

On sM -Prime Ideals in Commutative Rings

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Abstract

All rings considered are commutative with identity, and all modules are assumed to be unital. In this paper, we study R -modules in which every quasi-primary submodule is also primary; we refer to such modules as satisfying condition (*). We present several structural properties of these modules and investigate when the direct sum of two modules M_1 and M_2 inherits condition (*). In addition, we focus on prime ideals P of a ring R with the property that any P -quasi-primary submodule of an R -module M is automatically P -primary. Prime ideals exhibiting this behaviour are introduced as weak sM -prime ideals relative to M . Our results provide a framework for understanding the interaction between the quasi-primary structure of modules and the prime spectrum of the underlying ring.

Keywords: prime ideals; sM -prime ideals; primary submodules; multiplication module

MSC: 13A15; 13C05

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity ($1 \neq 0$), and all modules are unital. Let R be such a ring, and let M be an R -module.

The concept of quasi-primary ideals was first introduced by Fuchs [1], who defined an ideal I as quasi-primary if its radical $\sqrt{I}$ is prime. Around the same time, Gilmer called these ideals semi-primary [2]. Later, these ideas were extended to more general algebraic structures, including lattices. For instance, the authors of [3] studied s -prime elements in C -lattices and considered when semi-primary elements are actually primary. The idea is similar to quasi-primary ideals because it relies on the radical being prime.

Since the set of all ideals of a ring forms a C -lattice, these results apply to quasi-primary ideals as well. This inspired the definitions of *weak s -prime* and *s -prime* elements: a prime element p is weak s -prime if every p -semi-primary element is p -primary, and p is s -prime if all prime elements under p are weak s -prime.

For modules, we use some standard notation. For a submodule N of M , set

$$(N : M) = \{r \in R \mid rM \subseteq N\}.$$

In particular, $(0 : M)$ is the annihilator of M , denoted $\text{Ann}(M)$. A proper ideal P of R is prime if $ab \in P$ implies $a \in P$ or $b \in P$ and maximal if it is maximal among proper ideals. For a submodule N , $\sqrt{N}$ denotes the intersection of all prime submodules containing N .

A submodule N is *prime* if $rm \in N$ implies $r \in (N : M)$ or $m \in N$ and *primary* if $rm \in N$ implies $m \in N$ or $r \in \sqrt{(N : M)}$. In particular, N is P -primary if $\sqrt{(N : M)} = P$ [4–9].



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Several definitions of quasi-primary submodules exist. Moghimi and Samiei [10,11] defined N as quasi-primary if $rm \in N$ implies $r \in \sqrt{(N : M)}$ or $m \in \sqrt{N}$. This matches the classical definition when N has the primeful property, but not always. Another approach [12] defines N as quasi-primary when $\sqrt{(N : M)}$ is prime. If $\sqrt{(N : M)} = P$, we say N is P -quasi-primary. This makes it easy to compare quasi-primary and primary submodules.

More recent work has expanded these ideas. For example, ref. [13] studied 2-absorbing primary submodules, which relax primeness while keeping primary-like behavior. Celikel and Khashan [14] introduced quasi J -submodules, Moghimi and Samiei [15] studied quasi-primaryful modules, and Sönmez, Yeşilot, and Hila [16] examined weakly $\delta_{R,M}$ -semi-primary submodules. These works show that variations of primeness can affect module structure in interesting ways.

In this paper, we focus on modules in which every quasi-primary submodule is automatically primary. Such modules are said to satisfy condition $(*)$. Our main results are as follows:

- Proposition 2 shows that in faithful multiplication modules, a quasi-primary ideal I with $\sqrt{IM} \neq M$ gives a quasi-primary submodule IM .
- Proposition 3 states that in multiplication modules, a submodule N is quasi-primary if and only if $(N : M)$ is a quasi-primary ideal.
- Theorems 1 and 2 describe properties of P -primary submodules in modules satisfying $(*)$, including $N(K : M) = N$ and $(N : M) = P$ for nonmaximal primes.
- Corollary 2 extends this to direct sums: if one summand satisfies $(*)$, any quasi-primary submodule containing the other summand is primary.
- Finally, we introduce weak sM -prime and sM -prime ideals for modules. Theorem 4 shows that these properties are preserved under localization for finitely generated modules, and Theorem 6 proves that a nonmaximal weak sM -prime ideal has a unique P -quasi-primary submodule.

2. Basic Results on Modules Satisfying Condition $(*)$

We start this section by recalling the notion of a quasi-primary submodule, as introduced in [12].

Definition 1 ([12]). *Let R be a commutative ring and M an R -module. A proper submodule N of M is called a quasi-primary submodule if the radical of the annihilator of M/N , denoted $\sqrt{(N : M)}$, forms a prime ideal of R . In particular, if $\sqrt{(N : M)} = P$, we say that N is a P -quasi-primary submodule.*

It is immediate that any primary submodule automatically satisfies the quasi-primary condition, though the reverse is not always true. The following examples illustrate that a quasi-primary submodule may fail to be primary.

