

Article

Global Solutions to the Vlasov–Fokker–Planck Equation with Local Alignment Forces Under Specular Reflection Boundary Condition

Yanming Chang and Yingzhe Fan *

School of Mathematics and Statistics, Nanyang Normal University, Nanyang 473061, China;
ymchang@nynu.edu.cn

* Correspondence: yzf@nynu.edu.cn

Abstract

In this article, we establish the existence of global mild solutions to the Vlasov–Fokker–Planck equation with local alignment forces under specular reflection boundary conditions in the low-regularity function space $L_k^1 L_T^\infty L_v^2$. A key difficulty is that the macroscopic averaged velocity u does not directly possess a dissipative structure in the equation. To overcome this, we rely on the dissipation $u - b$ from the linear part, combined with the dissipation of the macroscopic component b derived from the associated macroscopic equation. Moreover, since no direct energy functional is available for u , we fully exploit the dissipative mechanisms of both $u - b$ and b when handling the estimates for the nonlinear terms.

Keywords: global existence; mild solutions; Vlasov–Fokker–Planck equation with local alignment forces; specular reflection

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1. Introduction

1.1. Equations

We study the Cauchy problem for the Vlasov–Fokker–Planck equation with local alignment forces

$$\begin{cases} \partial_t F + v \cdot \nabla_x F + \nabla_v \cdot ((u - v)F) = \Delta_v F, \\ F(0, x, v) = F_0(x, v), \end{cases} \quad (1)$$

where $F(t, x, v) \geq 0$ denotes the particle density located at position $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{T}^3$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$. The averaged local velocity u is defined by

$$u(t, x) = \frac{\int_{\mathbb{R}_v^3} v F(t, x, v) dv}{\int_{\mathbb{R}_v^3} F(t, x, v) dv}.$$

We investigate perturbations around the global Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Setting

$$F(t, x, v) = \mu + \mu^{\frac{1}{2}} g(t, x, v), \quad (2)$$

we recast the Cauchy problem (1) as

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + u \cdot \nabla_v g - \frac{1}{2} u \cdot v g - u \cdot v \mu^{\frac{1}{2}} = Lg, \\ g(0, x, v) = g_0(x, v) := \mu^{-1/2} (F_0(x, v) - \mu), \end{cases} \quad (3)$$

with the renormalized averaged velocity

$$u(x, t) = \frac{\int_{\mathbb{R}_v^3} v \mu^{\frac{1}{2}} g(t, x, v) dv}{1 + \int_{\mathbb{R}_v^3} \mu^{\frac{1}{2}} g(t, x, v) dv}, \quad (4)$$

and the linear Fokker–Planck operator

$$Lg = \mu^{-\frac{1}{2}} \nabla_v \cdot (\mu \nabla_v (\mu^{-\frac{1}{2}} g)) = \Delta_v g + \frac{1}{4} (6 - |v|^2) g. \quad (5)$$

Define the orthogonal projection operator $\mathbf{P}$ of velocity as

$$\mathbf{P} : L^2(\mathbb{R}_v^3) \rightarrow \text{Span}\left\{\mu^{\frac{1}{2}}, v_i \mu^{\frac{1}{2}} (1 \leq i \leq 3)\right\},$$

such that for any function $g(t, x, v) \in L^2(\mathbb{R}_v^3)$, it holds that

$$\mathbf{P}g = a(t, x) \mu^{\frac{1}{2}} + b(t, x) \cdot v \mu^{\frac{1}{2}}, \quad (6)$$

with

$$\begin{aligned} a &= \int_{\mathbb{R}_v^3} \mu^{\frac{1}{2}} g dv = \int_{\mathbb{R}_v^3} \mu^{\frac{1}{2}} \mathbf{P}g dv, \\ b_i &= \int_{\mathbb{R}_v^3} v_i \mu^{\frac{1}{2}} g dv = \int_{\mathbb{R}_v^3} v_i \mu^{\frac{1}{2}} \mathbf{P}g dv, \quad j = 1, 2, 3. \end{aligned} \quad (7)$$

Following [1], we decompose the solution $g(t, x, v)$ of (3) as

$$g(t, x, v) = \mathbf{P}g(t, x, v) + \{\mathbf{I} - \mathbf{P}\}g(t, x, v), \quad (8)$$

where $\mathbf{I}$ denotes the identity operator, and $\mathbf{P}g$ and $\{\mathbf{I} - \mathbf{P}\}g$ are referred to as the macroscopic and microscopic components of $g(t, x, v)$, respectively.

1.2. Boundary Condition

In this manuscript, we consider a finite channel

$$\Omega = I \times \mathbb{T}^2 = \left\{x = (x_1, \bar{x}), x_1 \in I := (-1, 1), \bar{x} := (x_2, x_3) \in \mathbb{T}^2 = [0, 2\pi]^2\right\}, \quad (9)$$

endowed with the specular reflection boundary condition at $x_1 = \pm 1$ [2]:

$$\begin{aligned} F(t, -1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} &= F(t, -1, \bar{x}, -v_1, \bar{v}), \\ F(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} &= F(t, 1, \bar{x}, -v_1, \bar{v}), \end{aligned}$$

where $\bar{v} = (v_2, v_3)$. Then $g(t, x, v)$ also satisfies

$$\begin{aligned} g(t, -1, \bar{x}, v_1, \bar{v})|_{v_1 > 0} &= g(t, -1, \bar{x}, -v_1, \bar{v}), \\ g(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} &= g(t, 1, \bar{x}, -v_1, \bar{v}). \end{aligned} \quad (10)$$

1.3. Macroscopic Equations

Multiplying the first equation of (3) by $\mu^{\frac{1}{2}}$ and $v\mu^{\frac{1}{2}}$, and integrating over velocity space $\mathbb{R}_v^3$, we can get the following hyperbolic–parabolic system for the macroscopic components a and b

$$\partial_t a + \nabla_x \cdot b = 0, \quad (11)$$

$$\partial_t b_i + \partial_{x_i} a + \sum_{j=1}^3 \partial_{x_j} A_{ij}(\{\mathbf{I} - \mathbf{P}\}g) = 0, \quad (12)$$

where the coefficients are defined by $A_{ij}(g) = \int_{\mathbb{R}^3} (v_i v_j - 1) \mu^{\frac{1}{2}} g(t, x, v) dv$. Furthermore, integrating these equations over the channel yields the conservation laws

$$\int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} g(t, x, v) dv d\bar{x} dx_1 = \int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} g(0, x, v) dv d\bar{x} dx_1 = 0,$$

$$\int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} v \mu^{\frac{1}{2}} g(t, x, v) dv d\bar{x} dx_1 = \int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} v \mu^{\frac{1}{2}} g(0, x, v) dv d\bar{x} dx_1 = 0.$$

1.4. Function Space

The global solvability with high regularity for the Vlasov–Fokker–Planck equation with local alignment forces in Sobolev spaces $H_{x,v}^4$ has been established in [3]. Inspired by [2], we adopt the low-regularity function space $L_k^1 L_T^\infty L_v^2$ to establish the global existence of the system. More specifically, the low-regularity function space is endowed with the norm defined as

$$\|g\|_{L_k^1 L_T^\infty L_{x_1,v}^2} := \int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{\bar{x}_1,v}^2} d\Sigma(\bar{k}) < \infty,$$

where $\bar{k} = (\bar{k}_2, \bar{k}_3) \in \mathbb{Z}^2$ and $\bar{x} = (x_2, x_3) \in \mathbb{T}^2$. The Fourier transform is taken as

$$\hat{g}(t, x_1, \bar{k}, v) = \mathcal{F}[g](t, x_1, \bar{k}, v) = \int_{\mathbb{T}^2} e^{-i\bar{k} \cdot \bar{x}} g(t, x_1, \bar{x}, v) d\bar{x}, \quad \bar{k} \in \mathbb{Z}^2,$$

where $d\Sigma(\bar{k})$ represents the discrete measure in $\mathbb{Z}^2$, i.e.,

$$\int_{\mathbb{Z}^2} g(\bar{k}) d\Sigma(\bar{k}) = \sum_{\bar{k} \in \mathbb{Z}^2} g(\bar{k}).$$

1.5. Notation

For clarity in presentation, we define the following notation.

- The relation $A \lesssim B$ denotes an inequality $A \leq CB$ for a universal constant $C > 0$. The notation $A \gtrsim B$ is defined analogously, and we write $A \sim B$ if both $A \lesssim B$ and $A \gtrsim B$ are satisfied;
- The inner product of two complex functions f and g is defined as $(f, g) = f \cdot \bar{g}$;
- The complex inner product L_v^2 over the velocity variable is denoted by $(\cdot, \cdot)_{L_v^2}$, defined as

$$(f, g)_{L_v^2} = \int_{\mathbb{R}_v^3} f(v) \overline{g(v)} dv,$$

with the L^2 norm denoted by $|\cdot|_{L_v^2}$;

- The complex inner product $L_{x_1}^2$ over the spatial variable is denoted by $(\cdot, \cdot)_{L_{x_1}^2}$, defined as

$$(f, g)_{L_{x_1}^2} = \int_{-1}^1 f(x_1) \overline{g(x_1)} dx_1,$$

with the L^2 norm denoted by $|\cdot|_{L^2_{x_1}}$;

- The combined inner product $L^2_{x_1,v}$ is

$$(f, g)_{L^2_{x_1,v}} = \int_{-1}^1 \int_{\mathbb{R}^3_v} f(v) \overline{g(v)} dv dx_1,$$

with the L^2 norm $\|\cdot\|_{L^2_{x_1,v}}$.

