

Article

Nonlinear Mixed bi-Skew Jordan-Type and Skew Jordan Higher Derivations on $\ast$ -Algebras

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Abstract

Let $\mathcal{A}$ be a unital $\ast$ -algebra over the complex field $\mathbb{C}$, and let $\Psi = \{\psi_m\}_{m \in \mathbb{N}}$ be a nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation satisfies the relation $\psi_m(L_1 \diamond L_2 \diamond \cdots \diamond L_{n-1} \bullet L_n) = \sum_{r_1 + \cdots + r_n = m} \psi_{r_1}(L_1) \diamond \cdots \diamond \psi_{r_{n-1}}(L_{n-1}) \bullet \psi_{r_n}(L_n)$, where $L \bullet N = LN + NL^*$ and $L \diamond N = L^*N + N^*L$ for all $L, N, L_i \in \mathcal{A}$ with $i \in \{1, 2, \dots, n\}$. We demonstrate that every such higher derivation $\Psi = \{\psi_m\}_{m \in \mathbb{N}}$ is an additive higher $\ast$ -derivation. As an application, we use this result to characterize the structure of nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations on a class of typical unital $\ast$ -algebras, including standard operator algebras and von Neumann factors. This result also generalizes several existing results, in particular those concerning nonlinear mixed bi-skew Jordan-type and skew Jordan derivations on unital $\ast$ -algebras.

Keywords: nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations; additive higher $\ast$ -derivation; unital $\ast$ -algebras

MSC: 16W25; 46L10

1. Introduction

In the study of operator theory and algebraic structures, derivations and their various generalizations play an indispensable role. Classical derivations reveal the intrinsic structure of algebraic operations, and this concept has been successfully extended from associative algebras to non-associative algebras (e.g., Jordan algebras and Lie algebras), giving rise to research directions such as Jordan derivations and Lie derivations. In recent years, the study of the structure of (bi-)skew Jordan derivations determined by the involution-characterized (bi-)skew Jordan product on unital $\ast$ -algebras has attracted considerable attention. This paper focuses on the structure of nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations on $\ast$ -algebras, which combine bi-skew Jordan-type and skew Jordan derivations. Let $\mathcal{A}$ be a unital $\ast$ -algebra, where the involution $\ast$ satisfies $(LN)^* = N^*L^*$ and $(L^*)^* = L$ for all $L, N \in \mathcal{A}$. Define the following products for all $L, N \in \mathcal{A}$: the Jordan product $L \circ N = LN + NL$; the skew Jordan product $L \bullet N = LN + NL^*$; and the bi-skew Jordan product $L \diamond N = L^*N + N^*L$. A map $\psi_1 : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is called:

(a) A nonlinear $\ast$ -derivation if, for all $L, N \in \mathcal{A}$,

$$\psi_1(LN) = \psi_1(L)N + L\psi_1(N) \text{ and } \psi_1(L^*) = \psi_1(L)^*;$$



Academic Editor: Rutwig Campoamor-Stursberg

Received: 5 January 2026

Revised: 11 February 2026

Accepted: 12 February 2026

Published: 13 February 2026

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(b) A nonlinear (bi-)skew Jordan n -derivation if, for all $L_1, \dots, L_n \in \mathcal{A}$,

$$\psi_1(L_1 \bullet \dots \bullet L_n) = \sum_{k=1}^n L_1 \bullet \dots \bullet \psi_1(L_k) \bullet \dots \bullet L_n;$$

$$(\text{resp. } \psi_1(L_1 \diamond \dots \diamond L_n) = \sum_{k=1}^n L_1 \diamond \dots \diamond \psi_1(L_k) \diamond \dots \diamond L_n).$$

In 2023, Zhao and colleagues [1] introduced the structure of nonlinear bi-skew Jordan-type derivations on $\ast$ -algebras and proved that they are additive $\ast$ -derivations, thereby generalizing the results in [2]. Additionally, Ashraf and colleagues [3] studied such mappings on von Neumann algebras. Furthermore, many mappings related to (bi-)skew Jordan derivations have been investigated (see [1,2,4–8]).

A common feature of the aforementioned studies is that they deal with nonlinear mappings defined solely by either the bi-skew Jordan product or the skew Jordan product. Recently, researchers have gradually shifted their focus to the study of the structure of nonlinear mappings determined by a mixture of the bi-skew Jordan product and the skew Jordan product. In 2024, Bhat [9] investigated the structure of nonlinear mixed bi-skew Jordan and skew Jordan derivations satisfying relation (d) for $m = 1$ and proved that they are additive $\ast$ -derivations. This result extends the authors' previous characterization of the structure of nonlinear mixed bi-skew and skew Jordan triple derivations [10] (Theorem 2.1). Recall that a unital $\ast$ -algebra is an associative algebra $\mathcal{A}$ with an involution operation $\ast$. In recent years, research on Jordan-type and multiplicative mappings has expanded considerably to broader algebraic frameworks, particularly to non-associative and alternative algebras (see [11–15]). Significant contributions in this area include Ferreira's work on the additivity of n -multiplicative maps over alternative rings [11] and the analysis of nonlinear mixed $\ast$ -Jordan-type derivations [13] and Jordan triple derivations [14]. Further studies on related types of mappings can be found in [11,15].