Example 1. *Let $R = k[X, Y]$, where k is a field, and consider $M = R$. Take the submodule $N = (X) \cap (X, Y)^2$ of M . Since $M = R$, the annihilator satisfies $(N : M) = (N : R) = N$, and hence $\sqrt{(N : M)} = \sqrt{N} = (X)$, which is a prime ideal. Consequently, N qualifies as a quasi-primary submodule. Nevertheless, N is not primary because $XY \in N$, yet $X \notin N$ and $Y \notin \sqrt{(N : M)}$.*

Example 2. *Let $R = k[X]$ be the polynomial ring over a field k , and define $M = R \times R$ with submodule $N = (X) \times 0$. One computes $(N : M) = (0)$, which is a prime ideal since R is an integral domain. This ensures that N is quasi-primary. However, N does not satisfy the primary condition: indeed, $X \cdot (1, 0) \in N$, while $(1, 0) \notin N$, and $X^n \notin (N : M)$ for any $n \geq 1$.*

In [2], the concept of a semi-primary ideal was introduced in the context of commutative rings. An ideal I of a ring R is called semi-primary if the radical of I , denoted $\sqrt{I}$, is a prime ideal. Clearly, every primary ideal satisfies this property, but the converse does not necessarily hold. Inspired by this notion, a ring R is said to satisfy condition (*) if all its semi-primary ideals are in fact primary.

Motivated by the behavior of semi-primary ideals, we turn our attention to modules and investigate submodules with a similar property. Specifically, we are interested in modules for which every quasi-primary submodule is primary. Recall that a quasi-primary submodule N of a module M is one for which $\sqrt{(N : M)}$ is a prime ideal; however, as discussed earlier, quasi-primary submodules are not always primary. In analogy with the notation in [2], we denote by (*) the condition that all quasi-primary submodules of a module are primary.

An important class of modules in this context is the class of multiplication modules. A module M over a commutative ring R is called a multiplication module if each submodule N of M can be expressed in the form $N = IM$ for some ideal I of R . In such modules, the ideal I coincides with the annihilator $(N : M)$ of the submodule (see [9] for details). This correspondence provides a natural bridge between the properties of ideals in R and the properties of submodules in M .

The following proposition illustrates how the condition (*) transfers from the ring to its multiplication modules.

Proposition 1. *Let M be a multiplication module over a commutative ring R . If the ring R satisfies condition (*), then the module M also satisfies condition (*).*

Proof. Take any quasi-primary submodule N of M . By definition, the radical of its annihilator, $\sqrt{(N : M)}$, is a prime ideal of R . Since R satisfies condition (*), it follows that $(N : M)$ is a primary ideal. Because M is a multiplication module, the structure of submodules ensures that $N = (N : M)M$, and thus N is a primary submodule. Therefore, every quasi-primary submodule of M is primary, and M inherits condition (*) from R . $\square$

Proposition 2. *Let M be a faithful multiplication R -module. If I is a quasi-primary ideal of R and $\sqrt{I}M \neq M$, then the submodule IM is quasi-primary in M .*

Proof. Suppose I is a quasi-primary ideal of R with $\sqrt{I}M \neq M$. Since $\sqrt{I}$ is a prime ideal, it follows from [9] that $\sqrt{I}M$ is a prime submodule of M . In particular, $(\sqrt{I}M : M)$ is a prime ideal.

We claim that

$$(\sqrt{I}M : M) = \sqrt{(IM : M)}.$$

First, let $x \in \sqrt{(IM : M)}$. Then, there exists $n \in \mathbb{N}$ such that $x^n \in (IM : M)$, which gives $x^n M \subseteq IM \subseteq \sqrt{I}M$. Let n be the smallest positive integer for which this inclusion holds. By [9] (Lemma 2.10), it follows that $x \in \sqrt{I}$, such that $xM \subseteq \sqrt{I}M$, i.e., $x \in (\sqrt{I}M : M)$.

Conversely, take $x \in (\sqrt{I}M : M)$. Then, $xM \subseteq \sqrt{I}M$. Because M is faithful, there exists $m \in M$ with $m \notin \sqrt{I}M$. Since $xm \in \sqrt{I}M$, again by [9] (Lemma 2.10), we conclude that $x \in \sqrt{I}$. Therefore, there exists a positive integer n such that $x^n \in I$, which implies $x^n M \subseteq IM$. Hence, $x \in \sqrt{(IM : M)}$.