- We write $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ for a standard multi-index. Only $\alpha = (0, 0, 0)$ and $\alpha = (1, 0, 0)$ are needed for this paper.
- The symbol $\Re$ denotes the real part of a complex number.
- We define the norm $\|\cdot\|_\nu$ for a function $f(x_1, v)$ as

$$\|g\|_{L^2_\nu}^2 = \|g\|_\nu^2 = \int_{-1}^1 \int_{\mathbb{R}^3_v} (\nu(v)|g|^2 + |\nabla_v g|^2) dv dx_1, \quad \nu(v) = 1 + |v|^2. \quad (13)$$

- The associated energy functional $\mathcal{E}(g(t))$ and dissipation functional $\mathcal{D}(g(t))$ are given by

$$\mathcal{E}(g(t)) := \sum_{|\alpha| \leq 1} \|\partial^\alpha g\|_{L^1_k L^\infty_t L^2_{x_1,v}}, \quad (14)$$

and

$$\begin{aligned} \mathcal{D}(g(t)) := & \sum_{|\alpha| \leq 1} \|\partial^\alpha \{\mathbf{I} - \mathbf{P}\} g\|_{L^1_k L^2_t L^2_v} + \sum_{|\alpha| \leq 1} \|\partial^\alpha (b - u)\|_{L^1_k L^2_t L^2_{x_1}} \\ & + \sum_{|\alpha| \leq 1} \|\partial^\alpha a\|_{L^1_k L^2_t L^2_{x_1}} + \sum_{|\alpha| \leq 1} \|\partial^\alpha b\|_{L^1_k L^2_t L^2_{x_1}}, \end{aligned} \quad (15)$$

respectively.

1.6. Related Works

We now briefly review related works on nonlinear Fokker–Planck equations. Karper et al. [4] proved the global existence of weak solutions to a class of kinetic flocking equations, including the kinetic Cucker–Smale equation by employing the velocity-averaging lemma and the Schauder fixed-point theorem. The hydrodynamic limit was investigated in [5] using a relative entropy method. Choi [6] investigated the global classical solutions and their large-time behavior for a coupled compressible Euler and incompressible Navier–Stokes system, which emerges as the hydrodynamic limit of a Vlasov–Navier–Stokes system subject to strong noise and local alignment forces.

The nonlinear energy method, pioneered in [1] for the Boltzmann equation, offers an effective perturbative framework for establishing global classical solutions, often termed the macro-micro decomposition. This methodology has since enabled the construction of global solutions with time-decay estimates for many kinetic models [7,8]. Specific to the Vlasov–Poisson–Fokker–Planck system, global existence and decay rates have been established in [9–11]. For the Vlasov–Fokker–Planck equation with local alignment forces, Choi [3] proved the global existence and uniqueness of classical solutions near the global Maxwellian in the whole space, along with an algebraic decay rate, under suitable initial conditions. For further related topics, we refer the readers to [9,12–20] and the references therein.

The specular reflection boundary condition is also a fundamental concept in kinetic theory, governing the interaction between gas particles and solid boundaries. Yang-Zhao [21] investigated the nonlinear stability of rarefaction waves for the Boltzmann equation. Duan et al. [22] proved the unique global solution and exponential convergence to equilibrium for the hard-sphere Boltzmann equation in convex C^3 domains. Guo et al. [23] establish the global stability of the Landau equation with Coulomb potential in general smooth domains under low-regularity initial data. Furthermore, Dong et al. [24] construct

global solutions and establish well-posedness for the Vlasov–Poisson–Landau system near Maxwellians in general bounded domains.

Recent work by Duan et al. [2] introduced the low-regularity function space $L_k^1 L_T^\infty L_v^2$ to address the Landau and non-cutoff Boltzmann equations, effectively weakening the initial data regularity requirements due to the embedding $H_x^2 \hookrightarrow L_k^1$. Capitalizing on this insight, we extend the application of this space to establish the global existence of solutions for the Vlasov–Fokker–Planck equation with local alignment forces.

2. Main Result

Theorem 1. Assume the non-negative initial data $F_0(x, v) = \mu + \mu^{\frac{1}{2}} g_0(x, v) \geq 0$ satisfies the mass and momentum conservation laws

$$\int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} g(t, x, v) dv d\bar{x} dx_1 = \int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} \mu^{\frac{1}{2}} g(0, x, v) dv d\bar{x} dx_1 = 0, \quad (16)$$

$$\int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} v \mu^{\frac{1}{2}} g(t, x, v) dv d\bar{x} dx_1 = \int_{-1}^1 \int_{\mathbb{T}^2} \int_{\mathbb{R}^3} v \mu^{\frac{1}{2}} g(0, x, v) dv d\bar{x} dx_1 = 0, \quad (17)$$

and the symmetry condition

$$g_0(t, -x_1, \bar{x}, -v_1, \bar{v}) = g_0(t, x_1, \bar{x}, v_1, \bar{v}),$$

also has sufficiently small energy

$$\mathcal{E}(g_0) = \|g_0\|_{L_k^1 L_{x_1}^2 L_v^2} \leq \epsilon_0,$$

for some $\epsilon_0 > 0$. Then the Cauchy problem (3) has a unique global mild solution $g(t, x, v)$, which also satisfies $F(t, x, v) = \mu + \mu^{\frac{1}{2}} g(t, x, v) \geq 0$ and obeys the uniform bound

$$\mathcal{E}(g(t)) + \mathcal{D}(g(t)) \lesssim \mathcal{E}(g_0), \quad (18)$$

for any $t \geq 0$.

Remark 1. The principal difficulty in the analysis lies in estimating

$$\sum_{|\alpha| \leq 1} \|\partial^\alpha u\|_{L_k^1 L_t^2 L_{x_1}^2}$$

which originates from the nonlinear expressions $u \cdot \nabla_v g$ and $u \cdot v g$. Thanks to the linear part of (3) and the macroscopic equations, we can obtain dissipation $u - b$ and b by an elliptic structure, respectively. Consequently, the difficult term can be estimated as

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\partial^\alpha u\|_{L_k^1 L_t^2 L_{x_1}^2} &= \sum_{|\alpha| \leq 1} \|\partial^\alpha (u - b + b)\|_{L_k^1 L_t^2 L_{x_1}^2} \\ &\lesssim \sum_{|\alpha| \leq 1} \|\partial^\alpha (u - b)\|_{L_k^1 L_t^2 L_{x_1}^2} + \sum_{|\alpha| \leq 1} \|\partial^\alpha b\|_{L_k^1 L_t^2 L_{x_1}^2} \\ &\lesssim \mathcal{D}(g(t)). \end{aligned}$$

Remark 2. Notice that the energy functional

$$\mathcal{E}(g(t)) = \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\widehat{\partial^\alpha g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{k}),$$

does not contain the macroscopic velocity component u , i.e.,

$$\sum_{|\alpha| \leq 1} \int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\widehat{\partial^\alpha u}(t, \cdot, \bar{k}, \cdot)\|_{L_{x_1}^2} d\Sigma(\bar{k}).$$

This structural difference forces us to treat the nonlinear terms in a way that is distinct from the approach of [25]. Details are provided in Lemma 2.

The structure of the rest of this paper is as follows: Section 3 presents fundamental lemmas essential for later sections, and the proofs of Theorem 1 are detailed in Section 4.

3. Preliminaries

We begin by recalling the coercivity property of the linear Fokker–Planck operator L , which is fundamental to the analysis. According to [10,15], there exists a constant $\lambda_0 > 0$ such that

$$-(Lg, g)_{L_v^2} \geq \lambda_0 \{|\mathbf{I} - \mathbf{P}|g|_v^2 + |b|^2\}. \quad (19)$$

Leveraging the linearity of L , we immediately obtain the following estimate.

Lemma 1. Let $|\alpha| \leq 1$. There exists a constant $\lambda_0 > 0$ such that

$$-\mathcal{R}(L\widehat{\partial^\alpha g}, \widehat{\partial^\alpha g})_{L_{x_1,v}^2} \geq \lambda_0 \{|\{\mathbf{I} - \mathbf{P}\}\widehat{\partial^\alpha g}|_v^2 + |\widehat{\partial^\alpha b}|_{L_{x_1}^2}^2\}. \quad (20)$$

The next two results provide estimates for the trilinear terms.

