

Article

Two-Sided Zero-Divisor Graphs of Order-Preserving and A-Decreasing Finite Transformation Semigroups

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Abstract

Let $\mathcal{O}_n(A)$ be the order-preserving and A-decreasing finite transformation semigroup on $X_n = \{1, 2, \dots, n\}$. It is known that $\mathcal{O}_n(A)$ has zero element if and only if $1 \in A$. In this paper, we investigate zero-divisor graphs of $\mathcal{O}_n(A)$ where $1 \in A$ and $n \geq 4$. First, we determine the set of right, left, and two-sided zero-divisors of $\mathcal{O}_n(A)$; and their cardinalities. Let $\Gamma(\mathcal{O}_n(A))$ be the undirected graph whose vertices are the two sided zero-divisors of $\mathcal{O}_n(A)$ excluding the zero element (θ) and distinct two vertices α and β joined by an edge if and only if $\alpha\beta = \theta = \beta\alpha$. In this paper, we prove that $\Gamma(\mathcal{O}_n(A))$ is a connected graph and find its diameter, girth, domination number and degrees of all vertices in $\Gamma(\mathcal{O}_n(A))$. Moreover, we prove that $\Gamma(\mathcal{O}_n(A))$ is a perfect graph and we calculate clique number and chromatic number of it.

Keywords: order-preserving; transformation semigroup; zero-divisor graph; diameter; domination number; clique number

MSC: 20M20; 97K30

1. Introduction

The zero-divisor graph of a commutative ring was defined by Beck [1]; however, the zero element is a vertex in Beck's definition. Later, the zero-divisor graph of commutative ring redefined by Anderson and Livingston which is now standart zero-divisor graph of a commutative ring [2]. Similarly, the zero-divisor graph of a commutative semigroup with zero, as defined by Demeyer et al. [3]. Let S be a commutative semigroup with zero and 0 be the zero element of S and $Z(S)$ be the zero-divisors of S . The zero-divisor graph of S is denoted by $\bar{\Gamma}(S)$ and it is an undirected graph with vertex set $Z(S)^* = Z(S) \setminus \{0\}$ and distinct two vertices α and β are adjacent vertices in $\bar{\Gamma}(S)$ if and only if $\alpha\beta = 0$. Recently, zero-divisor graphs of some commutative semigroups have been studied [4].

In 2002, Redmond defined four different zero-divisor graphs for noncommutative rings [5]. Therefore, these definitions can be considered for noncommutative semigroups with zero. For a non-commutative semigroup S with zero, let

$$L(S) = \{x \in S : xy = 0 \text{ for some } y \in S \setminus \{0\}\},$$

$$R(S) = \{x \in S : yx = 0 \text{ for some } y \in S \setminus \{0\}\} \text{ and}$$

$$T(S) = \{x \in S : xy = 0 = zx \text{ for some } y, z \in S \setminus \{0\}\}$$



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are called left, right, and two-sided zero divisors of S , respectively. Notice that $T(S) = L(S) \cap R(S)$. Moreover, let

$$Z(S) = \{x \in S : xy = 0 \text{ or } yx = 0 \text{ for some } y \in S \setminus \{0\}\},$$

$$Z(S)^* = Z(S) \setminus \{0\} \text{ and } T(S)^* = T(S) \setminus \{0\}.$$

Notice that $Z(S) = L(S) \cup R(S)$. As defined by Redmond, we define four different zero-divisor graphs for noncommutative semigroups with zero as follows:

- $\Gamma(S) = \Gamma_1(S)$ is the undirected graph with vertices $T(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0 = yx$;
- $\Gamma_2(S)$ is the undirected graph with vertices $Z(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0 = yx$;
- $\Gamma_3(S)$ is the undirected graph with vertices $Z(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0$ or $yx = 0$; and
- $\Gamma_4(S)$ is the directed graph with vertices $Z(S)^*$ and for distinct two vertices x and y , $x \rightarrow y$ is a directed edge if and only if $xy = 0$.

In this paper, we only consider $\Gamma(S)$. Notice that if S is a commutative semigroup, then $\Gamma(S) = \bar{\Gamma}(S)$. $\Gamma(S)$ is an undirected graph with vertex set $T(S)^* = T(S) \setminus \{0\}$, and distinct two vertices α and β in $T(S)^*$ are adjacent vertices in $\Gamma(S)$ if and only if $\alpha\beta = 0 = \beta\alpha$. It is called the two-sided zero-divisor graph of S .

Let $n \in \mathbb{Z}^+$ and $\mathcal{T}_n$ be the full transformation semigroup on $X_n = \{1, \dots, n\}$. For $\alpha \in \mathcal{T}_n$, if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, then α is called order-preserving transformation. The set of all order-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{O}_n$, which is a subsemigroup of $\mathcal{T}_n$. For $\alpha \in \mathcal{T}_n$, if $x\alpha \leq x$ for all $x \in X_n$, then α is called an order-decreasing element in $\mathcal{T}_n$. The set of all order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{D}_n$, which is a subsemigroup of $\mathcal{T}_n$. The set of all order-preserving and order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{C}_n$ and it is called the Catalan monoid on X_n . In 2021, two-sided zero-divisor graphs of Catalan monoid were investigated [6]. For a sequence $(x_1, x_2, \dots, x_r)$ on X_n , if there exists no more than one subscript i such that $x_i > x_{i+1}$ where $x_{r+1} = x_1$, then $(x_1, x_2, \dots, x_r)$ is called a cyclic. For α in $\mathcal{T}_n$, if $(1\alpha, 2\alpha, \dots, n\alpha)$ is a cyclic then α is called orientation-preserving. The subsemigroup consisting of all orientation-preserving and order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{OPD}_n$. In 2025, two-sided zero-divisor graphs of orientation-preserving and order-decreasing transformation semigroups were investigated [7].

