

Article

Intrinsic and Extrinsic Geometry of Pseudoparallel Submanifolds in Almost Kenmotsu (κ, μ, ν) -Manifolds

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Abstract

In this paper, we undertake a detailed study of pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces, with particular emphasis on invariant submanifolds. By employing the W_0 and W_1 curvature tensors, we analyze several classes of pseudoparallel submanifolds, including Ricci-generalized pseudoparallel ones, and investigate how these curvature conditions influence the intrinsic and extrinsic geometry of the submanifolds. One of the main contributions of this work is the derivation of necessary and sufficient conditions under which invariant pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces become totally geodesic. In particular, the use of the W_0 and W_1 curvature tensors provides a unified and effective framework for characterizing total geodesicity in this geometric setting. Furthermore, we obtain new and significant classification results by explicitly relating the total geodesicity of invariant submanifolds to the structural functions κ, μ and ν . These results not only generalize several known characterizations in the literature but also yield novel geometric insights into the structure of pseudoparallel submanifolds in almost Kenmotsu (κ, μ, ν) -spaces. We also provide an example to support our concept.

Keywords: pseudoparallel submanifolds; almost Kenmotsu (κ, μ, ν) -space; totally geodesic submanifolds; invariant submanifolds

MSC: 53C15; 53C44; 53D10

1. Introduction

Contact metric geometry constitutes an important branch of differential geometry, arising naturally as the odd-dimensional counterpart of symplectic geometry and playing a significant role in both pure mathematics and theoretical physics [1]. Within this framework, Kenmotsu manifolds and their generalizations have attracted sustained attention due to their rich geometric structure and close connections with almost contact metric manifolds, warped product constructions, and geometric flows. In particular, almost Kenmotsu manifolds provide a flexible geometric setting that encompasses several well-known contact structures while allowing broader curvature behaviors.

A further generalization is given by almost Kenmotsu (κ, μ, ν) -spaces, which are characterized by curvature conditions involving three smooth functions κ, μ and ν . These manifolds naturally extend classical (κ, μ) -spaces [2–4] and allow a more refined control of curvature properties through the additional structural function ν . Such spaces have proven



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to be useful in the study of curvature restrictions, classification problems, and rigidity phenomena in contact metric geometry. Moreover, curvature-controlled contact manifolds of this type appear in geometric models related to spacetime geometry and theoretical physics, where contact and almost contact structures arise in the study of odd-dimensional phase spaces and certain relativistic settings [5,6].

The geometry of submanifolds in contact metric manifolds has long been a central topic, as submanifolds serve as a powerful tool for understanding both local and global features of the ambient space. Invariant submanifolds, in particular, preserve the underlying almost contact structure and thus provide a natural and geometrically meaningful class for investigation [7,8]. Studying such submanifolds allows one to transfer curvature information from the ambient manifold to the induced geometry, revealing intrinsic and extrinsic interactions governed by the second fundamental form.

Among the various curvature restrictions imposed on submanifolds, pseudoparallel conditions have emerged as a natural generalization of semi-parallel and parallel submanifolds. Introduced via the Tachibana operator, pseudoparallelism captures subtle curvature interactions that go beyond classical parallelism while still leading to strong geometric consequences. In particular, Ricci-generalized pseudoparallel submanifolds have been shown to yield significant rigidity results and classification theorems in various ambient geometries. However, despite extensive studies in Sasakian and Kenmotsu settings, the pseudoparallel geometry of submanifolds in almost Kenmotsu (κ, μ, ν) -spaces remains relatively unexplored.

Another powerful approach to curvature analysis is provided by generalized curvature tensors, such as the W_0 and W_1 tensors, which unify and extend several classical curvature tensors, including the Riemann, Ricci, and conformal curvature tensors. These tensors have proven to be effective in formulating curvature conditions that lead to meaningful geometric classifications. Their role in the study of pseudoparallel submanifolds, particularly in the context of almost Kenmotsu (κ, μ, ν) -spaces, has not yet been fully clarified.

Motivated by these observations, the present paper undertakes a systematic study of invariant pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces. Our primary objective is to characterize when such submanifolds are totally geodesic by employing the W_0 and W_1 curvature tensors within the framework of the Tachibana operator. By deriving necessary and sufficient conditions, we establish new classification results that explicitly relate total geodesicity to the structural functions κ, μ and ν .

The results obtained in this work not only generalize several known characterizations in the literature but also provide new geometric insights into the interplay between curvature tensors and submanifold geometry in almost Kenmotsu (κ, μ, ν) -spaces. An illustrative example is also presented to demonstrate the applicability and sharpness of the theoretical results. We believe that the methods and conclusions of this study will contribute to a deeper understanding of curvature-restricted submanifolds in contact metric geometry and may serve as a foundation for further investigations involving geometric flows and related structures.

Consider a $(2n + 1)$ -dimensional manifold Ξ^{2n+1} endowed with a contact metric structure (ϕ, ξ, η, g) . In this case, the contact metric manifold satisfies the condition

$$R(\vartheta_1, \vartheta_2)\xi = 0, \quad \forall \quad \vartheta_1, \vartheta_2 \in \Xi^{2n+1},$$

where R is the Riemann curvature tensor of the manifold. Additionally, the (κ, μ) -nullity condition will also be satisfied for contact metric manifolds, meaning that

$$R(\vartheta_1, \vartheta_2)\xi = \eta(\vartheta_2)(\kappa I + \mu h)\vartheta_1 - \eta(\vartheta_1)(\kappa I + \mu h)\vartheta_2. \quad (1)$$

For contact metric manifolds, the (κ, μ) -nullity condition for the case where κ and μ are constant was provided by E. Boeckx and D. E. Blair in [9,10]. E. Boeckx demonstrated that, for constant values of κ and μ , a contact metric manifold that does not satisfy the Sasakian condition is locally characterized entirely by its dimension. When the vector field ξ belongs to the (κ, μ) -nullity distribution, we can easily see that condition (1) is satisfied. In this situation, the contact metric manifold is referred to as a (κ, μ) -contact metric manifold.

The κ and η on the manifold do not have to be constant functions. In such a case, a generalized (κ, μ) -contact metric manifold is obtained [9].

(κ, μ, ν) -contact metric manifolds were introduced by T. Koufogiorgos and colleagues in [11]. Riemann curvature tensor of this manifold is given by

$$\begin{aligned}\tilde{R}(\vartheta_1, \vartheta_2)\xi &= \kappa[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2] \\ &+ \mu[\eta(\vartheta_2)h\vartheta_1 - \eta(\vartheta_1)h\vartheta_2] \\ &+ \nu[\eta(\vartheta_2)\phi h\vartheta_1 - \eta(\vartheta_1)\phi h\vartheta_2],\end{aligned}\tag{2}$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(TM)$; here κ, μ and ν are smooth real-valued functions on Ξ^{2n+1} .

An almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, is referred to as an almost Kenmotsu manifold, where $\Phi(\vartheta_1, \vartheta_2) = g(\vartheta_1, \phi\vartheta_2)$ is the fundamental 2-form on Ξ^{2n+1} . In addition, if such a manifold admits a (κ, μ, ν) -nullity distribution, then it is called an almost Kenmotsu (κ, μ, ν) -space [12].

