathcal{F}(1 - \kappa) > \omega \cdot \mathcal{F}(1 - \kappa) \geq \mathcal{F}(1 - \kappa).$$

This implies that $\mathcal{F}(1 - \kappa) = 0$, contradicting the fact that $\mathcal{F}(1 - \kappa) > 0$. Thus, $\{\omega_n\}$ forms a Cauchy sequence. Given that (Ω, Ξ, Π) is complete, there exists $\nu \in \Omega$ such that $\omega_n \rightarrow \nu$ as $n \rightarrow +\infty$. We shall prove that ν is a fixed point of $\mathcal{T}$. Indeed, we have

$$\Xi_{\omega_{n+1}, \mathcal{T}\nu}(t) \geq \Xi_{\omega_n, \nu}(t) \geq \Pi(\Xi_{\omega_n, \omega_{n+1}}(\frac{t}{2}), \Xi_{\omega_{n+1}, \nu}(\frac{t}{2})) \text{ for any } n \in \mathbb{N}.$$

Letting $n \rightarrow +\infty$ and using the fact that Π is continuous, we have

$$\Xi_{\nu, \mathcal{T}\nu}(t) \geq 1.$$

Hence, $\mathcal{T}\nu = \nu$.

For establishing uniqueness, we assert the existence of $\eta \in \Omega$, distinct from ν , such that $\mathcal{T}\eta = \eta$. Utilizing the contractivity condition, we derive

$$\mathcal{F}(\Xi_{\nu, \eta}(t)) = \mathcal{F}(\Xi_{\mathcal{T}\nu, \mathcal{T}\eta}(t)) > \omega \cdot \mathcal{F}(\Xi_{\mathcal{T}\nu, \mathcal{T}\eta}(t)) \geq \mathcal{F}(\Xi_{\nu, \eta}(t)).$$

which leads to a contradiction. Therefore, $\nu = \eta$. $\square$

Example 1. Let $\Omega = \mathbb{R}$ and Ξ be a distance distribution function which is defined by

$$\Xi_{\rho, \sigma}(0) = 0 \text{ and } \Xi_{\rho, \sigma}(t) = e^{-\frac{|\rho - \sigma|}{t}} \text{ for all } \rho, \sigma \in \Omega \text{ and } t \geq 0.$$

We assert that (Ω, Ξ, Π_P) is a Menger space, a fact that is not difficult to demonstrate. We put $\mathcal{F}(0) = 0$, $\mathcal{F}(\rho) = -\frac{1}{\ln(\rho)}$ for $\rho \in (0, 1)$, and $\mathcal{T}(\rho) = \frac{1}{8}\rho$ for all $\rho \in \Omega$. We shall prove that $\mathcal{T}$ is a probabilistic $\mathcal{F}$ -contraction where $\omega = \frac{1}{4}$. In fact, for all $\rho, \sigma \in \Omega$ where $\rho \neq \sigma$ and $t > 0$, we have

$$\begin{aligned} \omega \cdot \mathcal{F}(\Xi_{\mathcal{T}\rho, \mathcal{T}\sigma}(t)) &= -\frac{1}{4} \cdot \frac{1}{\ln(e^{-\frac{|\rho - \sigma|}{8t}})} \\ &= \frac{2t}{|\rho - \sigma|} \\ &> \frac{t}{|\rho - \sigma|} \\ &= \mathcal{F}(\Xi_{\rho, \sigma}(t)). \end{aligned}$$

Therefore, $\mathcal{T}$ is a probabilistic $\mathcal{F}$ -contraction, and, since all the conditions of Theorem 1 are fulfilled, $\mathcal{T}$ has a unique fixed point.

Remark 1. It is noteworthy that our approach relies exclusively on property $(\mathcal{F}_1)$ and the continuity of the control function $\mathcal{F}$, distinguishing it from the methodologies typically employed in many studies focused on ordinary metric spaces [1,22].