Higher derivations are a natural generalization of derivations in the study of functional identities. Consequently, the structure of higher derivations defined by a single product on unital $\ast$ -algebras has attracted considerable attention. Rehman and colleagues [16] studied the structure of bi-skew Jordan triple higher derivations on unital $\ast$ -algebras. Around the same time, Liang and colleagues [17] investigated the structure of bi-skew Jordan-type higher derivations on unital $\ast$ -algebras, generalizing the findings of Rehman et al. [16]. Subsequently, Ren and colleagues [18] introduced the concept of mixed higher derivations. They studied the structure of mixed Jordan and bi-skew Jordan higher derivations and established that they are additive $\ast$ -higher derivations. Building on this and following the above concept of nonlinear higher derivations, we similarly define the notion of nonlinear mixed bi-skew and skew Jordan higher derivations. Let $\mathcal{N}$ denote the set of non-negative integers, and consider a family $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ of mappings $\psi_m : \mathcal{A} \rightarrow \mathcal{A}$ such that $\psi_0 = \text{id}_{\mathcal{A}}$. The family Ψ is said to be:

(c) An **additive higher $\ast$ -derivation** if, for all $L, N \in \mathcal{A}$ and every $m \in \mathcal{N}$,

$$\psi_m(LN) = \sum_{i+j=m} \psi_i(L)\psi_j(N), \quad \psi_m(L+N) = \psi_m(L) + \psi_m(N) \quad \text{and} \quad \psi_m(L^\ast) = \psi_m(N)^\ast; \quad (1)$$

(d) A **nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation** if, for all $L_1, \dots, L_n \in \mathcal{A}$ and each $m \in \mathcal{N}$ with $n \geq 3$,

$$\psi_m(L_1 \diamond L_2 \diamond \dots \diamond L_{n-1} \bullet L_n) = \sum_{r_1 + \dots + r_n = m} \psi_{r_1}(L_1) \diamond \dots \diamond \psi_{r_{n-1}}(L_{n-1}) \bullet \psi_{r_n}(L_n). \quad (2)$$

Note that $n \geq 3$ is imposed because the mixed product contains both bi-skew and skew products, and since each must appear at least once, at least three elements are needed in the corresponding setting. Clearly, when $m = 1$, definition (d) reduces to a nonlinear mixed bi-skew Jordan and skew Jordan derivation. Furthermore, when $n = 3$, it reduces to the nonlinear mixed $*$ -Jordan-type derivation studied by Siddeeqe et al. In cohomology theory, higher derivations are closely related to higher cocycles and deformation cohomology, providing concrete tools for computing and understanding the cohomology groups of algebras. Therefore, many researchers have studied the structure of various types of higher derivations on different algebras, obtaining numerous interesting results [17,19–24].

Inspired by [9,17,19], a natural question arises: Under what conditions is a nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation on a unital $*$ -algebra an additive higher $*$ -derivation? This paper provides an affirmative answer to this question. For $m = 1$, expression (2) simplifies to the nonlinear mixed bi-skew Jordan-type and skew Jordan derivation investigated by Bhat and colleagues [9]. Consequently, the principal result established here can be viewed as a higher-order extension of their result, and it additionally characterizes the structure of nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations on various typical operator algebras, such as standard operator algebras and von Neumann factors.

2. Main Results

In this section, we present the main results of this paper, namely the structure of nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations. We begin by reviewing some fundamental concepts related to unital $*$ -algebras $\mathcal{A}$.

Let $\mathcal{A}$ be a unital $*$ -algebra over the complex field $\mathbb{C}$, equipped with two nontrivial idempotents P_1 and P_2 such that $P_1 + P_2 = I$, where I denotes the identity. According to the Peirce decomposition, $\mathcal{A}$ admits the direct sum decomposition

$$\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22},$$

where $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ with $i, j \in \{1, 2\}$. The multiplicative structure among these components satisfies the following relations:

1. $\mathcal{A}_{ij} \mathcal{A}_{jl} \subseteq \mathcal{A}_{il}$ ($i, j, l = 1, 2$);
2. $\mathcal{A}_{ij} \mathcal{A}_{kl} = 0$ if $j \neq k$ ($k = 1, 2$);
3. $(\mathcal{A}_{ij})^* \subseteq \mathcal{A}_{ji}$.

In what follows, we assume that the algebra $\mathcal{A}$ satisfies

$$L \mathcal{A} P_i = 0 \text{ implies } L = 0 \text{ for } i \in \{1, 2\}, \quad (3)$$

where $L \in \mathcal{A}$. By virtue of the properties shown in [17], prime $*$ -algebras, standard operator algebras, and von Neumann factors all satisfy Assumption (3).

To state the main result precisely, we present the following theorem.

Theorem 1. *Let $\mathcal{A}$ be a unital $*$ -algebra with identity element I that satisfies condition (3). Suppose that a map $\psi_m : \mathcal{A} \rightarrow \mathcal{A}$ satisfies the condition*

$$\psi_m(L_1 \diamond L_2 \diamond \cdots \diamond L_{n-1} \bullet L_n) = \sum_{r_1 + \cdots + r_n = m} \psi_{r_1}(L_1) \diamond \cdots \diamond \psi_{r_{n-1}}(L_{n-1}) \bullet \psi_{r_n}(L_n)$$

for all $L_1, \dots, L_n \in \mathcal{A}$ and each $m \in \mathbb{N}$, where $n \geq 3$. Then $\Psi = \{\psi_m\}_{m \in \mathbb{N}}$ is additive. Moreover, if $\{\psi_m(\alpha I)\}_{m \in \mathbb{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, then $\Psi = \{\psi_m\}_{m \in \mathbb{N}}$ is an additive higher $*$ -derivation on $\mathcal{A}$.

The proof of Theorem 1 is divided into two steps. First, using mathematical induction, we show that the nonlinear higher derivation is additive (see Lemmas 1–7). Then, based on additivity and under the additional assumption that the set of elements $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, we prove that it is indeed a higher derivation (see Lemmas 8–11). Finally, combining these two steps, we ultimately prove that, under our assumptions, every nonlinear derivation is an additive $*$ -derivation.

Proof. Note that any $L \in \mathcal{A}$ can be written as $L = L_{11} + L_{12} + L_{21} + L_{22}$, where $L_{ij} \in \mathcal{A}_{ij}$ for $i, j = \{1, 2\}$. To show the additivity of ψ_m on $\mathcal{A}$, we use the inductive hypothesis and first establish several claims showing that ψ_m is additive on $\mathcal{A}_{ij}$ for $i, j = \{1, 2\}$.

