

Article

A Closed-Form Inverse Laplace Transform of Shifted Quasi-Rational Spectral Functions via Generalized Hypergeometric and Kampé de Fériet Functions

Slobodanka Galovic ^{1,*} , Aleksa Djordjevic ²  and Katarina Lj. Djordjevic ¹ ¹ Vinca Institute of Nuclear Sciences-National Institute of the Republic of Serbia, University of Belgrade, 11001 Belgrade, Serbia² Faculty of Physics, University of Belgrade, 11000 Belgrade, Serbia

* Correspondence: bobagal@vin.bg.ac.rs

Abstract

Closed-form analytic inverses allow explicit tracking of parameter effects, facilitate interpretation of experimental signals, and support solving inverse problems. Here, we derive a rigorous closed-form expression for the inverse Laplace transform of a class of shifted quasi-rational spectral functions with a square-root radical and a power-law decaying factor. These functions appear in coupled diffusion processes in physics and in the analysis of electromagnetic signal propagation through electrically cascaded networks, signal processing, and related areas. The transform is expressed as a finite sum of three generalized hypergeometric functions—two Kummer functions and one five-parameter Kampé de Fériet function—each multiplied by a monomial depending on the decay parameter. The validity of the result is confirmed by direct Laplace transformation, which recovers the original spectral function. Several known inverse transforms appear as limiting cases, illustrating the generality of the solution. Additionally, reduction formulas for a subclass of Kampé de Fériet functions demonstrate how the general solution encompasses previously known results and highlight the generality of the method.

Keywords: inverse Laplace transform; generalized hypergeometric functions; Kummer function; Kampé de Fériet function; integral transforms; reduction formulas; quasi-rational spectral functions; coupled diffusion

MSC: 44A10; 33C20; 33C65



Academic Editor: Hyun Soo Chung

Received: 21 January 2026

Revised: 12 February 2026

Accepted: 13 February 2026

Published: 19 February 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Laplace transforms represent a powerful analytical tool widely used to solve a broad class of problems in physics and engineering, from transient circuit behavior and control systems [1,2] to heat transfer, mass and particle transport, vibration analysis, and other physical phenomena [3–12]. The inverse Laplace transform is equally important, as it enables the return to the time domain, providing direct insight into the evolution of physical systems [6–12].

However, applying Laplace transform methods presents numerous mathematical and practical challenges. Inversion typically requires contour integration in the complex plane, which becomes complicated when the function exhibits multiple singularities or branch points [13,14]. The choice of an appropriate integration contour and the definition of branch cuts are especially delicate when radicals or other multivalued expressions are

present [14,15]. Moreover, no general criterion guarantees the convergence of the Laplace transform or its inverse in all cases.

Problems governed by fractional differential equations, systems of coupled equations of motion, or (fractional) partial differential equations—such as diffusion, anomalous diffusion, vibrational systems, fluid dynamics, and electromagnetic wave propagation—often lead to spectral functions that are irrational, quasi-rational, or transcendental in form when solved using Laplace transforms [6,7,12]. These spectral functions are generally not amenable to classical inversion techniques (circular, hyperbolic, or parabolic contours), and their inverse transforms typically require special analytical [15–17] or numerical methods [18–23]. Some analytically obtained inverses of such functions have been compiled into reference tables [24–26], while many are expressed in terms of special functions of mathematical physics [27–31], enabling deeper insights into the associated physical processes.

One important application where Laplace transforms yield complex spectral functions that are difficult to invert analytically by standard techniques is the interpretation of experimentally measured time-domain signals in photothermal (PT) and photoacoustic (PA) techniques [32–35]. These signals arise from the interplay of coupled physical processes, where one time-dependent phenomenon serves as a stimulus for another [36–40]. In PT measurements, the signal evolution depends on the temporal profile of optical excitation, heat transport properties, acoustic wave propagation in the surrounding medium, and the transfer function of the acoustic detector.

In such cases, finding an inverse Laplace transform in closed form is crucial for interpreting experimental data and understanding the underlying dynamics, yet for many of these functions no general analytical solution exists. Closed-form analytical inverses enable explicit tracking of parameter dependencies, facilitate physical interpretation of measured signals, and support the formulation and solution of inverse problems. This motivates the need for closed-form analytic inverses, as they allow explicit assessment of how individual system parameters affect observed signals.

The present work addresses a specific class of such spectral functions and derives, for the first time, a closed-form analytical expression for their inverse Laplace transform, expressed explicitly as a sum of two Kummer functions and one five-parameter Kampé de Fériet (KdF) function. Recent advances in the theory of generalized hypergeometric functions—particularly in reduction formulas and structural properties of the KdF function—have renewed interest in explicit analytical representations arising in integral transforms (see, e.g., [41–45]). This solution represents the main contribution of the paper, as no closed-form inverse exists for this class of spectral function in the literature.