This establishes the equality $(\sqrt{I}M : M) = \sqrt{(IM : M)}$, and therefore IM is a quasi-primary submodule of M . $\square$

At first glance, Proposition 2 may appear similar to Theorem 2.2 in [11]. However, the two results are conceptually distinct. In [11], quasi-primary submodules are defined

via the primeful property, and the theorem there is stated under the assumption that this property holds. In contrast, our definition of quasi-primary submodules (Definition 1) does not require the primeful property. Consequently, although the statements may seem similar superficially, the conclusions in Proposition 2 are not the same as those in Theorem 2.2 of [11].

Proposition 3. *Let M be a multiplication R -module and N a proper submodule of M . The following statements are equivalent:*

1. N is a quasi-primary submodule of M .
2. $(N : M)$ is a quasi-primary ideal of R .
3. $N = IM$ for some quasi-primary ideal I of R containing $\text{Ann}(M)$.

Proof. $(1 \Rightarrow 2)$: This is immediate from the definition of a quasi-primary submodule.

$(2 \Rightarrow 3)$: Let $I = (N : M)$. Since M is a multiplication module, we can write $N = IM$, which satisfies the desired conditions.

$(3 \Rightarrow 1)$: Assume $N = IM$ for a quasi-primary ideal I of R with $\text{Ann}(M) \subseteq I$.

Case 1: If I is a prime ideal, then by [9] (Corollary 2.11), N is a prime submodule of M . In particular, $(N : M)$ is prime, so N is quasi-primary.

Case 2: If I is not prime, then by the properties of radicals,

$$\sqrt{I} \subseteq \sqrt{(N : M)}.$$

If equality holds, the statement is proven. Suppose instead that the inclusion is strict, i.e., $\sqrt{I} \subsetneq \sqrt{(N : M)}$.

Since M is a multiplication module, we have

$$(N : M)M = IM \subseteq \sqrt{I}M.$$

Because I is proper and not prime, $\sqrt{I}M \neq IM$ and also $\sqrt{I}M \neq M$. Therefore, there exists $m \in M$ such that $m \notin \sqrt{I}M$.

From this element m , we observe that

$$m(N : M) \subseteq \sqrt{I}M.$$

Using the standard fact that if P is a prime ideal with $\text{Ann}(M) \subseteq P$, then for any ideal J of R , the inclusion $JM \subseteq PM$ implies either $J \subseteq P$ or $M = PM$ (see [8]). We deduce

$$\sqrt{(N : M)} \subseteq \sqrt{I}.$$

Combining the inclusions, we obtain

$$\sqrt{I} = \sqrt{(N : M)},$$

which shows that N is indeed a quasi-primary submodule of M . $\square$

Proposition 4. *Let R be a valuation ring and M an R -module. Then, M satisfies condition (*).*

Proof. This follows directly from the fact that, over a valuation ring, every submodule of M is automatically quasi-primary [5] (Theorem 5.10). Hence, M trivially satisfies (*). $\square$

Proposition 5. *Let M be an R -module that satisfies condition (*). Then, the following properties hold:*

1. *For any proper submodule N of M , the quotient module M/N also satisfies (*).*

2. Let K and N be proper submodules of M such that

$$(N : M) \subseteq (K : M) \subseteq \sqrt{(N : M)}$$

and let $\sqrt{(N : M)} = P$ be a prime ideal of R . If N is P -primary, then K is also P -primary.

Proof. Assume M satisfies (*).

1. Consider a proper submodule N of M . For any submodule K containing N , we have

$$\sqrt{(K/N : M/N)} = \sqrt{(K : M)},$$

which shows that quasi-primary submodules of M/N correspond to quasi-primary submodules of M . Therefore, M/N satisfies (*).

2. Suppose K and N are as stated. Since

$$\sqrt{(K : M)} = \sqrt{(N : M)} = P,$$

and N is P -primary, the inclusion $(N : M) \subseteq (K : M) \subseteq \sqrt{(N : M)}$ ensures that K is also P -primary.

□

Theorem 1. Let M be an R -module satisfying condition (*), and let P be a prime ideal of R . If N is a P -primary submodule and K is a submodule with $P \subset (K : M)$, then

$$N(K : M) = N.$$

Proof. Let N be a P -primary submodule of M , and let K be a submodule with $P \subset (K : M)$. We first show that $N(K : M)$ is P -primary. Consider the radical of its annihilator:

$$\sqrt{(N(K : M) : M)}.$$

Since $N \subseteq N(K : M)$, it is immediate that

$$\sqrt{(N(K : M) : M)} \subseteq \sqrt{(N : M)} = P.$$

For the reverse inclusion, take any $x \in P = \sqrt{(N : M)}$. By definition, there exists a positive integer n such that

$$x^n M \subseteq N.$$

Multiplying by x gives

$$x^{n+1} M \subseteq xN.$$

Since $x \in P \subset (K : M)$, it follows that $xM \subseteq K$, and therefore $xN \subseteq N(K : M)$. This implies