Proceeding by induction on $m \in \mathcal{N}$ using (2), we first consider the base case $m = 1$. In this case, ψ_1 is a nonlinear mixed $*$ -Jordan n -derivation on $\mathcal{A}$. According to [9] (Theorem 2.1), any such map is an additive $*$ -derivation on $\mathcal{A}$ and satisfies

$$\mathbb{H}_1 = \begin{cases} \psi_1(0) = 0; \psi_1(L + N) = \psi_1(L) + \psi_1(N) \text{ for all } L, N \in \mathcal{A}; \\ \psi_1(I) = 0; \psi_1(L)^* = \psi_1(L^*); \psi_1(iI) = 0; \psi_1(-iI) = 0; \\ \psi_1(-iL) = -i\psi_1(L); \psi_1(iL) = i\psi_1(L); \\ \psi_1(LN) = \psi_1(L)N + L\psi_1(N) \text{ for all } L, N \in \mathcal{A}. \end{cases}$$

Assume inductively that for some integer s with $1 < s < m$, the statement of Theorem 1 holds; that is, the finite sequence $\{\psi_s\}_{s=0}^{m-1}$ satisfies the following properties:

$$\mathbb{H}_s = \begin{cases} \psi_s(0) = 0; \psi_s(L + N) = \psi_s(L) + \psi_s(N) \text{ for all } L, N \in \mathcal{A}; \\ \psi_s(I) = 0; \psi_s(L)^* = \psi_s(L^*); \psi_s(iI) = 0; \psi_s(-iI) = 0; \\ \psi_s(-iL) = -i\psi_s(L); \psi_s(iL) = i\psi_s(L); \\ \psi_s(LN) = \psi_s(L)N + L\psi_s(N) \text{ for all } L, N \in \mathcal{A}. \end{cases}$$

The goal of this section is to prove that the family $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ satisfies condition $\mathbb{H}_m$ (i.e., the case $s = m$). Once this is established, it follows that Ψ is an additive higher $*$ -derivation. To simplify notation, for fixed n , we define $V_n(a, \dots, a, x_{n-1}, x_n) = a \diamond a \diamond \dots \diamond a \diamond x_{n-1} \bullet x_n$ by $\xi_a(x_{n-1}, x_n)$. The inductive argument will be carried out via a sequence of lemmas. $\square$

Lemma 1. $\psi_m(0) = 0$ for each $m \in \mathcal{N}$.

Proof. By the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$) that $\psi_s(0) = 0$, we have

$$\begin{aligned} \psi_m(0) &= \psi_m(\xi_I(0, 0)) \\ &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(I), \psi_{r_{n-1}}(0), \psi_{r_n}(0)) \\ &= V_n(\psi_m(I), I, \dots, I, 0, 0) + V_n(I, \psi_m(I), \dots, I, 0, 0) + \dots + V_n(I, I, \dots, I, 0, \psi_m(0)) \\ &+ \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1, r_2, \dots, r_n < m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(I), \psi_{r_{n-1}}(0), \psi_{r_n}(0)) \\ &= 0. \end{aligned}$$

$\square$

Lemma 2. Let $L_{12} \in \mathcal{A}_{12}$, $L_{21} \in \mathcal{A}_{21}$, and $m \in \mathcal{N}$. Then

$$\psi_m(L_{12} + L_{21}) = \psi_m(L_{12}) + \psi_m(L_{21}).$$

Proof. Let $K = \psi_m(L_{12} + L_{21}) - \psi_m(L_{12}) - \psi_m(L_{21})$. By the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$) that $\psi_s(L_{12} + L_{21}) = \psi_s(L_{12}) + \psi_s(L_{21})$ and $\xi_I(\frac{P_2 - P_1}{2^{n-2}}, L_{12}) = \xi_I(\frac{P_2 - P_1}{2^{n-2}}, L_{21}) = 0$, we have

$$\begin{aligned} \psi_m(\xi_I(\frac{P_2 - P_1}{2^{n-2}}, (L_{12} + L_{21}))) &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), \psi_{r_n}(L_{12} + L_{21})) \\ &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), \psi_{r_n}(L_{12})) \\ &\quad + \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), \psi_{r_n}(L_{21})) \\ &= \sum_{\substack{r_1 + r_2 + \dots + r_n = m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), (\psi_{r_n}(L_{12}) + \psi_{r_n}(L_{21}))) \\ &\quad + \xi_I(\frac{P_2 - P_1}{2^{n-2}}, (\psi_m(L_{12}) + \psi_m(L_{21}))). \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi_m(\xi_I(\frac{P_2 - P_1}{2^{n-2}}, (L_{12} + L_{21}))) &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), \psi_{r_n}(L_{12} + L_{21})) \\ &= \sum_{\substack{r_1 + r_2 + \dots + r_n = m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2 - P_1}{2^{n-2}}), \psi_{r_n}(L_{12} + L_{21})) \\ &\quad + \xi_I(\frac{P_2 - P_1}{2^{n-2}}, \psi_m(L_{12} + L_{21})). \end{aligned}$$

It follows from the above two equations that $\xi_I(\frac{P_2 - P_1}{2^{n-2}}, K) = 0$, which implies that $P_2K - P_1K + KP_2 - KP_1 = 0$. Multiplying both sides by P_1 , we get $K_{11} = 0$. Similarly, $K_{22} = 0$. It follows from $I \diamond \dots \diamond P_1 \diamond L_{21} \bullet P_2 = 0$ that

$$\begin{aligned} &\psi_m(I \diamond \dots \diamond P_1 \diamond (L_{12} + L_{21}) \bullet P_2) \\ &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), \psi_{r_{n-1}}(L_{12} + L_{21}), \psi_{r_n}(P_2)) \\ &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), \psi_{r_{n-1}}(L_{12}), \psi_{r_n}(P_2)) \\ &\quad + \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), \psi_{r_{n-1}}(L_{21}), \psi_{r_n}(P_2)) \\ &= \sum_{\substack{r_1 + r_2 + \dots + r_n = m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), (\psi_{r_{n-1}}(L_{12}) + \psi_{r_{n-1}}(L_{21})), \psi_{r_n}(P_2)) \\ &\quad + I \diamond \dots \diamond P_1 \diamond (\psi_m(L_{12}) + \psi_m(L_{21})) \bullet P_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\psi_m(I \diamond \dots \diamond P_1 \diamond (L_{12} + L_{21}) \bullet P_2) \\ &= \sum_{r_1 + r_2 + \dots + r_n = m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), \psi_{r_{n-1}}(L_{12} + L_{21}), \psi_{r_n}(P_2)) \\ &= \sum_{\substack{r_1 + r_2 + \dots + r_n = m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-2}}(P_1), \psi_{r_{n-1}}(L_{12} + L_{21}), \psi_{r_n}(P_2)) \\ &\quad + I \diamond \dots \diamond P_1 \diamond \psi_m(L_{12} + L_{21}) \bullet P_2. \end{aligned}$$

