

Article

Nonlinear Mixed Skew Lie-Type Derivations on $\ast$ -Algebras

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Abstract

Consider $\mathcal{A}$ as a unital $\ast$ -algebra. Given elements $A, B \in \mathcal{A}$, the operations $A \bullet B = AB + BA^\ast$ and $[A, B]_\ast = AB - BA^\ast$ represent the skew Jordan product and the skew Lie product, correspondingly. Within this study, we demonstrate that a nonlinear mapping $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ adhering to $\Phi([A_1 \bullet A_2 \bullet \dots \bullet A_{n-1}, A_n]_\ast) = \sum_{j=1}^n [A_1 \bullet A_2 \bullet \dots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \dots \bullet A_{n-1}, A_n]_\ast$ for every $A_1, A_2, \dots, A_n \in \mathcal{A}$ with $n \geq 3$, then Φ is an additive $\ast$ -derivation. As an application of our main results, we establish characterizations of prime $\ast$ -algebras, standard operator algebras, von Neumann algebras without central summands of type I_1 , and factor von Neumann algebras.

Keywords: mixed skew Lie-type derivations; $\ast$ -algebras; von Neumann algebras

MSC: 16W25; 16W10; 16N60; 47B47

1. Introduction

Throughout this article, an algebra $\mathcal{A}$ refers to associative algebra over the complex numbers $\mathbb{C}$. Associative algebras equipped with an involution, commonly known as $\ast$ -algebras, play a fundamental role in various branches of mathematics and mathematical physics, including operator theory, ring theory, and quantum mechanics. These structures generalize the familiar algebras of operators on Hilbert spaces and provide a rich framework for studying symmetries, derivations, and automorphisms. In this paper, we consider a unital associative $\ast$ -algebra $\mathcal{A}$. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is called *additive* if $\Phi(A + B) = \Phi(A) + \Phi(B)$ for all $A, B \in \mathcal{A}$ and an *involution* on $\mathcal{A}$ is an additive anti-automorphism $\ast : \mathcal{A} \rightarrow \mathcal{A}$ satisfying $(A^\ast)^\ast = A$ and $(AB)^\ast = B^\ast A^\ast$ for all $A, B \in \mathcal{A}$. Algebras endowed with such an involution are known as $\ast$ -algebras. For elements $A, B \in \mathcal{A}$, several bilinear operations arise naturally in the study of quadratic and sesquilinear forms, as well as in operator algebras and the ring theory. These include the *Jordan product* $A \circ B = AB + BA$, the *Lie product* $[A, B] = AB - BA$, the *skew Jordan product* $A \bullet B = AB + BA^\ast$, and the *skew Lie product* $[A, B]_\ast = AB - BA^\ast$. These operations have been extensively investigated in the literature; for more details, we refer the reader to [1–8].

The historical development of derivations and automorphisms in algebras traces back to foundational questions about the additivity of nonlinear(or multiplicative) maps. Martindale [9] provided partial results using sets of idempotents, while Daif [10] introduced the concept of multiplicative derivations and proved their additivity in rings with nontrivial idempotents.

An additive map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is an *additive derivation* if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{A}$. If, in addition, $\Phi(A^\ast) = \Phi(A)^\ast$ for all $A \in \mathcal{A}$, then Φ is an *additive $\ast$ -derivation*.



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A related concept is that of a *Lie derivation*, defined by $\Phi([A, B]) = [\Phi(A), B] + [A, \Phi(B)]$ for all $A, B \in \mathcal{A}$. The relationship between Lie derivations and additive derivations has been a subject of significant interest (see, e.g., [11–13]).

Relaxing the additivity assumption leads to *nonlinear Lie derivations*, also known as *Lie derivable maps*, which satisfy the same functional equation without requiring linearity. Such maps have been studied on triangular algebras and operator algebras by Chen and Zhang [14], Yu and Zhang [15], Lu and Liu [11], and Jing and Lu [13].

Parallel to these developments, the skew Lie product has received attention. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is a *nonlinear skew Lie derivation* if $\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$ for all $A, B \in \mathcal{A}$. Recently, Yu and Zhang [16] and Jing [17] showed that nonlinear skew Lie derivations on factor von Neumann algebras and standard operator algebras reduce to additive $*$ -derivations.

The notion extends to *nonlinear Lie triple derivations*, where $\Phi([A, B], C) = [[\Phi(A), B], C] + [[A, \Phi(B)], C] + [[A, B], \Phi(C)]$ for all $A, B, C \in \mathcal{A}$. These have been examined on triangular algebras, parabolic subalgebras, generalized matrix algebras, and von Neumann algebras (see [14,18–21]).

The skew analog gives rise to *nonlinear skew Lie triple derivations*, satisfying $\Phi([A, B]_*, C) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C]_* + [[A, B]_*, \Phi(C)]_*$. Li et al. [22] demonstrated that on factor von Neumann algebras, these maps are precisely additive $*$ -derivations, highlighting the connection between skew Lie-type maps and $*$ -derivations.

A related concept is the *nonlinear mixed Lie triple derivation* [23–25], which combines Lie, skew Lie and skew Jordan products and typically reduces to additive $*$ -derivations under appropriate conditions.