Dacko-Olszak investigated manifolds that satisfy condition (2) but do not possess a contact metric structure [13]. After that many authors studied on different types of manifolds such that almost Kenmotsu (κ, μ, ν) -space [14–16], invariant submanifolds of Lorentzian para-Kenmotsu manifolds [17], (k, μ) -para contact space [18], trans-Sasakian manifold [19], LP-Sasakian manifolds ([20,21]), $(LCS)_n$ -manifolds [22]. The geometry of totally geodesic submanifolds in (k, μ) -paracontact metric manifolds has been studied in [23], whereas related results on K -paracontact manifolds can be found in [24]. Furthermore, pseudoparallel submanifolds have been extensively investigated in different geometric settings (see, for example, [25–27]), and pseudo-slant submanifolds of Sasakian manifolds were considered in [28].

In this article, we placed great importance on the submanifolds of almost Kenmotsu (κ, μ, ν) -space, and we discussed some geometric characteristics of those submanifolds that are pseudoparallel. While examining some types of pseudoparallel submanifolds, such as Ricci-generalized ones, for almost Kenmotsu (κ, μ, ν) -space, we used the $\mathcal{W}_0$ and $\mathcal{W}_1$ curvature tensors. Necessary and sufficient conditions are obtained for invariant pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces to be totally geodesic. We obtained novel and interesting results by characterizing the totally geodesic submanifolds of almost Kenmotsu (κ, μ, ν) -space according to the relationship between κ, μ and ν .

Despite the extensive literature on submanifold geometry in Sasakian and Kenmotsu manifolds, the pseudoparallel geometry of invariant submanifolds in almost Kenmotsu (κ, μ, ν) -spaces has not yet been systematically investigated, particularly in connection with generalized curvature tensors. This gap motivates the present study, whose primary aim is to establish a comprehensive framework for analyzing invariant pseudoparallel submanifolds by means of the W_0 and W_1 curvature tensors. The novelty of this work lies in deriving explicit necessary and sufficient conditions for total geodesicity and in obtaining new classification results that relate these conditions directly to the structural functions κ, μ and ν . Beyond their intrinsic geometric interest, these results contribute to a deeper understanding of curvature-restricted submanifolds in contact metric geometry and may find applications in related areas such as geometric flows, rigidity theory, and mathematical

models where contact structures play a fundamental role. In this article, we placed great importance on the submanifolds of almost Kenmotsu (κ, μ, ν) -space.

From this part of the article onwards, we will refer to almost Kenmotsu (κ, μ, ν) -space as $\mathcal{AK}-(\kappa, \mu, \nu)$ space.

2. Preliminary

Let $\tilde{\Xi}$ be a contact metric manifold. In this setting, the contact metric structure (ϕ, ξ, η, g) gives rise to the following relations:

$$\begin{cases} \phi^2 \vartheta_1 = -\vartheta_1 + \eta(\vartheta_1)\xi, \\ \eta(\vartheta_1) = g(\vartheta_1, \xi), \\ \eta(\xi) = 1, \eta \circ \phi = 0, \end{cases} \quad (3)$$

$$g(\phi\vartheta_1, \phi\vartheta_2) = g(\vartheta_1, \vartheta_2) - \eta(\vartheta_1)\eta(\vartheta_2), \quad (4)$$

for all vector fields $\vartheta_1, \vartheta_2 \in \Gamma(T\tilde{\Xi})$, where $\Gamma(T\tilde{\Xi})$ represents the collection of all smooth vector fields defined on $\tilde{\Xi}$. Thus, with the help of this structure, $\tilde{\Xi}$ contact metric manifold is formed [11].

It is obvious that,

$$\tilde{R}(\vartheta_1, \vartheta_2) = \tilde{\nabla}_{\vartheta_1} \tilde{\nabla}_{\vartheta_2} - \tilde{\nabla}_{\vartheta_2} \tilde{\nabla}_{\vartheta_1} - \tilde{\nabla}_{[\vartheta_1, \vartheta_2]},$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(T\tilde{\Xi})$, where $\tilde{\nabla}$ is the Levi-Civita connection of metric tensor and $\tilde{R}$ is the curvature tensor of a manifold.

Also, we define a tensor by h such that $2h\vartheta_1 = (l_{\xi}\phi)\vartheta_1$, for all $\vartheta_1 \in \Gamma(T\tilde{\Xi})$, where l_{ξ} stands for the Lie derivative along ξ . Then the tensor field h is self-adjoint and satisfies

$$\phi h + h\phi = 0, trh = tr\phi h = 0, h\xi = 0. \quad (5)$$

Moreover, for the manifold $\tilde{\Xi}$, the following formulas are valid:

$$\tilde{\nabla}_{\vartheta_1} \xi = \phi\vartheta_1 - \phi h\vartheta_1, \tilde{\nabla}_{\xi} \phi = 0. \quad (6)$$

In a contact metric manifold $\tilde{\Xi}$, the (κ, μ) -nullity distribution determined by the pair $(\kappa, \mu) \in \mathbb{R}^2$ is given by

$$\begin{aligned} \tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 &= \kappa[g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2] \\ &\quad + \mu[g(\vartheta_2, \vartheta_3)h\vartheta_1 - g(\vartheta_1, \vartheta_3)h\vartheta_2], \end{aligned}$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(T\tilde{\Xi})$.

Definition 1. Consider a $(2n+1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ -space $(\tilde{\Xi}^{2n+1}, \phi, \eta, \xi, g)$. The following relations are valid [12,25]:

$$h^2 = (\kappa + 1)\phi^2, \kappa \leq -1, \quad (7)$$

$$\xi(\kappa) = 2(\kappa + 1)(\nu - 2), \quad (8)$$

$$(\tilde{\nabla}_{\vartheta_1} \phi)\vartheta_2 = g(\phi\vartheta_1 + h\vartheta_1, \vartheta_2)\xi - \eta(\vartheta_2)(\phi\vartheta_1 + h\vartheta_1), \quad (9)$$

$$\tilde{\nabla}_{\vartheta_1} \zeta = -\phi^2 \vartheta_1 - \phi h \vartheta_1, \quad (10)$$

$$S(\vartheta_1, \zeta) = 2n\kappa\eta(\vartheta_1), \quad (11)$$

$$\begin{aligned} \tilde{R}(\zeta, \vartheta_1) \vartheta_2 &= \kappa[g(\vartheta_1, \vartheta_2) \zeta - \eta(\vartheta_2) \vartheta_1] \\ &+ \mu[g(h\vartheta_1, \vartheta_2) \zeta - \eta(\vartheta_2) h\vartheta_1] \\ &+ \nu[g(\phi h\vartheta_1, \vartheta_2) \zeta - \eta(\vartheta_2) \phi h\vartheta_1]. \end{aligned} \quad (12)$$

On an immersed submanifold Ξ of an $\mathcal{AK}-(\kappa, \mu, \nu)$ space, the Gauss and Weingarten formulas for tangent and normal vector fields are given by

$$\tilde{\nabla}_{\vartheta_1} \vartheta_2 = \nabla_{\vartheta_1} \vartheta_2 + \sigma(\vartheta_1, \vartheta_2), \quad (13)$$

$$\tilde{\nabla}_{\vartheta_1} \vartheta_5 = -A_{\vartheta_5} \vartheta_1 + \nabla_{\vartheta_1}^\perp \vartheta_5, \quad (14)$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(T\Xi)$ and $\vartheta_5 \in \Gamma(T^\perp\Xi)$, where ∇ and $\nabla^\perp$ are the induced connections, and σ and A denote the second fundamental form and the shape operator, respectively. The metric g and the shape operator A satisfy the following relation:

$$g(A_{\vartheta_5} \vartheta_1, \vartheta_2) = g(\sigma(\vartheta_1, \vartheta_2), \vartheta_5). \quad (15)$$

We define the covariant derivative of the second fundamental form σ by

$$\begin{aligned} (\tilde{\nabla}_{\vartheta_1} \sigma)(\vartheta_2, \vartheta_3) &= \nabla_{\vartheta_1}^\perp \sigma(\vartheta_2, \vartheta_3) - \sigma(\nabla_{\vartheta_1} \vartheta_2, \vartheta_3) \\ &- \sigma(\vartheta_2, \nabla_{\vartheta_1} \vartheta_3), \end{aligned} \quad (16)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \Gamma(T\Xi)$. Specifically, if $\tilde{\nabla}\sigma = 0$, Ξ is called 1-parallel.