$$x^{n+1} \in (N(K : M) : M),$$

so $x \in \sqrt{(N(K : M) : M)}$. Consequently, we obtain

$$P \subseteq \sqrt{(N(K : M) : M)},$$

and combining with the previous inclusion gives

$$\sqrt{(N(K : M) : M)} = P.$$

Thus, $N(K : M)$ is indeed a P -primary submodule. Finally, since $(K : M) \not\subseteq P = \sqrt{(N(K : M) : M)}$, the chain of inclusions

$$N \subseteq N(K : M) \subseteq N$$

forces equality, and we conclude that

$$N(K : M) = N.$$

□

Theorem 2. Let M be an R -module satisfying condition (*), let P be a nonmaximal prime ideal of R , and let N be a P -primary submodule of M . Then,

$$(N : M) = P.$$

Proof. Since N is P -primary, it is immediate that $(N : M) \subseteq P$. We now show the reverse inclusion $P \subseteq (N : M)$.

Because P is nonmaximal, there exists a maximal ideal P_1 of R such that $P \subset P_1$. Take any $p_1 \in P_1 \setminus P$ and any $p \in P$. For an arbitrary $m \in M$, consider the submodule

$$N + pp_1(m).$$

Its annihilator satisfies

$$(N + pp_1(m) : M) \subseteq P,$$

so that

$$\sqrt{(N + pp_1(m) : M)} = P.$$

By assumption (*), $N + pp_1(m)$ is therefore a P -primary submodule.

Now, since $pp_1m \in N + pp_1(m)$ and $p_1 \notin P$, we have $pm \in N + pp_1(m)$. Hence, there exist $n \in N$ and $r \in R$ such that

$$pm = n + pp_1rm.$$

Rewriting gives

$$pm(1 - p_1r) = n \in N.$$

Because $1 - p_1r \notin P$, it follows that $pm \in N$. Since m was arbitrary, this shows $pM \subseteq N$, i.e., $p \in (N : M)$.

As $p \in P$ was arbitrary, we conclude that

$$P \subseteq (N : M),$$

and combining with the previous inclusion yields $(N : M) = P$. □

It is known that a submodule N of M is called *idempotent* if

$$N = (N : M)N$$

(see [4] for further details). In the following result, we investigate idempotent submodules in modules satisfying condition (*).

Corollary 1. Let M be an R -module satisfying condition (*), and let P be a nonmaximal prime ideal of R . If N is a P -primary submodule of M , then N is idempotent.

Proof. Let N be a P -primary submodule of M , with P nonmaximal. Clearly,

$$N(N : M) \subseteq N,$$

so it remains to show the reverse inclusion, $N \subseteq N(N : M)$.

First, we show that $N(N : M)$ is P -quasi-primary. From Theorem 2, we know that $(N : M) = P$, so for any $x \in P$, we have $xM \subseteq N$. Consequently,

$$x(N : M)M \subseteq N(N : M),$$

which implies $x^2M \subseteq N(N : M)$, and thus $x^2 \in (N(N : M) : M)$. Therefore,

$$x \in \sqrt{(N(N : M) : M)},$$

showing that

$$P \subseteq \sqrt{(N(N : M) : M)} \subseteq P.$$

Hence, $N(N : M)$ is a P -quasi-primary submodule.

Since M satisfies (*), any quasi-primary submodule is primary, so $N(N : M)$ is P -primary. By Theorem 2, we then have

$$(N(N : M) : M) = P.$$

Now, suppose there exists $m \in N \setminus N(N : M)$. Then, for some $r \in P = (N : M)$, we would have $rm \notin N(N : M)$. However, since $r \in (N(N : M) : M)$, this would imply $rm \in N(N : M)$ for all $m \in M$, which is a contradiction. Therefore,

$$N \subseteq N(N : M) \subseteq N,$$

and we conclude that

$$N(N : M) = N.$$

Thus, N is idempotent. $\square$

Proposition 6. Let $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of an R -module M , and let Q be a proper submodule of M with $M_2 \subseteq Q$. Then, Q is quasi-primary in M if and only if $M_1 \cap Q$ is quasi-primary in M_1 .

Proof. First, observe that for a submodule Q with $M_2 \subseteq Q$, the annihilators satisfy

$$(Q : M) = (Q : M_1).$$

Indeed, if $r \in (Q : M)$, then $r(m_1 + m_2) \in Q$ for all $m_1 \in M_1$ and $m_2 \in M_2$. Since $M_2 \subseteq Q$, it follows that $rm_1 \in Q$ for every $m_1 \in M_1$, so $r \in (Q : M_1)$. Conversely, if $r \in (Q : M_1)$, then $rM_1 \subseteq Q$, and using $M_2 \subseteq Q$, we also have $rM \subseteq Q$; hence, $r \in (Q : M)$.