In exploring the connections between the results obtained in Menger spaces and those in ordinary metric spaces, we present the following result:

Corollary 1. Let $(\Omega, \mathcal{D})$ be a complete metric space and $T : \Omega \rightarrow \Omega$ be a mapping that satisfies the following inequality:

$$t + \mathcal{D}(T\rho, T\sigma) \leq \omega(t + \mathcal{D}(\rho, \sigma)) \quad \text{for all } \rho, \sigma \in \Omega \quad \text{and } t \geq 0. \quad (9)$$

Then, T admits a unique fixed point.

Proof. Let $(\Omega, \mathcal{D})$ be a complete metric space. Define the probabilistic metric Ξ on Ω by setting

$$\Xi_{\rho, \sigma}(t) = \frac{t}{t + \mathcal{D}(\rho, \sigma)}, \quad (10)$$

for all $\rho, \sigma \in \Omega$ and $t > 0$. By Lemma 1, this converts $(\Omega, \mathcal{D})$ into a Menger space (Ω, Ξ, Π_P) .

Given the mapping $T : \Omega \rightarrow \Omega$ that satisfies

$$t + \mathcal{D}(T\rho, T\sigma) \leq \omega(t + \mathcal{D}(\rho, \sigma)) \quad (11)$$

for all $\rho, \sigma \in \Omega$ and $t \geq 0$, we need to show that this implies the probabilistic $\mathcal{F}$ -contraction condition with $\mathcal{F}$ as the identity function.

Rewriting the inequality, we obtain

$$\frac{1}{t + \mathcal{D}(T\rho, T\sigma)} \geq \frac{1}{\omega} \frac{1}{t + \mathcal{D}(\rho, \sigma)}. \quad (12)$$

Using the definition of Ξ , this becomes

$$\omega \cdot \frac{t}{t + \mathcal{D}(T\rho, T\sigma)} \geq \frac{t}{t + \mathcal{D}(\rho, \sigma)}. \quad (13)$$

Recognizing that,

$$\omega \cdot \Xi_{T\rho, T\sigma}(t) \geq \Xi_{\rho, \sigma}(t). \quad (14)$$

Since $\mathcal{F}$ is the identity function,

$$\omega \cdot \mathcal{F}(\Xi_{T\rho, T\sigma}(t)) = \omega \cdot \Xi_{T\rho, T\sigma}(t) \geq \Xi_{\rho, \sigma}(t) = \mathcal{F}(\Xi_{\rho, \sigma}(t)). \quad (15)$$

Thus, T satisfies the probabilistic $\mathcal{F}$ -contraction condition in the Menger space setting with $\mathcal{F}$ as the identity function.

Since (Ω, Ξ, Π_P) is a complete Menger space, and T is a probabilistic $\mathcal{F}$ -contraction with $\mathcal{F}$ being continuous (identity function is trivially continuous), by Theorem 1, T has a unique fixed point in Ω . $\square$

4. Application of the Generalized Probabilistic $\mathcal{F}$ -Contraction Theorems

Probabilistic fixed-point theorems are powerful tools in analysis that are particularly useful in dealing with stochastic processes, random operator theory, and models where uncertainty and randomness play crucial roles. These theorems provide conditions under which mappings in probabilistic metric spaces have unique fixed points. Applications span

various fields, including mathematical biology, economics, and engineering, where systems often exhibit probabilistic behavior or involve random parameters.

In this context, let us consider a more complex version of the probabilistic $\mathcal{F}$ -contraction theorem and demonstrate its application in solving a system of Fredholm integral equations.

We consider the following system of Fredholm integral equations:

$$\begin{cases} \omega_1(t) = \lambda_1 \int_a^b K_1(t,s)\omega_1(s) ds + \lambda_2 \int_a^b K_2(t,s)\omega_2(s) ds + f_1(t), \\ \omega_2(t) = \lambda_3 \int_a^b K_3(t,s)\omega_1(s) ds + \lambda_4 \int_a^b K_4(t,s)\omega_2(s) ds + f_2(t), \end{cases}$$

where $K_i(t,s)$ are given kernel functions, λ_i are constants, and $f_i(t)$ are given functions for $t \in [a, b]$.