For $\emptyset \neq A \subseteq X_n$, $\mathcal{O}_n(A) = \{\alpha \in \mathcal{O}_n : x\alpha \leq x \text{ } (\forall x \in A)\}$ is called order-preserving and A -decreasing transformation semigroup on X_n . From the definitions, $\mathcal{C}_n \subseteq \mathcal{O}_n(A) \subseteq \mathcal{O}_n$.

If $B = A \cup \{n\}$, then it is clear that $\mathcal{O}_n(A) = \mathcal{O}_n(B)$. Moreover, it is known that $\mathcal{O}_n(A)$ has a zero element if and only if $1 \in A$ [8]. Throughout this paper, we suppose $1 \in A$ and $n \notin A$.

In this paper, we investigate some properties of $\Gamma(\mathcal{O}_n(A))$. The vertex set of $\Gamma(\mathcal{O}_n(A))$ is denoted by $V(\Gamma(\mathcal{O}_n(A)))$ and the edge set of $\Gamma(\mathcal{O}_n(A))$ is denoted by $E(\Gamma(\mathcal{O}_n(A)))$. $\Gamma(\mathcal{O}_n(A))$ is an undirected graph, $V(\Gamma(\mathcal{O}_n(A))) = T^*(\mathcal{O}_n(A))$ and $E(\Gamma(\mathcal{O}_n(A))) = \{\alpha - \beta : \alpha\beta = \theta = \beta\alpha\}$. It is clear that $\Gamma(\mathcal{C}_n)$ is a subgraph of $\Gamma(\mathcal{O}_n(A))$.

Throughout this paper, we suppose $n \geq 3$ since $\mathcal{O}_n(A) = \mathcal{C}_n$ for $n \in \{1, 2\}$. Since $n \notin A$, notice that for $n \geq 3$, $\mathcal{O}_n(A) = \mathcal{C}_n$ if and only if $A = X_{n-1}$.

For $n \geq 3$, recall that $\mathcal{O}_n(A)$ is a non-commutative semigroup with the zero element. We denote the zero element of $\mathcal{O}_n(A)$ by θ and $\theta = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$.

We refer to [9–11] for other terms in semigroup and graph theories, which are not explained here.

2. Primaries

Definition 1. Let K be a non-empty subset of $\mathbb{Z}^d$ and L be a path in $\mathbb{Z}^d$. Moreover, let L be a path length of k with steps $v_0, v_1, \dots, v_k$ such that each consecutive difference $v_i - v_{i+1}$ lies in K for all $i \in \{0, 1, \dots, k-1\}$. Then, L is called a lattice path.

Definition 2. Let L be a lattice path in $\mathbb{Z}^2$, where $K = \{(1, 0), (0, 1)\}$. Then, L is called a North-East lattice path.

Proposition 1. Let $\mathcal{NE}$ be the set of all North-East lattice paths from $(0, 0)$ to $(n, n-1)$. Then, there is a bijection from $\mathcal{O}_n$ to $\mathcal{NE}$.

Proof. For any $\alpha \in \mathcal{O}_n$ has the following tabular form,

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}.$$

Let $i, j \in X_n$ such that $i < j$. Then, we have $c_i \leq c_j$ from the definition. Let h_i be the horizontal line such that $(i-1, c_i-1) - (i, c_i-1)$ for all $i \in X_n$. Moreover, there is a unique North-East lattice path in $\mathcal{NE}$, which has horizontal lines $h_1, h_2, \dots, h_n$. Lets call this path P_α . We define a function $f : \mathcal{O}_n \rightarrow \mathcal{NE}$ as follows: $\alpha f = P_\alpha$. It is clear that f is a one-to-one mapping. Let P be a North-East lattice path from $(0, 0)$ to $(n, n-1)$. There are horizontal lines $h'_1, h'_2, \dots, h'_n$. Then, we have $h'_i = (i-1, j_i) - (i, j_i)$ for all $i \in X_n$. Moreover, let

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1+1 & j_2+1 & \cdots & j_n+1 \end{pmatrix}.$$

It is clear that $\beta \in \mathcal{O}_n$ and $\beta f = P$. Thus f is onto. It follows that f is a bijection. $\square$

Example 1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 5 & 5 \end{pmatrix} \in \mathcal{O}_5$ and f be the function defined above. Then the horizontal lines are $h_1 : (0, 1) - (1, 1), h_2 : (1, 1) - (2, 1), h_3 : (2, 3) - (3, 3), h_4 : (3, 4) - (4, 4), h_5 : (4, 4) - (5, 4)$. Moreover, $\alpha f = (0, 0) - (0, 1) - (1, 1) - (2, 1) - (2, 2) - (2, 3) - (3, 3) - (3, 4) - (4, 4) - (5, 4)$ which is a North-East lattice path from $(0, 0)$ to $(5, 4)$.

3. Zero-Divisors of $\mathcal{O}_n(A)$

For $n \geq 3$, let $\mathcal{O}_n(A)^* = \mathcal{O}_n(A) \setminus \{\theta\}$, and we define the following sets:

$$\begin{aligned} L &= L(\mathcal{O}_n(A)) = \{\alpha \in \mathcal{O}_n(A) : \alpha\beta = \theta \text{ for some } \beta \in \mathcal{O}_n(A)^*\}, \\ R &= R(\mathcal{O}_n(A)) = \{\alpha \in \mathcal{O}_n(A) : \gamma\alpha = \theta \text{ for some } \gamma \in \mathcal{O}_n(A)^*\} \text{ and} \\ T &= T(\mathcal{O}_n(A)) = \{\alpha \in \mathcal{O}_n(A) : \alpha\beta = \theta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{O}_n(A)^*\} = L \cap R \end{aligned}$$

which are called the set of left, right and two sided zero-divisors of $\mathcal{O}_n(A)$, respectively. In this section, we find the left, right and two sided zero-divisors of $\mathcal{O}_n(A)$, and their cardinalities.