Lemma 2. Let $|\alpha| \leq 1$. There exists a small constant $\eta > 0$ such that

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \left(\mathcal{F}[\partial^\alpha(u \nabla_v g)], \widehat{\partial^\alpha g} \right)_{L_{x_1,v}^2} \right| d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)), \quad (21)$$

and

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \left(\mathcal{F}[\partial^\alpha(u \nabla_v g)], \mu^{\frac{1}{4}} \right)_{L_{x_1,v}^2} \right|^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \mathcal{E}(g(t)) \mathcal{D}(g(t)). \quad (22)$$

Proof. Case 1: $\alpha = (0, 0, 0)$. Applying Fubini's theorem yields

$$\begin{aligned} \left| (\hat{u} * \nabla_v \hat{g}, \hat{g})_{L_{x_1,v}^2} \right| &= \left| \int_{-1}^1 \int_{\mathbb{R}_v^3} \left(\int_{\mathbb{Z}_l^2} \hat{u}(\tau, x_1, \bar{l}) \cdot \nabla_v \hat{g}(\tau, x_1, \bar{k} - \bar{l}, v) d\Sigma(\bar{l}) \right) \hat{g}(\tau, x_1, \bar{k}, v) dv dx_1 \right| \\ &= \left| \int_{\mathbb{Z}_l^2} \int_{-1}^1 \hat{u}(\tau, x_1, \bar{l}) \left(\int_{\mathbb{R}_v^3} \nabla_v \hat{g}(\tau, x_1, \bar{k} - \bar{l}, v) \hat{g}(\tau, x_1, \bar{k}, v) dv \right) dx_1 d\Sigma(\bar{l}) \right| \\ &= \left| \int_{\mathbb{Z}_l^2} \int_{-1}^1 \hat{u}(\tau, x_1, \bar{l}) \left(\int_{\mathbb{R}_v^3} \hat{g}(\tau, x_1, \bar{k} - \bar{l}, v) \nabla_v \hat{g}(\tau, x_1, \bar{k}, v) dv \right) dx_1 d\Sigma(\bar{l}) \right| \\ &\lesssim \left| \int_{\mathbb{Z}_l^2} \int_{-1}^1 \hat{u}(\tau, x_1, \bar{l}) \|\hat{g}(\tau, x_1, \bar{k} - \bar{l}, \cdot)\|_{L_v^2} \|\nabla_v \hat{g}(\tau, x_1, \bar{k}, \cdot)\|_{L_v^2} dx_1 d\Sigma(\bar{l}) \right| \\ &\lesssim \int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{L_{x_1}^\infty} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}) \\ &\lesssim \int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}), \end{aligned} \quad (23)$$

where the embedding inequality $H_{x_1}^1 \hookrightarrow L_{x_1}^\infty$ is also employed. We then apply the Cauchy–Schwarz and Young's inequalities to deduce that

$$\begin{aligned}
& \int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \hat{u} * \nabla_v \hat{g}, \hat{g} \right|_{L_{x_1, v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}) d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/4} \\
& \quad \times \left(\int_0^t \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/4} d\Sigma(\bar{k}) \\
& \lesssim \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \quad + \frac{1}{4\eta} \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}),
\end{aligned} \tag{24}$$

where $\eta > 0$ is an arbitrarily small constant. Using the Minkowski's inequality, i.e.,

$$\|\|\cdot\|_{L_l^1}\|_{L_t^2} \leq \|\|\cdot\|_{L_t^2}\|_{L_l^1},$$

we can further deduce that

$$\begin{aligned}
& \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} \\
& \leq \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}).
\end{aligned} \tag{25}$$

Accordingly, the upper bound for the second term in (24) can be computed as

$$\begin{aligned}
& \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \leq \int_{\mathbb{Z}_k^2} \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) d\Sigma(\bar{k}) \\
& \leq \int_{\mathbb{Z}_k^2} \int_{\mathbb{Z}_l^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) d\Sigma(\bar{k}) \\
& = \int_{\mathbb{Z}_l^2} \int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) d\Sigma(\bar{l}) \\
& = \int_{\mathbb{Z}_l^2} \left(\int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 d\Sigma(\bar{k}) \right) \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) \\
& = \left(\int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2}^2 d\Sigma(\bar{k}) \right) \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}).
\end{aligned} \tag{26}$$

Regarding the dissipation of u , we verify that

$$\begin{aligned} \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) &= \int_{\mathbb{Z}_l^2} \left(\int_0^t |(\hat{u} - \hat{b} + \hat{b})(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) \\ &\leq \sqrt{2} \int_{\mathbb{Z}_l^2} \left(\int_0^t |(\hat{u} - \hat{b})(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) + \sqrt{2} \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{b}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) \\ &\lesssim \mathcal{D}(f). \end{aligned} \quad (27)$$

Given that

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \{\mathbf{I} - \mathbf{P}\} \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_v^2 d\tau \right)^{1/2} d\Sigma(\bar{k}),$$

and

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \mathbf{P} \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t |(\hat{a}, \hat{b})(\tau, \bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}),$$

we can deduce that

$$\begin{aligned} \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) &\leq \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \{\mathbf{I} - \mathbf{P}\} \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ &\quad + \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\nabla_v \mathbf{P} \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \mathcal{D}(g(t)). \end{aligned} \quad (28)$$

Gathering (24), (26)–(28) with the definition of $\mathcal{E}(g(t))$ in (14), the desired inequality (21) is obtained. Since

$$\left| \left(\hat{u} * \nabla_v \hat{g}, \mu^{\frac{1}{4}} \right)_{L_v^2} \right| \lesssim \int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \bar{l})| \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} d\Sigma(\bar{l}),$$

this concludes the desired result (22) with (26)–(28).

Case 2: $\alpha = (1, 0, 0)$. The reasoning closely follows that leading to (23)

$$\begin{aligned} \left| \left(\widehat{\partial^\alpha u} * \nabla_v \hat{g}, \widehat{\partial^\alpha g} \right)_{L_{x_1, v}^2} \right| &\lesssim \int_{\mathbb{Z}_l^2} |\widehat{\partial^\alpha u}(\tau, \cdot, \bar{l})|_{L_{x_1}^2} \|\widehat{\partial^\alpha g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1}^\infty L_v^2} d\Sigma(\bar{l}) \\ &\lesssim \int_{\mathbb{Z}_l^2} |\widehat{\partial^\alpha u}(\tau, \cdot, \bar{l})|_{L_{x_1}^2} \|\widehat{\partial^\alpha g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1, v}^2} \|\nabla_v \hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{H_{x_1}^1 L_v^2} d\Sigma(\bar{l}). \end{aligned}$$

Therefore, an estimate analogous to (26) is obtained:

$$\begin{aligned}
& \int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \widehat{\partial^\alpha u} * \nabla_v \widehat{g}, \widehat{\partial^\alpha g} \right|_{L_{\bar{x}_1, v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t \left\| \nabla_v \widehat{g}(\tau, \cdot, \bar{k}, \cdot) \right\|_{H_{\bar{x}_1}^1 L_v^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \quad + \frac{1}{4\eta} \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} \left| \widehat{\partial^\alpha u}(\tau, \cdot, \bar{l}) \right|_{L_{\bar{x}_1}^2} \left\| \widehat{\partial^\alpha g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot) \right\|_{L_{\bar{x}_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t \left\| \nabla_v \widehat{g}(\tau, \cdot, \bar{k}, \cdot) \right\|_{H_{\bar{x}_1}^1 L_v^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \quad + \frac{1}{4\eta} \left(\int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \left\| \widehat{\partial^\alpha g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot) \right\|_{L_{\bar{x}_1, v}^2}^2 d\Sigma(\bar{k}) \right) \int_{\mathbb{Z}_l^2} \left(\int_0^t \left| \widehat{\partial^\alpha u}(\tau, \cdot, \bar{l}) \right|_{L_{\bar{x}_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}) \\
& \lesssim \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)).
\end{aligned} \tag{29}$$

Consequently, we again obtain a bound similar to the following

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \widehat{u} * \nabla_v \widehat{\partial^\alpha g}, \widehat{\partial^\alpha g} \right|_{L_{\bar{x}_1, v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \tag{30}$$

Collecting (29) and (30) yields the desired inequality (21). $\square$

Remark 3. Note that the following estimate

$$\begin{aligned}
& \int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \widehat{u} * \nabla_v \widehat{g}, \widehat{g} \right|_{L_{\bar{x}_1, v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t \left\| \nabla_v \widehat{g}(\tau, \cdot, \bar{k}, \cdot) \right\|_{L_{\bar{x}_1, v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \quad + \frac{1}{4\eta} \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} \left| \widehat{u}(\tau, \cdot, \bar{l}) \right|_{H_{\bar{x}_1}^1} \left\| \widehat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot) \right\|_{L_{\bar{x}_1, v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}),
\end{aligned}$$

differs from the one used in [25] because the energy component u is absent. In [25], the corresponding bound reads

$$\begin{aligned}
& \int_{\mathbb{Z}_k^3} \left(\int_0^t \left| \widehat{u} * \nabla_v \widehat{g}, \widehat{g} \right|_{L_v^2} d\tau \right)^{1/2} d\Sigma(k) \\
& \lesssim \eta \int_{\mathbb{Z}_k^3} \left(\int_0^t \left\| \widehat{g}(\tau, k, \cdot) \right\|_{L_v^2}^2 d\tau \right)^{1/2} d\Sigma(k) \\
& \quad + \frac{1}{4\eta} \int_{\mathbb{Z}_k^3} \left(\int_0^t \left(\int_{\mathbb{Z}_l^3} \left| \widehat{u}(\tau, k - l) \right| \left\| \nabla_v \widehat{g}(\tau, l, \cdot) \right\|_{L_v^2} d\Sigma(l) \right)^2 d\tau \right)^{1/2} d\Sigma(k).
\end{aligned}$$

The next lemma addresses the term involving $u \cdot v g$; the argument parallels that of Lemma 2.