From the last two equations, we have

$$I \diamond \dots \diamond P_1 \diamond K \bullet P_2 = 0.$$

From the above equation, we can derive

$$P_1 K P_2 + P_2 K^* P_1 = 0.$$

Multiplying the equation from the left by P_1 yields $K_{12} = 0$. Similarly, we can show that $K_{21} = 0$. Hence, $K = 0$. This completes the proof. $\square$

Lemma 3. For any $L_{11} \in \mathcal{A}_{11}$, $L_{12} \in \mathcal{A}_{12}$, $L_{21} \in \mathcal{A}_{21}$, $L_{22} \in \mathcal{A}_{22}$, and $m \in \mathcal{N}$, we have

$$\psi_m(L_{11} + L_{12} + L_{21}) = \psi_m(L_{11}) + \psi_m(L_{12}) + \psi_m(L_{21});$$

$$\psi_m(L_{12} + L_{21} + L_{22}) = \psi_m(L_{12}) + \psi_m(L_{21}) + \psi_m(L_{22}).$$

Proof. To prove the lemma, define $K = \psi_m(L_{11} + L_{12} + L_{21}) - \psi_m(L_{11}) - \psi_m(L_{12}) - \psi_m(L_{21})$. Since $\xi_I(\frac{P_2}{2^{n-2}}, L_{11}) = 0$ and by the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$), which asserts that $\psi_s(L_{11} + L_{12} + L_{21}) = \psi_s(L_{11}) + \psi_s(L_{12}) + \psi_s(L_{21})$ for all $L_{11} \in \mathcal{A}_{11}$, $L_{12} \in \mathcal{A}_{12}$, and $L_{21} \in \mathcal{A}_{21}$, using Lemma 2, we have

$$\begin{aligned} & \psi_m(\xi_I(\frac{P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21}))) \\ &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11})) \\ &+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{12})) \\ &+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{21})) \\ &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), (\psi_{r_n}(L_{11}) + \psi_{r_n}(L_{12}) + \psi_{r_n}(L_{21}))) \\ &+ \xi_I(\frac{P_2}{2^{n-2}}, (\psi_m(L_{11}) + \psi_m(L_{12}) + \psi_m(L_{21}))). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \psi_m(\xi_I(\frac{P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21}))) \\ &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21})) \\ &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21})) \\ &+ \xi_I(\frac{P_2}{2^{n-2}}, \psi_m(L_{11} + L_{12} + L_{21})). \end{aligned}$$

From the above two equations, we obtain $\xi_I(\frac{P_2}{2^{n-2}}, K) = 0$, which implies that

$$P_2 K + K P_2 = 0.$$

This implies that $K_{22} = K_{21} = K_{12} = 0$. Meanwhile, we have

$$\xi_I(\frac{P_1 - P_2}{2^{n-2}}, L_{12}) = \xi_I(\frac{P_1 - P_2}{2^{n-2}}, L_{21}) = 0.$$

It follows from Lemma 1 that

$$\begin{aligned}
 & \psi_m(\xi_I(\frac{P_1 - P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21}))) \\
 &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21})) \\
 &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{11})) \\
 &+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{12})) \\
 &+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{21})) \\
 &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), (\psi_{r_n}(L_{11}) + \psi_{r_n}(L_{12}) + \psi_{r_n}(L_{21}))) \\
 &+ \xi_I(\frac{P_1 - P_2}{2^{n-2}}, (\psi_m(L_{11}) + \psi_m(L_{12}) + \psi_m(L_{21}))).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \psi_m(\xi_I(\frac{P_1 - P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21}))) \\
 &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21})) \\
 &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_1 - P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21})) \\
 &+ \xi_I(\frac{P_1 - P_2}{2^{n-2}}, \psi_m(L_{11} + L_{12} + L_{21}))
 \end{aligned}$$

From the last two equations, we get

$$\xi_I(\frac{P_1 - P_2}{2^{n-2}}, K) = 0.$$

The above relation simplifies to $P_1K - P_2K + KP_1 - KP_2 = 0$. Multiplying the last relation by P_1 from both left and right, we get $K_{11} = 0$. Similarly, The second conclusion follows by an analogous argument. Therefore, the desired result follows. $\square$

Lemma 4. For any $L_{ij} \in \mathcal{A}_{ij}$ with $1 \leq i, j \leq 2$, we have

$$\psi_m(\sum_{i,j=1}^2 L_{ij}) = \sum_{i,j=1}^2 \psi_m(L_{ij}).$$

Proof. The statement holds for $m=1$. Assume inductively that it holds for all s with $1 \leq s < m$; that is, $\psi_s(L_{11} + L_{12} + L_{21} + L_{22}) = \psi_s(L_{11}) + \psi_s(L_{12}) + \psi_s(L_{21}) + \psi_s(L_{22})$. It remains to prove the statement for $s = m$. Let $K = \psi_m(L_{11} + L_{12} + L_{21} + L_{22}) - \psi_m(L_{11}) - \psi_m(L_{12}) - \psi_m(L_{21}) - \psi_m(L_{22})$. Since $\xi_I(\frac{P_2}{2^{n-2}}, L_{11}) = 0$, it follows from Lemma 3 that

$$\begin{aligned}
& \psi_m(\xi_I(\frac{P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21} + L_{22}))) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21} + L_{22})) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11})) \\
&+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{12})) \\
&+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{21})) \\
&+ \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{22})) \\
&= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), (\psi_{r_n}(L_{11}) + \psi_{r_n}(L_{12}) + \psi_{r_n}(L_{21}) + \psi_{r_n}(L_{22}))) \\
&+ \xi_I(\frac{P_2}{2^{n-2}}, (\psi_m(L_{11}) + \psi_m(L_{12}) + \psi_m(L_{21}) + \psi_m(L_{22}))).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \psi_m(\xi_I(\frac{P_2}{2^{n-2}}, (L_{11} + L_{12} + L_{21} + L_{22}))) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21} + L_{22})) \\
&= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_2}{2^{n-2}}), \psi_{r_n}(L_{11} + L_{12} + L_{21} + L_{22})) \\
&+ \xi_I(\frac{P_2}{2^{n-2}}, \psi_m(L_{11} + L_{12} + L_{21} + L_{22})).
\end{aligned}$$