Proposition 2 ([12]). For any $\alpha, \beta \in \mathcal{T}_n$, $\alpha\beta = \theta$ if and only if $\text{im}(\alpha) \subseteq 1\beta^{-1}$. In particular, $\alpha^2 = \theta$ if and only if $\text{im}(\alpha) \subseteq 1\alpha^{-1}$.

We now define a specific mapping that is useful throughout this paper. Let

$$\pi = \begin{pmatrix} 1 & \cdots & n-1 & n \\ 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Theorem 1. For $n \geq 3$, $L = \{\alpha \in \mathcal{O}_n(A) : n\alpha < n\}$.

Proof. Let $n \geq 3$ and $\alpha \in \mathcal{O}_n(A)$. If $n\alpha = n$ and $\alpha\beta = \theta$ for any $\beta \in \mathcal{O}_n(A)$, then it is clear that $n\beta = 1$ and $\beta = \theta$. Let $n\alpha < n$. Then we have $\alpha\pi = \theta$. It follows that $L = \{\alpha \in \mathcal{O}_n(A) : n\alpha < n\}$. $\square$

Theorem 2. For $n \geq 3$, $R = \{\alpha \in \mathcal{O}_n(A) : 2\alpha = 1\}$.

Proof. Let $n \geq 3$ and $\alpha \in \mathcal{O}_n(A)$. Let $2\alpha = 1$ and $\beta = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & 2 \end{pmatrix}$. Then, it is clear that $\beta \in \mathcal{O}_n(A)$ and $\beta\alpha = \theta$. Suppose that $2\alpha = k \neq 1$ and $\gamma\alpha = \theta$ for any $\gamma \in \mathcal{O}_n(A)$. Then, we have $\text{im}(\gamma) \subseteq 1\alpha^{-1} = \{1\}$ from Proposition 2. It follows that $\gamma = \theta$. So, $R = \{\alpha \in \mathcal{O}_n(A) : 2\alpha = 1\}$. $\square$

Corollary 1. For $n \geq 3$, $T = \{\alpha \in \mathcal{O}_n(A) : 2\alpha = 1 \text{ and } n\alpha < n\}$.

Theorem 3.

(i) Let $A = \{1 = a_1 < a_2 < \cdots < a_k < n\}$ for $n \geq 4$ and $k \geq 3$. Then, we have

$$|L| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \cdots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-3+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

(ii) Let $A = \{1 = a_1 < a_2 < n\}$ for $n \geq 3$. Then, we have

$$|L| = \sum_{r_2=1}^{a_2} \binom{a_2-3+r_2}{r_2-1} \cdot \binom{2n-1-a_2-r_2}{n-1-r_2}.$$

(iii) Let $A = \{1\}$ for $n \geq 3$. Then, we have

$$|L| = \binom{2n-3}{n-2}.$$

Proof. (i) Let $A = \{1 = a_1 < a_2 < \cdots < a_k < n\}$ for $n \geq 4$ and $k \geq 3$. If $\alpha \in L$, then $a_i\alpha \leq a_i$ for all $1 \leq i \leq k$ and $n\alpha < n$. We can use Proposition 1 to calculate $|L|$. Therefore, the number of North-East lattice paths in the following is equal to $|L|$, and the path is

$$(0,0) - (1,0) - \cdots - (a_2-1, r_2-1) - (a_2, r_2-1) - \cdots - (a_3-1, r_3-1) - (a_3, r_3-1) - \cdots - (a_k-1, r_k-1) - (a_k, r_k-1) - \cdots - (n, n-2) - (n, n-1)$$

where $1 \leq r_2 \leq a_2, r_2 \leq r_3 \leq a_3, \dots, r_{k-1} \leq r_k \leq a_k$. It follows that

$$|L| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \cdots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-3+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

Similarly, (ii) and (iii) can be shown. $\square$

Theorem 4.

(i) Let $A = \{1 = a_1 < a_2 < \dots < a_k < n\}$ and $a_2 \neq 2$ for $n \geq 4, k \geq 3$. Then, we have

$$|R| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-4+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-r_k-a_k}{n-r_k}.$$

(ii) Let $A = \{1 = a_1 < a_2 < n\}$ and $a_2 \neq 2$ for $n \geq 3$. Then, we have

$$|R| = \sum_{r_2=1}^{a_2} \binom{a_2-4+r_2}{r_2-1} \cdot \binom{2n-r_2-a_2}{n-r_2}.$$

(iii) Let $A = \{1\}$ for $n \geq 3$. Then, we have

$$|R| = \binom{2n-3}{n-2}.$$

Proof. (i) Let $A = \{1 = a_1 < a_2 < \dots < a_k < n\}$ and $a_2 \neq 2$ for $n \geq 4, k \geq 3$. If $\alpha \in R$, then $a_i \alpha \leq a_i$ for all $1 \leq i \leq k$ and $2\alpha = 1$. We can use Proposition 1 to calculate $|R|$. Therefore, the number of North-East lattice paths in the following is equal to $|R|$, and the path is

$$(0,0) - (1,0) - (2,0) - \dots - (a_2-1, r_2-1) - (a_2, r_2-1) - \dots - (a_3-1, r_3-1) - (a_3, r_3-1) - \dots - (a_k-1, r_k-1) - (a_k, r_k-1) - \dots - (n, n-1)$$

where $1 \leq r_2 \leq a_2, r_2 \leq r_3 \leq a_3, \dots, r_{k-1} \leq r_k \leq a_k$. It follows that

$$|R| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-4+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-r_k-a_k}{n-r_k}.$$

Similarly, (ii) and (iii) can be shown. $\square$

Theorem 5.