Lemma 3. Let $|\alpha| \leq 1$. There exists a small constant $\eta > 0$ such that

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \mathcal{F}[v \cdot \partial^\alpha (u g)], \widehat{\partial^\alpha g} \right|_{L_{\bar{x}_1, v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)), \tag{31}$$

and

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \left(\mathcal{F}[v \cdot \partial^\alpha(ug)], \mu^{\frac{1}{4}} \right)_{L_{x_1,v}^2} \right|^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \mathcal{E}(g(t)) \mathcal{D}(g(t)). \quad (32)$$

Proof. When $\alpha = (0, 0, 0)$, we proceed in the same spirit as the proof of Lemma 2 and obtain

$$\begin{aligned} & \int_{\mathbb{Z}_k^2} \left(\int_0^t \left| (v \cdot \hat{u} * \hat{g}, \hat{g})_{L_{x_1,v}^2} \right|^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} \|v\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}) d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/4} \\ & \quad \times \left(\int_0^t \|v\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2}^2 d\tau \right)^{1/4} d\Sigma(\bar{k}) \\ & \lesssim \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t \|v\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \quad + \frac{1}{4\eta} \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}). \end{aligned}$$

For the second term on the right-hand side of the preceding inequality, a technique analogous to the one used to derive (26) gives

$$\begin{aligned} & \int_{\mathbb{Z}_k^2} \left(\int_0^t \left(\int_{\mathbb{Z}_l^2} |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2} d\Sigma(\bar{l}) \right)^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \left(\int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\hat{g}(\tau, \cdot, \bar{k} - \bar{l}, \cdot)\|_{L_{x_1,v}^2}^2 d\Sigma(\bar{k}) \right) \int_{\mathbb{Z}_l^2} \left(\int_0^t |\hat{u}(\tau, \cdot, \bar{l})|_{H_{x_1}^1}^2 d\tau \right)^{1/2} d\Sigma(\bar{l}). \end{aligned}$$

Using the fact that

$$\begin{aligned} & \int_{\mathbb{Z}_k^2} \left(\int_0^t \|v\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \|v\{\mathbf{I} - \mathbf{P}\}\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t \|v\mathbf{P}\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_{L_{x_1,v}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\}\hat{g}(\tau, \cdot, \bar{k}, \cdot)\|_v^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |(\hat{a}, \hat{b})(\tau, \bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \mathcal{D}(g(t)), \end{aligned}$$

we immediately get the estimate (31). When $\alpha = (1, 0, 0)$, the same line of reasoning employed in Lemma 2 again yields (31). Estimate (32) is obtained in a similar way. $\square$

Next, we turn to the bound for $ua \cdot u$.

Lemma 4. Let $|\alpha| \leq 1$. There exists a small constant $\eta > 0$ such that

$$\int_{\mathbb{Z}_k^2} \left(\int_0^t \left| \left(\mathcal{F}[\partial^\alpha(au)], \widehat{\partial^\alpha u} \right)_{L_{x_1,v}^2} \right|^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \lesssim \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \quad (33)$$

The final estimate concerns specular reflection at the boundary.

Lemma 5. Let $|\alpha| \leq 1$. There exists a small constant $\eta > 0$ such that

$$\int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, 1, \bar{k}, v)|^2 dv = \int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, -1, \bar{k}, v)|^2 dv = 0. \quad (34)$$

Proof. First, the boundary condition at $x_1 = 1$, given by (10)

$$g(t, 1, \bar{x}, v_1, \bar{v})|_{v_1 < 0} = g(t, 1, \bar{x}, -v_1, \bar{v}),$$

implies

$$\begin{aligned} \hat{b}_1(t, 1, \bar{k}) &= \int_{\mathbb{R}_v^3} \hat{g}(t, 1, \bar{k}, v) v_1 \mu^{\frac{1}{2}}(v) dv \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 \hat{g}(t, 1, \bar{k}, v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} \hat{g}(t, 1, \bar{k}, v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 \hat{g}(t, 1, \bar{k}, -v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} \hat{g}(t, 1, \bar{k}, v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} \\ &= - \int_{\mathbb{R}_v^2} \int_0^{+\infty} \hat{g}(t, 1, \bar{k}, v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} \hat{g}(t, 1, \bar{k}, v_1, \bar{v}) v_1 \mu^{\frac{1}{2}}(v) dv_1 d\bar{v} \\ &= 0. \end{aligned}$$

by changing the variable $v_1 \rightarrow -v_1$ in the first integral. This implies the averaged local velocity

$$\hat{u}_1(t, 1, \bar{k}) = 0, \quad (35)$$

by the definition (4) $u(x, t) = \frac{b(t, x)}{1 + a(t, x)}$.

Next, taking the Fourier transform of (3) with respect to $\bar{x} = (x_2, x_3)$ yields

$$\partial_t \hat{g} + v_1 \partial_{x_1} \hat{g} + i\bar{k} \cdot \bar{v} \hat{g} - \hat{u} \cdot v \mu^{\frac{1}{2}} - L \hat{g} = \hat{H}, \quad (36)$$

with the initial datum $\hat{g}(0, x_1, \bar{k}, v) = \hat{g}_0(x_1, \bar{k}, v)$, where $H = -u \cdot \nabla_v g + \frac{1}{2} v \cdot u g$. Therefore, we have that

$$\begin{aligned} &\partial_t \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) + v_1 \partial_{x_1} \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) + i\bar{k} \cdot \bar{v} \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) \\ &- \hat{u}_1(t, 1, \bar{x}) v_1 \mu^{\frac{1}{2}} - \hat{u}_2(t, 1, \bar{x}) v_2 \mu^{\frac{1}{2}} - \hat{u}_3(t, 1, \bar{x}) v_3 \mu^{\frac{1}{2}} - L \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) = \hat{H}(t, 1, \bar{x}, v_1, \bar{v}). \end{aligned} \quad (37)$$

Applying the change of variable $v_1 \rightarrow -v_1$ gives

$$\begin{aligned} &\partial_t \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) - v_1 \partial_{x_1} \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) + i\bar{k} \cdot \bar{v} \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &+ \hat{u}_1(t, 1, \bar{x}) v_1 \mu^{\frac{1}{2}} - \hat{u}_2(t, 1, \bar{x}) v_2 \mu^{\frac{1}{2}} - \hat{u}_3(t, 1, \bar{x}) v_3 \mu^{\frac{1}{2}} - L \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) = \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}). \end{aligned} \quad (38)$$

Using the boundary condition at $x_1 = 1$ (10) for $v_1 < 0$, we deduce from (38) that

$$\begin{aligned} &\partial_t \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) - v_1 \partial_{x_1} \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) + i\bar{k} \cdot \bar{v} \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) \\ &+ \hat{u}_1(t, 1, \bar{x}) v_1 \mu^{\frac{1}{2}} - \hat{u}_2(t, 1, \bar{x}) v_2 \mu^{\frac{1}{2}} - \hat{u}_3(t, 1, \bar{x}) v_3 \mu^{\frac{1}{2}} - L \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) = \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}). \end{aligned} \quad (39)$$

Combining (37) and (39), we obtain that

$$\begin{aligned} &v_1 \partial_{x_1} \hat{g}(t, 1, \bar{x}, v_1, \bar{v}) + v_1 \partial_{x_1} \hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) - 2\hat{u}_1(t, 1, \bar{x}) v_1 \mu^{\frac{1}{2}} \\ &= \hat{H}(t, 1, \bar{x}, v_1, \bar{v}) - \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}). \end{aligned} \quad (40)$$

We next claim that

$$\hat{H}(t, 1, \bar{x}, v_1, \bar{v}) - \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}) = 0. \quad (41)$$

Since

$$\begin{aligned} \hat{H}(t, 1, \bar{x}, v_1, \bar{v}) &= \frac{1}{2}v_1u_1(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) + \frac{1}{2}v_2u_2(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) \\ &\quad + \frac{1}{2}v_3u_3(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) - u_1(t, 1, \bar{x})\partial_{v_1}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) \\ &\quad - u_2(t, 1, \bar{x})\partial_{v_2}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) - u_3(t, 1, \bar{x})\partial_{v_3}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}), \end{aligned} \quad (42)$$

applying the change of variable $v_1 \rightarrow -v_1$ together with the boundary condition (10) at $x_1 = 1$, we obtain

$$\begin{aligned} \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}) &= -\frac{1}{2}v_1u_1(t, 1, \bar{x})g(t, 1, \bar{x}, -v_1, \bar{v}) + \frac{1}{2}v_2u_2(t, 1, \bar{x})g(t, 1, \bar{x}, -v_1, \bar{v}) \\ &\quad + \frac{1}{2}v_3u_3(t, 1, \bar{x})g(t, 1, \bar{x}, -v_1, \bar{v}) - u_1(t, 1, \bar{x})\partial_{v_1}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &\quad - u_2(t, 1, \bar{x})\partial_{v_2}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) - u_3(t, 1, \bar{x})\partial_{v_3}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &= -\frac{1}{2}v_1u_1(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) + \frac{1}{2}v_2u_2(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) \\ &\quad + \frac{1}{2}v_3u_3(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) - u_1(t, 1, \bar{x})\partial_{v_1}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &\quad - u_2(t, 1, \bar{x})\partial_{v_2}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) - u_3(t, 1, \bar{x})\partial_{v_3}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}). \end{aligned} \quad (43)$$