Theorem 2. Consider the system of Fredholm integral equations as described above. Assume the following conditions hold:

1. The kernel functions $K_i(t,s)$ are continuous and bounded on the domain $[a, b] \times [a, b]$.
2. The constants λ_i satisfy

$$\max\{|\lambda_1|\|K_1\| + |\lambda_2|\|K_2\|, |\lambda_3|\|K_3\| + |\lambda_4|\|K_4\|\} < 1.$$

3. The functions $f_i(t)$ are continuous on $[a, b]$.

Then, there exists a unique solution $(\omega_1(t), \omega_2(t))$ to the system of Fredholm integral equations.

Proof. Let $\Omega = C([a, b], \mathbb{R}) \times C([a, b], \mathbb{R})$ be the space of continuous function pairs from $[a, b]$ to $\mathbb{R}$ with the supremum norm $\|(\omega_1, \omega_2)\| = \max\{\|\omega_1\|, \|\omega_2\|\}$.

Define the probabilistic metric Ξ on Ω such that for $(\omega_1, \omega_2), (v_1, v_2) \in \Omega$ and $t > 0$,

$$\Xi_{(\omega_1, \omega_2), (v_1, v_2)}(t) = \min\left\{e^{-\frac{\|\omega_1 - v_1\|}{t}}, e^{-\frac{\|\omega_2 - v_2\|}{t}}\right\}.$$

Define $\mathcal{T} : \Omega \rightarrow \Omega$ by

$$\mathcal{T}(\omega_1, \omega_2) = (\mathcal{T}_1(\omega_1, \omega_2), \mathcal{T}_2(\omega_1, \omega_2)),$$

where

$$\mathcal{T}_1(\omega_1, \omega_2)(t) = \lambda_1 \int_a^b K_1(t,s)\omega_1(s) ds + \lambda_2 \int_a^b K_2(t,s)\omega_2(s) ds + f_1(t),$$

and

$$\mathcal{T}_2(\omega_1, \omega_2)(t) = \lambda_3 \int_a^b K_3(t,s)\omega_1(s) ds + \lambda_4 \int_a^b K_4(t,s)\omega_2(s) ds + f_2(t).$$

To verify that $\mathcal{T}$ is continuous, we need to show that, if $(\omega_{1n}, \omega_{2n}) \rightarrow (\omega_1, \omega_2)$ in the supremum norm, then $\mathcal{T}(\omega_{1n}, \omega_{2n}) \rightarrow \mathcal{T}(\omega_1, \omega_2)$ in the supremum norm.

For $(\omega_{1n}, \omega_{2n}), (\omega_1, \omega_2) \in \Omega$,

$$\|\mathcal{T}_1(\omega_{1n}, \omega_{2n}) - \mathcal{T}_1(\omega_1, \omega_2)\| \leq |\lambda_1|\|K_1\|\|\omega_{1n} - \omega_1\| + |\lambda_2|\|K_2\|\|\omega_{2n} - \omega_2\|,$$

and

$$\|\mathcal{T}_2(\omega_{1n}, \omega_{2n}) - \mathcal{T}_2(\omega_1, \omega_2)\| \leq |\lambda_3|\|K_3\|\|\omega_{1n} - \omega_1\| + |\lambda_4|\|K_4\|\|\omega_{2n} - \omega_2\|,$$

where $\|K_i\|$ denotes the supremum norm of the kernel K_i . Since K_i are continuous and bounded, and $\omega_{1n} \rightarrow \omega_1, \omega_{2n} \rightarrow \omega_2$, it follows that $\mathcal{T}(\omega_{1n}, \omega_{2n}) \rightarrow \mathcal{T}(\omega_1, \omega_2)$.