It follows from the above two equations that $\xi_I(\frac{P_2}{2^{n-2}}, K) = 0$, which implies that $K_{22} = K_{21} = K_{12} = 0$. Similarly, we can show that $K_{11} = 0$. This completes the proof. $\square$

Lemma 5. For any $L_{ij}, N_{ij} \in \mathcal{A}_{ij}$ with $i \neq j$ and $m \in \mathcal{N}$, we have

$$\psi_m(L_{ij} + N_{ij}) = \psi_m(L_{ij}) + \psi_m(N_{ij}).$$

Proof. By the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$), we have $\psi_s(L_{ij} + N_{ij}) = \psi_s(L_{ij}) + \psi_s(N_{ij})$. Furthermore, from the definition of the product, we have

$$\xi_I((\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), (P_j + N_{ij})) = L_{ij} + N_{ij} + L_{ij}^* N_{ij} + L_{ij}^* + N_{ij} L_{ij}^*.$$

Invoking Lemmas 2–4, we obtain

$$\begin{aligned}
& \psi_m(L_{ij} + N_{ij}) + \psi_m(L_{ij}^* N_{ij}) + \psi_m(L_{ij}^*) + \psi_m(N_{ij} L_{ij}^*) \\
&= \psi_m(L_{ij} + N_{ij} + L_{ij}^* N_{ij} + L_{ij}^* + N_{ij} L_{ij}^*) \\
&= \psi_m(\xi_I((\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), (P_j + N_{ij}))) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), \psi_{r_n}(P_j + N_{ij})) \\
&= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1}, r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), \psi_{r_n}(P_j + N_{ij})) \\
&\quad + \xi_I(\psi_m(\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), (P_j + N_{ij})) + \xi_I((\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), \psi_m(P_j + N_{ij})) \\
&= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1}, r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, (\psi_{r_{n-1}}(\frac{P_i}{2^{n-2}}) + \psi_{r_{n-1}}(\frac{L_{ij} + L_{ij}^*}{2^{n-2}})), (\psi_{r_n}(P_j) + \psi_{r_n}(N_{ij}))) \\
&\quad + \xi_I((\psi_m(\frac{P_i}{2^{n-2}}) + \psi_m(\frac{L_{ij} + L_{ij}^*}{2^{n-2}})), (P_j + N_{ij})) + \xi_I((\frac{P_i + L_{ij} + L_{ij}^*}{2^{n-2}}), (\psi_m(P_j) + \psi_m(N_{ij}))) \\
&= \psi_m(\xi_I(\frac{P_i}{2^{n-2}}, P_j)) + \psi_m(\xi_I(\frac{P_i}{2^{n-2}}, N_{ij})) \\
&\quad + \psi_m(\xi_I(\frac{L_{ij} + L_{ij}^*}{2^{n-2}}, P_j)) + \psi_m(\xi_I(\frac{L_{ij} + L_{ij}^*}{2^{n-2}}, N_{ij})) \\
&= \psi_m(N_{ij}) + \psi_m(L_{ij} + L_{ij}^*) + \psi_m(L_{ij}^* N_{ij} + N_{ij} L_{ij}^*) \\
&= \psi_m(N_{ij}) + \psi_m(L_{ij}) + \psi_m(L_{ij}^*) + \psi_m(L_{ij}^* N_{ij}) + \psi_m(N_{ij} L_{ij}^*).
\end{aligned}$$

Therefore, $\psi_m(L_{ij} + N_{ij}) = \psi_m(L_{ij}) + \psi_m(N_{ij})$. This completes the proof. $\square$

Lemma 6. For any $L_{ii}, N_{ii} \in \mathcal{A}_{ii}$, $1 \leq i \leq 2$ and $m \in \mathcal{N}$, we have

$$\psi_m(L_{ii} + N_{ii}) = \psi_m(L_{ii}) + \psi_m(N_{ii}).$$

Proof. Let $K = \psi_m(L_{ii} + N_{ii}) - \psi_m(L_{ii}) - \psi_m(N_{ii})$. Applying Lemma 1 and the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$), which asserts that $\psi_s(L_{ii} + N_{ii}) = \psi_s(L_{ii}) + \psi_s(N_{ii})$ for all $L_{ii}, N_{ii} \in \mathcal{A}_{ii}$, it follows from $\xi_I(\frac{P_j}{2^{n-2}}, L_{ii}) = \xi_I(\frac{P_j}{2^{n-2}}, N_{ii}) = 0$ that

$$\begin{aligned}
& \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_j}{2^{n-2}}), \psi_{r_n}(L_{ii} + N_{ii})) \\
&\quad + \xi_I(\frac{P_j}{2^{n-2}}, \psi_m(L_{ii} + N_{ii})) \\
&= \psi_m(\xi_I(\frac{P_j}{2^{n-2}}, (L_{ii} + N_{ii}))) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_j}{2^{n-2}}), \psi_{r_n}(L_{ii} + N_{ii})) \\
&= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_j}{2^{n-2}}), \psi_{r_n}(L_{ii})) \\
&\quad + \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_j}{2^{n-2}}), \psi_{r_n}(N_{ii})) \\
&= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{P_j}{2^{n-2}}), (\psi_{r_n}(L_{ii}) + \psi_{r_n}(N_{ii}))) \\
&\quad + \xi_I(\frac{P_j}{2^{n-2}}, (\psi_m(L_{ii}) + \psi_m(N_{ii}))).
\end{aligned}$$