To illustrate the generality of the solution, we show that in certain limiting cases the general expression reduces to previously known forms expressed via Bessel functions, error functions (erf, erfc), and even a case involving the Mittag-Leffler (ML) function. Establishing these connections required the derivation of two reduction formulas for the five-parameter Kampé de Fériet function. These reduction formulas are secondary, serving solely to demonstrate that the general solution encompasses known results, and are not the main contribution.

Such quasi-rational spectral functions with shifted square-root radicals naturally arise in multistage diffusion processes, where the diffusion of one subsystem generates the source term for the subsequent process. A representative example occurs in photoacoustic signal formation in semiconductors, where the diffusion of photogenerated charge carriers produces a distributed heat source, the diffusion of heat creates surface temperature variations, and these fluctuations induce pressure oscillations in a closed cell. In these cascaded diffusion chains, each stage leaves a distinct structural signature in the Laplace

domain [36–39]. Obtaining an explicit analytical inverse transform, therefore, allows one to track the contribution of each stage, to understand the evolution of the final measured signal and to formulate and solve inverse problems at different levels of the process chain.

Overall, the derived solution provides a unified analytical framework linking inverse Laplace transforms and generalized hypergeometric functions, while indicating broader connections to coupled diffusion-type phenomena. The structure of the paper is as follows. Section 2 summarizes the preliminary mathematical results and definitions used throughout the paper. Section 3 presents the derivation of the inverse Laplace transform by reformulating the complex contour integral as a convolution and expressing it in terms of generalized hypergeometric functions. Section 4 verifies the result by reapplying the Laplace transform. Section 5 demonstrates several known results as special cases, and Section 6 concludes the work and outlines directions for future research.

2. Preliminaries

In this section, we summarize the basic properties of Laplace transform, definitions of special functions and series and other mathematical remainders that will be used throughout the paper. For convenience, all theorems of Laplace transform are stated in the form of equivalences, as they will be employed in both directions, particularly when calculating inverse Laplace transforms.

2.1. Laplace Transform and Inverse Relations

For a sufficiently regular function $f(t)$, the Laplace transform and its inverse are connected by the pairs of equivalent relations Equations (1) and (2), where Equation (2) denotes the so-called Mellin formula, Bromwich integral, or Mellin inversion formula [1–3]:

$$\bar{F}(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t) dt; \quad \operatorname{Re}\{s\} > 0 \quad (1)$$

$$f(t) = \mathcal{L}^{-1}\{\bar{F}(s)\}(s) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{F}(s) ds; \quad t > 0 \quad (2)$$

The following equivalences will be frequently used in the sequel (often in the inverse direction to compute explicit time–domain expressions):

Linearity

$$\mathcal{L}\left\{\sum_{k=1}^n c_k f_k(t)\right\}(s) = \sum_{k=1}^n c_k \bar{F}_k(s) \Leftrightarrow \mathcal{L}^{-1}\left\{\sum_{k=1}^n c_k \bar{F}_k(s)\right\}(t) = \sum_{k=1}^n c_k f_k(t), \quad c_k \in \mathbb{R} \quad (3)$$

Shift in the s -domain (complex/frequency shift):

$$\mathcal{L}\{e^{At} f(t)\}(s) = \bar{F}(s - A) \Leftrightarrow \mathcal{L}^{-1}\{\bar{F}_k(s - A)\}(t) = e^{At} f(t) \quad (4)$$

Convolution theorem:

$$\mathcal{L}\{(f * g)\}(s) = \bar{F}(s) \bar{G}(s) \Leftrightarrow \mathcal{L}^{-1}\{\bar{F}(s) \bar{G}(s)\} = \int_0^t f(\tau) g(\tau - s) d\tau \quad (5)$$

In this paper, we also use selected table values for causal systems ($\operatorname{Re}\{s\} > 0$) [24,26]:
for $q > 0, q \in \mathbb{R}$

$$\mathcal{L}\left\{\frac{t^{q-1}}{\Gamma(q)} h(t)\right\} = \frac{1}{s^q} \Leftrightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^q}\right\} = \frac{t^{q-1}}{\Gamma(q)} h(t) \quad (6)$$

where $\Gamma(\dots)$ denotes the Gamma function [27].