(i) Let $A = \{1 = a_1 < 2 = a_2 < a_3 < \dots < a_k < n\}$ for $n \geq 5, k \geq 4$. Then, we have

$$|R| = \sum_{r_3=1}^{a_3} \sum_{r_4=r_3}^{a_4} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_3-4+r_3}{r_3-1} \cdot \left[\prod_{j=4}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-r_k-a_k}{n-r_k}.$$

(ii) Let $A = \{1 = a_1 < 2 = a_2 < a_3 < n\}$ for $n \geq 4$. Then, we have

$$|R| = \sum_{r_3=1}^{a_3} \binom{a_3-4+r_3}{r_3-1} \cdot \binom{2n-r_3-a_3}{n-r_3}.$$

(iii) Let $A = \{1 = a_1 < 2 = a_2\}$ for $n \geq 3$. Then, we have

$$|R| = \binom{2n-3}{n-2}.$$

Proof. (i) $A = \{1 = a_1 < 2 = a_2 < a_3 < \dots < a_k < n\}$ for $n \geq 5, k \geq 4$. If $\alpha \in R$, then $a_i \alpha \leq a_i$ for all $1 \leq i \leq k$ and $2\alpha = 1$. We can use Proposition 1 to calculate $|R|$. Therefore, the number of North-East lattice paths in the following is equal to $|R|$, and the path is

$$(0,0) - (1,0) - (2,0) - \dots - (a_3-1, r_3-1) - (a_3, r_3-1) - \dots - (a_4-1, r_4-1) - (a_4, r_4-1) - \dots - (a_k-1, r_k-1) - (a_k, r_k-1) - \dots - (n, n-1)$$

where $1 \leq r_3 \leq a_3, r_3 \leq r_4 \leq a_4, \dots, r_{k-1} \leq r_k \leq a_k$. It follows that

$$|R| = \sum_{r_3=1}^{a_3} \sum_{r_4=r_3}^{a_4} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_3-4+r_3}{r_3-1} \cdot \left[\prod_{j=4}^k \binom{r_j-r_{j-1}+a_j-1-a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-r_k-a_k}{n-r_k}.$$

(ii) and (iii) can be shown similarly. $\square$

Theorem 6.

(i) Let $A = \{1 = a_1 < a_2 < \dots < a_k < n\}$ for $n \geq 4, a_2 \neq 2$ and $k \geq 3$. Then, we have

$$|T| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-4+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1+a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

(ii) Let $A = \{1 = a_1 < a_2 < n\}$ and $a_2 \neq 2$ for $n \geq 3$. Then, we have

$$|T| = \sum_{r_2=1}^{a_2} \binom{a_2-4+r_2}{r_2-1} \cdot \binom{2n-1-a_2-r_2}{n-1-r_2}.$$

(iii) Let $A = \{1 = a_1\}$ for $n \geq 2$. Then, we have

$$|T| = \binom{2n-4}{n-2}.$$

Proof. (i) Let $A = \{1 = a_1 < a_2 < \dots < a_k < n\}$ for $n \geq 4, a_2 \neq 2$ and $k \geq 3$. If $\alpha \in T$, then $a_i \alpha \leq a_i$ for all $1 \leq i \leq k, 2\alpha = 1$ and $n\alpha < n$. We can use Proposition 1 to calculate $|T|$. Therefore, the number of North-East lattice paths in the following is equal to $|T|$, and the path is

$$(0,0) - (1,0) - (2,0) - \dots - (a_2-1, r_2-1) - (a_2, r_2-1) - \dots - (a_3-1, r_3-1) - (a_3, r_3-1) - \dots - (a_k-1, r_k-1) - (a_k, r_k-1) - \dots - (n, n-2) - (n, n-1)$$

where $1 \leq r_2 \leq a_2, r_2 \leq r_3 \leq a_3, \dots, r_{k-1} \leq r_k \leq a_k$. It follows that

$$|T| = \sum_{r_2=1}^{a_2} \sum_{r_3=r_2}^{a_3} \cdots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_2-4+r_2}{r_2-1} \cdot \left[\prod_{j=3}^k \binom{r_j-r_{j-1}+a_j-1+a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

Similarly, (ii) and (iii) can be shown. $\square$

Example 2. Let $A = \{1, 4\}$ and $n = 5$. Then, we have

$$|T| = \sum_{r_2=1}^4 \binom{r_2}{r_2-1} \cdot \binom{5-r_2}{4-r_2} = 20.$$

Moreover,

$$T = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}, \right. \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 3 & 4 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 4 \end{pmatrix}, \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 4 & 4 \end{pmatrix} \right\}.$$

Theorem 7.

(i) Let $A = \{1 = a_1 < 2 = a_2 < \cdots < a_k < n\}$ for $n \geq 5$ and $k \geq 4$. Then, we have

$$|T| = \sum_{r_3=1}^{a_3} \sum_{r_4=r_3}^{a_4} \cdots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_3-4+r_3}{r_3-1} \cdot \left[\prod_{j=4}^k \binom{r_j-r_{j-1}+a_j-1+a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

(ii) Let $A = \{1 = a_1 < 2 = a_2 < a_3 < n\}$ for $n \geq 4$. Then, we have

$$|T| = \sum_{r_3=1}^{a_3} \binom{a_3-4+r_3}{r_3-1} \binom{2n-1-a_3-r_3}{n-1-r_3}.$$