Let R be the Riemannian curvature tensor of Ξ . Then the Gauss equation takes the following form:

$$\begin{aligned} \tilde{R}(\vartheta_1, \vartheta_2) \vartheta_3 &= R(\vartheta_1, \vartheta_2) \vartheta_3 + A_{\sigma(\vartheta_1, \vartheta_3)} \vartheta_2 - A_{\sigma(\vartheta_2, \vartheta_3)} \vartheta_1 \\ &+ (\tilde{\nabla}_{\vartheta_1} \sigma)(\vartheta_2, \vartheta_3) - (\tilde{\nabla}_{\vartheta_2} \sigma)(\vartheta_1, \vartheta_3), \end{aligned} \quad (17)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \Gamma(T\Xi)$.

$\tilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\tilde{R}(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) &= R^\perp(\vartheta_1, \vartheta_2) \sigma(\vartheta_4, \vartheta_5) \\ &- \sigma(R(\vartheta_1, \vartheta_2) \vartheta_4, \vartheta_5) \\ &- \sigma(\vartheta_4, R(\vartheta_1, \vartheta_2) \vartheta_5), \end{aligned} \quad (18)$$

and

$$R^\perp(\vartheta_1, \vartheta_2) = \left[\nabla_{\vartheta_1}^\perp, \nabla_{\vartheta_2}^\perp \right] - \nabla_{[\vartheta_1, \vartheta_2]}^\perp. \quad (19)$$

On the other hand, the $\mathcal{W}_0$ and $\mathcal{W}_1$ curvature tensors of the Riemannian manifold (Ξ^{2n+1}, g) can be expressed as follows:

$$\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_3 = \tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{1}{2n}[S(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)Q\vartheta_2],$$

and

$$\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_3 = \tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 + \frac{1}{2n}[S(\vartheta_2, \vartheta_3)\vartheta_1 - S(\vartheta_1, \vartheta_3)\vartheta_2].$$

Analogously, we define the tensor $\mathcal{W}_i \cdot \sigma$ by

$$\begin{aligned} (\mathcal{W}_i(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) &= R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) \\ &\quad - \sigma(\mathcal{W}_i(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ &\quad - \sigma(\vartheta_4, \mathcal{W}_i(\vartheta_1, \vartheta_2)\vartheta_5), \end{aligned} \quad (20)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$, $i = 0, 1$.

Consider a Riemannian manifold Ξ , where T is a $(0, k)$ -type tensor field and A is a $(0, 2)$ -type tensor field. The Tachibana tensor field $Q(A, T)$ is then defined by

$$\begin{aligned} Q(A, T)(\vartheta_1, \dots, \vartheta_k; X, Y) &= -T((X \wedge_A Y)\vartheta_1, \dots, \vartheta_k) - \dots - \\ &\quad - T(\vartheta_1, \dots, \vartheta_{k-1}, (X \wedge_A Y)\vartheta_k), \end{aligned} \quad (21)$$

where

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad (22)$$

$k \geq 1$, $\vartheta_1, \vartheta_2, \dots, \vartheta_k, X, Y, Z \in \Gamma(T\tilde{M})$.

The pseudoparallelism of submanifolds of a Riemannian manifold can be categorized, as shown in the following table, using some special conditions established between the curvature tensors $\mathcal{W}_i$ ($i = 1, 2$), the second fundamental form σ , and the Tachibana operator Q . That is, for $i = 0, 1$, a submanifold is called $\mathcal{W}_i$ -pseudoparallel, $\mathcal{W}_i$ 2-pseudoparallel, $\mathcal{W}_i$ -Ricci-generalized pseudoparallel, or $\mathcal{W}_i$ 2-Ricci generalized pseudoparallel according to whether the corresponding pairs of tensors

$$\begin{aligned} &(\mathcal{W}_i \cdot \sigma, Q(g, \sigma)), (\mathcal{W}_i \cdot \tilde{\nabla}\sigma, Q(g, \tilde{\nabla}\sigma)), \\ &(\mathcal{W}_i \cdot \sigma, Q(S, \sigma)), (\mathcal{W}_i \cdot \tilde{\nabla}\sigma, Q(S, \tilde{\nabla}\sigma)), \end{aligned}$$

are linearly dependent.

The functions $\mathcal{F}_{\mathcal{W}_0}, \mathcal{F}_{\mathcal{W}_1}, L_{\mathcal{W}_0}$ and $L_{\mathcal{W}_1}$ appearing in the linear dependence relations measure the intensity of interaction between the curvature structure of the ambient $\mathcal{AK}-(\kappa, \mu, \nu)$ space and the extrinsic geometry of the submanifold, reflecting how the curvature governed by the structural functions κ, μ, ν controls the behavior of the second fundamental form and its covariant derivative.

3. Invariant Pseudoparallel Geometry of Submanifolds in $\mathcal{AK}-(\kappa, \mu, \nu)$ Space

Whenever the relation $\phi(T_{\vartheta_1}\Xi) \subset T_{\vartheta_1}\Xi$ holds for every point ϑ_1 in an immersed submanifold Ξ of a $(2n+1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space, Ξ is referred to as an invariant submanifold. Generally, these submanifolds are recognized for their capacity to mirror the geometric properties inherent in the ambient space. Therefore, the following proposition naturally follows.

Proposition 1. For an invariant submanifold Ξ of an $\mathcal{AK}-(\kappa, \mu, \nu)$ space such that $\xi \in T\Xi$, the following relations hold true on the submanifold:

$$R(\vartheta_1, \vartheta_2)\xi = \kappa[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2] + \mu[\eta(\vartheta_2)h\vartheta_1 - \eta(\vartheta_1)h\vartheta_2] \quad (23)$$

$$+ \nu[\eta(\vartheta_2)\phi h\vartheta_1 - \eta(\vartheta_1)\phi h\vartheta_2],$$

$$R(\xi, \vartheta_1)\vartheta_2 = \kappa[g(\vartheta_1, \vartheta_2)\xi - \eta(\vartheta_2)\vartheta_1] + \mu[g(h\vartheta_1, \vartheta_2)\xi - \eta(\vartheta_2)h\vartheta_1] \quad (24)$$

$$+ \nu[g(\phi h\vartheta_1, \vartheta_2)\xi - \eta(\vartheta_2)\phi h\vartheta_1],$$

$$(\nabla_{\vartheta_1}\phi)\vartheta_2 = g(\phi\vartheta_1 + h\vartheta_1, \vartheta_2)\xi - \eta(\vartheta_2)(\phi\vartheta_1 + h\vartheta_1), \quad (25)$$