For $B > 0$:

$$\mathcal{L}\{e^{B^2 t} \operatorname{erfc}(b\sqrt{t})\} = \frac{1}{\sqrt{s+B}} \Leftrightarrow \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+B}}\right\} = e^{B^2 t} \operatorname{erfc}(b\sqrt{t}) \quad (7)$$

where $\operatorname{erfc}(\dots)$ denotes the complementary error function [27,31] that is defined by

$$\operatorname{erfc}(y) = 1 - \operatorname{erf}(y) \quad (8)$$

2.2. Kummer's Confluent Hypergeometric Function and KdF Function: Definitions

The confluent hypergeometric function of the first kind (Kummer's function) is defined by the infinite series [41]:

$$M(r_1, r_2, y) = \sum_{k=0}^{\infty} \frac{r_1^{(k)} y^k}{r_2^{(k)} k!} \quad (9)$$

valid for $y \in \mathbb{R}$. It also admits the integral representation:

$$M(r_1, r_2, y) = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2-r_1)} \int_0^1 e^{yz} y^{r_1-1} (1-y)^{r_2-r_1-1} dy \quad (10)$$

The Kummer transformation relates the Kummer function of positive and negative arguments:

$$M(r_1, r_2, y) = e^y M(r_2 - r_1, r_2, -y) \quad (11)$$

The KdF function is defined as a double hypergeometric series [42–46]:

$$F_{1;1;1}^{1;1;1}\left(\begin{matrix} a & b & c \\ d & e & f \end{matrix}; y_1, y_2\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{y_1^n}{n!} \frac{a^{(n+k)}}{d^{(n+k)}} \frac{b^{(n)}}{e^{(n)}} \frac{c^{(k)}}{f^{(k)}} \frac{y_2^k}{k!} \quad (12)$$

This function generalizes many known bivariate hypergeometric functions.

In Equations (5) and (6) the Pochhammer symbol $\alpha^{(n)}$ represents the rising factorial [47]:

$$\alpha^{(0)} = 1, \quad \alpha^{(n)} = \alpha(\alpha+1) \cdots (\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (13)$$

Similarly, the falling factorial is defined as [47]:

$$\alpha_{(n)} = \alpha(\alpha-1) \cdots (\alpha-n+1), \text{ with } \alpha_{(0)} = 1 \quad (14)$$

and appears in the definition of binomial coefficients [48]:

$$\binom{\alpha}{n} = \frac{\alpha_{(n)}}{n!} \quad (15)$$

The relationship between rising and falling factorials is given in [49]:

$$\alpha^{(n)} = (-1)^n (-\alpha)_{(n)} \quad (16)$$

Additionally, the following identity holds:

$$\alpha^{(n+k)} = (\alpha+n)^{(k)} \quad (17)$$

Useful contiguous relations for Kummer function that are used in the paper are [41,49]

$$r_2 M(r_1, r_2, y) - r_2 M(r_1 - 1, r_2, y) - y M(r_1, r_2 + 1, y) = 0 \quad (18)$$

$$(r_1 - r_2 + 1) M(r_1, r_2, y) - r_1 M(r_1 + 1, r_2, y) + (r_2 - 1) M(r_1, r_2 - 1, y) = 0 \quad (19)$$

Other identities used from the literature are given by the following equations [49]:

$$M(r_1, r_2, 0) = 1 \quad (20)$$

$$M(1/2, 1, y) = e^{y/2} I_0(y/2) \quad (21)$$

$$M(3/2, 3, y) = e^{y/2} \frac{4}{y} I_1(y/2) \quad (22)$$

$$M(1/2, 3/2, -y) = \frac{\sqrt{\pi}}{2\sqrt{y}} \operatorname{erf}(\sqrt{y}) \quad (23)$$

$$M(1, 3/2, y) = e^y \frac{\pi}{2y} \operatorname{erf}(y) \quad (24)$$

$$M(1, 2, y) = \frac{e^y - 1}{y} \quad (25)$$

where I_0 and I_1 are modified Bessel functions of zero and first order [27], respectively. For background on the KdF function, its reduction formulas, and related summation identities, we refer to classical sources and recent contributions, such as [42–46]. Some reduction identities used in the paper that reduce specific KdF functions to Kummer's functions are derived in Appendix B and they are given by following equations:

$$F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ 2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -y, y \right) = M(1, 2, y) \quad (26)$$

$$F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ 5/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -y, y \right) = \frac{3}{2y} (M(1, 3/2, y) - 1) \quad (27)$$

2.3. Other Special Functions Used in the Paper, Defined by Infinite Series

The one-parameter ML function is defined by the series [50–53]:

$$E_{1/2}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k/2 + 1)k!} \quad (28)$$

There exists a relationship between the ML function and the complementary error function [50]:

$$E_{1/2}(\sqrt{y}) = e^y \operatorname{erfc}(-\sqrt{y}) \quad (29)$$

2.4. Series Reminders

We recall the definitions of binomial and infinite geometric series [54,55]:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = (1+z)^\alpha, \quad \alpha \in \mathbb{R}, z \in \mathbb{C}, |z| < 1, n \in \mathbb{N} \quad (30)$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in \mathbb{C}, |z| < 1, n \in \mathbb{N} \quad (31)$$

3. Main Results

This section provides a detailed derivation of the inverse Laplace transform for a combined shifted quasi-rational functions with square-root radical and power-law decay term.