More generally, Lie-type derivations involve higher-order commutators. Define the $(n - 1)$ -th commutator recursively by $p_1(A_1) = A_1$, $p_2(A_1, A_2) = [A_1, A_2]$, and $p_n(A_1, \dots, A_n) = [p_{n-1}(A_1, \dots, A_{n-1}), A_n]$ for $n \geq 3$. A nonlinear map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is a *nonlinear Lie n -derivation* if $\Phi(p_n(A_1, \dots, A_n)) = \sum_{k=1}^n p_n(A_1, \dots, A_{k-1}, \Phi(A_k), A_{k+1}, \dots, A_n)$. This concept was introduced by Abdullaev [26], who provided decomposition theorems for von Neumann algebras. Subsequent works extended these results to triangular rings and generalized matrix algebras [27,28]. In similar way, we can define a nonlinear skew Lie (skew Jordan) n -derivation. A nonlinear skew Lie (skew Jordan) 2-derivation coincides with a nonlinear skew Lie (skew Jordan) derivation, while a nonlinear skew Lie (skew Jordan) 3-derivation is exactly a nonlinear skew Lie (skew Jordan) triple derivation. In general, nonlinear skew Lie (skew Jordan) 2-derivations, nonlinear skew Lie (skew Jordan) 3-derivations, and nonlinear skew Lie (skew Jordan) n -derivations are collectively referred to as nonlinear skew Lie-type (skew Jordan-type) derivations. Zhang [29] proved that every nonlinear skew Jordan derivation on a factor von Neumann algebra must be an additive $*$ -derivation. Later, Zhao and Li [30] showed that if A is a von Neumann algebra with no central summands of type I_1 , then a nonlinear mapping $\Phi : A \rightarrow A$ is a nonlinear $*$ -Jordan triple derivation if and only if Φ is an additive $*$ -derivation. This result was further extended by Li et al. [31] to nonlinear $*$ -Jordan-type derivations on arbitrary $*$ -algebras.

Motivated by these advancements, recent studies have focused on nonlinear skew Lie-type derivations on $*$ -algebras, showing that they often reduce to additive $*$ -derivations [31–33]. Building upon this foundation, we extend the findings of Li and Zhang [25]. the present work introduces and investigates the *nonlinear mixed skew Lie-type derivation*. For a unital $*$ -algebra $\mathcal{A}$ and elements $A, B \in \mathcal{A}$, recall $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$. For $n \geq 3$, we define a nonlinear mixed skew Lie-type derivation $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ by $\Phi([A_1 \bullet A_2 \bullet \dots \bullet A_{n-1}, A_n]_*) = \sum_{j=1}^n [A_1 \bullet A_2 \bullet \dots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \dots \bullet A_{n-1}, A_n]_*$ for every $A_1, A_2, \dots, A_n \in \mathcal{A}$.

Our motivation stems from the recurring theme in the literature that nonlinear Lie-type, skew Lie-type and skew Jordan-type maps often becomes additive $*$ -derivations under mild algebraic conditions, such as primeness or the absence of certain central projections. This pattern suggests a deeper structural rigidity in $*$ -algebras, prompting us to explore whether similar results hold for the mixed skew variant, which combines elements of both skew Jordan and skew Lie products. By introducing this new class of maps, we aim to unify and extend previous findings on mixed Lie triple derivations [25].

This research fills a gap in the existing literature by addressing nonlinear mixed skew Lie-type derivations specifically in the associative $*$ -algebra setting, where previous works have focused on Lie-type or skew Lie-type or skew Jordan-type maps separately. While related functional identities have been studied in broader nonassociative structures, such as alternative $*$ -algebras (see, e.g., [34] for nonlinear mixed $*$ -Jordan-type derivations on alternative $*$ -algebras), our work provides a precise characterization in the associative case, leveraging Peirce decomposition techniques that are particularly effective here. The identity we consider shares a similar "mixed" philosophy with that in [34], combining Jordan-type and Lie-type products, but our recursive skew Lie-type nesting and hypotheses are tailored to the associative framework, yielding new insights into the rigidity of such maps.

In this manuscript, we prove that, under moderate conditions, nonlinear mixed skew Lie-type derivations on $*$ -algebras are necessarily additive $*$ -derivations. As corollaries, we verify this characterization for important classes of prime $*$ -algebras, including standard operator algebras, factor von Neumann algebras, and von Neumann algebras without central summands of type I_1 . Our approach relies on the Peirce decomposition associated with idempotents, a fundamental tool in the structural theory of rings and algebras.

The Peirce decomposition, named after Benjamin Peirce, originates from the study of idempotents in associative rings and has been widely applied in nonassociative algebras, Jordan algebras, and operator algebras. Given a unital algebra $\mathcal{A}$ and a nonzero idempotent $P_1 \in \mathcal{A}$ (i.e., $P_1^2 = P_1$), let $P_2 = I - P_1$, where I is the unity element. Then P_2 is also an idempotent, and P_1, P_2 are orthogonal in the sense that $P_1 P_2 = P_2 P_1 = 0$. The algebra $\mathcal{A}$ decomposes as a direct sum of vector spaces $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ for $i, j = 1, 2$.

This decomposition satisfies the following multiplication rules:

1. $\mathcal{A}_{ij} \mathcal{A}_{jl} \subseteq \mathcal{A}_{il}$ for $i, j, l \in \{1, 2\}$,
2. $\mathcal{A}_{ij} \mathcal{A}_{kl} = 0$ if $j \neq k$, for $i, j, k, l \in \{1, 2\}$.

In the context of $*$ -algebras, if the idempotent P_1 is self-adjoint (i.e., $P_1^* = P_1$), then $P_2^* = P_2$, and the Peirce spaces often exhibit additional symmetry: $\mathcal{A}_{ij}^* \subseteq \mathcal{A}_{ji}$. This property is particularly useful in von Neumann algebras and C^* -algebras, where idempotents are typically projections (self-adjoint idempotents). The Peirce decomposition allows for the reduction of global properties of maps on $\mathcal{A}$ to local behaviors on these subspaces, facilitating proofs of additivity and other structural results by analyzing the action of Φ on each $\mathcal{A}_{ij}$.