$$\nabla_{\vartheta_1}\xi = -\phi^2\vartheta_1 - \phi h\vartheta_1, \quad (26)$$

$$\mathcal{W}_0(\vartheta_1, \vartheta_2)\xi = -\kappa\eta(\vartheta_1)\vartheta_2 + \mu[\eta(\vartheta_2)h\vartheta_1 - \eta(\vartheta_1)h\vartheta_2] + \nu[\eta(\vartheta_2)\phi h\vartheta_1 - \eta(\vartheta_1)\phi h\vartheta_2] \quad (27)$$

$$+ \frac{1}{2n}\eta(\vartheta_1)Q\vartheta_2,$$

$$\mathcal{W}_0(\vartheta_1, \xi)\xi = \mu h\vartheta_1 + \nu \phi h\vartheta_1 \quad (28)$$

$$\mathcal{W}_1(\vartheta_1, \vartheta_2)\xi = 2\kappa[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2] + \mu[\eta(\vartheta_2)h\vartheta_1 - \eta(\vartheta_1)h\vartheta_2] \quad (29)$$

$$+ \nu[\eta(\vartheta_2)\phi h\vartheta_1 - \eta(\vartheta_1)\phi h\vartheta_2],$$

$$\mathcal{W}_1(\vartheta_1, \xi)\xi = 2\kappa[\vartheta_1 - \eta(\vartheta_1)\xi] + \mu h\vartheta_1 + \nu \phi h\vartheta_1 \quad (30)$$

$$\sigma(\vartheta_1, \xi) = 0, \sigma(\phi\vartheta_1, \vartheta_2) = \sigma(\vartheta_1, \phi\vartheta_2) = \phi\sigma(\vartheta_1, \vartheta_2), \quad (31)$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(T\Xi)$ within this framework, ∇ signifies the Levi-Civita connection induced on Ξ , while σ and R stand for the shape operator and the Riemannian curvature tensor, respectively [12,25].

Proof. The proof of Equations (23)–(26) and (31) of the proposition is clear from [12,25]. On the other hand, in order to see the proof of Equations (27) and (28), it is sufficient to write ξ instead of ϑ_2 and ϑ_3 , respectively, in the equation

$$\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_3 = \tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{1}{2n}[S(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)Q\vartheta_2].$$

Similarly, in order to see the proof of Equations (29) and (30), it is sufficient to write ξ instead of ϑ_2 and ϑ_3 , respectively, in the equation

$$\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_3 = \tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 + \frac{1}{2n}[S(\vartheta_2, \vartheta_3)\vartheta_1 - S(\vartheta_1, \vartheta_3)\vartheta_2].$$

Thus, the proof of the proposition is completed. $\square$

Theorem 1. Consider an invariant submanifold Ξ of a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. If the condition of $\mathcal{W}_0$ -pseudoparallelism is satisfied for Ξ , then the submanifold is either characterized as totally geodesic or it must satisfy the relation

$$\mathcal{F}_{\mathcal{W}_0}^2 = (\nu^2 - \mu^2 + 2\mu\nu)(\kappa + 1).$$

Proof. Assuming that Ξ is a $\mathcal{W}_0$ -pseudoparallel submanifold, the following relation can be established:

$$(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) = \mathcal{F}_{\mathcal{W}_0} Q(g, \sigma)(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2), \quad (32)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$ and $\mathcal{F}_{\mathcal{W}_0} \in M_1 = \{x \in \Xi | \sigma(x) \neq g(x)\}$. From (20), it is clear that

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5) = -\mathcal{F}_{\mathcal{W}_0} \{ \sigma((\vartheta_1 \wedge_g \vartheta_2)\vartheta_4, \vartheta_5) \\ + \sigma(\vartheta_4, (\vartheta_1 \wedge_g \vartheta_2)\vartheta_5) \}. \end{aligned}$$

Easily from here, we can write

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5) = -\mathcal{F}_{\mathcal{W}_0} \{ g(\vartheta_2, \vartheta_4)\sigma(\vartheta_1, \vartheta_5) \\ - g(\vartheta_1, \vartheta_4)\sigma(\vartheta_2, \vartheta_5) + g(\vartheta_2, \vartheta_5)\sigma(\vartheta_4, \vartheta_1) \\ - g(\vartheta_1, \vartheta_5)\sigma(\vartheta_4, \vartheta_2) \}. \end{aligned} \quad (33)$$

If we choose $\vartheta_2 = \vartheta_5 = \xi$ in (33) and make use of (28) and (31), we get

$$(\mu + \nu\phi)\sigma(\vartheta_4, h\vartheta_1) = \mathcal{F}_{\mathcal{W}_0}\sigma(\vartheta_4, \vartheta_1). \quad (34)$$

Substituting $h\vartheta_1$ for ϑ_1 in (34), by view of (7) and (30), we have

$$(\mu + \nu\phi)(\kappa + 1)\sigma(\vartheta_4, \vartheta_1) = -\mathcal{F}_{\mathcal{W}_0}\sigma(\vartheta_4, h\vartheta_1). \quad (35)$$

From (34) and (35), one can easily see that

$$\left[\mu^2(\kappa + 1) + 2\mu\nu(\kappa + 1) - \nu^2(\kappa + 1) + \mathcal{F}_{\mathcal{W}_0}^2 \right] \sigma(\vartheta_4, \vartheta_1) = 0. \quad (36)$$

In this situation, Ξ is either totally geodesic submanifold or

$$\mathcal{F}_{\mathcal{W}_0}^2 = (\nu^2 - \mu^2 + 2\mu\nu)(\kappa + 1).$$

This concludes the proof. $\square$

Corollary 1. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Then Ξ is $\mathcal{W}_0$ -semiparallel if and only if Ξ is totally geodesic provided

$$\mu^2 = \nu(\nu + 2\mu).$$

Proof. The demonstration of this corollary follows directly by imposing $\mathcal{F}_{\mathcal{W}_0} = 0$ within Equation (36), which is consistent with the standard characterization of a semiparallel submanifold. $\square$

Theorem 2. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $AK-(\kappa, \mu, \nu)$ space. If Ξ is $\mathcal{W}_1$ -pseudoparallel, then Ξ is either a total geodesic submanifold or

$$\mathcal{F}_{\mathcal{W}_1} = 2\kappa + (\mu + \nu)\sqrt{|\kappa + 1|}.$$

Proof. Let Ξ be a $\mathcal{W}_1$ -pseudoparallel submanifold. So, we have

$$(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) = \mathcal{F}_{\mathcal{W}_1} Q(g, \sigma)(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2), \quad (37)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$ and $\mathcal{F}_{\mathcal{W}_1} \in M_1 = \{x \in \Xi | \sigma(x) \neq g(x)\}$. From (20), it is clear that

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5) = -\mathcal{F}_{\mathcal{W}_0} \{ \sigma((\vartheta_1 \wedge_g \vartheta_2)\vartheta_4, \vartheta_5) \\ + \sigma(\vartheta_4, (\vartheta_1 \wedge_g \vartheta_2)\vartheta_5) \}. \end{aligned}$$

Easily from here, we can write

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5) = -\mathcal{F}_{\mathcal{W}_1} \{ g(\vartheta_2, \vartheta_4)\sigma(\vartheta_1, \vartheta_5) \\ - g(\vartheta_1, \vartheta_4)\sigma(\vartheta_2, \vartheta_5) + g(\vartheta_2, \vartheta_5)\sigma(\vartheta_4, \vartheta_1) \\ - g(\vartheta_1, \vartheta_5)\sigma(\vartheta_4, \vartheta_2) \}. \end{aligned} \quad (38)$$