The transformation introduced here is not a matter of algebraic convenience, but a methodological necessity. The original spectral form contains shifted radicals for which no direct inversion formula exists. By mapping the problem into an auxiliary quasi-rational form, the structure becomes amenable to analytical treatment via generalized hypergeometric and KdF functions, enabling the derivation of an exact inverse transform.

Theorem 1. Let complex function $\bar{G}(s)$ is given by

$$G(s) = \frac{1}{s^q(\sqrt{s+A+B})} \quad (32)$$

where s is a complex variable, A and B are shift parameters (internal and external, respectively), and $q > 0$. Then, the inverse Laplace transform of $\bar{G}(s)$ is given by

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} = & \frac{t^{q-1/2}}{\Gamma(q+1/2)} M\left(1/2, q+1/2, -At\right) + \\ & + \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1}\left(\frac{3}{2}; \frac{1}{2}; -B^2 t, (B^2 - A)t\right) - \\ & - \frac{B t^q}{\Gamma(q+1)} M(1, q+1, (B^2 - A)t) \end{aligned} \quad (33)$$

where $M(r_1, r_2, y)$ denotes the confluent hypergeometric function (Kummer's function), the generalized two-argument five-parameter KdF function is

$$F_{1;1}^{1;1}\left(\frac{a}{d}; \frac{b}{e}; -B^2 t, (B^2 - A)t\right) = F_{1;1}^{1;1}\left(\frac{a}{d}; \frac{b}{e}; \frac{c}{c}; -B^2 t, (B^2 - A)t\right).$$

Proof of the Theorem 1 ($\Rightarrow$). The proof proceeds as follows: firstly, it is shown that the inverse Laplace transform of the given function can be reduced to solving a specific convolution integral. The resulting convolution integral is then solved analytically.

The given complex function Equation (32) can be written as a product of two complex functions:

$$\bar{G}(s) = \frac{1}{s^q} \frac{1}{\sqrt{s+A+B}} = \bar{G}_1(s) \bar{G}_2(s) \quad (34)$$

where:

$$\bar{G}_1(s) = \frac{1}{s^q}, \quad (35)$$

$$\bar{G}_2(s) = \frac{1}{\sqrt{s+A+B}}. \quad (36)$$

Using the Convolution Theorem (Equation (5)), the inverse Laplace transform of $\bar{G}(s)$ can be found as:

$$g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} = \int_0^t g_1(t-u) g_2(u) du \quad (37)$$

where

$$g_1(t) = \mathcal{L}^{-1}\{\bar{G}_1(s)\}, \quad (38)$$

$$g_2(t) = \mathcal{L}^{-1}\{\bar{G}_2(s)\}. \quad (39)$$

The inverse Laplace transform for $\bar{G}_1(s)$ (Equation (35)) can be found in many reference tables [24,26] and is given in Equation (6) of the Preliminaries section.

The inverse Laplace transform of $\bar{G}_2(s)$ (Equation (36)) can be obtained using the complex shifting property (Equation (4)) and solutions in tables of inverse Laplace transforms of irrational functions [25]:

$$g_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+A+B}}\right\} = \frac{1}{\pi t} e^{-At} - B e^{(B^2-A)t} \operatorname{erfc}(B\sqrt{t}). \quad (40)$$

By substituting Equations (6) and (40) into Equation (37), we obtain:

$$g(t) = \mathcal{L}^{-1}\{\bar{G}(s)\} = \int_0^t \frac{(t-u)^{q-1}}{\Gamma(q)} \left(\frac{1}{\sqrt{\pi u}} e^{-Au} - B e^{(B^2-A)u} \operatorname{erfc}(B\sqrt{u}) \right) du. \quad (41)$$