This work not only advances the understanding of functional equations in $*$ -algebras but also opens avenues for future research. Potential extensions include examining these maps on non-prime $*$ -algebras, algebras over other fields, or in the context of Banach $*$ -algebras with continuity assumptions. Open problems may involve classifying such derivations in finite-dimensional settings or exploring their connections to automorphisms and inner derivations.

2. The Main Result and Its Proof

The principal theorem of this manuscript is articulated as follows:

Theorem 1. Let $\mathcal{A}$ be a unital $*$ -algebra featuring the unit I . Suppose $\mathcal{A}$ includes a nontrivial projection P fulfilling

$$XAP = 0 \quad \text{implies} \quad X = 0, \quad (1)$$

for all $X \in \mathcal{A}$, and

$$XA(I - P) = 0 \quad \text{implies} \quad X = 0, \quad (2)$$

for all $X \in \mathcal{A}$. Then a mapping $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ obeys

$$\begin{aligned} & \Phi([A_1 \bullet A_2 \bullet \cdots \bullet A_{n-1}, A_n]_*) \\ &= \sum_{j=1}^n [A_1 \bullet A_2 \bullet \cdots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \cdots \bullet A_{n-1}, A_n]_* \end{aligned}$$

for every $A_1, A_2, \dots, A_n \in \mathcal{A}$ precisely when Φ constitutes an additive $*$ -derivation.

Remark 1. Assumption (1) guarantees that $\mathcal{A}$ has no nontrivial left annihilators on P , which is invoked in Lemmas 6 and 10 to conclude vanishing of certain Peirce components. Similarly for (2) on $I - P$. See Example 1 for necessity.

Proof. Let $P_1 = P$ and $P_2 = I - P$ be two nontrivial projections in $\mathcal{A}$, and define $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, $1 \leq i, j \leq 2$. By Peirce decomposition, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Accordingly, every element $A \in \mathcal{A}$ can be written uniquely as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{A}_{ij}$. Moreover, if P_1 is self-adjoint, then the involution satisfies $\mathcal{A}_{ij}^* \subseteq \mathcal{A}_{ji}$.

Since the sufficiency part is straightforward, we focus on proving the necessity. The argument proceeds through a sequence of auxiliary lemmas. For clarity, we briefly describe the strategy. First, we establish that $\Phi(0) = 0$. Next, we prove the additivity of Φ on each Peirce component and then extend it to the whole algebra. After that, we show that $\Phi(I) = 0$ and verify compatibility with the involution. Finally, we establish the derivation property. The projection assumptions (1) and (2) play a crucial role in eliminating nontrivial annihilators, thereby forcing certain Peirce components to vanish when necessary.

The above theorem follows from the subsequent lemmas, which provide a structured roadmap for the proof.

Lemmas 1–7 establish additivity component-wise on the Peirce decomposition. Lemmas 8–11 address the centrality of $\Phi(I)$ and show vanishing on projections and the identity element. Lemmas 12–15 prove that Φ is a skew Lie derivation, is compatible with the involution, and is $\mathbb{C}$ -linear (in particular, linear over i). Lemma 16 completes the argument by verifying the full derivation property.

The assumptions (1) and (2) are specifically employed in Lemmas 6 and 10 to deduce triviality of certain components via annihilator arguments. $\square$

Lemma 1. $\Phi(0) = 0$.

Proof. Observe that

$$\begin{aligned} \Phi(0) &= \Phi([0 \bullet \cdots \bullet 0, 0]_*) \\ &= [\Phi(0) \bullet \cdots \bullet 0, 0]_* + \cdots + [0 \bullet \cdots \bullet \Phi(0), 0]_* + [0 \bullet \cdots \bullet 0, \Phi(0)]_* = 0. \end{aligned}$$

$\square$

Lemma 2. For any $A_{12} \in \mathcal{A}_{12}$ and $B_{21} \in \mathcal{A}_{21}$, we obtain

$$\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21}).$$

Proof. Let $M = \Phi(A_{12} + B_{21}) - \Phi(A_{12}) - \Phi(B_{21}) = 0$. It suffices to verify $M = 0$. Evidently,

$$\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} \right]_* = \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), B_{21} \right]_* = 0.$$

Via Lemma 1, we derive

$$\begin{aligned} & \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), \Phi(A_{12} + B_{21}) \right]_* \\ & = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_*\right) \\ & = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} \right]_*\right) + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), B_{21} \right]_*\right) \\ & = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(i(P_2 - P_1)), A_{12} + B_{21} \right]_* \\ & + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), \Phi(A_{12}) + \Phi(B_{21}) \right]_* . \end{aligned}$$

Thus, $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), M \right]_* = 0$. This yields $M_{11} = M_{22} = 0$.

Given $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet A_{12}, P_1 \right]_* = 0$, we deduce

$$\begin{aligned} & \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), P_1 \right]_* + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), P_1 \right]_* \\ & + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (A_{12} + B_{21}), P_1 \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(A_{12} + B_{21}), P_1 \right]_* \\ & + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), \Phi(P_1) \right]_* \\ & = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), P_1 \right]_*\right) \\ & = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet A_{12}, P_1 \right]_*\right) + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet B_{21}, P_1 \right]_*\right) \\ & = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), P_1 \right]_* + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), P_1 \right]_* \end{aligned}$$

$$\begin{aligned}
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (A_{12} + B_{21}), P_1 \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (\Phi(A_{12}) + \Phi(B_{21})), P_1 \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{12} + B_{21}), \Phi(P_1) \right]_*
\end{aligned}$$

Consequently, $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet M, P_1 \right]_* = 0$, implying $M_{12} = 0$ and $M_{21} = 0$. Thus $M = 0$. $\square$

Lemma 3. For any $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$ we obtain

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$$

and

$$\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Proof. Let $M = \Phi(A_{11} + B_{12} + C_{21}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21})$.