Thus, we have $\xi_I(\frac{P_j}{2^{n-2}}, K) = 0$, which implies that $P_j K + K P_j = 0$. Hence, $K_{jj} = K_{ij} = K_{ji} = 0$. By Lemma 5 and the inductive hypothesis $\mathbb{H}_s$ ($1 \leq s < m$), and for any $B_{ij} \in \mathcal{A}_{ij}$ with $i \neq j$, we obtain

$$\begin{aligned} & \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{B_{ij}}{2^{n-3}}), \psi_{r_n}(L_{ii} + N_{ii})) \\ & + \xi_I(\frac{B_{ij}}{2^{n-3}}, \psi_m(L_{ii} + N_{ii})) \\ & = \psi_m(\xi_I(\frac{B_{ij}}{2^{n-3}}, (L_{ii} + N_{ii}))) \\ & = \psi_m(\xi_I(\frac{B_{ij}}{2^{n-3}}, L_{ii})) + \psi_m(\xi_I(\frac{B_{ij}}{2^{n-3}}, N_{ii})) \\ & = \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{B_{ij}}{2^{n-3}}), \psi_{r_n}(L_{ii})) \\ & + \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{B_{ij}}{2^{n-3}}), \psi_{r_n}(N_{ii})) \\ & = \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{B_{ij}}{2^{n-3}}), (\psi_{r_n}(L_{ii}) + \psi_{r_n}(N_{ii}))) \\ & + \xi_I(\frac{B_{ij}}{2^{n-3}}, (\psi_m(L_{ii}) + \psi_m(N_{ii}))). \end{aligned}$$

This implies that $\xi_I(\frac{B_{ij}}{2^{n-3}}, K) = 0$. Consequently, $B_{ij}K + B_{ij}^*K + KB_{ij}^* + KB_{ij} = 0$, i.e., $B_{ij}K_{ii} + B_{ij}^*K_{ii} + K_{ii}K_{ij}^* + K_{ii}B_{ij} = 0$. Thus, we have $K_{ii}B_{ij} = 0$. It follows from condition (3) in Theorem 1 that $K_{ii} = 0$. Therefore, $\psi_m(L_{ii} + N_{ii}) = \psi_m(L_{ii}) + \psi_m(N_{ii})$. $\square$

Lemma 7. ψ_m is additive.

Proof. For any $L, N \in \mathcal{A}$, we write $L = L_{11} + L_{12} + L_{21} + L_{22}$ and $N = N_{11} + N_{12} + N_{21} + N_{22}$. By Lemmas 4–6, we have

$$\begin{aligned} \psi_m(L + N) & = \psi_m(L_{11} + L_{12} + L_{21} + L_{22} + N_{11} + N_{12} + N_{21} + N_{22}) \\ & = \psi_m(L_{11} + N_{11}) + \psi_m(L_{12} + N_{12}) + \psi_m(L_{21} + N_{21}) + \psi_m(L_{22} + N_{22}) \\ & = \psi_m(L_{11}) + \psi_m(N_{11}) + \psi_m(L_{12}) + \psi_m(N_{12}) + \psi_m(L_{21}) + \psi_m(N_{21}) + \psi_m(L_{22}) + \psi_m(N_{22}) \\ & = \psi_m(L_{11} + L_{12} + L_{21} + L_{22}) + \psi_m(N_{11} + N_{12} + N_{21} + N_{22}) \\ & = \psi_m(L) + \psi_m(N). \end{aligned}$$

Hence, ψ_m is additive. $\square$

Lemma 8. The following hold for each $m \in \mathcal{N}$ with $m > 1$:

- (1) $\psi_m(I) = 0$;
- (2) $\psi_m(iI) = 0$;
- (3) $\psi_m(-iI) = 0$.

Proof. The statement holds for $1 \leq s < m$ by the inductive hypothesis, i.e., $\psi_s(I) = 0$. It remains to prove the case $s = m$.

$$\begin{aligned}
 2^{n-1}\psi_m(I) &= \psi_m(\xi_I(I, I)) \\
 &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(I), \psi_{r_n}(I)) \\
 &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_1, r_2, \dots, r_n \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(I), \psi_{r_n}(I)) \\
 &\quad + \sum_{k=1}^{n-1} V_n(I, I, \dots, I, \underbrace{\psi_m(I)}_{k\text{-component}}, I, \dots, I) + \xi_I(I, \psi_m(I)). \\
 &= \sum_{k=1}^{n-1} V_n(I, I, \dots, I, \underbrace{\psi_m(I)}_{k\text{-component}}, I, \dots, I) + \xi_I(I, \psi_m(I)). \\
 &= (n-1)2^{n-2}((\psi_m(I) + \psi_m(I)^*) \bullet I) + 2^{n-1}\psi_m(I).
 \end{aligned}$$

This implies that $\psi_m(I) + \psi_m(I)^* = 0$. By the given hypothesis, $\psi_m(I)$ is self-adjoint for $m \in \mathcal{N}$. Hence, $\psi_m(I) = 0$. Applying Lemma 1, we have

$$\psi_m(\xi_I(iI, \frac{I}{2^{n-2}})) = 0.$$

On the other hand, we have

$$\begin{aligned}
 \psi_m(\xi_I(iI, \frac{I}{2^{n-2}})) &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(iI), \psi_{r_n}(\frac{I}{2^{n-2}})) \\
 &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(iI), \psi_{r_n}(\frac{I}{2^{n-2}})) \\
 &\quad + \xi_I(\psi_m(iI), \frac{I}{2^{n-2}}) \\
 &= \psi_m(iI) + \psi_m(iI)^*.
 \end{aligned}$$

By hypothesis, $\psi_m(iI)$ is self-adjoint for $m \in \mathcal{N}$. Thus, we obtain $\psi_m(iI) = 0$.

By a similar method, we can obtain that equation $\psi_m(iI) = 0$ holds. $\square$

Lemma 9. $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ preserves the involution $*$, i.e., $\psi_m(L^*) = \psi_m(L)^*$ for all $L \in \mathcal{A}$.