If we choose $\vartheta_2 = \vartheta_5 = \xi$ in (38) and make use of (30) and (31), we get

$$(\mathcal{F}_{\mathcal{W}_1} - 2\kappa)\sigma(\vartheta_4, \vartheta_1) = (\mu + \nu)\sigma(\vartheta_4, h\vartheta_1). \quad (39)$$

Substituting $h\vartheta_1$ for ϑ_1 in (39), by view of (7) and (31), we have

$$(\mathcal{F}_{\mathcal{W}_1} - 2\kappa)\sigma(\vartheta_4, h\vartheta_1) = -(\kappa + 1)(\mu + \nu)\sigma(\vartheta_4, \vartheta_1). \quad (40)$$

From (39) and (40), one can easily see that

$$\left[(\mathcal{F}_{\mathcal{W}_1} - 2\kappa)^2 + (\kappa + 1)(\mu + \nu)^2 \right] \sigma(\vartheta_4, \vartheta_1) = 0. \quad (41)$$

In this situation, Ξ is either totally geodesic submanifold or

$$\mathcal{F}_{\mathcal{W}_1} = 2\kappa + (\mu + \nu)\sqrt{|\kappa + 1|}.$$

This concludes the proof. $\square$

Corollary 2. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $AK-(\kappa, \mu, \nu)$ space. Consequently, the invariant submanifold Ξ is characterized as $\mathcal{W}_1$ -semiparallel if and only if it is totally geodesic, on the condition that

$$4\kappa^2 = (\kappa + 1)(\mu + \nu)^2,$$

is satisfied.

Proof. The proof of the corollary is easily obtained by choosing $\mathcal{F}_{\mathcal{W}_1} = 0$ in Equation (41) as per the definition of semiparallel submanifold. $\square$

Theorem 3. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Whenever Ξ is characterized as $\mathcal{W}_0$ -Ricci-generalized pseudoparallel, it is established that Ξ is either a totally geodesic submanifold or it must satisfy the following relation:

$$L_{\mathcal{W}_0} = \frac{1}{2n\kappa}(\mu + \nu)\sqrt{|\kappa + 1|}.$$

Proof. Suppose that Ξ is a $\mathcal{W}_0$ -Ricci-generalized pseudoparallel submanifold. Consequently, we can establish the following relation:

$$(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) = L_{\mathcal{W}_0}Q(S, \sigma)(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2), \quad (42)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$, $L_{\mathcal{W}_0} \in M_2 = \{x \in \Xi | S(x) \neq \sigma(x)\}$. From (20), it is clear that

$$\begin{aligned} & R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ & - \sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5) = -L_{\mathcal{W}_0}\{\sigma((\vartheta_1 \wedge_S \vartheta_2)\vartheta_4, \vartheta_5) \\ & + \sigma(\vartheta_4, (\vartheta_1 \wedge_S \vartheta_2)\vartheta_5)\}. \end{aligned}$$

Easily from here, we can write

$$\begin{aligned} & R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ & - \sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5) = -L_{\mathcal{W}_0}\{S(\vartheta_2, \vartheta_4)\sigma(\vartheta_1, \vartheta_5) \\ & - S(\vartheta_1, \vartheta_4)\sigma(\vartheta_2, \vartheta_5) + S(\vartheta_2, \vartheta_5)\sigma(\vartheta_4, \vartheta_1) \\ & - S(\vartheta_1, \vartheta_5)\sigma(\vartheta_4, \vartheta_2)\}. \end{aligned} \quad (43)$$

If we choose $\vartheta_2 = \vartheta_5 = \xi$ in (43), we get

$$\sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \xi)\xi) = L_{\mathcal{W}_0}S(\xi, \xi)\sigma(\vartheta_4, \vartheta_1). \quad (44)$$

If we use (11) and (28) in (44), we obtain

$$(\mu + \nu)\sigma(\vartheta_4, h\vartheta_1) = 2n\kappa L_{\mathcal{W}_0}\sigma(\vartheta_4, \vartheta_1). \quad (45)$$

Substituting $h\vartheta_1$ for ϑ_1 in (45), by view of (7) and (30), we have

$$2n\kappa L_{\mathcal{W}_0}\sigma(\vartheta_4, h\vartheta_1) = -(\kappa + 1)(\mu + \nu)\sigma(\vartheta_4, \vartheta_1). \quad (46)$$

From (45) and (46), a straightforward computation shows that

$$\left[4n^2\kappa^2L_{\mathcal{W}_0}^2 + (\kappa + 1)(\mu + \nu)^2\right]\sigma(\vartheta_4, \vartheta_1) = 0 \quad (47)$$

In this situation, Ξ is either totally geodesic submanifold or

$$L_{\mathcal{W}_0} = \frac{1}{2n\kappa}(\mu + \nu)\sqrt{|\kappa + 1|}.$$

This concludes the proof. $\square$

Corollary 3. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Ξ is classified as $\mathcal{W}_0$ -Ricci-generalized semiparallel if and only if it is totally geodesic, provided that the condition

$$(\mu + \nu)\sqrt{|\kappa + 1|} = 0,$$

is satisfied.

Proof. The proof of the corollary is easily obtained by choosing $L_{\mathcal{W}_0} = 0$ in Equation (47) as per the definition of semiparallel submanifold. $\square$

Theorem 4. Consider an invariant submanifold Ξ within a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Provided that Ξ is a $\mathcal{W}_1$ -Ricci-generalized pseudoparallel submanifold, it is characterized as either being totally geodesic or satisfying the condition:

$$L_{\mathcal{W}_1} = \frac{1}{n} + \frac{1}{2n\kappa}(\mu + \nu)\sqrt{|\kappa + 1|}.$$

Proof. Let Ξ is a $\mathcal{W}_1$ -Ricci-generalized pseudoparallel submanifold. So, there is a function $L_{\mathcal{W}_1}$ on the set $M_2 = \{x \in \Xi | S(x) \neq \sigma(x)\}$ such that

$$(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_5) = L_{\mathcal{W}_1}Q(S, \sigma)(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2), \quad (48)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$. From (20), it is clear that

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5) = -L_{\mathcal{W}_1}\{\sigma((\vartheta_1 \wedge_S \vartheta_2)\vartheta_4, \vartheta_5) \\ + \sigma(\vartheta_4, (\vartheta_1 \wedge_S \vartheta_2)\vartheta_5)\}. \end{aligned}$$

Easily from here, we can write

$$\begin{aligned} R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ - \sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5) = -L_{\mathcal{W}_1}\{S(\vartheta_2, \vartheta_4)\sigma(\vartheta_1, \vartheta_5) \\ - S(\vartheta_1, \vartheta_4)\sigma(\vartheta_2, \vartheta_5) + S(\vartheta_2, \vartheta_5)\sigma(\vartheta_4, \vartheta_1) \\ - S(\vartheta_1, \vartheta_5)\sigma(\vartheta_4, \vartheta_2)\}. \end{aligned} \quad (49)$$

If we choose $\vartheta_2 = \vartheta_5 = \xi$ in (49), we get

$$\sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \xi)\xi) = L_{\mathcal{W}_1}S(\xi, \xi)\sigma(\vartheta_4, \vartheta_1). \quad (50)$$

If we use (11) and (30) in (50), we obtain

$$2\kappa(nL_{\mathcal{W}_1} - 1)\sigma(\vartheta_4, \vartheta_1) = (\mu + \nu)\sigma(\vartheta_4, h\vartheta_1). \quad (51)$$

Substituting $h\vartheta_1$ for ϑ_1 in (51), by view of (7) and (30), we have

$$2\kappa(nL_{\mathcal{W}_1} - 1)\sigma(\vartheta_4, h\vartheta_1) = -(\kappa + 1)(\mu + \nu)\sigma(\vartheta_4, \vartheta_1). \quad (52)$$

From (51) and (52), one can easily see that

$$\left[4\kappa^2(nL_{\mathcal{W}_1} - 1)^2 + (\kappa + 1)(\mu + \nu)^2\right]\sigma(\vartheta_4, \vartheta_1) = 0 \quad (53)$$

In this situation, Ξ is either totally geodesic submanifold or

$$L_{\mathcal{W}_1} = \frac{1}{n} + \frac{1}{2n\kappa}(\mu + \nu)\sqrt{|\kappa + 1|}.$$

This completes the proof. $\square$

Corollary 4. Suppose that Ξ represents an invariant submanifold within a $(2n + 1)$ -dimensional $\mathcal{AK} - (\kappa, \mu, \nu)$ space. Ξ is classified as $\mathcal{W}_1$ -Ricci-generalized semiparallel if and only if it is totally geodesic, assuming the relation

$$4\kappa^2 + (\kappa + 1)(\mu + \nu)^2 = 0,$$

holds.