From Lemmas 1 and 2, we derive

$$\begin{aligned}
& \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(A_{11} + B_{12} + C_{21}) \right]_* \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_*\right) \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} \right]_*\right) + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), (B_{12} + C_{21}) \right]_*\right) \\
& = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) \right]_*.
\end{aligned}$$

Hence $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), M \right]_* = 0$. Thus $M_{12} = M_{21} = M_{22} = 0$.
As

$$\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), B_{12} \right]_* = \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), C_{21} \right]_* = 0.$$

We infer

$$\left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_*$$

$$\begin{aligned}
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), \Phi(A_{11} + B_{12} + C_{21}) \right]_* \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_*\right) \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} \right]_*\right) + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), B_{12} \right]_*\right) \\
& + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), C_{21} \right]_*\right) \\
& = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(i(P_2 - P_1)), A_{11} + B_{12} + C_{21} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) \right]_* .
\end{aligned}$$

Thus $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (i(P_2 - P_1)), M \right]_* = 0$. Hence $M_{11} = 0$, so $M = 0$. Analogously, $\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$. $\square$

Lemma 4. For every $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$, we derive

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Proof. Let $M = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22})$. Via Lemmas 1 and 3, we obtain

$$\begin{aligned}
& \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(A_{11} + B_{12} + C_{21} + D_{22}) \right]_* \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_*\right) \\
& = \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} \right]_*\right) + \Phi\left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), (B_{12} + C_{21} + D_{22}) \right]_*\right) \\
& = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} \right]_* + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} \right]_*
\end{aligned}$$

$$\begin{aligned}
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), A_{11} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), A_{11} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(A_{11}) \right]_* + \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), B_{12} + C_{21} + D_{22} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), B_{12} + C_{21} + D_{22} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(B_{12} + C_{21} + D_{22}) \right]_* \\
& = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} + D_{22} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}) \right]_* .
\end{aligned}$$

Hence $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_2), M \right]_* = 0$. Thus $M_{12} = M_{21} = M_{22} = 0$. Likewise, $M_{11} = 0$ follows. $\square$

Lemma 5. For any $A_{jk}, B_{jk} \in \mathcal{A}_{jk}$, where $1 \leq j \neq k \leq 2$, we derive

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

Proof. Observe that

$$\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk} \right]_* = (A_{jk} + B_{jk}) - A_{jk}^* - B_{jk}A_{jk}^*,$$

so via Lemma 4,

$$\begin{aligned}
& \Phi(A_{jk} + B_{jk}) + \Phi(-A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*) \\
& = \Phi\left(\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_*\right) \\
& = \left[\Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* \\
& + \cdots + \left[\frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* \\
& + \left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(P_j + A_{jk}), P_k + B_{jk}\right]_* \\
& + \left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), \Phi(P_k + B_{jk})\right]_* \\
& = \left[\Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* \\
& + \cdots + \left[\frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_*
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (\Phi(P_j) + \Phi(A_{jk})), P_k + B_{jk} \right]_* \\
& + \left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (P_j + A_{jk}), (\Phi(P_k) + \Phi(B_{jk})) \right]_* \\
& = \Phi \left(\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet P_j, P_k \right]_* \right) + \Phi \left(\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet P_j, B_{jk} \right]_* \right) \\
& + \Phi \left(\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet A_{jk}, P_k \right]_* \right) + \Phi \left(\left[\frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet A_{jk}, B_{jk} \right]_* \right) \\
& = \Phi(B_{jk}) + \Phi(A_{jk} - A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*) \\
& = \Phi(B_{jk}) + \Phi(A_{jk}) + \Phi(-A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*).
\end{aligned}$$

Thus $\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$. $\square$

Lemma 6. For any $A_{jj}, B_{jj} \in \mathcal{A}_{jj}$, where $1 \leq j \leq 2$, we obtain

$$\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Proof. Let $M = \Phi(A_{jj} + B_{jj}) - \Phi(A_{jj}) - \Phi(B_{jj})$. For $1 \leq j \neq k \leq 2$, we derive

$$\begin{aligned}
& \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} + B_{jj} \right]_* + \left[I \bullet \Phi \left(\frac{I}{2} \right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} + B_{jj} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi \left(\frac{I}{2} \right) \bullet (iP_k), A_{jj} + B_{jj} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_k), A_{jj} + B_{jj} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), \Phi(A_{jj} + B_{jj}) \right]_* \\
& = \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} + B_{jj} \right]_* \right) \\
& = \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} \right]_* \right) + \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), B_{jj} \right]_* \right) \\
& = \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} + B_{jj} \right]_* + \left[I \bullet \Phi \left(\frac{I}{2} \right) \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), A_{jj} + B_{jj} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi \left(\frac{I}{2} \right) \bullet (iP_k), A_{jj} + B_{jj} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(iP_k), A_{jj} + B_{jj} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), \Phi(A_{jj}) + \Phi(B_{jj}) \right]_*.
\end{aligned}$$

Thus $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (iP_k), M \right]_* = 0$. Hence $M_{jk} = M_{kj} = M_{kk} = 0$. Now $M = M_{jj}$. For all $C_{jk} \in \mathcal{A}_{jk}$, $j \neq k$, via Lemma 5,

$$\begin{aligned}
& \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* + \left[I \bullet \Phi \left(\frac{I}{2} \right) \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* \\
& + \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi \left(\frac{I}{2} \right) \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* \\
& + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet \Phi(A_{jj} + B_{jj}), C_{jk} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), \Phi(C_{jk}) \right]_* \\
& = \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* \right) \\
& = \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet A_{jj}, C_{jk} \right]_* \right) + \Phi \left(\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet B_{jj}, C_{jk} \right]_* \right)
\end{aligned}$$

$$\begin{aligned}
&= \left[\Phi(I) \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* + \left[I \bullet \Phi\left(\frac{I}{2}\right) \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* \\
&+ \cdots + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \Phi\left(\frac{I}{2}\right) \bullet (A_{jj} + B_{jj}), C_{jk} \right]_* \\
&+ \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (\Phi(A_{jj}) + \Phi(B_{jj})), C_{jk} \right]_* + \left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet (A_{jj} + B_{jj}), \Phi(C_{jk}) \right]_*.
\end{aligned}$$