Hence, from the two preceding equations we have

$$\begin{aligned} &\hat{H}(t, 1, \bar{x}, v_1, \bar{v}) - \hat{H}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &= v_1u_1(t, 1, \bar{x})g(t, 1, \bar{x}, v_1, \bar{v}) - u_1(t, 1, \bar{x})\partial_{v_1}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) + u_1(t, 1, \bar{x})\partial_{v_1}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}) \\ &= 0, \end{aligned} \quad (44)$$

where we have used the fact that $\hat{u}_1(t, 1, \bar{k}) = 0$ (35). Combining (35) and (40) then yields

$$v_1\partial_{x_1}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) = -v_1\partial_{x_1}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}),$$

that is,

$$\partial_{x_1}\hat{g}(t, 1, \bar{x}, v_1, \bar{v}) = -\partial_{x_1}\hat{g}(t, 1, \bar{x}, -v_1, \bar{v}), \quad (45)$$

for $v_1 < 0$. For the case $\alpha = (0, 0, 0)$,

$$\begin{aligned} &\int_{\mathbb{R}_v^3} v_1|\hat{g}(t, 1, \bar{k}, v)|^2 dv \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 v_1|\hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1|\hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 v_1|\hat{g}(t, 1, \bar{k}, -v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1|\hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= -\int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1|\hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1|\hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= 0, \end{aligned}$$

by the change in variable $v_1 \rightarrow -v_1$ in the first term. A similar argument shows that

$$\int_{\mathbb{R}_v^3} v_1|\hat{g}(t, -1, \bar{k}, v)|^2 dv = 0.$$

For the case $\alpha = (1, 0, 0)$, employing (45) and the change in variable $v_1 \rightarrow -v_1$, we find

$$\begin{aligned} & \int_{\mathbb{R}_v^3} v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v)|^2 dv \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= \int_{\mathbb{R}_v^2} \int_{-\infty}^0 v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, -v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= - \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} + \int_{\mathbb{R}_v^2} \int_0^{+\infty} v_1 |\partial_{x_1} \hat{g}(t, 1, \bar{k}, v_1, \bar{v})|^2 dv_1 d\bar{v} \\ &= 0. \end{aligned}$$

An analogous computation leads to

$$\int_{\mathbb{R}_v^3} v_1 |\partial_{x_1} \hat{g}(t, -1, \bar{k}, v)|^2 dv = 0.$$

This completes the proof. $\square$

4. The Proofs of Theorem 1

In this section, we first derive uniform a priori estimates, which are essential for establishing the global well-posedness of the Cauchy problem (3).

4.1. The Microscopic Estimate

We begin by estimating the microscopic dissipation.

Lemma 6. Let $|\alpha| \leq 1$, and suppose $g(\tau, x, v)$ is a smooth solution of (3) on $[0, t]$, then

$$\begin{aligned} & \mathcal{E}(g(t)) + \sum_{|\alpha| \leq 1} \|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha g\|_{L_k^1 L_t^2 L_v^2} + \sum_{|\alpha| \leq 1} \|\partial^\alpha (b - u)\|_{L_k^1 L_t^2 L_{x_1}^2} \\ & \lesssim \mathcal{E}(g_0) + \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)), \end{aligned} \quad (46)$$

where $\eta > 0$ is a sufficiently small constant.

Proof. We focus on the case $\alpha = (1, 0, 0)$. Applying ∂^α to the first equation of (3) and taking the Fourier transform with respect to $\bar{x} = (x_2, x_3)$ yields

$$\partial_t \widehat{\partial^\alpha g} + v_1 \partial_{x_1} \widehat{\partial^\alpha g} + i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha g} + \mathcal{F}[\partial^\alpha (u \nabla_v g)] - \frac{1}{2} v \cdot \mathcal{F}[\partial^\alpha (u g)] - \widehat{\partial^\alpha u} \cdot v \mu^{\frac{1}{2}} = L \widehat{\partial^\alpha g}. \quad (47)$$

Taking the inner product of (47) with the complex conjugate of $\widehat{\partial^\alpha g}$ with respect to (x_1, v) gives

$$\begin{aligned} & (\partial_t \widehat{\partial^\alpha g}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2} + (v_1 \partial_{x_1} \widehat{\partial^\alpha g}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2} + (i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha g}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2} - (\widehat{\partial^\alpha u} \cdot v \mu^{\frac{1}{2}}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2} \\ & - (L \widehat{\partial^\alpha g}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2} = (\widehat{\partial^\alpha H}, \widehat{\partial^\alpha g})_{L_{x_1, v}^2}, \end{aligned} \quad (48)$$

where

$$H = -u \cdot \nabla_v g + \frac{1}{2} v \cdot u g. \quad (49)$$

Taking the real part of the equation and integrating in time from 0 to t yields

$$\begin{aligned}
& \frac{1}{2} \|\widehat{\partial^\alpha g}\|_{L_{x_1,v}^2}^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, 1, \bar{k}, v)|^2 dv d\tau - \frac{1}{2} \int_0^t \int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, -1, \bar{k}, v)|^2 dv d\tau \\
& - \int_0^t \mathcal{R}(\widehat{\partial^\alpha u}, \widehat{\partial^\alpha b})_{L_{x_1}^2} d\tau + \lambda_0 \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau + \int_0^t |\widehat{\partial^\alpha b}|_{L_{x_1}^2}^2 d\tau \\
& \leq \frac{1}{2} \|\widehat{\partial^\alpha g_0}\|_{L_{x_1,v}^2}^2 + \int_0^t \mathcal{R}(\widehat{\partial^\alpha H}, \widehat{\partial^\alpha g})_{L_{x_1,v}^2} d\tau.
\end{aligned} \tag{50}$$

Here, the coercivity property (20) of L has been used. Since $b - u = au$ by (4), we have that

$$\begin{aligned}
(\widehat{\partial^\alpha b}, \widehat{\partial^\alpha b})_{L_{x_1}^2} - (\widehat{\partial^\alpha u}, \widehat{\partial^\alpha b})_{L_{x_1}^2} &= (\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}, \widehat{\partial^\alpha b})_{L_{x_1}^2} \\
&= (\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}, \widehat{\partial^\alpha b} - \widehat{\partial^\alpha u} + \widehat{\partial^\alpha u})_{L_{x_1}^2} \\
&= (\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}, \widehat{\partial^\alpha b} - \widehat{\partial^\alpha u})_{L_{x_1}^2} + (\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}, \widehat{\partial^\alpha u})_{L_{x_1}^2} \\
&= |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 + (\widehat{\partial^\alpha (au)}, \widehat{\partial^\alpha u})_{L_{x_1}^2}.
\end{aligned}$$

Subsequently, from (50) we deduce that

$$\begin{aligned}
& \frac{1}{2} \|\widehat{\partial^\alpha g}\|_{L_{x_1,v}^2}^2 + \lambda_0 \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau + \int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \\
& \leq \frac{1}{2} \|\widehat{\partial^\alpha g_0}\|_{L_{x_1,v}^2}^2 + \int_0^t \mathcal{R}(\widehat{\partial^\alpha H}, \widehat{\partial^\alpha g})_{L_{x_1,v}^2} d\tau + \int_0^t \mathcal{R}(\widehat{\partial^\alpha (au)}, \widehat{\partial^\alpha u})_{L_{x_1}^2} d\tau,
\end{aligned}$$

where we have used the identity (34)

$$\frac{1}{2} \int_0^t \int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, 1, \bar{k}, v)|^2 dv d\tau - \frac{1}{2} \int_0^t \int_{\mathbb{R}_v^3} v_1 |\widehat{\partial^\alpha g}(t, -1, \bar{k}, v)|^2 dv d\tau = 0.$$

Taking the square root of the preceding inequality then produces

$$\begin{aligned}
& \|\widehat{\partial^\alpha g}\|_{L_{x_1,v}^2} + \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau \right)^{1/2} + \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} \\
& \lesssim \|\widehat{\partial^\alpha g_0}\|_{L_{x_1,v}^2} + \left(\int_0^t \mathcal{R}(\widehat{\partial^\alpha H}, \widehat{\partial^\alpha g})_{L_{x_1,v}^2} d\tau \right)^{1/2} + \left(\int_0^t \mathcal{R}(\widehat{\partial^\alpha (au)}, \widehat{\partial^\alpha u})_{L_{x_1}^2} d\tau \right)^{1/2}.
\end{aligned}$$

We then take the supremum over $0 \leq \tau \leq t$ and integrate with respect to $d\Sigma(\bar{k})$ over $\mathbb{Z}^2$. This results in the following estimate

$$\begin{aligned}
& \int_{\mathbb{Z}_k^2} \sup_{0 \leq \tau \leq t} \|\widehat{\partial^\alpha g}\|_{L_{x_1,v}^2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \|\widehat{\partial^\alpha g_0}\|_{L_k^1 L_{x_1,v}^2} + \int_{\mathbb{Z}_k^2} \left(\int_0^t \mathcal{R}(\widehat{\partial^\alpha H}, \widehat{\partial^\alpha g})_{L_{x_1,v}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t \mathcal{R}(\widehat{\partial^\alpha (au)}, \widehat{\partial^\alpha u})_{L_{x_1}^2} d\tau \right)^{1/2} d\Sigma(\bar{k}) \\
& \lesssim \|\widehat{\partial^\alpha g_0}\|_{L_k^1 L_{x_1,v}^2} + \eta \mathcal{D}(g(t)) + \mathcal{E}(g(t)) \mathcal{D}(g(t)),
\end{aligned}$$

where Lemmas 2 and 3 are employed. This completes the proof. $\square$

4.2. The Macroscopic Estimate

We now turn to the macroscopic components $\hat{a}(\tau, k)$ and $\hat{b}(\tau, k)$, following the strategy outlined in [2] with the dual argument.