Proof. It follows from Lemma 7 that

$$\begin{aligned}
 \psi_m(\xi_I(\frac{L}{2^{n-2}}, I)) &= \psi_m(L + L^*) \\
 &= \psi_m(L) + \psi_m(L^*).
 \end{aligned}$$

On the other hand, applying Lemmas 7 and 8, we have

$$\begin{aligned}
 \psi_m(\xi_I(\frac{L}{2^{n-2}}, I)) &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{L}{2^{n-2}}), \psi_{r_n}(I)) \\
 &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{L}{2^{n-2}}), \psi_{r_n}(I)) \\
 &\quad + \xi_I(\psi_m(\frac{L}{2^{n-2}}), I) \\
 &= \psi_m(L) + \psi_m(L)^*.
 \end{aligned}$$

According to the above two relations, we obtain $\psi_m(L^*) = \psi_m(L)^*$. $\square$

Lemma 10. For each $m \in \mathcal{N}$ and all $L \in \mathcal{A}$, the following hold:

- (1) $\psi_m(iL) = i\psi_m(L)$;
- (2) $\psi_m(-iL) = -i\psi_m(L)$.

Proof. For any $L \in \mathcal{A}$, applying Lemmas 7 and 8, we have

$$\begin{aligned}\psi_m(\xi_I(\frac{iL}{2^{n-2}}, iI)) &= \psi_m(L^* - L) \\ &= \psi_m(L^*) - \psi_m(L).\end{aligned}$$

Furthermore,

$$\begin{aligned}\psi_m(\xi_I(\frac{iL}{2^{n-2}}, iI)) &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{iL}{2^{n-2}}), \psi_{r_n}(iI)) \\ &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{iL}{2^{n-2}}), \psi_{r_n}(iI)) \\ &\quad + \xi_I(\psi_m(\frac{iL}{2^{n-2}}), iI) \\ &= i(\psi_m(iL) + \psi_m(iL)^*).\end{aligned}$$

Therefore, from the above two relations, we have

$$i\psi_m(L)^* - i\psi_m(L) = -\psi_m(iL) - \psi_m(iL)^*. \quad (4)$$

Also, we have

$$\psi_m(\xi_I(\frac{L}{2^{n-2}}, iI)) = \psi_m(iL^*) + \psi_m(iL).$$

On the other hand,

$$\begin{aligned}\psi_m(\xi_I(\frac{L}{2^{n-2}}, iI)) &= \sum_{r_1+r_2+\dots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{L}{2^{n-2}}), \psi_{r_n}(iI)) \\ &= \sum_{\substack{r_1+r_2+\dots+r_n=m \\ r_{n-1} \neq m}} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \dots, \psi_{r_{n-1}}(\frac{L}{2^{n-2}}), \psi_{r_n}(iI)) \\ &\quad + \xi_I(\psi_m(\frac{L}{2^{n-2}}), iI) \\ &= i(\psi_m(L) + \psi_m(L)^*).\end{aligned}$$

From the above two relations, we obtain

$$\psi_m(iL) - \psi_m(iL)^* = i\psi_m(L) + i\psi_m(L)^*. \quad (5)$$

Thus, from (4) and (5), we have

$$\psi_m(iL) = i\psi_m(L).$$

Similarly, we can show that $\psi_m(-iL) = -i\psi_m(L)$. $\square$

Lemma 11. $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is a higher derivation.

Proof. For arbitrary $L, N \in \mathcal{A}$, using the inductive hypothesis $\mathbb{H}_s (1 \leq s < m)$ together with Lemmas 7–9, we have

$$\begin{aligned}
2\psi_m(LN + L^*N + N^*L^* + N^*L) &= \psi_m(I \diamond I \cdots \diamond I \diamond \frac{L^*}{2^{n-4}} \diamond N \bullet I) \\
&= \sum_{r_1+r_2+\cdots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \cdots, \psi_{r_{n-2}}(\frac{L^*}{2^{n-4}}), \psi_{r_{n-1}}(N), \psi_{r_n}(I)) \\
&= \sum_{r_{n-2}+r_{n-1}=m} V_n(I, I, \cdots, \psi_{r_{n-2}}(\frac{L^*}{2^{n-4}}), \psi_{r_{n-1}}(N), I) \\
&= \sum_{r_{n-2}+r_{n-1}=m} 2(\psi_{r_{n-2}}(L)^* \psi_{r_{n-1}}(N) + \psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) \\
&\quad + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L^*) + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)),
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi_m(LN + L^*N + N^*L^* + N^*L) &= \sum_{r_{n-2}+r_{n-1}=m} (\psi_{r_{n-2}}(L)^* \psi_{r_{n-1}}(N) + \psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) \\
&\quad + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L^*) + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
2\psi_m(iLN - iL^*N - iN^*L^* + iN^*L) &= \psi_m(I \diamond I \cdots \diamond I \diamond \frac{iL}{2^{n-4}} \diamond N \bullet I) \\
&= \sum_{r_1+r_2+\cdots+r_n=m} V_n(\psi_{r_1}(I), \psi_{r_2}(I), \cdots, \psi_{r_{n-2}}(\frac{iL}{2^{n-4}}), \psi_{r_{n-1}}(N), \psi_{r_n}(I)) \\
&= \sum_{r_{n-2}+r_{n-1}=m} V_n(I, I, \cdots, \psi_{r_{n-2}}(\frac{iL}{2^{n-4}}), \psi_{r_{n-1}}(N), I) \\
&= \sum_{r_{n-2}+r_{n-1}=m} 2(i\psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) - i\psi_{r_{n-2}}(L)^* \psi_{r_{n-1}}(N) \\
&\quad + i\psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L) - i\psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)^*),
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi_m(LN + N^*L - N^*L^* - L^*N) &= \sum_{r_{n-2}+r_{n-1}=m} (\psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) - \psi_{r_{n-2}}(L)^* \psi_{r_{n-1}}(N) \\
&\quad + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L) - \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)^*).
\end{aligned}$$

From the last two equations, we have

$$\psi_m(LN + N^*L) = \sum_{r_{n-2}+r_{n-1}=m} (\psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) + \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)). \quad (6)$$