(iii) Let $A = \{1 = a_1 < 2 = a_2\}$ for $n \geq 3$. Then, we have

$$|T| = \binom{2n-4}{n-2}.$$

Proof. (i) Let $A = \{1 = a_1 < 2 = a_2 < \dots < a_k < n\}$ for $n \geq 5$ and $k \geq 4$. If $\alpha \in T$, then $a_i \alpha \leq a_i$ for all $1 \leq i \leq k$, $2\alpha = 1$ and $n\alpha < n$. We can use Proposition 1 to calculate $|T|$. Therefore, the number of North-East lattice paths in the following is equal to $|T|$, and the path is

$$(0, 0) - (1, 0) - (2, 0) - \dots - (a_3 - 1, r_3 - 1) - (a_3, r_3 - 1) - \dots - (a_4 - 1, r_4 - 1) \\ - (a_4, r_4 - 1) - \dots - (a_k - 1, r_k - 1) - (a_k, r_k - 1) - \dots - (n, n - 2) - (n, n - 1)$$

where $1 \leq r_3 \leq a_3, r_3 \leq r_4 \leq a_4, \dots, r_{k-1} \leq r_k \leq a_k$. It follows that

$$|T| = \sum_{r_3=1}^{a_3} \sum_{r_4=r_3}^{a_4} \dots \sum_{r_k=r_{k-1}}^{a_k} \binom{a_3-4+r_3}{r_3-1} \cdot \left[\prod_{j=4}^k \binom{r_j-r_{j-1}+a_j-1+a_{j-1}}{r_j-r_{j-1}} \right] \cdot \binom{2n-1-a_k-r_k}{n-1-r_k}.$$

Similarly, (ii) and (iii) can be shown. $\square$

4. Zero-Divisor Graph of $\mathcal{O}_n(A)$

Let G be a graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. If G does not contain any loops or multiple edges, then G is called a simple graph. In this section, we consider only simple undirected graphs. Let $u_1, u_2, \dots, u_n \in V(G)$. If $u_i - u_{i+1} \in E(G)$ for each $1 \leq i \leq n-1$, then $u_1 - u_2 - \dots - u_n$ is called a path between u_1 and u_n , moreover the length of this path is defined as $n-1$. For $u, v \in V(G)$, the length of the shortest path between u and v in G is denoted by $d_G(u, v)$. G is connected if there exists a path for all two distinct vertices in G ; otherwise, G is called disconnected. The eccentricity of a vertex $v \in V(G)$ is denoted by $\text{ecc}(v)$ and defined by

$$\text{ecc}(v) = \max\{d_G(u, v) : u \in V(G)\}.$$

The diameter of G is denoted by $\text{diam}(G)$ and defined by

$$\text{diam}(G) = \max\{\text{ecc}(v) : v \in V(G)\}.$$

For simple graph G and $v \in V(G)$, the cardinality of $\{u \in V(G) : u - v \in E(G)\}$ is called the degree of v , denoted by $\deg_G(v)$. The minimum degree of G is denoted by $\delta(G)$ and the maximum degree of G is denoted by $\Delta(G)$.

In a graph G , the length of a shortest cycle is called girth of G , denoted by $\text{gr}(G)$. Moreover, if G does not contain any cycles, then its girth is defined as infinity.

Let G be a graph and $\emptyset \neq D \subseteq V(G)$. D is called a dominating set for G if, for each $v \in V(G)$, either $v \in D$ or there exists $u \in D$ such that $u - v \in E(G)$. The domination number of G is denoted by $\gamma(G)$ and defined by

$$\gamma(G) = \min\{|D| : D \text{ is a dominating set for } G\}.$$

Let G be a simple graph. If any two vertices are adjacent in G , then G is called a complete graph. Moreover, let $\emptyset \neq C \subseteq V(G)$ and $G(C)$ be the subgraph of G whose

vertex set is C and whose edge set consists of all edges of G which have both ends in C . $G(C)$ is called subgraph of G induced by C . If $G(C)$ is a complete graph, then C is called a clique of G . The cardinality of vertices in any maximal clique of G is called clique number of G , denoted by $\omega(G)$.

Let G be a simple graph. If we colour all the vertices in G with the rule of no two adjacent vertices have the same colour, then the cardinality of colours needed to colour of G is called chromatic number of G , denoted by $\chi(G)$. If $\omega(H) = \chi(H)$ for every induced subgraph H of G , then G is called a perfect graph. Notice that $1 \in A$, $n \notin A$. Observe that $V(\Gamma(\mathcal{O}_3(A))) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \right\}$ and $E(\Gamma(\mathcal{O}_3(A))) = \emptyset$. Since $\Gamma(\mathcal{O}_3(A))$ is not connected, from now on, we consider $\Gamma(\mathcal{O}_n(A))$ for $n \geq 4$. Moreover, we use Γ instead of $\Gamma(\mathcal{O}_n(A))$ for convenience.

It is known that, if S is a commutative semigroup with zero element, then $\bar{\Gamma}(S)$ is connected graph and $\text{diam}(\bar{\Gamma}(S)) \leq 3$ [3]. Moreover, if S is a noncommutative semigroup with zero element, then $\Gamma(S)$ need not be connected [5] and $\text{diam}(\Gamma(S))$ can be greater than 3 [7]. In the following theorem, we prove that Γ is connected a graph and calculate $\text{diam}(\Gamma)$ for $n \geq 4$.

Theorem 8. Γ is connected graph and $\text{diam}(\Gamma) = 2$ for $n \geq 4$.

Proof. Let $n \geq 4$. Notice that $\pi \in V(\Gamma)$. Let $\alpha \in V(\Gamma) \setminus \{\pi\}$. Then, we have $2\alpha = 1$ and $n\alpha < n$. So, $\text{im}(\pi) = \{1, 2\} \subseteq 1\alpha^{-1}$ and $\text{im}(\alpha) \subseteq X_{n-1} = 1\pi^{-1}$. It follows that $\alpha - \pi \in E(\Gamma)$ from Proposition 2. Thus, Γ is a connected graph. Since every vertex without π is adjacent to π in Γ , it follows that $\text{diam}(\Gamma) \leq 2$. Moreover, let $\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & 3 \end{pmatrix}$. Then, it is clear that $\alpha_1, \alpha_2 \in \mathcal{O}_n(A)$, α_1 and α_2 are non-adjacent vertices in Γ . So, $\text{diam}(\Gamma) = 2$. $\square$

Since π is adjacent to all other vertices in Γ , we have the following corollary.