Thus $\left[I \bullet \frac{I}{2} \bullet \cdots \bullet \frac{I}{2} \bullet M, C_{jk} \right]_* = 0$, for every $C_{jk} \in \mathcal{A}_{jk}$; that is, $M_{jj}CP_k = 0$ for all $C \in \mathcal{A}$. From (1) and (2), $M = M_{jj} = 0$. $\square$

Given Lemmas 4–6, the subsequent lemma follows readily:

Lemma 7. Φ is additive on $\mathcal{A}$.

Lemma 8. $\Phi(I)$ constitutes a self-adjoint central element within $\mathcal{A}$.

Proof. As $[I \bullet I \bullet \cdots \bullet I, I] = 0$, via Lemma 1,

$$\begin{aligned}
0 &= \Phi([I \bullet I \bullet \cdots \bullet I, I]) \\
&= [\Phi(I) \bullet I \bullet \cdots \bullet I, I] + [I \bullet \Phi(I) \bullet \cdots \bullet I, I] \\
&+ \cdots + [I \bullet I \bullet \cdots \bullet \Phi(I), I] + [I \bullet I \bullet \cdots \bullet I, \Phi(I)] \\
&= 2^{(n-2)}(\Phi(I) - \Phi(I)^*).
\end{aligned}$$

This forces $\Phi(I)$ to be self-adjoint in $\mathcal{A}$.

Moreover, for every $A \in \mathcal{A}$ we obtain

$$\begin{aligned}
0 &= \Phi([I \bullet I \bullet \cdots \bullet I, A]) \\
&= [\Phi(I) \bullet I \bullet \cdots \bullet I, A] + [I \bullet \Phi(I) \bullet \cdots \bullet I, A] \\
&+ \cdots + [I \bullet I \bullet \cdots \bullet \Phi(I), A] + [I \bullet I \bullet \cdots \bullet I, \Phi(A)] \\
&= 2^{(n-2)}(n-1)(\Phi(I)A - A\Phi(I)).
\end{aligned}$$

Hence $\Phi(I)$ is central in $\mathcal{A}$. $\square$

Lemma 9. $P_1\Phi(P_1)P_2 = -P_1\Phi(P_2)P_2$, $P_2\Phi(P_1)P_1 = -P_2\Phi(P_2)P_1$, and $P_1\Phi(P_2)P_1 = P_2\Phi(P_1)P_2 = 0$.

Proof. Firstly, for $1 \leq j \neq k \leq 2$, via Lemma 8,

$$\begin{aligned}
0 &= \Phi([I \bullet I \bullet \cdots \bullet I \bullet P_j, P_k]) \\
&= [\Phi(I) \bullet I \bullet \cdots \bullet I \bullet P_j, P_k] + [I \bullet \Phi(I) \bullet \cdots \bullet I \bullet P_j, P_k] \\
&+ \cdots + [I \bullet I \bullet \cdots \bullet \Phi(I) \bullet P_j, P_k] + [I \bullet I \bullet \cdots \bullet I \bullet \Phi(P_j), P_k] \\
&+ [I \bullet I \bullet \cdots \bullet I \bullet P_j, \Phi(P_k)] \\
&= [I \bullet I \bullet \cdots \bullet I \bullet \Phi(P_j), P_k] + [I \bullet I \bullet \cdots \bullet I \bullet P_j, \Phi(P_k)] \\
&= 2^{(n-2)}(\Phi(P_j)P_k - P_k\Phi(P_j)^* + P_j\Phi(P_k) - \Phi(P_k)P_j).
\end{aligned}$$

Multiplying both sides by P_j leftward and P_k rightward yields $P_j\Phi(P_j)P_k + P_j\Phi(P_k)P_k = 0$, implying $P_j\Phi(P_j)P_k = -P_j\Phi(P_k)P_k = 0$. Thus $P_1\Phi(P_1)P_2 = -P_1\Phi(P_2)P_2$ and $P_2\Phi(P_1)P_1 = -P_2\Phi(P_2)P_1$. Secondly,

$$0 = \Phi([I \bullet I \bullet \cdots \bullet I \bullet (iP_j), P_k])$$

$$\begin{aligned}
&= [\Phi(I) \bullet I \bullet \dots \bullet I \bullet (iP_j), P_k] + [I \bullet \Phi(I) \bullet \dots \bullet I \bullet (iP_j), P_k] \\
&+ \dots + [I \bullet I \bullet \dots \bullet \Phi(I) \bullet (iP_j), P_k] + [I \bullet I \bullet \dots \bullet I \bullet \Phi(iP_j), P_k] \\
&+ [I \bullet I \bullet \dots \bullet I \bullet (iP_j), \Phi(P_k)] \\
&= [I \bullet I \bullet \dots \bullet I \bullet \Phi(iP_j), P_k] + [I \bullet I \bullet \dots \bullet I \bullet (iP_j), \Phi(P_k)] \\
&= 2^{(n-2)} (\Phi(iP_j)P_k - P_k\Phi(iP_j)^* + iP_j\Phi(P_k) + i\Phi(P_k)P_j).
\end{aligned}$$

Multiplying leftward and rightward by P_j yields $P_j\Phi(P_k)P_j = 0$. Thus $P_1\Phi(P_2)P_1 = P_2\Phi(P_1)P_2 = 0$. $\square$

Lemma 10. $P_1\Phi(P_1)P_1 = P_2\Phi(P_2)P_2 = 0$.