From (6), we obtain that

$$\begin{aligned}
\psi_m(LN - N^*L) &= \psi_m((-iL)(iN) + (iN)^*(-iL)) \\
&= \sum_{r_{n-2}+r_{n-1}=m} (\psi_{r_{n-2}}(-iL) \psi_{r_{n-1}}(iN) + \psi_{r_{n-1}}(iN)^* \psi_{r_{n-2}}(-iL)) \\
&= \sum_{r_{n-2}+r_{n-1}=m} (\psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N) - \psi_{r_{n-1}}(N)^* \psi_{r_{n-2}}(L)).
\end{aligned}$$

According to the above two relations, we have

$$\psi_m(LN) = \sum_{r_{n-2}+r_{n-1}=m} \psi_{r_{n-2}}(L) \psi_{r_{n-1}}(N).$$

Hence, $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is a higher derivation. $\square$

Proof of Theorem 1:

With the help of Lemma 7, we obtain that the nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation is additive. This result, together with the additional assumption that $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, allows us to derive Lemma 9 and Lemma 11. These lemmas show that every nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation is, in fact, a higher $*$ -derivation. Consequently, Theorem 1 holds.

Let $\mathcal{A}$ be a unital prime $*$ -algebra containing a non-trivial projection. In a prime algebra, the condition $L\mathcal{A}N = 0$ for $L, N \in \mathcal{A}$ implies that $L = 0$ or $N = 0$. Consequently, such an algebra automatically satisfies condition 3 stated in Theorem 1. So, we have the following corollary.

Corollary 1. *Let $\mathcal{A}$ be a prime $*$ -algebra containing a non-trivial projection. If $\psi_m : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\psi_m(L_1 \diamond L_2 \diamond \cdots \diamond L_{n-1} \bullet L_n) = \sum_{r_1 + \cdots + r_n = m} \psi_{r_1}(L_1) \diamond \cdots \diamond \psi_{r_{n-1}}(L_{n-1}) \bullet \psi_{r_n}(L_n).$$

for all $L_1, \dots, L_n \in \mathcal{A}$ and each $m \in \mathcal{N}$, where $n \geq 3$, then $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is additive. Moreover, if $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, then $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is an additive higher $$ -derivation on $\mathcal{A}$.*

It follows from [17] that standard operator algebras and von Neumann factors serve as standard examples of prime $*$ -algebras, and therefore of unital $*$ -algebras. Hence, the subsequent corollary is an immediate consequence.

Corollary 2. *Let $\mathcal{A}$ be a standard operator algebra on an infinite-dimensional complex Hilbert space $\mathcal{H}$ that contains the identity operator I and is closed under the adjoint operation. Then, every nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation is additive. Moreover, if $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, then $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is an additive higher $*$ -derivation on $\mathcal{A}$.*

Corollary 3. *Let $\mathcal{A}$ be a von Neumann factor with $\dim \mathcal{A} > 2$. Then, every nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivation is additive. Moreover, if $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, then $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is an additive higher $*$ -derivation on $\mathcal{A}$.*

3. Topics for Further Research

This paper focuses on investigating the structure of nonlinear mixed Jordan-type and skew Jordan higher derivations on a unital $*$ -algebra $\mathcal{A}$, where a unital $*$ -algebra is an associative algebra $\mathcal{A}$ equipped with an involution $*$ -operation. In parallel, the study of Herstein Lie-type mappings on non-associative algebras has also attracted considerable interest, such as those on alternative $*$ -algebras. Let $\mathcal{B}$ be an alternative $*$ -algebra containing nontrivial idempotents P_1 and P_2 with $P_1 + P_2 = I$, where I denotes the identity element. For further details, we refer the reader to [4]. Ferreira has played a significant role in advancing the study of Herstein Lie-type mappings on alternative $*$ -algebras and related algebraic structures. The structures of nonlinear mixed $*$ -Jordan-type derivations, Jordan triple derivations, and Lie multiplicative derivations have all been examined. Inspired by [4] and the main theorem of this paper (see Theorem 1, we propose the following interesting conjecture.

Conjecture 1. *Let $\mathcal{B}$ be an alternative unital $*$ -algebra with identity element I satisfying the condition*

$$(L\mathcal{B})P_i = 0 \text{ implies } L = 0, \text{ for } i \in \{1, 2\},$$

where $L \in \mathcal{B}$. Suppose that a family of maps $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$, with $\psi_m : \mathcal{B} \rightarrow \mathcal{B}$, satisfies

$$\psi_m(L_1 \diamond L_2 \diamond \cdots \diamond L_{n-1} \bullet L_n) = \sum_{r_1 + \cdots + r_n = m} \psi_{r_1}(L_1) \diamond \cdots \diamond \psi_{r_{n-1}}(L_{n-1}) \bullet \psi_{r_n}(L_n).$$

for all $L_1, \dots, L_n \in \mathcal{B}$, for each $m \in \mathcal{N}$, and $n \geq 3$. Then Ψ is additive. Moreover, if $\{\psi_m(\alpha I)\}_{m \in \mathcal{N}}$ is self-adjoint for $\alpha \in \{i, 1\}$, then $\Psi = \{\psi_m\}_{m \in \mathcal{N}}$ is an additive higher $*$ -derivation on $\mathcal{B}$.

Author Contributions: Methodology, D.R., J.Z., X.L. and Y.S.; Investigation, D.R., J.Z., X.L. and Y.S.; Writing—review & editing, X.L. and Y.S.; Funding acquisition, X.L. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (No. 11901030); the Natural Science Foundation of Beijing Municipality (No. 1232017); the Beijing Institute of Technology Research Fund Program for Young Scholars; the Open Research Fund of Hubei Key Laboratory of Mathematical Sciences (Central China Normal University) (No. MPL2024ORG004), Wuhan, 430079, P.R. China; the Youth Fund of Anhui Natural Science Foundation (Grant No. 2008085QA01); and the Key Projects of the University Natural Science Research Project of Anhui Province (Grant No. KJ2019A0107).

Data Availability Statement: No new data were generated or analyzed during this study.

Acknowledgments: The authors would like to thank the referees for their careful reading of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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