($\Rightarrow$) Assume Q is quasi-primary in M . Then, $\sqrt{(Q : M)} = \sqrt{(Q : M_1)}$ is a prime ideal. Take any $a, b \in R$ with $ab \in \sqrt{((M_1 \cap Q) : M_1)}$. By definition, there exists $n \in \mathbb{N}$ such that $(ab)^n M_1 \subseteq M_1 \cap Q$, which implies $(ab)^n \in (Q : M_1)$. Since Q is quasi-primary in

M , we deduce that $a \in \sqrt{(Q : M_1)}$ or $b \in \sqrt{(Q : M_1)}$. Therefore, $a \in \sqrt{((M_1 \cap Q) : M_1)}$ or $b \in \sqrt{((M_1 \cap Q) : M_1)}$, showing that $M_1 \cap Q$ is quasi-primary in M_1 .

($\Leftarrow$) Conversely, assume $M_1 \cap Q$ is quasi-primary in M_1 . Let $ab \in \sqrt{(Q : M)}$. Then, $(ab)^n M \subseteq Q$ for some $n \in \mathbb{N}$. Since $M_2 \subseteq Q$, it follows that $(ab)^n M_1 \subseteq M_1 \cap Q$, i.e., $ab \in \sqrt{((M_1 \cap Q) : M_1)}$. By the quasi-primary property of $M_1 \cap Q$, we have $a \in \sqrt{((M_1 \cap Q) : M_1)}$ or $b \in \sqrt{((M_1 \cap Q) : M_1)}$, which implies $a \in \sqrt{(Q : M_1)} = \sqrt{(Q : M)}$ or $b \in \sqrt{(Q : M)}$. Hence, Q is quasi-primary in M . $\square$

The next corollary follows directly from the previous proposition.

Corollary 2. Let $M = M_1 \oplus M_2$ for submodules M_1 and M_2 of M . If M_1 satisfies condition (*), then any quasi-primary submodule of M that contains M_2 is, in fact, primary.

Proof. Suppose M_1 satisfies (*), and let Q be a quasi-primary submodule of M with $M_2 \subseteq Q$.

Take any $r \in R$ and $m \in M$ such that $rm \in Q$. Decompose m as $m = m_1 + m_2$ with $m_1 \in M_1$ and $m_2 \in M_2$. Then,

$$rm = rm_1 + rm_2 \in Q.$$

Since $m_2 \in M_2 \subseteq Q$, it follows that $rm_1 \in Q$.

By Proposition 3 and the assumption that M_1 satisfies (*), we have

$$r \in \sqrt{(Q : M_1)} \quad \text{or} \quad m_1 \in Q.$$

Finally, because $(Q : M) = (Q : M_1)$ and $M_2 \subseteq Q$, this implies

$$r \in \sqrt{(Q : M)} \quad \text{or} \quad m \in Q,$$

showing that Q is primary in M . $\square$

3. sM-Prime Ideals

In this section, we introduce the notions of weak sM-prime ideals and sM-prime ideals in the context of an R -module M .

Definition 2. Let M be an R -module and P a prime ideal of R . Then, we define the following:

1. P is called a weak sM-prime ideal of M if there exists at least one P -quasi-primary submodule of M and, moreover, every P -quasi-primary submodule of M is actually P -primary.
2. P is called an sM-prime ideal of M if every prime ideal Q contained in P is a weak sM-prime ideal of M .

These definitions allow us to explore the interactions between prime ideals, minimal ideals, and maximal ideals in R via concrete examples, highlighting how the sM-prime structure influences the behavior of submodules in M .

Example 3. Let M be an R -module, and let P be a maximal ideal of R . Suppose N is a submodule of M such that

$$\sqrt{(N : M)} = P.$$

Take any $r \in R$ and $m \in M$ with $rm \in N$ and $m \notin N$. Then, clearly $r \in (N : m) \subseteq \sqrt{(N : m)}$. Since $\sqrt{(N : M)} \subseteq \sqrt{(N : m)}$ and $\sqrt{(N : M)}$ is maximal, it follows that $r \in \sqrt{(N : M)} = P$. This shows that N is P -primary.

Consequently, maximal ideals of R are weak sM -prime. Moreover, one can easily deduce the following equivalences: every prime ideal of R is weak sM -prime for M if and only if every maximal ideal is sM -prime for M , which is also equivalent to the statement that every quasi-primary submodule of M is primary; in other words, M satisfies condition $(*)$.

Example 4. Let P be a minimal prime ideal of R . Then, P is an sM -prime ideal for M if and only if it is a weak sM -prime ideal for M . This follows directly from the definitions, as no smaller prime ideals exist under P .

Proposition 7. Let $N_1, N_2, \dots, N_k$ be proper submodules of an R -module M , and let P be a weak sM -prime ideal of R . If

$$\bigcap_{i=1}^k N_i$$

is a P -quasi-primary submodule of M , then there exists some $j \in \{1, 2, \dots, k\}$ such that N_j is P -quasi-primary.