Lemma 7. Under the assumptions of Theorem 1, it holds that

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|\partial^\alpha [a, b]\|_{L_k^1 L_t^2 L_{x_1}^2} &\lesssim \mathcal{E}(g(t)) + \mathcal{E}(g_0) + \sum_{|\alpha| \leq 1} \|\partial^\alpha (b - u)\|_{L_k^1 L_t^2 L_{x_1}^2} \\ &+ \sum_{|\alpha| \leq 1} \|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha g\|_{L_k^1 L_t^2 L_v^2} + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \end{aligned} \quad (51)$$

Proof. We again restrict attention to the case $\alpha = (1, 0, 0)$, i.e., $\partial^\alpha = \partial_{x_1}$. Taking the inner product of (47) with the test function $\hat{\Psi}(t, x_1, \bar{k}, v)$ and integrating in time over t over $[0, t]$ yields

$$\begin{aligned} &(\widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2}(t) - (\widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2}(0) - \int_0^t (\widehat{\partial^\alpha g}, \partial_\tau \hat{\Psi})_{L_{x_1, v}^2} d\tau + \int_0^t (i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2} d\tau \\ &+ \int_0^t (v_1 \widehat{\partial^\alpha g}(1), \hat{\Psi}(1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial^\alpha g}(-1), \hat{\Psi}(-1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial^\alpha g}, \partial_{x_1} \hat{\Psi})_{L_{x_1, v}^2} d\tau \\ &- \int_0^t (\widehat{\partial^\alpha u} \cdot v \mu^{\frac{1}{2}}, \hat{\Psi})_{L_{x_1, v}^2} d\tau - \int_0^t (L \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2} d\tau = \int_0^t (\widehat{\partial^\alpha H}, \hat{\Psi})_{L_{x_1, v}^2} d\tau \end{aligned} \quad (52)$$

where $\hat{\Psi}(t, x_1, \bar{k}, v) \in C^1((0, +\infty) \times (-1, 1) \times \mathbb{Z}^2 \times \mathbb{R}^3)$ is a smooth test function, and the shorthand $g(\pm 1) = g(t, \pm 1, \bar{k}, v_1, \bar{v})$, $\hat{\Psi}(\pm 1) = \hat{\Psi}(t, \pm 1, \bar{k}, v)$ is used. Decomposing g into its macroscopic and microscopic parts, we obtain

$$\begin{aligned} \int_0^t (\widehat{\partial^\alpha \mathbf{P}g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau &= \underbrace{(\widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2}(t) - (\widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2}(0) - \int_0^t (\widehat{\partial^\alpha g}, \partial_\tau \hat{\Psi})_{L_{x_1, v}^2} d\tau}_{J_1} \\ &- \underbrace{\int_0^t (\widehat{\partial^\alpha u} \cdot v \mu^{\frac{1}{2}}, \hat{\Psi})_{L_{x_1, v}^2} d\tau}_{J_2} - \underbrace{\int_0^t (\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau}_{J_3} \\ &- \underbrace{\int_0^t (L \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2} d\tau}_{J_4} - \underbrace{\int_0^t (\widehat{\partial^\alpha H}, \hat{\Psi})_{L_{x_1, v}^2} d\tau}_{J_5} \\ &+ \underbrace{\int_0^t (v_1 \widehat{\partial^\alpha g}(1), \hat{\Psi}(1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial^\alpha g}(-1), \hat{\Psi}(-1))_{L_v^2} d\tau}_{J_6}, \end{aligned}$$

where

$$\begin{aligned} \int_0^t (\widehat{\partial^\alpha g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau &= \int_0^t (v_1 \widehat{\partial^\alpha g}, \partial_{x_1} \hat{\Psi})_{L_{x_1, v}^2} d\tau - \int_0^t (i\bar{k} \cdot \bar{v} \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2} d\tau \\ &= \int_0^t (v_1 \widehat{\partial^\alpha g}, \partial_{x_1} \hat{\Psi})_{L_{x_1, v}^2} d\tau + \int_0^t (\widehat{\partial^\alpha g}, \bar{v} \cdot \nabla_{\bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau, \end{aligned}$$

and

$$\int_0^t (\widehat{\partial^\alpha g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau = \int_0^t (\widehat{\partial^\alpha \mathbf{P}g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau + \int_0^t (\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}, v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi})_{L_{x_1, v}^2} d\tau.$$

For the estimate of a , we take the test function

$$\hat{\Psi} = \hat{\Psi}_a = (|v|^2 - 10) \{v \cdot \nabla_{x_1, \bar{x}} \hat{\Psi}_a(t, x_1, \bar{k})\} \mu^{\frac{1}{2}}, \quad (53)$$

with

$$-\partial_{x_1}^2 \hat{\Psi}_a + |\bar{k}|^2 \hat{\Psi}_a(\bar{k}) = \widehat{\partial^\alpha a}(\bar{k}), \text{ and } \partial_{x_1} \hat{\Psi}_a(\pm 1, \bar{k}) = 0. \quad (54)$$

A standard elliptic estimate then gives (see [2,26])

$$|\Psi_a|_{H_{x_1}^2} + |\bar{k}| |\Psi_a|_{L_{x_1}^2} \lesssim |\widehat{\partial^\alpha a}|_{L_{x_1}^2}, \quad (55)$$

and

$$|\partial_t \Psi_a|_{H_{x_1}^1} \lesssim |\partial_t \widehat{\partial^\alpha a}|_{H_{x_1}^{-1}}, \quad |\bar{k}| |\partial_t \Psi_a|_{L_{x_1}^2} \lesssim |\bar{k}|^{-1} |\partial_t \widehat{\partial^\alpha a}|_{L_{x_1}^2}, \quad k \neq 0. \quad (56)$$

With this choice of the test function Ψ (53), we see that

$$\begin{aligned} & \int_0^t (\widehat{\partial^\alpha \mathbf{P}g}, v \cdot \widehat{\nabla_{x_1, \bar{x}} \Psi})_{L_{x_1, v}^2} d\tau \\ &= \sum_j \int_0^t (\{\widehat{\partial^\alpha a} + \widehat{\partial^\alpha b} \cdot v\} \mu^{\frac{1}{2}}, v_j \widehat{\partial_j \Psi_a})_{L_{x_1, v}^2} d\tau \\ &= \sum_{j, n} \int_0^t (\{\widehat{\partial^\alpha a} + \widehat{\partial^\alpha b} \cdot v\} \mu^{\frac{1}{2}}, v_j v_n (|v|^2 - 10) \mu^{\frac{1}{2}} \widehat{\partial_j \partial_n \Psi_a})_{L_{x_1, v}^2} d\tau \\ &= 5 \sum_j \int_0^t (\widehat{\partial^\alpha a}, -\widehat{\partial_j^2 \Psi_a})_{L_{x_1}^2} d\tau = 5 \int_0^t |\widehat{\partial^\alpha a}(\bar{k})|_{L_{x_1}^2}^2 d\tau. \end{aligned}$$

The upper bounds for J_1 - J_3 are obtained exactly as in [2,26]

$$\begin{aligned} |J_1| &\lesssim |\widehat{\partial^\alpha g}(t)|_{L_{x_1, v}^2}^2 + |\widehat{\partial^\alpha g_0}|_{L_{x_1, v}^2}^2 + \eta \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau, \\ |J_3| &\lesssim \eta \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau, \end{aligned}$$

where $\eta > 0$ is a small enough constant. Using Young's inequality together with the elliptic estimate (55), we find

$$\begin{aligned} |J_2| &= \int_0^t (\widehat{\partial^\alpha u} \cdot v \mu^{\frac{1}{2}}, (|v|^2 - 10) \{v \cdot \widehat{\nabla_{x_1, \bar{x}} \Psi_a}(\tau, x_1, \bar{k})\} \mu^{\frac{1}{2}})_{L_{x_1, v}^2} d\tau \\ &\lesssim \eta \int_0^t |\partial_{x_1} \Psi_a(\bar{k})|_{L_{x_1}^2}^2 d\tau + \eta |\bar{k}|^2 \int_0^t |\Psi_a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |\widehat{\partial^\alpha u}(\bar{k})|_{L_{x_1}^2}^2 d\tau \\ &\lesssim \eta \int_0^t |\widehat{\partial^\alpha a}(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |\widehat{\partial^\alpha u}(\bar{k})|_{L_{x_1}^2}^2 d\tau. \end{aligned}$$

As for the term J_4 , one can derive that

$$\begin{aligned} |J_4| &= \left| \int_0^t (L \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right| = \left| \int_0^t (\Delta_v \widehat{\partial^\alpha g} + \frac{1}{4} (6 - |v|^2) \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right| \\ &\leq \underbrace{\left| \int_0^t (\Delta_v \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right|}_{J_{4,1}} + \underbrace{\left| \int_0^t (\frac{1}{4} (6 - |v|^2) \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right|}_{J_{4,2}}, \end{aligned}$$

by employing (5). Applying the macro-micro decomposition, (6) and (8) yields

$$\begin{aligned} J_{4,1} &\leq \left| \int_0^t (\Delta_v \mathbf{P} \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right| + \left| \int_0^t (\Delta_v \{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right| \\ &\leq \underbrace{\left| \int_0^t (\widehat{\partial^\alpha a} \Delta_v \mu^{\frac{1}{2}}, \Psi)_{L_{x_1, v}^2} d\tau \right|}_{J_{4,1}^1} + \underbrace{\left| \int_0^t (\widehat{\partial^\alpha b} \Delta_v (v \mu^{\frac{1}{2}}), \Psi)_{L_{x_1, v}^2} d\tau \right|}_{J_{4,1}^2} + \underbrace{\left| \int_0^t (\Delta_v \{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}, \Psi)_{L_{x_1, v}^2} d\tau \right|}_{J_{4,1}^3}. \end{aligned}$$