Corollary 2. $\gamma(\Gamma) = 1$ for $n \geq 4$.

In the following theorem, we find the girth of Γ .

Theorem 9. For $n \geq 4$,

$$\text{gr}(\Gamma) = \begin{cases} \infty & \text{if } n = 4 \\ 3 & \text{if } n \geq 5. \end{cases}$$

Proof. If $n = 4$, then we have $V(\Gamma) = \left\{ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} \right\}$ and in this case Γ does not have any cycle. Thus, if $n = 4$, then $\text{gr}(\Gamma) = \infty$. Let $n \geq 5$ and

$$\begin{aligned} \alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 2 & 2 & \cdots & 2 \end{pmatrix}, \\ \beta &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 1 & 1 & 2 & \cdots & 2 \end{pmatrix}. \end{aligned}$$

Then it is clear that $\pi, \alpha, \beta \in V(\Gamma)$ and $\pi - \alpha - \beta - \pi$ is a cycle in Γ . Thus, if $n \geq 5$, then $\text{gr}(\Gamma) = 3$. $\square$

In the following three theorems, we find the degree of any vertex in Γ for all the different cases.

Theorem 10. Let $n \geq 4$ and $\alpha \in V(\Gamma)$. α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ 1 & b_2 & \cdots & b_r \end{pmatrix}$ where $1 < b_2 < \cdots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover, let $Y = \{b_r + 1, \dots, n\}$ and $B_1 \cap A \cap Y = \emptyset$. Then we have

$$\deg_{\Gamma}(\alpha) = \begin{cases} \binom{n - b_r + t - 1}{t - 1} - 1 & \text{if } \alpha^2 \neq \theta \\ \binom{n - b_r + t - 1}{t - 1} - 2 & \text{if } \alpha^2 = \theta. \end{cases}$$

Proof. Let $n \geq 4$ and $\alpha \in V(\Gamma)$. α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ 1 & b_2 & \cdots & b_r \end{pmatrix}$ where $1 < b_2 < \cdots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover, let $Y = \{b_r + 1, \dots, n\}$, $B_1 \cap A \cap Y = \emptyset$ and $X = \{\beta \in V(\Gamma) : \alpha - \beta \in E(\Gamma)\}$. It is clear that $|X| = \deg_{\Gamma}(\alpha)$. Let $\beta \in \mathcal{O}_n \setminus \{\theta\}$ and $\alpha\beta = \theta = \beta\alpha$. Then, we have $\text{im}(\beta) \subseteq B_1$ and $b_r(\beta) = 1$. Thus, we have $a\beta = 1$ for all $1 \leq a \leq b_r$. Assume that $x \in X_n$ such that $x \in A$ and $x(\beta) > x$. It follows that $t \geq x(\beta) > x$. Thus, $x > b_r$ and $x \in B_1 \cap A \cap Y$ which contradicts $B_1 \cap A \cap Y = \emptyset$. Therefore, we have $x(\beta) \leq x$ which means $\beta \in V(\Gamma)$. If $\alpha^2 \neq \theta$, then $\beta \neq \alpha$. Moreover, if $\beta \in X$, then we have $\beta \neq \theta$. We can use Proposition 1 to find $\deg_{\Gamma}(\alpha)$. In addition, the number of following North-East lattice paths is equal to $\deg_{\Gamma}(\alpha) + 1$, and one of these paths is θf where f is the function defined in the proof of Proposition 1, and the path is

$$(0, 0) - (1, 0) - \cdots - (b_r, 0) - \cdots - (n, t - 1) - (n, t) - \cdots - (n, n - 1).$$

So, we have $\deg_{\Gamma}(\alpha) = \binom{n - b_r + t - 1}{t - 1} - 1$. Similarly, if $\alpha^2 = \theta$, then it is easy to see that $\deg_{\Gamma}(\alpha) = \binom{n - b_r + t - 1}{t - 1} - 2$. $\square$

Example 3. Let $A = \{1, 3\}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 \end{pmatrix} \in \mathcal{O}_8(A)$. Then, we have $B_1 = \{1, 2, 3, 4\}$, $Y = \{4, 5, 6, 7, 8\}$ and $B_1 \cap A \cap Y = \emptyset$. Moreover $\alpha^2 = \theta$. It follows that $\deg_{\Gamma}(\alpha) = \binom{8 - 3 + 4 - 1}{4 - 1} - 2 = 54$. Notice that, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \in \mathcal{O}_8(A)$ and it is adjacent to α in Γ , however $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \notin C_8$.

Let $n \geq 4$ and $\alpha \in V(\Gamma)$. Then, α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ 1 & b_2 & \cdots & b_r \end{pmatrix}$ where $1 < b_2 < \cdots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover, let

$Y = \{b_r + 1, \dots, n\}$, $B_1 \cap A \cap Y = \{a_p, a_{p+1}, \dots, a_s\}$ and $s > p$. In this case we define an integer z such that

$$z = \sum_{r_1=1}^{a_p} \sum_{r_2=r_1}^{a_{p+1}} \dots \sum_{r_{s-p+1}=r_{s-p}}^{a_s} \binom{a_p - b_r + r_1 - 2}{r_1 - 1} \cdot \left[\prod_{k=1}^{s-p} \binom{a_{k+p} - 1 - a_{k+p-1} + r_{k+1} - r_k}{r_{k+1} - r_k} \right] \cdot \binom{n - a_s + t - r_{s-p+1}}{t - r_{s-p+1}}.$$

Theorem 11. Let $n \geq 4$ and $\alpha \in V(\Gamma)$. Then, α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \dots & B_r \\ 1 & b_2 & \dots & b_r \end{pmatrix}$ where $1 < b_2 < \dots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover let $Y = \{b_r + 1, \dots, n\}$, $B_1 \cap A \cap Y = \{a_p, a_{p+1}, \dots, a_s\}$ and $s > p$. Let z be an integer defined as above. Then we have

$$\deg_{\Gamma}(\alpha) = \begin{cases} z - 1 & \text{if } \alpha^2 \neq \theta \\ z - 2 & \text{if } \alpha^2 = \theta. \end{cases}$$