Proof. Assume that P is weak sM -prime and that

$$\bigcap_{i=1}^k N_i$$

is P -quasi-primary. Then, by definition,

$$\sqrt{\left(\bigcap_{i=1}^k N_i : M\right)} = P.$$

Using standard properties of annihilators and radicals, we have

$$\sqrt{\left(\bigcap_{i=1}^k N_i : M\right)} = \sqrt{\bigcap_{i=1}^k (N_i : M)} = \bigcap_{i=1}^k \sqrt{(N_i : M)} = P.$$

Since $P = \bigcap_{i=1}^k \sqrt{(N_i : M)}$, it follows by [8] (Lemma 3.55) that there exists some index $j \in \{1, 2, \dots, k\}$ such that

$$\sqrt{(N_j : M)} = P.$$

Hence, N_j is a P -quasi-primary submodule of M , as claimed. $\square$

Proposition 8. Let P be a nonmaximal prime ideal of R . If P is a weak sM -prime ideal of M , then for every P -quasi-primary submodule N of M , we have

$$(N : M) = P.$$

Conversely, if M is a multiplication R -module and $(N : M) = P$ holds for all P -quasi-primary submodules N of M , then P is a weak sM -prime ideal of M .

Proof. Assume that P is a weak sM -prime ideal of M and let N be a P -quasi-primary submodule. We aim to show $P \subseteq (N : M)$.

Since P is nonmaximal, there exists a maximal ideal P_1 such that $P \subset P_1$. Pick $p \in P$ and $p_1 \in P_1 \setminus P$. For any $m \in M$, we have

$$(N : M) \subseteq (N + pp_1(m) : M) \subseteq P,$$

which implies

$$\sqrt{(N + pp_1(m) : M)} = P.$$

Because P is weak sM -prime, the submodule $N + pp_1(m)$ is P -primary. Since $pp_1m \in N + pp_1(m)$ and $p_1 \notin P$, we get $pm \in N + pp_1(m)$.

Thus, there exist $n \in N$ and $r \in R$ such that

$$pm = n + pp_1rm \Rightarrow pm(1 - p_1r) = n \in N.$$

As $1 - p_1r \notin P$, it follows that $pm \in N$, and therefore $pM \subseteq N$, i.e., $p \in (N : M)$. Since $p \in P$ was arbitrary, we conclude $(N : M) = P$.

For the converse, assume M is a multiplication module and that $(N : M) = P$ for every P -quasi-primary submodule N of M . Then, $N = (N : M)M = PM$. As $\text{Ann}(M) \subseteq P$, it follows from [8] that PM is a prime submodule. Therefore, N is P -primary, and hence P is weak sM -prime. $\square$

Theorem 3. Let M_1 and M_2 be R -modules, P a prime ideal of R , and $g : M_1 \rightarrow M_2$ an R -epimorphism. Then, the following are true:

1. If N is a P -quasi-primary submodule of M_2 , then the preimage $g^{-1}(N)$ is a P -quasi-primary submodule of M_1 .
2. If K is a P -quasi-primary submodule of M_1 containing $\ker(g)$, then the image $g(K)$ is a P -quasi-primary submodule of M_2 .

Proof. Assume $g : M_1 \rightarrow M_2$ is surjective and let P be a prime ideal of R .

1. Let us show that $g^{-1}(N)$ is P -quasi-primary. Consider any $r \in \sqrt{(g^{-1}(N) : M_1)}$. Then, $r^n M_1 \subseteq g^{-1}(N)$ for some positive integer n , which implies

$$g(r^n M_1) \subseteq g(g^{-1}(N)) \subseteq N.$$

Since g is surjective, $g(r^n M_1) = r^n M_2 \subseteq N$, so $r^n \in (N : M_2)$ and therefore $r \in \sqrt{(N : M_2)} = P$. Conversely, if $r \in P = \sqrt{(N : M_2)}$, then $r^v M_2 \subseteq N$ for some $v \in \mathbb{N}$. Using surjectivity again, $r^v M_1 \subseteq g^{-1}(N)$, showing $r \in \sqrt{(g^{-1}(N) : M_1)}$. Hence, $g^{-1}(N)$ is P -quasi-primary.

2. Next, assume K is P -quasi-primary and $\ker(g) \subseteq K$. To show $g(K)$ is P -quasi-primary, we prove

$$\sqrt{(g(K) : M_2)} = \sqrt{(K : M_1)}.$$

Let $r \in \sqrt{(g(K) : M_2)}$. Then, $r^n M_2 \subseteq g(K)$ for some n . For each $m \in M_1$, surjectivity gives $r^n g(m) = g(r^n m) \in g(K)$, so $r^n m \in K$ because $r^n m - k \in \ker(g) \subseteq K$ for some $k \in K$. Thus, $r^n M_1 \subseteq K$, and $r \in \sqrt{(K : M_1)}$.