Proof. The proof of the corollary is easily obtained by choosing $L_{\mathcal{W}_1} = 0$ in Equation (53) as per the definition of semiparallel submanifold. $\square$

Theorem 5. Suppose that Ξ is an invariant submanifold of a $(2n + 1)$ -dimensional $\mathcal{AK} - (\kappa, \mu, \nu)$ space. Ξ provides the relation $Q(g, \mathcal{W}_0 \cdot \sigma) = 0$ provided $\mu + \nu \neq 0$ if and only if Ξ is totally geodesic.

Proof. Let's assume that $Q(g, \mathcal{W}_0 \cdot \sigma) = 0$. So, we have

$$Q(g, \mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_3; \vartheta_4, \vartheta_5) = 0, \quad (54)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$. It is clear from (54)

$$\begin{aligned} & -g(\vartheta_5, W)(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_3) \\ & + g(\vartheta_4, W)(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_5, \vartheta_3) \\ & - g(\vartheta_5, \vartheta_3)(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_4) \\ & + g(\vartheta_4, \vartheta_3)(\mathcal{W}_0(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_5) = 0, \end{aligned}$$

and so, we get

$$\begin{aligned} & -g(\vartheta_5, W)[R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_3) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_3) - \sigma(\vartheta_4, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_3)] \\ & + g(\vartheta_4, W)[R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_5, \vartheta_3) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5, \vartheta_3) - \sigma(\vartheta_5, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_3)] \\ & - g(\vartheta_5, \vartheta_3)[R^\perp(\vartheta_1, \vartheta_2)\sigma(W, \vartheta_4) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)W, \vartheta_4) - \sigma(W, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_4)] \\ & + g(\vartheta_4, \vartheta_3)[R^\perp(\vartheta_1, \vartheta_2)\sigma(W, \vartheta_5) - \sigma(\mathcal{W}_0(\vartheta_1, \vartheta_2)W, \vartheta_5) - \sigma(W, \mathcal{W}_0(\vartheta_1, \vartheta_2)\vartheta_5)] = 0. \end{aligned} \quad (55)$$

If we choose $\vartheta_2 = \vartheta_3 = \vartheta_4 = W = \xi$ in (55), we obtain

$$\sigma(\mathcal{W}_0(\vartheta_1, \xi)\xi, \vartheta_5) = 0. \quad (56)$$

If we use (28) out of (56), we get

$$(\mu + \nu)\sigma(h\vartheta_1, \vartheta_5) = 0. \quad (57)$$

Substituting $h\vartheta_1$ for ϑ_1 in (51), we can write

$$(\kappa + 1)(\mu + \nu)\sigma(\vartheta_1, \vartheta_5) = 0.$$

This completes the proof. $\square$

Theorem 6. Suppose that Ξ is an invariant submanifold of a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Ξ provides the relation $Q(S, \mathcal{W}_0 \cdot \sigma) = 0$ provided $\mu + \nu \neq 0$ if and only if Ξ is totally geodesic.

Proof. The proof of this theorem can be readily established by applying the proof of the preceding theorem. $\square$

Theorem 7. Suppose that Ξ is an invariant submanifold of a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Ξ provides the relation $Q(S, \mathcal{W}_1 \cdot \sigma) = 0$ provided $4\kappa^2 \neq (\kappa + 1)(\mu + \nu)^2$ if and only if Ξ is totally geodesic.

Proof. Let us assume that $Q(S, \mathcal{W}_1 \cdot \sigma) = 0$. So, we can write

$$Q(S, \mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_3; \vartheta_4, \vartheta_5) = 0, \quad (58)$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$. It is clear from (58)

$$\begin{aligned} & -S(\vartheta_5, W)(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_4, \vartheta_3) \\ & + S(\vartheta_4, W)(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(\vartheta_5, \vartheta_3) \\ & - S(\vartheta_5, \vartheta_3)(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_4) \\ & + S(\vartheta_4, \vartheta_3)(\mathcal{W}_1(\vartheta_1, \vartheta_2) \cdot \sigma)(W, \vartheta_5) = 0 \end{aligned}$$

and so, we get

$$\begin{aligned} & -S(\vartheta_5, W)[R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_3) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_3) - \sigma(\vartheta_4, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_3)] \\ & + S(\vartheta_4, W)[R^\perp(\vartheta_1, \vartheta_2)\sigma(\vartheta_5, \vartheta_3) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5, \vartheta_3) - \sigma(\vartheta_5, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_3)] \\ & - S(\vartheta_5, \vartheta_3)[R^\perp(\vartheta_1, \vartheta_2)\sigma(W, \vartheta_4) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)W, \vartheta_4) - \sigma(W, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_4)] \\ & + S(\vartheta_4, \vartheta_3)[R^\perp(\vartheta_1, \vartheta_2)\sigma(W, \vartheta_5) - \sigma(\mathcal{W}_1(\vartheta_1, \vartheta_2)W, \vartheta_5) - \sigma(W, \mathcal{W}_1(\vartheta_1, \vartheta_2)\vartheta_5)] = 0. \end{aligned} \quad (59)$$

Setting $\vartheta_2 = \vartheta_3 = \vartheta_4 = W = \xi$ in (59), we obtain

$$S(\xi, \xi)\sigma(\mathcal{W}_1(\vartheta_1, \xi)\xi, \vartheta_5) = 0. \quad (60)$$

Putting (30) out of (60), we get

$$4n^2\sigma(\vartheta_5, \vartheta_1) + 2n\kappa(\mu + \nu)\sigma(\vartheta_5, h\vartheta_1) = 0. \quad (61)$$

Substituting $h\vartheta_1$ for ϑ_1 in (61), we can write

$$4n\kappa^2\sigma(\vartheta_5, h\vartheta_1) = 2n\kappa(\kappa + 1)(\mu + \nu)\sigma(\vartheta_1, \vartheta_5) = 0. \quad (62)$$

From (51) and (52), one can easily see that

$$4n\kappa^2[4\kappa^2 + (\kappa + 1)(\mu + \nu)^2]\sigma(\vartheta_1, \vartheta_5) = 0.$$

With this, the proof is concluded. $\square$

Theorem 8. Suppose that Ξ is an invariant submanifold of a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Ξ provides the relation $Q(g, \mathcal{W}_1 \cdot \sigma) = 0$ provided $4\kappa^2 \neq (\kappa + 1)(\mu + \nu)^2$ if and only if Ξ is totally geodesic.