Proof. For each $B_{12} \in \mathcal{A}_{12}$, clearly $[I \bullet I \bullet \dots \bullet P_1, B_{12}] = 2^{(n-2)}B_{12}$, so via Lemma 7,

$$\begin{aligned}
2^{(n-2)}\Phi(B_{12}) &= \Phi([I \bullet I \bullet \dots \bullet I \bullet P_1, B_{12}]) \\
&= [\Phi(I) \bullet I \bullet \dots \bullet I \bullet P_1, B_{12}] + [I \bullet \Phi(I) \bullet \dots \bullet I \bullet P_1, B_{12}] \\
&+ \dots + [I \bullet I \bullet \dots \bullet \Phi(I) \bullet P_1, B_{12}] + [I \bullet I \bullet \dots \bullet I \bullet \Phi(P_1), B_{12}] \\
&+ [I \bullet I \bullet \dots \bullet I \bullet P_1, \Phi(B_{12})] \\
&= 2^{(n-2)} \{ (n-2)\Phi(I)B_{12} + \Phi(P_1)B_{12} - B_{12}\Phi(P_1)^* + P_1\Phi(B_{12}) - \Phi(B_{12})P_1 \}.
\end{aligned}$$

Multiplying leftward by P_1 and rightward by P_2 , via Lemma 9, yields

$$P_1\Phi(P_1)B_{12} + (n-2)\Phi(I)B_{12} = 0. \quad (3)$$

Moreover,

$$\begin{aligned}
2^{(n-2)}\Phi(B_{12}) &= \Phi([P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet P_1, B_{12}]) \\
&= [\Phi(P_1) \bullet P_1 \bullet \dots \bullet P_1 \bullet P_1, B_{12}] + [P_1 \bullet \Phi(P_1) \bullet \dots \bullet P_1 \bullet P_1, B_{12}] \\
&+ \dots + [P_1 \bullet P_1 \bullet \dots \bullet \Phi(P_1) \bullet P_1, B_{12}] + [P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet \Phi(P_1), B_{12}] \\
&+ [P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet P_1, \Phi(B_{12})] \\
&= 2^{(n-3)}(n-2)\{P_1\Phi(P_1)B_{12} + P_1\Phi(P_1)^*B_{12}\} \\
&+ 2^{(n-3)}\{P_2\Phi(P_1)B_{12} - B_{12}\Phi(P_1)P_1\} \\
&+ 2^{(n-2)}P_1\Phi(P_1)B_{12} - 2^{(n-3)}(P_2\Phi(P_1)B_{12} - B_{12}\Phi(P_1)^*P_1) \\
&+ 2^{(n-2)}(P_1\Phi(B_{12}) - \Phi(B_{12})P_1).
\end{aligned}$$

Multiplying leftward by P_1 and rightward by P_2 yields

$$nP_1\Phi(P_1)B_{12} + (n-2)P_1\Phi(P_1)^*B_{12} = 0. \quad (4)$$

Lastly,

$$\begin{aligned}
2^{(n-2)}\Phi(B_{12}) &= \Phi([P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet I, B_{12}]) \\
&= [\Phi(P_1) \bullet P_1 \bullet \dots \bullet P_1 \bullet I, B_{12}] + [P_1 \bullet \Phi(P_1) \bullet \dots \bullet P_1 \bullet I, B_{12}] \\
&+ \dots + [P_1 \bullet P_1 \bullet \dots \bullet \Phi(P_1) \bullet I, B_{12}] + [P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet \Phi(I), B_{12}] \\
&+ [P_1 \bullet P_1 \bullet \dots \bullet P_1 \bullet I, \Phi(B_{12})] \\
&= 2^{(n-3)}(n-2)\{P_1\Phi(P_1)B_{12} + P_1\Phi(P_1)^*B_{12}\} \\
&+ 2^{(n-3)}\{P_2\Phi(P_1)B_{12} - B_{12}\Phi(P_1)P_1\} \\
&+ 2^{(n-4)}\{P_2\Phi(P_1)B_{12} - B_{12}\Phi(P_1)P_1 + P_2\Phi(P_1)^*B_{12} - B_{12}\Phi(P_1)^*P_1\}
\end{aligned}$$

$$\begin{aligned}
 &+ 2^{(n-3)}(P_1\Phi(I)B_{12} + \Phi(I)B_{12} - B_{12}\Phi(I)^*P_1) \\
 &+ 2^{(n-2)}(P_1\Phi(B_{12}) - \Phi(B_{12})P_1).
 \end{aligned}$$

Multiplying leftward by P_1 and rightward by P_2 yields

$$2^{(n-3)}(n-2)(P_1\Phi(P_1)B_{12} + P_1\Phi(P_1)^*B_{12}) + 2^{(n-2)}\Phi(I)B_{12} = 0. \quad (5)$$

From (4) and (5),

$$P_1\Phi(P_1)B_{12} - \Phi(I)B_{12} = 0. \quad (6)$$

By (3) and (6), $P_1\Phi(P_1)B_{12} = 0$; i.e., $P_1\Phi(P_1)P_1AP_2 = 0$ for every $A \in \mathcal{A}$. From (2), $P_1\Phi(P_1)P_1 = 0$. Similarly, $P_2\Phi(P_2)P_2 = 0$. $\square$

Lemma 11. $\Phi(I) = 0$.

Proof. Via Lemmas 7, 9, and 10,

$$\begin{aligned}
 \Phi(I) &= \Phi(P_1) + \Phi(P_2) \\
 &= P_1\Phi(P_1)P_2 + P_2\Phi(P_1)P_1 + P_1\Phi(P_2)P_2 + P_2\Phi(P_2)P_1 = 0.
 \end{aligned}$$

$\square$

Lemma 12. $\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$ for every $A, B \in \mathcal{A}$.