Proof. Let $n \geq 4$ and $\alpha \in V(\Gamma)$. Then, α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \dots & B_r \\ 1 & b_2 & \dots & b_r \end{pmatrix}$ where $1 < b_2 < \dots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover let $Y = \{b_r + 1, \dots, n\}$, $B_1 \cap A \cap Y = \{a_p, a_{p+1}, \dots, a_s\}$ and $s > p$. Let z be an integer defined as above. Let $X = \{\beta \in V(\Gamma) : \alpha - \beta \in E(\Gamma)\}$. It is clear that $|X| = \deg_{\Gamma}(\alpha)$. Moreover, let $\beta \in X$. So we have $\text{im}(\beta) \subseteq B_1$, $b_r(\beta) = 1$ and $x(\beta) \leq x$ for all $x \in \{a_p, a_{p+1}, \dots, a_s\}$. If $\alpha^2 \neq \theta$, then we have $\beta \neq \alpha$. Moreover, we have $\beta \neq \theta$ since $\beta \in X$. We can use Proposition 1 to find $\deg_{\Gamma}(\alpha)$. The number of following North-East lattice paths is equal to $\deg_{\Gamma}(\alpha) + 1$, and one of these paths is θf where f is the function defined in the proof of Proposition 1, and the path is

$$(0, 0) - (1, 0) - \dots - (b_r, 0) - \dots - (a_p - 1, r_1 - 1) - (a_p, r_1 - 1) - \dots - (a_{p+1} - 1, r_2 - 1) - (a_{p+1}, r_2 - 1) - \dots - (a_s - 1, r_{s-p+1} - 1) - (a_s, r_{s-p+1} - 1) - \dots - (n, t - 1) - (n, t) - \dots - (n, n - 1)$$

where $1 \leq r_1 \leq a_p$, $r_1 \leq r_2 \leq a_{p+1}, \dots, r_{s-p} \leq r_{s-p+1} \leq a_s$. So, we have $\deg_{\Gamma}(\alpha) = z - 1$. Similarly, if $\alpha^2 = \theta$, then we have $\deg_{\Gamma}(\alpha) = z - 2$. $\square$

The following theorem can be proved in a similar manner.

Theorem 12. Let $n \geq 4$ and $\alpha \in V(\Gamma)$. Then, α has the following tabular form $\alpha = \begin{pmatrix} B_1 & B_2 & \dots & B_r \\ 1 & b_2 & \dots & b_r \end{pmatrix}$ where $1 < b_2 < \dots < b_r$ and $B_1 = \{1, 2, \dots, t\}$. Moreover let $Y = \{b_r + 1, \dots, n\}$, $B_1 \cap A \cap Y = \{a_p\}$. Then we have

$$\deg_{\Gamma}(\alpha) = \begin{cases} \left[\sum_{r_1=1}^{a_p} \binom{a_p - b_r + r_1 - 2}{r_1 - 1} \cdot \binom{n - a_p + t - r_1}{t - r_1} \right] - 1 & \text{if } \alpha^2 \neq \theta \\ \left[\sum_{r_1=1}^{a_p} \binom{a_p - b_r + r_1 - 2}{r_1 - 1} \cdot \binom{n - a_p + t - r_1}{t - r_1} \right] - 2 & \text{if } \alpha^2 = \theta. \end{cases}$$

Example 4. Let $A = \{1, 4\}$ and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \end{pmatrix} \in \mathcal{O}_9(A)$. Then, we have $B_1 = \{1, 2, 3, 4, 5, 6\}$, $Y = \{4, 5, 6, 7, 8, 9\}$ and $B_1 \cap A \cap Y = \{4\}$. Moreover $\alpha^2 = \theta$. It follows that $\deg_\Gamma(\alpha) = \left[\sum_{r_1=1}^4 \binom{4-3+r_1-2}{r_1-1} \cdot \binom{9-4+6-r_1}{6-r_1} \right] - 2 = 453$.

Let $\varepsilon = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}$. Then it is clear that $\varepsilon \in V(\Gamma)$ and $\deg_\Gamma(\varepsilon) = 1$ and moreover since π is adjacent to all other vertices in Γ . Therefore, we have the following corollary.

Corollary 3. For $n \geq 4$, $\Delta(\Gamma) = |T| - 2$ and $\delta(\Gamma) = 1$.

A chordal graph is a simple graph, it does not contain an induced cycle of length 4 or more. Moreover, every chordal graph is a perfect graph [11]. The following theorem can be proved using the same technique as Theorem 4.8 of [6].

Theorem 13. Γ is chordal for $n \geq 4$. Therefore, Γ is a perfect graph for $n \geq 4$ and $\chi(\Gamma) = \omega(\Gamma)$.

Now, our aim is to find $\omega(\Gamma)$ and so $\chi(\Gamma)$. For $n \geq 4$ and $\alpha \in V(\Gamma)$, let $k_\alpha, t_\alpha \in X_{n-1}$ such that $k_\alpha \alpha = 1$ and $(k_\alpha + 1)\alpha \neq 1$ and $n\alpha = t_\alpha$. Moreover, let $X(n, k, t) = \{\alpha \in \mathcal{O}_n(A) : k_\alpha = k \text{ and } t_\alpha = t\}$. If we use Proposition 1, then we can calculate $|X(n, k, t)|$.