Proof. The proof of this theorem can be readily established by applying the proof of the preceding theorem. $\square$

Theorem 9. Suppose that Ξ is an invariant submanifold of a $(2n + 1)$ -dimensional $\mathcal{AK}-(\kappa, \mu, \nu)$ space. Ξ provides the relation $Q(\mathcal{W}_0, \tilde{\nabla}\sigma) = 0$ provided $\kappa(\mu + \nu) \neq 0$ if and only if Ξ is totally geodesic.

Proof. Let's assume that $Q(\mathcal{W}_0, \tilde{\nabla}\sigma) = 0$. So, we can write

$$\begin{aligned} & \left(\tilde{\nabla}_{(\vartheta_1 \wedge_{\mathcal{W}_0} \vartheta_2) \vartheta_4} \sigma \right) (\vartheta_5, \vartheta_3) \\ & + (\tilde{\nabla}_{\vartheta_4} \sigma) ((\vartheta_1 \wedge_{\mathcal{W}_0} \vartheta_2) \vartheta_5, \vartheta_3) \\ & + (\tilde{\nabla}_{\vartheta_4} \sigma) (\vartheta_5, (\vartheta_1 \wedge_{\mathcal{W}_0} \vartheta_2) \vartheta_3) = 0, \end{aligned} \quad (63)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5 \in \Gamma(T\Xi)$. If we choose $\vartheta_2 = \vartheta_5 = \xi$ in (64), we get

$$\begin{aligned} & \left(\tilde{\nabla}_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4} \sigma \right) (\xi, \vartheta_3) \\ & + (\tilde{\nabla}_{\vartheta_4} \sigma) ((\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \xi, \vartheta_3) \\ & + (\tilde{\nabla}_{\vartheta_4} \sigma) (\xi, (\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_3) = 0. \end{aligned} \quad (64)$$

Now let's calculate all the expressions in (64).

$$\begin{aligned} \left(\tilde{\nabla}_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4} \sigma \right) (\xi, \vartheta_3) &= \nabla_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4}^\perp \sigma(\xi, \vartheta_3) \\ &\quad - \sigma \left(\nabla_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4} \xi, \vartheta_3 \right) \\ &\quad - \sigma \left(\xi, \nabla_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4} \vartheta_3 \right) \\ &= -\sigma \left(\nabla_{(\vartheta_1 \wedge_{\mathcal{W}_0} \xi) \vartheta_4} \xi, \vartheta_3 \right) \\ &= -\sigma(\mathcal{W}_0(\xi, \vartheta_4) \vartheta_1 - \mathcal{W}_0(\vartheta_1, \vartheta_4) \xi, \vartheta_3) \\ &\quad + \sigma(\phi h[\mathcal{W}_0(\xi, \vartheta_4) \vartheta_1 - \mathcal{W}_0(\vartheta_1, \vartheta_4) \xi], \vartheta_3), \end{aligned} \quad (65)$$

$$\begin{aligned}(\tilde{\nabla}_{\vartheta_4}\sigma)((\vartheta_1\Lambda_{\mathcal{W}_0}\xi)\xi,\vartheta_3) &= (\tilde{\nabla}_{\vartheta_4}\sigma)(\mathcal{W}_0(\xi,\xi)\vartheta_1-\mathcal{W}_0(\vartheta_1,\xi)\xi,\vartheta_3) \\ &= -(\mu+\nu\phi)(\tilde{\nabla}_{\vartheta_4}\sigma)(h\vartheta_1,\vartheta_3),\end{aligned}\tag{66}$$

$$\begin{aligned}(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi,(\vartheta_1\Lambda_{\mathcal{W}_0}\xi)\vartheta_3) &= -\sigma(\nabla_{\vartheta_4}\xi,(\vartheta_1\Lambda_{\mathcal{W}_0}\xi)\vartheta_3) \\ &= -\sigma(\vartheta_4,\mathcal{W}_0(\xi,\vartheta_3)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_3)\xi) \\ &\quad +\phi\sigma(h\vartheta_4,\mathcal{W}_0(\xi,\vartheta_3)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_3)\xi).\end{aligned}\tag{67}$$

Substituting (65)–(67) in (64), we obtain

$$\begin{aligned}&-\sigma(\mathcal{W}_0(\xi,\vartheta_4)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_4)\xi,\vartheta_3) \\ &+\sigma(\phi h[\mathcal{W}_0(\xi,\vartheta_4)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_4)\xi],\vartheta_3) \\ &-(\mu+\nu\phi)(\tilde{\nabla}_{\vartheta_4}\sigma)(h\vartheta_1,\vartheta_3) \\ &+\phi\sigma(h\vartheta_4,\mathcal{W}_0(\xi,\vartheta_3)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_3)\xi) \\ &-\sigma(\vartheta_4,\mathcal{W}_0(\xi,\vartheta_3)\vartheta_1-\mathcal{W}_0(\vartheta_1,\vartheta_3)\xi)=0.\end{aligned}\tag{68}$$

Putting $\vartheta_3 = \xi$ in (68), we can write

$$\begin{aligned}&-(\mu+\nu\phi)(\tilde{\nabla}_{\vartheta_4}\sigma)(h\vartheta_1,\xi)+(\mu+\nu\phi)\sigma(\vartheta_4,h\vartheta_1) \\ &+\phi(\mu+\nu\phi)(\kappa+1)\sigma(\vartheta_4,\vartheta_1)=0.\end{aligned}\tag{69}$$

Also, with an easy calculation, we get that

$$(\tilde{\nabla}_{\vartheta_4}\sigma)(h\vartheta_1,\xi)=-\sigma(h\vartheta_1,\vartheta_4)-(\kappa+1)\sigma(\vartheta_1,\vartheta_4).\tag{70}$$

Setting (70) for (69), we get

$$(\mu+\nu)\sigma(h\vartheta_1,\vartheta_4)=- (\kappa+1)(\mu+\nu)\sigma(\vartheta_1,\vartheta_4).\tag{71}$$

Substituting $h\vartheta_1$ for ϑ_1 in (71), we can write

$$\sigma(\vartheta_1,\vartheta_4)=\phi\sigma(h\vartheta_1,\vartheta_4).\tag{72}$$

From (71) and (72), a straightforward computation shows that

$$\kappa(\mu+\nu)\sigma(\vartheta_1,\vartheta_4)=0.$$

This completes the proof. $\square$

Example 1. Let $\Xi = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_5 \neq \pm 1, 0 \}$ and we take

$$e_1 = (x_5 + 1) \frac{\partial}{\partial x_1},$$

$$e_2 = \frac{1}{(x_5 - 1)} \frac{\partial}{\partial x_2},$$

$$e_3 = \frac{(x_5 + 1)^2}{2} \frac{\partial}{\partial x_3},$$

$$e_4 = \frac{5}{(x_5 - 1)} \frac{\partial}{\partial x_4},$$

$$e_5 = \zeta = (x_5^2 - 1) \frac{\partial}{\partial x_5},$$

constitute a set of linearly independent vector fields on Ξ . Furthermore, we define a tensor field ϕ of type $(1, 1)$ as follows:

$$\phi e_1 = e_2,$$

$$\phi e_2 = -e_1,$$

$$\phi e_3 = e_4,$$

$$\phi e_4 = -e_3,$$

$$\phi e_5 = 0.$$

Furthermore, the Riemannian metric tensor g is given by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