Conversely, if $r \in \sqrt{(K : M_1)}$, then $r^n M_1 \subseteq K$ for some n , which implies

$$r^n M_2 = r^n g(M_1) = g(r^n M_1) \subseteq g(K),$$

so $r \in \sqrt{(g(K) : M_2)}$. This proves $g(K)$ is P -quasi-primary.

$\square$

Corollary 3. Let M_1 and M_2 be R -modules, P a prime ideal of R , and $g : M_1 \rightarrow M_2$ an R -epimorphism. Then, the following hold:

1. If P is weak sM -prime for M_2 , then P is weak sM -prime for M_1 .
2. If P is weak sM -prime for M_1 , then P is weak sM -prime for M_2 .

Proof. These claims follow directly from Theorem 3.

For (1), if P is weak sM -prime for M_2 , then every P -quasi-primary submodule N of M_2 is P -primary. By Theorem 3(1), $g^{-1}(N)$ is P -quasi-primary in M_1 , and since M_1 inherits the quasi-primary property from M_2 , it is P -primary. Hence, P is weak sM -prime for M_1 .

Similarly, for (2), if P is weak sM -prime for M_1 , then for any P -quasi-primary submodule K of M_1 with $\ker(g) \subseteq K$, $g(K)$ is P -quasi-primary in M_2 by Theorem 3(2). As K is P -primary in M_1 , $g(K)$ is P -primary in M_2 . Thus, P is weak sM -prime for M_2 . $\square$

Theorem 4. Let M be a finitely generated R -module, P a prime ideal of R , and S a multiplicatively closed subset of R with $P \cap S = \emptyset$. If P is a weak sM -prime ideal of M , then the extended ideal P_S is a weak sM -prime ideal of the localized module M_S .

Proof. Assume that P is weak sM -prime for M . Then, P_S is a prime ideal of the localized ring R_S .

Let N_S be a P_S -quasi-primary submodule of M_S , where N is a proper submodule of M . By properties of localization (see [8] (Theorem 5.31)), we have

$$\sqrt{(N_S :_{R_S} M_S)} = \sqrt{(N : M)_S}.$$

Since N_S is P_S -quasi-primary, it follows that $\sqrt{(N : M)} = P$ by [8] (Theorem 5.12), so N is a P -quasi-primary submodule of M . By assumption, every P -quasi-primary submodule of M is P -primary; hence, N is P -primary.

Now, it is well-known that localization preserves the primariness of submodules, so N_S is P_S -primary in M_S . Therefore, every P_S -quasi-primary submodule of M_S is P_S -primary, which shows that P_S is weak sM -prime for M_S . $\square$

Corollary 4. Let M be a finitely generated R -module, P a prime ideal of R , and S a multiplicatively closed subset of R with $P \cap S = \emptyset$. If P is an sM -prime ideal of M , then the localized ideal P_S is an sM -prime ideal of M_S .

Proof. Suppose that P is an sM -prime ideal of M . Consider any prime ideal $Q_S \subseteq P_S$ in R_S . It is known that such Q_S corresponds to a prime ideal $Q \subseteq P$ in R via contraction.

Since P is sM -prime, Q is weak sM -prime for M . By Theorem 4, the extended ideal Q_S is weak sM -prime for M_S .

Because this holds for every prime ideal $Q_S \subseteq P_S$, it follows that P_S is sM -prime for the localized module M_S . $\square$

Theorem 5. Let P be a nonmaximal prime ideal of R . If P is a weak sM -prime ideal for M , then every P -quasi-primary submodule of M is idempotent.

Proof. Let N be a P -quasi-primary submodule of M . By the definition of weak sM -prime, N is in fact P -primary. It is clear that

$$N(N : M) \subseteq N.$$

To prove the reverse inclusion, we first show that $N(N : M)$ is itself P -quasi-primary. By Proposition 8, we have $(N : M) = P$. Take any $x \in P = (N : M)$. Then,

$$xM \subseteq N \implies x(N : M)M \subseteq N(N : M).$$

This yields $x^2M \subseteq N(N : M)$, so $x^2 \in (N(N : M) : M)$, and therefore $x \in \sqrt{(N(N : M) : M)}$. Hence,

$$P \subseteq \sqrt{(N(N : M) : M)} \subseteq P,$$

which shows that $N(N : M)$ is P -quasi-primary. By the weak sM -prime property, $N(N : M)$ is actually P -primary, and Proposition 8 gives $(N(N : M) : M) = P$.