Since $\Delta_v \mu^{\frac{1}{2}} = (\frac{1}{4}v^2 - \frac{3}{2})\mu^{\frac{1}{2}}$ and (53), we see that

$$\begin{aligned} J_{4,1}^1 &= \left| \int_0^t \left(\widehat{\partial^\alpha a} \Delta_v \mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik \hat{\Psi}_a(\tau, k) \mu^{\frac{1}{2}} \right)_{L_{x_1, v}^2} d\tau \right| \\ &= \left| \int_0^t \left(\widehat{\partial^\alpha a} \left(\frac{1}{4}v^2 - \frac{3}{2} \right) \mu^{\frac{1}{2}}, (|v|^2 - 10)v \cdot ik \hat{\Psi}_a(\tau, k) \mu^{\frac{1}{2}} \right)_{L_{x_1, v}^2} d\tau \right| = 0, \end{aligned}$$

because the integrand is odd in v . By employing Young's inequality and elliptic estimate (55), we have

$$J_{4,1}^2 \lesssim \eta \int_0^t |\widehat{\partial^\alpha a}(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |\widehat{\partial^\alpha b}(\bar{k})|_{L_{x_1}^2}^2 d\tau.$$

Proceeding analogously, we also obtain

$$J_{4,1}^3 \lesssim \eta \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau.$$

For the estimation of $J_{4,2}$, it is also demonstrated that

$$J_{4,2} \lesssim \eta \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau.$$

Applying Young's inequality to J_5 gives

$$\begin{aligned} |J_5| &= \int_0^t (\widehat{\partial^\alpha H}, (|v|^2 - 10)v \cdot ik \hat{\Psi}_a(\tau, k) \mu^{\frac{1}{2}})_{L_{x_1, v}^2} d\tau \\ &\lesssim \eta \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |(|\widehat{\partial^\alpha H}(\bar{k})|, \mu^{\frac{1}{4}})_{L_{x_1, v}^2}|^2 d\tau. \end{aligned}$$

Regarding the estimate of J_6 , we can easily see that $\hat{g}(t, -1, \bar{k}, -v_1, \bar{v}) = \hat{g}(t, 1, \bar{k}, v_1, \bar{v})$ from the Equation (36) by using $\hat{u}_1(t, \pm 1, \bar{k}) = 0$. Moreover, we also can see that $\hat{g}(t, -1, \bar{k}, -v_1, \bar{v}) = \hat{g}(t, -1, \bar{k}, v_1, \bar{v}) = \hat{g}(t, 1, \bar{k}, v_1, \bar{v})$ from the boundary conditions. Similar to obtaining the results (45), one can also find that

$$\widehat{\partial_{x_1} g}(t, -1, \bar{k}, -v_1, \bar{v}) = \widehat{\partial_{x_1} g}(t, -1, \bar{k}, v_1, \bar{v}) = \widehat{\partial_{x_1} g}(t, 1, \bar{k}, v_1, \bar{v}), \text{ for } v_1 \neq 0,$$

by using the Equation (36). On the other hand, since

$$\hat{a}(t, x_1, \bar{k}) = \int_{\mathbb{R}^3} \hat{g}(t, x_1, \bar{k}, v) \mu^{\frac{1}{2}}(v) dv,$$

we know that $\hat{a}(1, \bar{k}) = \hat{a}(-1, \bar{k})$, which further implies that $\hat{\Psi}_a(t, -1, \bar{k}, v_1) = \hat{\Psi}_a(t, 1, \bar{k}, v_1)$ according to (53) and (54). Consequently, we obtain that

$$\begin{aligned} J_6 &= \int_0^t (v_1 \widehat{\partial_{x_1} g}(1), \hat{\Psi}_a(1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial_{x_1} g}(-1), \hat{\Psi}_a(-1))_{L_v^2} d\tau \\ &= \int_0^t (v_1 \widehat{\partial_{x_1} g}(1, v_1), \hat{\Psi}_a(1, v_1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial_{x_1} g}(-1, v_1), \hat{\Psi}_a(-1, v_1))_{L_v^2} d\tau \\ &= \int_0^t (v_1 \widehat{\partial_{x_1} g}(1, v_1), \hat{\Psi}_a(1, v_1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial_{x_1} g}(1, v_1), \hat{\Psi}_a(1, v_1))_{L_v^2} d\tau = 0. \end{aligned}$$

Collecting the above estimates and selecting $\eta > 0$ small enough, we arrive at

$$\begin{aligned} \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau &\lesssim |\widehat{\partial^\alpha g}(t)|_{L_{x_1, v}^2}^2 + |\widehat{\partial^\alpha g_0}|_{L_{x_1, v}^2}^2 + \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + \int_0^t |\widehat{\partial^\alpha u}(\bar{k})|_{L_{x_1}^2}^2 d\tau \\ &\quad + \int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_v^2 d\tau + \int_0^t |(|\widehat{\partial^\alpha H}(\bar{k})|, \mu^{\frac{1}{4}})_{L_{x_1, v}^2}|^2 d\tau. \end{aligned}$$

Furthermore, utilizing (22), (27), (32) and (49), we obtain that

$$\begin{aligned} \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) &\lesssim \mathcal{E}(g(t)) + \mathcal{E}(g_0) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ &+ \int_{\mathbb{Z}_k^2} \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ &+ \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_{\mathbb{V}}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \end{aligned} \quad (57)$$

We now turn to the macroscopic component b . Choose the test function

$$\hat{\Psi} = \hat{\Psi}_b = \sum_{m=1}^3 \hat{\Psi}_b^{J,m}, \quad J = 1, 2, 3,$$

defined as

$$\hat{\Psi}_b^{J,m} = \begin{cases} \left\{ |v|^2 v_m v_J i k_m \hat{\Psi}_J(t, k) - \frac{7}{2} (v_m^2 - 1) i k_J \hat{\Psi}_J(t, k) \right\} \mu^{\frac{1}{2}}, & J \neq m, \\ \frac{7}{2} (v_J^2 - 1) i k_J \hat{\Psi}_J(t, k) \mu^{\frac{1}{2}}, & J = m, \end{cases}$$

with the elliptic problem

$$-\partial_{x_1}^2 \hat{\Psi}_J + |\bar{k}|^2 \hat{\Psi}_J(\bar{k}) = \widehat{\partial b_J}(\bar{k}), \quad \hat{\Psi}_J(\pm 1, \bar{k}) = 0. \quad (58)$$

Standard elliptic regularity (see [2,26]) gives

$$|\hat{\Psi}_J|_{H_{x_1}^2} + |\bar{k}| |\hat{\Psi}_J|_{L_{x_1}^2} \lesssim |\widehat{\partial^\alpha b_J}|_{L_{x_1}^2}, \quad (59)$$

and

$$|\partial_t \hat{\Psi}_J|_{H_{x_1}^1} \lesssim |\partial_t \widehat{\partial^\alpha b}|_{H_{x_1}^{-1}}, \quad |\bar{k}| |\partial_t \hat{\Psi}_J|_{L_{x_1}^2} \lesssim |\bar{k}|^{-1} |\partial_t \widehat{\partial^\alpha b}|_{L_{x_1}^2}, \quad k \neq 0. \quad (60)$$

With this choice, one immediately has (see [2])

$$-\sum_{m=1}^3 \int_0^t (\mathbf{P} \hat{g}, v \cdot i k \hat{\Psi}_b^{J,m})_{L_v^2} d\tau = 7 \int_0^t |\hat{b}_J(\tau, k)|_{L_{x_1}^2}^2 dt.$$

As for the term J_4 , we also have that

$$\begin{aligned} |J_4| &= \left| \int_0^t (L \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1,v}^2} d\tau \right| = \left| \int_0^t (\Delta_v \widehat{\partial^\alpha g} + \frac{1}{4} (6 - |v|^2) \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1,v}^2} d\tau \right| \\ &\leq \underbrace{\left| \int_0^t (\Delta_v \mathbf{P} \widehat{\partial^\alpha g} + \frac{1}{4} (6 - |v|^2) \mathbf{P} \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1,v}^2} d\tau \right|}_{J_{4,3}} \\ &\quad + \underbrace{\left| \int_0^t (\Delta_v \{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g} + \frac{1}{4} (6 - |v|^2) \{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1,v}^2} d\tau \right|}_{J_{4,4}} \end{aligned}$$