Case 1: Let $A \cap \{k+1, \dots, n\} \cap \{1, 2, \dots, t\} = \{a_p, a_{p+1}, \dots, a_s\}$ and $s > p$. The number of following North-East lattice paths is equal to $|X(n, k, t)|$, and the path is

$$\begin{aligned} & (0, 0) - (1, 0) - \cdots - (k, 0) - (k, 1) - \cdots - (a_p - 1, r_1 - 1) - (a_p, r_1 - 1) - \cdots \\ & - (a_{p+1} - 1, r_2 - 1) - (a_{p+1}, r_2 - 1) - \cdots - (a_s - 1, r_{s-p+1} - 1) - (a_s, r_{s-p+1} - 1) \\ & - \cdots - (n - 1, t - 1) - (n, t - 1) - (n, t) - \cdots - (n, n - 1) \end{aligned}$$

where $2 \leq r_1 \leq a_p, r_1 \leq r_2 \leq a_{p+1}, \dots, r_{s-p} \leq r_{s-p+1} \leq a_s$. Thus, we have

$$\begin{aligned} |X(n, k, t)| &= \sum_{r_1=2}^{a_p} \sum_{r_2=r_1}^{a_{p+1}} \cdots \sum_{r_{s-p+1}=r_{s-p}}^{a_s} \binom{a_p - k + r_1 - 3}{r_1 - 2} \cdot \\ & \left[\prod_{j=1}^{s-p} \binom{a_{j+p} - 1 - a_{j+p-1} + r_{j+1} - r_j}{r_{j+1} - r_j} \right] \cdot \binom{n - a_s + t - r_{s-p+1} - 1}{t - r_{s-p+1}}. \end{aligned}$$

Case 2: Let $A \cap \{k+1, \dots, n\} \cap \{1, 2, \dots, t\} = \{a_p\}$. The number of following North-East lattice paths is equal to $|X(n, k, t)|$, and the path is

$$\begin{aligned} & (0, 0) - (1, 0) - \cdots - (k, 0) - (k, 1) - \cdots - (a_p - 1, r_1 - 1) - (a_p, r_1 - 1) - \cdots \\ & - \cdots - (n - 1, t - 1) - (n, t - 1) - (n, t) - \cdots - (n, n - 1) \end{aligned}$$

where $2 \leq r_1 \leq a_p$. Thus, we have

$$|X(n, k, t)| = \sum_{r_1=2}^{a_p} \binom{a_p - k + r_1 - 3}{r_1 - 2} \cdot \binom{n - a_p + t - r_1 - 1}{t - r_1}.$$

Case 3: Let $A \cap \{k+1, \dots, n\} \cap \{1, 2, \dots, t\} = \emptyset$. The number of following North-East lattice paths is equal to $|X(n, k, t)|$, and the path is

$$(0, 0) - (1, 0) - \dots - (k, 0) - (k, 1) - \dots - (n-1, t-1) - (n, t-1) - (n, t) - \dots - (n, n-1).$$

Thus, we have

$$|X(n, k, t)| = \binom{n-k+t-3}{t-2}.$$

Example 5. Let $A = \{1, 3, 4\}$ and $n = 8$. Then

$$X(8, 2, 7) = \{\alpha \in \mathcal{O}_8(A) : 2\alpha = 1, 3\alpha \geq 2 \text{ and } 8\alpha = 7\}$$

and we have $A \cap \{3, 4, 5, 6, 7, 8\} \cap \{1, 2, 3, 4, 5, 6, 7\} = \{3, 4\}$. So,

$$\begin{aligned} |X(8, 2, 7)| &= \sum_{r_1=2}^3 \sum_{r_2=r_1}^4 \binom{3-2+r_1-3}{r_1-2} \cdot \left[\prod_{j=1}^1 \binom{4-1-3+r_{j+1}-r_j}{r_{j+1}-r_j} \right] \\ &\quad \binom{8-4+7-r_2-1}{7-r_2} \\ &= 166. \end{aligned}$$

The following theorem can be proved using the same technique as Theorem 4.12 of [6].

Theorem 14. For $n \geq 4$, $\chi(\Gamma) = \omega(\Gamma) = \max\left\{\sum_{k=i}^{n-1} \sum_{t=2}^i |X(n, k, t)| : 2 \leq i \leq n-1\right\}$.

Example 6. Let $A = \{1, 3\}$ and $\Gamma = \Gamma(\mathcal{O}_7(A))$. Then, we have $|T| = 231$. Γ is a perfect graph and $|V(\Gamma)| = 230$, $\text{diam}(\Gamma) = 2$, $\text{gr}(\Gamma) = 3$, $\Delta(\Gamma) = 229$, $\delta(\Gamma) = 1$, $\gamma(\Gamma) = 1$, $\omega(\Gamma) = \chi(\Gamma) = 19$.

5. Conclusions

In this paper, the zero-divisor graphs of $\Gamma(\mathcal{O}_n(A))$ has been investigated for $n \geq 4$ where $1 \in A$ and $n \notin A$. We give a proposition about lattice paths and $\mathcal{O}_n$. Moreover, this proposition is a significant tool for understanding $\mathcal{O}_n$, which simplifies the calculations of graph invariants of $\Gamma(\mathcal{O}_n(A))$. We find the structure of the left, right, and two-sided zero divisors of $\mathcal{O}_n(A)$ and their cardinalities. We calculate some graph invariants such as the diameter, domination number, girth, minimum degree, and maximum degree of this graph. Moreover, we prove that $\Gamma(\mathcal{O}_n(A))$ is a chordal graph, and hence it is a perfect graph. Finally, we find the clique number of $\Gamma(\mathcal{O}_n(A))$ and so chromatic number of $\Gamma(\mathcal{O}_n(A))$. These properties can help us understand not only the structure of zero-divisor graphs but also the characteristics of transformation semigroups. The topic of zero-divisor graphs establishes a link between algebra and graph theory. One may also characterize undirected graph $\Gamma_3(\mathcal{O}_n(A))$ and the directed graph $\Gamma_4(\mathcal{O}_n(A))$.

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