Proof. Via Lemmas 7 and 11,

$$\begin{aligned}
 2^{(n-2)}\Phi([A, B]_*) &= \Phi([I \bullet I \bullet \dots \bullet I \bullet A, B]_*) \\
 &= [I \bullet I \bullet \dots \bullet I \bullet \Phi(A), B]_* + [I \bullet I \bullet \dots \bullet I \bullet A, \Phi(B)]_* \\
 &= [2^{(n-2)}\Phi(A), B]_* + [2^{(n-2)}A, \Phi(B)]_* \\
 &= 2^{(n-2)}([\Phi(A), B]_* + [A, \Phi(B)]_*).
 \end{aligned}$$

Hence $\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$. $\square$

Lemma 13. $\Phi(A^*) = \Phi(A)^*$ for every $A \in \mathcal{A}$.

Proof. For each $A \in \mathcal{A}$, via Lemmas 7, 11, and 12,

$$\Phi(A) - \Phi(A^*) = \Phi([A, I]_*) = [\Phi(A), I]_* = \Phi(A) - \Phi(A)^*$$

Thus $\Phi(A^*) = \Phi(A)^*$. $\square$

Lemma 14. $\Phi(iI) = 0$.

Proof. Via Lemmas 7 and 13, $\Phi(iI)^* = -\Phi(iI)$. Thus

$$0 = -2\Phi(I) = \Phi([iI, iI]_*) = [\Phi(iI), iI]_* + [iI, \Phi(iI)]_* = 4i\Phi(iI).$$

Hence $\Phi(iI) = 0$. $\square$

Lemma 15. $\Phi(iA) = i\Phi(A)$ for every $A \in \mathcal{A}$.

Proof. Via Lemmas 7 and 14,

$$2\Phi(iA) = \Phi(2iA) = \Phi([iI, A]_*) = [\Phi(iI), A]_* + [iI, \Phi(A)]_* = 2i\Phi(A).$$

Hence $\Phi(iA) = i\Phi(A)$. $\square$

Lemma 16. Φ becomes an additive $*$ -derivation.

Proof. Firstly, via Lemma 12,

$$\begin{aligned}\Phi(AB) - \Phi(BA^*) &= \Phi(AB - BA^*) = \Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_* \\ &= \Phi(A)B - B\Phi(A)^* + A\Phi(B) - \Phi(B)A^*.\end{aligned}\quad (7)$$

Secondly, via Lemmas 7, 12, and 15,

$$\begin{aligned}-\Phi(AB) - \Phi(BA^*) &= \Phi([iA, iB]_*) = [i\Phi(A), iB]_* + [iA, i\Phi(B)]_* \\ &= -\Phi(A)B - B\Phi(A)^* - A\Phi(B) - \Phi(B)A^*.\end{aligned}\quad (8)$$

From (7) and (8), $\Phi(AB) = \Phi(A)B + A\Phi(B)$.

Now, via Lemmas 7, 13, and 16, Φ is an additive $*$ -derivation. This concludes the proof of the principal theorem. $\square$

Example 1. Consider $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ as a unital $*$ -algebra with componentwise operations and involution $(z_1, z_2)^* = (\overline{z_1}, \overline{z_2})$. As $(1, 0)\mathcal{A}(0, 1) = \{0\}$, $\mathcal{A}$ lacks primeness. For $P = (1, 0)$, the nonzero $X = (0, 1)$ satisfies $XAP = 0$, breaching condition (1). The skew Jordan and skew Lie products read $(z_1, z_2) \bullet (w_1, w_2) = (2\operatorname{Re}(z_1\overline{w_1}), 2\operatorname{Re}(z_2\overline{w_2}))$, $[(z_1, z_2), (w_1, w_2)]_* = (2i\operatorname{Im}(z_1\overline{w_1}), 2i\operatorname{Im}(z_2\overline{w_2}))$. Since $\mathcal{A}$ commutes, higher nested brackets p_n nullify for $n \geq 3$. Hence, the functional identity holds vacuously for any $\Phi : \mathcal{A} \rightarrow \mathcal{A}$. For example, $\Phi(z_1, z_2) = (z_1, z_2^2)$ complies yet lacks additivity or $*$ -derivation property. This instance underscores the necessity of assumptions (1) and (2) for the theorem's outcome.

3. Applications of Theorem 1

Recall that if $AAB = 0$ for all $A, B \in \mathcal{A}$ implies that either $A = 0$ or $B = 0$, then the $*$ -algebra $\mathcal{A}$ is said to be *prime*. It is well known that prime $*$ -algebras satisfy the annihilator conditions (1) and (2). Hence, Theorem 1 applies to a broad class of operator algebras.

We first record the following general consequence.

Corollary 1. Let $\mathcal{A}$ be a unital prime $*$ -algebra containing a nontrivial projection P . A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies

$$\begin{aligned}&\Phi([A_1 \bullet A_2 \bullet \cdots \bullet A_{n-1}, A_n]_*) \\ &= \sum_{j=1}^n [A_1 \bullet \cdots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \cdots \bullet A_{n-1}, A_n]_*\end{aligned}$$

for all $A_1, A_2, \dots, A_n \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation.

Proof. Since $\mathcal{A}$ is a unital prime $*$ -algebra with a nontrivial projection, all assumptions of Theorem 1 are satisfied. The conclusion follows immediately. $\square$

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. If $\mathcal{F}(\mathcal{H})$, the ideal of finite-rank operators, is contained in a subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$,

then $\mathcal{A}$ is called a *standard operator algebra*. Every standard operator algebra is a prime $*$ -algebra. Thus, we obtain the following result.