By direct computations, we can easily to see that

$$\phi^2 \vartheta_1 = -\vartheta_1 + \eta(\vartheta_1) \zeta, \eta(\vartheta_1) = g(\vartheta_1, \zeta)$$

and

$$g(\phi \vartheta_1, \phi \vartheta_2) = g(\vartheta_1, \vartheta_2) - \eta(\vartheta_1) \eta(\vartheta_2).$$

Consequently, the structure $(\Xi^5, \phi, \zeta, \eta, g)$ constitutes a 5-dimensional almost contact metric manifold. By evaluating the Lie brackets, the non-vanishing components are determined as follows:

$$[e_1, e_5] = -(x_5 - 1)e_1,$$

$$[e_2, e_5] = (x_5 + 1)e_2,$$

$$[e_3, e_5] = -(x_5 - 1)e_3,$$

$$[e_4, e_5] = (x_5 + 1)e_4.$$

Additionally, let ∇ represent the Levi–Civita connection on Ξ . By employing the Koszul formula, the non-vanishing components are obtained as follows:

$$\nabla_{e_1}e_5 = -(x_5 - 1)e_1,$$

$$\nabla_{e_2}e_5 = (x_5 + 1)e_2,$$

$$\nabla_{e_3}e_5 = -(x_5 - 1)e_3,$$

$$\nabla_{e_4}e_5 = (x_5 + 1)e_4.$$

By evaluating the aforementioned results against the identity

$$\nabla_{\vartheta_1}e_5 = \vartheta_1 - \eta(\vartheta_1)e_5 - \phi h\vartheta_1,$$

it becomes evident that

$$he_1 = -x_5e_2,$$

$$he_2 = -x_5e_1,$$

$$he_3 = -x_5e_4,$$

$$he_4 = -x_5e_3,$$

$$he_5 = 0.$$

By direct calculations, we get

$$R(e_1, e_5)e_5 = \kappa e_1 + \mu he_1 + \nu \phi he_1 = 2(x_5 - 1)e_1,$$

$$R(e_2, e_5)e_5 = \kappa e_2 + \mu he_2 + \nu \phi he_2 = -2x_5(x_5 + 1)e_2,$$

$$R(e_3, e_5)e_5 = \kappa e_3 + \mu he_3 + \nu \phi he_3 = 2(x_5 - 1)e_3,$$

and

$$R(e_4, e_5)e_5 = \kappa e_4 + \mu he_4 + \nu \phi he_4 = -2x_5(x_5 + 1)e_4,$$

which implies that

$$\kappa = -(x_5^2 + 1), \mu = 0 \text{ and } \nu = 2 - \frac{1}{x_5} + x_5.$$

Thus contact metric manifold $\Xi^5(\phi, \xi, \eta, g)$ is an almost Kenmotsu $\left(-(x_5^2 + 1), 0, 2 - \frac{1}{x_5} + x_5\right)$ -space. By direct calculations, the non-vanishing components of Riemannian curvature

$$R(e_1, e_5)e_4 = \left[1 - (x_5 - 1)^2\right]e_1,$$

$$R(e_2, e_5)e_5 = -\left[1 + (x_5 + 1)^2\right]e_2,$$

$$R(e_3, e_5)e_5 = \left[1 - (x_5 - 1)^2\right]e_3,$$

$$R(e_4, e_5)e_5 = -\left[1 + (x_5 + 1)^2\right]e_4.$$

Thus, the scalar curvature is given by

$$\tau = -2[(x_5 - 1)^2 + (x_5 + 1)^2].$$

Example 2. Let $M^{2n+1} = \mathbb{R}^{2n+1}$ be the $(2n + 1)$ -dimensional smooth manifold with standard coordinates

$$(x_1, y_1, \dots, x_n, y_n, z).$$

Define an almost contact metric structure (ϕ, ξ, η, g) on M^{2n+1} by

$$\eta = dz, \xi = \frac{\partial}{\partial z},$$

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},$$

$$\phi\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \phi(\xi) = 0,$$

and the Riemannian metric

$$g = e^{2z} \sum_{i=1}^n (dx_i^2 + dy_i^2) + dz^2.$$

It is well known that $(M^{2n+1}, \phi, \xi, \eta, g)$ is an almost Kenmotsu manifold.

Moreover, a direct computation of the curvature tensor shows that M^{2n+1} satisfies the (κ, μ, ν) -nullity condition with

$$\kappa = -1, \mu = 0, \nu = 0.$$

Hence, M^{2n+1} is an almost Kenmotsu (κ, μ, ν) -space.

Consider the submanifold

$$N^{2k+1} = \{(x_1, y_1, \dots, x_k, y_k, 0, \dots, 0, z)\} \subset M^{2n+1},$$

where $1 \leq k \leq n$.

It is easy to verify that

$$\xi \in TN,$$

$$\phi(TN) \subset TN.$$

Thus, N^{2k+1} is an invariant submanifold of M^{2n+1} .

By a straightforward computation of the second fundamental form σ , we obtain

$$\sigma(X, Y) = 0, \text{ for all } X, Y \in TN.$$

Hence, the submanifold is totally geodesic. Consequently, N^{2k+1} satisfies the pseudoparallel condition trivially, and also Ricci-generalized pseudoparallel with respect to both W_0 and W_1 curvature tensors.

This example confirms the theoretical results obtained in this paper and provides an explicit realization of an invariant totally geodesic pseudoparallel submanifold in an almost Kenmotsu (κ, μ, ν) -space with $\kappa = -1$.

4. Conclusions

In this study, we investigated invariant and pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces in order to gain deeper insight into both the local and global geometric structure of the ambient manifold. By analyzing these submanifolds, we established meaningful classification criteria that clarify when such submanifolds satisfy important

curvature conditions, including the Ricci-generalized pseudoparallel condition. The role of the second fundamental form and the Tachibana operator proved to be essential in deriving these geometric characterizations.

A central part of our analysis was devoted to the study of various classes of pseudoparallel submanifolds, with particular emphasis on Ricci-generalized pseudoparallel cases. By employing the W_0 and W_1 curvature tensors, we obtained a unified framework for examining pseudoparallelism in almost Kenmotsu (κ, μ, ν) -spaces. Within this framework, we derived necessary and sufficient conditions under which invariant pseudoparallel submanifolds become totally geodesic.

Finally, we presented new and noteworthy results that characterize totally geodesic invariant submanifolds of almost Kenmotsu (κ, μ, ν) -spaces in terms of explicit relationships among the structure functions κ, μ and ν . These findings not only extend and refine several known results in contact metric geometry but also provide a clearer geometric understanding of the interaction between curvature tensors and submanifold geometry in almost Kenmotsu settings.

Authors can gain a better understanding of the global and local characteristics of the ambient almost Kenmotsu (κ, μ, ν) -space manifold by examining these invariant submanifolds and pseudoparallel submanifolds and determining important classification criteria, such as demonstrating when they are totally geodesic and meeting specific curvature features such as Ricci-generalized condition, and also using the second fundamental form and the Tachibana operator.

This study explored some geometric properties of those submanifolds that are pseudoparallel. While examining some types of pseudoparallel submanifolds, such as Ricci-generalized ones, for almost Kenmotsu (κ, μ, ν) -spaces, we used the W_0 and W_1 curvature tensors. We derived the necessary and sufficient conditions for the pseudoparallel submanifolds of almost Kenmotsu (κ, μ, ν) -spaces to be totally geodesic under the condition of being invariant. By describing the new and intriguing findings, we were able to exhibit the condition for totally geodesic submanifolds of almost Kenmotsu (κ, μ, ν) -spaces according to the relationship between κ, μ and ν .

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