We need to show there exists a strictly increasing function $\mathcal{F}$ and a constant $\omega \in (0, 1)$ such that for all $(\omega_1, \omega_2), (v_1, v_2) \in \Omega$ and $t > 0$,

$$\omega \cdot \mathcal{F}(\Xi_{\mathcal{T}(\omega_1, \omega_2), \mathcal{T}(v_1, v_2)}(t)) \geq \mathcal{F}(\Xi_{(\omega_1, \omega_2), (v_1, v_2)}(t)).$$

For $(\omega_1, \omega_2), (v_1, v_2) \in \Omega$,

$$\|\mathcal{T}(\omega_1, \omega_2) - \mathcal{T}(v_1, v_2)\| = \max\{\|\mathcal{T}_1(\omega_1, \omega_2) - \mathcal{T}_1(v_1, v_2)\|, \|\mathcal{T}_2(\omega_1, \omega_2) - \mathcal{T}_2(v_1, v_2)\|\}.$$

Given the inequalities,

$$\|\mathcal{T}_1(\omega_1, \omega_2) - \mathcal{T}_1(v_1, v_2)\| \leq |\lambda_1| \|K_1\| \|\omega_1 - v_1\| + |\lambda_2| \|K_2\| \|\omega_2 - v_2\|,$$

and

$$\|\mathcal{T}_2(\omega_1, \omega_2) - \mathcal{T}_2(v_1, v_2)\| \leq |\lambda_3| \|K_3\| \|\omega_1 - v_1\| + |\lambda_4| \|K_4\| \|\omega_2 - v_2\|,$$

we let $M_1 = |\lambda_1| \|K_1\| + |\lambda_2| \|K_2\|$ and $M_2 = |\lambda_3| \|K_3\| + |\lambda_4| \|K_4\|$.

Define $M = \max\{M_1, M_2\}$. Then,

$$\|\mathcal{T}(\omega_1, \omega_2) - \mathcal{T}(v_1, v_2)\| \leq M \max\{\|\omega_1 - v_1\|, \|\omega_2 - v_2\|\} = M \|(\omega_1, \omega_2) - (v_1, v_2)\|.$$

Let $\omega = M$. Since $M < 1$, $\mathcal{T}$ is a contraction. Using $\Xi_{(\omega_1, \omega_2), (v_1, v_2)}(t) = \min\left\{e^{-\frac{\|\omega_1 - v_1\|}{t}}, e^{-\frac{\|\omega_2 - v_2\|}{t}}\right\}$,

$$\Xi_{\mathcal{T}(\omega_1, \omega_2), \mathcal{T}(v_1, v_2)}(t) = \min\left\{e^{-\frac{\|\mathcal{T}_1(\omega_1, \omega_2) - \mathcal{T}_1(v_1, v_2)\|}{t}}, e^{-\frac{\|\mathcal{T}_2(\omega_1, \omega_2) - \mathcal{T}_2(v_1, v_2)\|}{t}}\right\} \leq e^{-\frac{M \|(\omega_1, \omega_2) - (v_1, v_2)\|}{t}}.$$

Choosing $\mathcal{F}(t) = t$, we obtain

$$\omega \cdot \mathcal{F}(\Xi_{\mathcal{T}(\omega_1, \omega_2), \mathcal{T}(v_1, v_2)}(t)) = \omega \cdot \Xi_{\mathcal{T}(\omega_1, \omega_2), \mathcal{T}(v_1, v_2)}(t) \leq \omega \cdot e^{-\frac{M \|(\omega_1, \omega_2) - (v_1, v_2)\|}{t}}.$$

For this inequality to hold for all $t > 0$, we observe that

$$\omega \cdot e^{-\frac{M \|(\omega_1, \omega_2) - (v_1, v_2)\|}{t}} \geq e^{-\frac{\|(\omega_1, \omega_2) - (v_1, v_2)\|}{t}}.$$