Corollary 2. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a standard operator algebra containing the identity operator I and closed under the adjoint operation. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\begin{aligned} & \Phi([A_1 \bullet A_2 \bullet \cdots \bullet A_{n-1}, A_n]_*) \\ &= \sum_{j=1}^n [A_1 \bullet \cdots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \cdots \bullet A_{n-1}, A_n]_* \end{aligned}$$

for all $A_1, A_2, \dots, A_n \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation. Moreover, Φ is inner.

Proof. By Corollary 1, Φ is an additive $*$ -derivation. It follows from [3] that Φ is a linear inner derivation, i.e., there exists $\mathcal{K} \in \mathcal{B}(\mathcal{H})$ such that

$$\Phi(A) = A\mathcal{K} - \mathcal{K}A \quad \text{for all } A \in \mathcal{A}.$$

Since $\Phi(A^*) = \Phi(A)^*$, we have $\mathcal{K} + \mathcal{K}^* = \beta I$ for some $\beta \in \mathbb{R}$. Setting $\mathcal{L} = \mathcal{K} - \frac{\beta}{2}I$ yields $\mathcal{L} + \mathcal{L}^* = 0$, completing the proof. $\square$

A von Neumann algebra $\mathcal{A}$ is a weakly closed self-adjoint operator algebra on a Hilbert space $\mathcal{H}$ containing the identity. If the center of $\mathcal{A}$ is trivial, then $\mathcal{A}$ is called a *factor von Neumann algebra*. Since factor von Neumann algebras are prime, we obtain the following.

Corollary 3. *Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim(\mathcal{A}) \geq 2$. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\begin{aligned} & \Phi([A_1 \bullet A_2 \bullet \cdots \bullet A_{n-1}, A_n]_*) \\ &= \sum_{j=1}^n [A_1 \bullet \cdots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \cdots \bullet A_{n-1}, A_n]_* \end{aligned}$$

for all $A_1, A_2, \dots, A_n \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation.

Proof. The conclusion follows directly from Corollary 1. $\square$

Finally, von Neumann algebras with no central summands of type I_1 are also prime $*$ -algebras. Therefore, we have the following result.

Corollary 4. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 . A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\begin{aligned} & \Phi([A_1 \bullet A_2 \bullet \cdots \bullet A_{n-1}, A_n]_*) \\ &= \sum_{j=1}^n [A_1 \bullet \cdots \bullet A_{j-1} \bullet \Phi(A_j) \bullet A_{j+1} \bullet \cdots \bullet A_{n-1}, A_n]_* \end{aligned}$$

for all $A_1, A_2, \dots, A_n \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation.

Proof. Since $\mathcal{A}$ is prime, the assertion follows from Corollary 1. $\square$

The study of nonlinear Lie-type constraints leading to additivity or derivation properties is a well-established topic in the theory of functional identities. The use of nontrivial projections and Peirce decomposition techniques plays a central role in this setting. Recent advances extend this framework from associative $*$ -algebras to alternative $*$ -algebras;

see [34]. Motivated by these developments, we propose the following conjecture for alternative unital $\ast$ -algebras.

Conjecture 1. Let $\mathcal{R}$ be an alternative unital $\ast$ -algebra with identity I . Assume that $\mathcal{R}$ contains a nontrivial projection P such that $X\mathcal{R}P = 0$ implies $X = 0$, and $X\mathcal{R}(I - P) = 0$ implies $X = 0$, for all $X \in \mathcal{R}$. Let $n \geq 3$ and let $\Phi : \mathcal{R} \rightarrow \mathcal{R}$ be a (not a priori additive) map satisfying

$$\Phi([x_1 \bullet x_2 \bullet \cdots \bullet x_{n-1}, x_n]_{\ast}) = \sum_{j=1}^n [x_1 \bullet \cdots \bullet x_{j-1} \bullet \Phi(x_j) \bullet x_{j+1} \bullet \cdots \bullet x_{n-1}, x_n]_{\ast}$$

for all $x_1, \dots, x_n \in \mathcal{R}$. Then Φ is an additive $\ast$ -derivation.

4. Conclusions and Future Work

In this paper, we proved that every mapping $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ on a unital $\ast$ -algebra $\mathcal{A}$ satisfying a higher-order mixed skew Lie-type functional identity is necessarily an additive $\ast$ -derivation, provided that $\mathcal{A}$ admits a nontrivial projection P such that the left actions of P and $I - P$ are faithful. The proof relies on a Peirce decomposition with respect to P , combined with a recursive reduction of the defining identity, which forces additivity, preservation of the involution, and the derivation property. As applications, our main theorem yields new characterizations of additive $\ast$ -derivations on several important classes of operator algebras, including standard operator algebras on infinite-dimensional Hilbert spaces, factor von Neumann algebras of dimension at least 2, and von Neumann algebras without central summands of type I_1 . These results demonstrate that higher-order nonlinear identities impose strong linear and algebraic rigidity on mappings of operator algebras.

Future Work

Motivated by recent advances on nonlinear η - $\ast$ -Jordan-type derivations and higher-order preserver problems (see, for example, [34,35]), several related directions remain open. A natural problem is to extend the present results to functional identities generated by the skew Jordan product and the bi-skew η -Jordan product, as well as their iterated forms for $n \geq 3$, including the corresponding Lie-type identities involving mixed products. An important question is whether such identities necessarily force additivity and $\ast$ -derivation behavior under suitable projection faithfulness assumptions. Further directions include relaxing the faithfulness conditions on projections, studying automatic continuity and boundedness of such mappings in Banach $\ast$ -algebras and C^* -algebras, and extending the theory to nonassociative settings such as alternative $\ast$ -algebras.

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