Exploiting the identities

$$\Delta_v (v \mu^{\frac{1}{2}}) + \frac{1}{4} (6 - |v|^2) v \mu^{\frac{1}{2}} = -\frac{5}{2} v \mu^{\frac{1}{2}} + \frac{1}{4} v |v|^2 \mu^{\frac{1}{2}} + \frac{3}{2} v \mu^{\frac{1}{2}} - \frac{1}{4} v |v|^2 \mu^{\frac{1}{2}} = -v \mu^{\frac{1}{2}},$$

$$\Delta_v \mu^{\frac{1}{2}} + \frac{1}{4} (6 - |v|^2) \mu^{\frac{1}{2}} = \left(\frac{1}{4} v^2 - \frac{3}{2} \right) \mu^{\frac{1}{2}} + \frac{1}{4} (6 - |v|^2) \mu^{\frac{1}{2}} = 0,$$

we find that

$$\begin{aligned} \int_0^t (\widehat{\partial^\alpha b} \Delta_v (v \mu^{\frac{1}{2}}) + \frac{1}{4} (6 - |v|^2) v \mu^{\frac{1}{2}} \widehat{\partial^\alpha b}, \hat{\Psi})_{L_{x_1, v}^2} d\tau &= \int_0^t (-\widehat{\partial^\alpha b} v \mu^{\frac{1}{2}}, \hat{\Psi})_{L_{x_1, v}^2} d\tau = 0, \\ \int_0^t (\widehat{\partial^\alpha a} \Delta_v \mu^{\frac{1}{2}} + \frac{1}{4} (6 - |v|^2) \mu^{\frac{1}{2}} \widehat{\partial^\alpha a}, \hat{\Psi})_{L_{x_1, v}^2} d\tau &= 0. \end{aligned}$$

Consequently, the potentially troublesome contributions in $J_{4,3}$ vanish. For the estimate of $J_{4,4}$, we have that

$$\begin{aligned} |J_{4,4}| &= \left| \int_0^t (\Delta_v \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g} + \frac{1}{4} (6 - |v|^2) \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g}, \hat{\Psi})_{L_{x_1, v}^2} d\tau \right| \\ &\lesssim \eta \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \| \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g} \|_v^2 d\tau. \end{aligned}$$

By using the Equation (12), we obtain

$$|\partial_t \widehat{\partial^\alpha b}|_{H_{x_1}^{-1}} \lesssim |\partial_{x_1} \hat{a}(\bar{k})|_{L_{x_1}^2} + |\bar{k}| |\hat{a}(\bar{k})|_{L_{x_1}^2} + \|\partial_{x_1} \{ \mathbf{I} - \mathbf{P} \} \hat{g}\|_v + |\bar{k}| \| \{ \mathbf{I} - \mathbf{P} \} \hat{g} \|_v,$$

and

$$|\bar{k}|^{-1} |\partial_t \widehat{\partial^\alpha b}(\bar{k})|_{L_{x_1}^2} \lesssim |\partial_{x_1} \hat{a}(\bar{k})|_{L_{x_1}^2} + |\bar{k}| |\hat{a}(\bar{k})|_{L_{x_1}^2} + \|\partial_{x_1} \{ \mathbf{I} - \mathbf{P} \} \hat{g}\|_v + |\bar{k}| \| \{ \mathbf{I} - \mathbf{P} \} \hat{g} \|_v,$$

for $k \neq 0$. Combining (59) with (60) and the preceding inequalities yields

$$|J_1| \lesssim |\widehat{\partial^\alpha g}(t)|_{L_{x_1, v}^2}^2 + |\widehat{\partial^\alpha g_0}|_{L_{x_1, v}^2}^2 + \eta \int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \| \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g} \|_v^2 d\tau.$$

The remaining terms are controlled similarly with the estimates of a as follows

$$\begin{aligned} |J_2| &\lesssim \eta \int_0^t |\widehat{\partial^\alpha b}(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |\widehat{\partial^\alpha u}(\bar{k})|_{L_{x_1}^2}^2 d\tau, \\ |J_3| &\lesssim \eta \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t \| \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g} \|_v^2 d\tau, \\ |J_5| &\lesssim \eta \int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau + C_\eta \int_0^t |(\widehat{\partial^\alpha H}(\bar{k}), \mu^{\frac{1}{4}})_{L_{x_1, v}^2}|^2 d\tau. \end{aligned}$$

For J_6 , we get that $\hat{\Psi}_b^{J, m}(t, -1, \bar{k}, v_1) = \hat{\Psi}_b^{J, m}(t, 1, \bar{k}, v_1) = 0$ from $\hat{\Psi}_J(\pm 1, \bar{k}) = 0$ in (58). Consequently, we obtain that

$$J_6 = \int_0^t (v_1 \widehat{\partial_{x_1} g}(1), \hat{\Psi}_b^{J, m}(1))_{L_v^2} d\tau - \int_0^t (v_1 \widehat{\partial_{x_1} g}(-1), \hat{\Psi}_b^{J, m}(-1))_{L_v^2} d\tau = 0.$$

Choosing $\eta > 0$ to be sufficiently small, we finally obtain

$$\begin{aligned} \int_{\mathbb{Z}_{\bar{k}}^2} \left(\int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) &\lesssim \mathcal{E}(g(t)) + \mathcal{E}(g_0) + \eta \int_{\mathbb{Z}_{\bar{k}}^2} \left(\int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ &\quad + \int_{\mathbb{Z}_{\bar{k}}^2} \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ &\quad + \int_{\mathbb{Z}_{\bar{k}}^2} \left(\int_0^t \| \{ \mathbf{I} - \mathbf{P} \} \widehat{\partial^\alpha g} \|_v^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \end{aligned} \quad (61)$$

Combining a positive constant $\lambda \times (57)$ with (61) yields

$$\begin{aligned} & \lambda \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim (1 + \lambda) \mathcal{E}(g(t)) + (1 + \lambda) \mathcal{E}(g_0) + \eta \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \quad + (1 + \lambda) \int_{\mathbb{Z}_k^2} \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \lambda \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \quad + (1 + \lambda) \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_{L_v^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + (1 + \lambda) \mathcal{E}(g(t)) \mathcal{D}(g(t)). \end{aligned}$$

Choosing $\eta > 0$ to be sufficiently small and then $\lambda > 0$ to be sufficiently small, we obtain

$$\begin{aligned} & \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha a(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |\partial^\alpha b(\bar{k})|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \lesssim \mathcal{E}(g(t)) + \mathcal{E}(g_0) + \int_{\mathbb{Z}_k^2} \left(\int_0^t |\widehat{\partial^\alpha b} - \widehat{\partial^\alpha u}|_{L_{x_1}^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) \\ & \quad + \int_{\mathbb{Z}_k^2} \left(\int_0^t \|\{\mathbf{I} - \mathbf{P}\} \widehat{\partial^\alpha g}\|_{L_v^2}^2 d\tau \right)^{1/2} d\Sigma(\bar{k}) + \mathcal{E}(g(t)) \mathcal{D}(g(t)). \end{aligned}$$

which completes the proof of Lemma 7. $\square$

4.3. The Uniform-in-Time Estimate

This section culminates in the proof of the main uniform in time energy estimate on $g(t, x, v)$, contingent on the following a priori assumption.

Proposition 1. *There exists a sufficiently small constant $\delta > 0$ such that the condition*

$$\sup_{0 \leq t \leq T} \mathcal{E}(g(t)) \leq \delta, \quad 0 < T < \infty,$$

then the solution satisfies

$$\mathcal{E}(g(t)) + \mathcal{D}(g(t)) \lesssim \mathcal{E}(g_0). \quad (62)$$

Proof. Fix a small $\lambda > 0$ and add $\lambda \times (51)$ to (46); we get that

$$\begin{aligned} & \mathcal{E}(g(t)) + \sum_{|\alpha| \leq 1} \|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha g\|_{L_k^1 L_t^2 L_v^2} + \sum_{|\alpha| \leq 1} \|\partial^\alpha (b - u)\|_{L_k^1 L_t^2 L_{x_1}^2} + \lambda \sum_{|\alpha| \leq 1} \|\partial^\alpha [a, b]\|_{L_k^1 L_t^2 L_{x_1}^2} \\ & \lesssim \lambda \mathcal{E}(g(t)) + (\lambda + 1) \mathcal{E}(g_0) + \eta \mathcal{D}(g(t)) + (\lambda + 1) \mathcal{E}(g(t)) \mathcal{D}(g(t)) \\ & \quad + \lambda \sum_{|\alpha| \leq 1} \|\partial^\alpha (b - u)\|_{L_k^1 L_t^2 L_{x_1}^2} + \lambda \sum_{|\alpha| \leq 1} \|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha g\|_{L_k^1 L_t^2 L_v^2}. \end{aligned}$$

Selecting $\lambda > 0$ and subsequently $\eta > 0$ to be small enough, and invoking the definitions of $\mathcal{E}(g(t))$ and $\mathcal{D}(g(t))$, i.e., (14) and (15), we arrive at

$$\mathcal{E}(g(t)) + \mathcal{D}(g(t)) \lesssim \mathcal{E}(g_0) + \mathcal{E}(g(t)) \mathcal{D}(g(t)),$$

and the desired conclusion (62) follows by taking $\delta > 0$ to be small enough, which completes the proof. $\square$

4.4. Proof of Theorem 1

Combining the uniform energy estimates (62) with a standard local existence argument as in [2,27], the global existence and uniqueness of mild solutions follow immediately from a continuation argument, provided $\mathcal{E}(g_0)$ is sufficiently small. This finishes the proof of Theorem 1.

5. Conclusions

This paper discusses the Cauchy problem for the Vlasov–Fokker–Planck equation with local alignment forces near a global Maxwellian under the specular reflection boundary condition in a finite channel. The global-in-time existence and uniqueness of mild solutions to the system are established in the low-regularity space $L_k^1 L_T^\infty L_v^2$, provided the initial data is sufficiently small. A key difficulty: the lack of direct dissipation for the macroscopic velocity u is overcome by exploiting the dissipative structures of $u - b$ and the macroscopic component b derived from the fluid system. Future research will consider the global existence of solutions in the whole space or under more general boundary conditions.

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