If we choose $\omega = M < 1$, this is satisfied, implying that

$$\omega \cdot \mathcal{F}(\Xi_{\mathcal{T}(\omega_1, \omega_2), \mathcal{T}(v_1, v_2)}(t)) \geq \mathcal{F}(\Xi_{(\omega_1, \omega_2), (v_1, v_2)}(t)).$$

Given that $\mathcal{T}$ is a continuous probabilistic $\mathcal{F}$ -contraction with $\omega < 1$, Theorem 1 guarantees the existence of a unique fixed point $(\omega_1^*, \omega_2^*) \in \Omega$ such that $\mathcal{T}(\omega_1^*, \omega_2^*) = (\omega_1^*, \omega_2^*)$. $\square$

By ensuring the kernels $K_i(t, s)$ and the parameters λ_i satisfy the conditions $M < 1$, we can apply the generalized probabilistic $\mathcal{F}$ -contraction theorem using the metric $\Xi_{(\omega_1, \omega_2), (v_1, v_2)}(t) = \min\left\{e^{-\frac{\|\omega_1 - v_1\|}{t}}, e^{-\frac{\|\omega_2 - v_2\|}{t}}\right\}$ to guarantee a unique solution to the system of Fredholm integral equations. This demonstrates the theorem's robust framework for solving more complex integral equations in a probabilistic setting.

5. Conclusions

In this paper, we have introduced the concept of the probabilistic $\mathcal{F}$ -contraction, a broader notion in the fixed-point theory literature aimed at providing a generalized

framework applicable to Menger spaces. Our goal was to establish fixed-point results that encompass various types of theorems within this extended setting. Additionally, we have illustrated the practical relevance of our findings through real-life applications. Specifically, we have demonstrated how our developed techniques in fixed-point theory offer a direct and effective approach to solving Fredholm integral equations. This application highlights the versatility and utility of our proposed framework in addressing practical problems across diverse contexts.

However, two unresolved issues merit further investigation:

1. *Understanding the Connection Between Probabilistic $\mathcal{F}$ -Contraction and Conventional $\mathcal{F}$ -Contraction:* While the probabilistic $\mathcal{F}$ -contraction extends the classical $\mathcal{F}$ -contraction into the probabilistic framework, the precise relationship between these two notions remains unclear. Specifically, under what additional assumptions or structural similarities can the probabilistic $\mathcal{F}$ -contraction recover the behavior of a classical $\mathcal{F}$ -contraction? Exploring this connection could provide deeper insight into how randomness and probabilistic settings influence the fixed-point results. Future research could investigate whether specific probabilistic parameters or conditions on the associated functions in Menger spaces bridge this gap, possibly leading to a unified theory that seamlessly integrates deterministic and probabilistic contractions.
2. *Relaxing the Continuity Condition of $\mathcal{F}$ Using Properties of the t -Norm:* The requirement of continuity for the function $\mathcal{F}$ is a common assumption in proving the existence of fixed points. However, it raises the question of whether alternative conditions, particularly related to the t -norm structure in Menger spaces, might suffice. By imposing weaker or more specialized conditions on the t -norm, we may broaden the applicability of the probabilistic $\mathcal{F}$ -contraction framework. Future work could explore the minimal requirements on t -norms, such as left continuity, monotonicity, or specific boundedness properties, that still guarantee the fixed-point existence. Such an approach could extend the framework to more generalized or non-standard probabilistic settings.

Investigating these unresolved issues will enhance the theoretical foundations of the probabilistic $\mathcal{F}$ -contraction and expand its applicability across various domains.

In conclusion, this research marks a significant advancement in fixed-point theory by bridging probabilistic structures with classical methods and extending the scope of applications to complex problems, including Fredholm integral equations. By laying the groundwork for a probabilistic framework in Menger spaces, we have opened new avenues for theoretical exploration and practical problem-solving, making this study a cornerstone for future developments in the field.

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