Suppose, for contradiction, that there exists $m \in N \setminus N(N : M)$. Then, there would exist $r \in P = (N : M)$ such that $rm \notin N(N : M)$. But $r \in (N(N : M) : M)$ implies $rm \in N(N : M)$ for all $m \in M$, a contradiction. Therefore, no such m exists, and we conclude that

$$N \subseteq N(N : M) \subseteq N.$$

Consequently, $N = N(N : M)$, so N is idempotent. $\square$

Theorem 6. *Let P be a nonmaximal prime ideal of R . If P is a weak sM -prime ideal for an R -module M , then M possesses a unique P -quasi-primary submodule.*

Proof. Suppose, for contradiction, that M contains two distinct P -quasi-primary submodules N and K . By definition, we have

$$\sqrt{(N : M)} = \sqrt{(K : M)} = P.$$

Proposition 8 then ensures that

$$(N : M) = (K : M) = P.$$

Consider the submodules $(N : M)N$ and $(K : M)K$. By the same reasoning used in Theorem 5, these are P -quasi-primary submodules. Since P is weak sM -prime, they are actually P -primary. Proposition 5 then guarantees that they are idempotent, i.e.,

$$(N : M)N = N \quad \text{and} \quad (K : M)K = K.$$

Using these identities, we get

$$K = (K : M)K = (N : M)K \subseteq (N : M)N = N.$$

By symmetry, the same argument yields $N \subseteq K$. Therefore,

$$N = K,$$

contradicting the assumption that they were distinct. Hence, M can have at most one P -quasi-primary submodule. $\square$

4. Conclusions

In this paper, we explored sM -prime ideals in commutative rings and their connection to submodules of modules. We concentrated on modules where every quasi-primary submodule is also primary, showing that this condition leads to a strong control over the structure of the module. In particular, we established that in multiplication modules, quasi-primary ideals give rise to quasi-primary submodules, and that for nonmaximal prime ideals, the annihilator of a primary submodule coincides with the prime ideal. We also proved that weak sM -prime ideals are preserved under localization in finitely generated modules and that nonmaximal weak sM -prime ideals correspond to a unique quasi-primary submodule.

These findings suggest several directions for future research. One could consider generalizing sM -prime ideals to other types of submodules, such as n -absorbing or t -primary submodules. It would also be interesting to examine these ideals in more general settings, including non-commutative rings or modules with additional structure. Studying how sM -prime ideals affect module decompositions, endomorphism rings, or other module properties may provide further insights. Finally, these ideas could help define new classes of submodules or better understand their role in module theory.

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References

1. Fuchs, L. On quasi-primary ideals. *Acta Sci. Math. (Szeged)* **1947**, *11*, 174–183.
2. Gilmer, R.W. Rings in which semi-primary ideals are primary. *Pac. J. Math.* **1962**, *12*, 185–190.
3. Jayaram, C.; Johnson, E.W. s -Prime elements in multiplicative lattices. *Period. Math. Hung.* **1995**, *31*, 201–208.
4. Ali, M.M. Idempotent and nilpotent submodules of multiplication modules. *Commun. Algebra* **2008**, *36*, 4620–4642.
5. Larsen, M.D.; McCarthy, P.J. *Multiplicative Theory of Ideals*; Academic Press: New York, NY, USA; London, UK, 1971.
6. Lu, C.P. Prime submodules of modules. *Comment. Math. Univ. St. Pauli* **1984**, *33*, 61–69.
7. Moore, M.E.; Smith, S.J. Prime and radical submodules of modules over commutative rings. *Commun. Algebra* **2002**, *30*, 5037–5064.
8. Sharp, R.Y. *Steps in Commutative Algebra*; Cambridge University Press: Cambridge, UK, 2000.
9. El-Bast, Z.A.; Smith, P.P. Multiplication modules. *Commun. Algebra* **1988**, *16*, 755–779.
10. Moghimi, H.F.; Samiei, M. Quasi-primary submodules satisfying the primeful property I. *Hacet. J. Math. Stat.* **2016**, *45*, 1421–1434.
11. Moghimi, H.F.; Samiei, M. Quasi-primary submodules satisfying the primeful property II. *Hacet. J. Math. Stat.* **2015**, *44*, 801–811.
12. Atani, S.E.; Darani, A.Y. On quasi-primary submodules. *Chiang Mai J. Sci.* **2006**, *33*, 249–254.
13. Mostafanasab, H.; Yetkin, E.; Tekir, Ü.; Darani, A.Y. On 2-Absorbing Primary Submodules of Modules over Commutative Rings. *An. Stiint. Univ. Ovidius Constanța Ser. Mat.* **2016**, *24*, 335–351.
14. Celikel, E.Y.; Khashan, H. Quasi J -submodules. *Turk. J. Math.* **2022**, *46*, 2610–2624.
15. Moghimi, H.F.; Samiei, M. Quasi-primaryful modules. *Asian-Eur. J. Math.* **2015**, *8*, 1550051.
16. Sönmez, D.; Yeşilot, G.; Hila, K. On weakly $\delta_{R,M}$ -semiprimary submodules. *Rend. Circ. Mat. Palermo Ser. 2* **2022**, *71*, 119–130.

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