The convolution integral in Equation (41) can be solved by replacing Equation (8) and decomposing the integral into three definite integrals:

$$g(t) = i_1(t) + i_2(t) - i_3(t) \quad (42)$$

breaking the problem into smaller parts:

$$i_1(t) = \frac{1}{\Gamma(q)\sqrt{\pi}} \int_0^t \frac{(t-u)^{q-1}}{\sqrt{u}} e^{-Au} du, \quad (43)$$

$$i_2(t) = \frac{B}{\Gamma(q)} \int_0^t (t-u)^{q-1} e^{(B^2-A)u} \operatorname{erf}(B\sqrt{u}) du, \quad (44)$$

$$i_3(t) = \frac{B}{\Gamma(q)} \int_0^t (t-u)^{q-1} e^{(B^2-A)u} du. \quad (45)$$

Each integral is solved by introducing the substitution:

$$x = \frac{u}{t}, \quad dx = \frac{du}{t} \quad (46)$$

Then the solution for the integral $i_1(t)$ becomes:

$$i_1(t) = \frac{1}{\sqrt{\pi}} \frac{t^{q-1/2}}{\Gamma(q)} \int_0^1 e^{-Atx} x^{-1/2} (1-x)^{q-1} dx = \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2; q+1/2; -At) \quad (47)$$

where a confluent hypergeometric function (Kummer's function, Equation (10)) is identified in the definite integral over the interval $[0, 1]$.

In the computation of the integral Equation (43), the series representation of the $\operatorname{erf}(y)$ function through its Taylor expansion [54–56] is used:

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{n!(2n+1)} \quad (48)$$

Remark 1. Appendix A shows that the series in Equation (48) has an infinite radius of convergence, implying that the function is entire and Equation (48) is valid over the entire domain.

By substituting Equation (48) into Equation (44), changing the order of summation and integration, and recognizing confluent hypergeometric functions under the infinite sum, the function $i_2(t)$ becomes:

$$\begin{aligned} i_2(t) &= \frac{2B^2}{\Gamma(q)\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n B^{2n}}{n!(2n+1)} t^{q+n+1/2} \int_0^1 e^{(B^2-A)tx} x^{n+1/2} (1-x)^{q-1} dx \right) = \\ &= \frac{2B^2 t^{q+1/2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-B^2 t)^n}{n!(2n+1)} \frac{\Gamma(n+3/2)}{\Gamma(n+q+3/2)} M(n+3/2; n+q+3/2; (B^2-A)t) \end{aligned} \quad (49)$$

Using the infinite series representation of confluent hypergeometric functions (Equation (9)) and substituting into Equation (49), the solution is expressed as a double infinite series:

$$i_2(t) = \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-B^2 t)^n}{n!} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \frac{(3/2)^{(n+k)}}{(q+3/2)^{(n+k)}} \frac{((B^2-A)t)^k}{k!} \quad (50)$$

In this series, the specific five-parameter KdF hypergeometric function is identified (Equation (12)).

The final analytical solution for the integral $i_2(t)$ is:

$$i_2(t) = \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2 - A)t \right) \quad (51)$$

where the solution does not depend on the parameter c .

By substituting Equation (46) into Equation (45) and using Equation (10), the solution for $i_3(t)$ is obtained:

$$\begin{aligned} i_3(t) &= \frac{Bt^q}{\Gamma(q)} \int_0^1 (1-x)^{q-1} e^{(B^2-A)tx} dx = \\ &= \frac{Bt^q}{\Gamma(q)} \int_0^1 (1-x)^{q-1} x^{1-1} e^{(B^2-A)tx} dx = \\ &= \frac{Bt^q}{\Gamma(q+1)} M(1, q+1, (B^2 - A)t) \end{aligned} \quad (52)$$

Based on Equations (42), (47), (51) and (52), the analytical solution of the convolution integral, i.e., the function $g(t)$ is:

$$\begin{aligned} g(t) &= \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2, q+1/2, -At) + \\ &+ \frac{B^2 t^{q+1/2}}{\Gamma(q+3/2)} F_{1;1}^{1;1} \left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2 - A)t \right) - \\ &- \frac{Bt^q}{\Gamma(q+1)} M(1, q+1, (B^2 - A)t) \end{aligned} \quad (53)$$

Equation (53) completes the proof of Theorem 1 (Equation (33)) in the forward direction ($\Rightarrow$). $\square$

4. Verification of the Solution

Let us now prove Theorem 1 on the inverse Laplace transform for a combined shifted quasi-rational functions with square-root radical and power-law decay term in the opposite direction, by finding the Laplace transform of Equation (53).

Proof of the Theorem 1 ($\Leftarrow$). Since the function $g(t)$ given by Equation (53) is represented as the sum of three functions, by using the linearity of the Laplace transform (Equation (3)), its Laplace transform can be written as:

$$\bar{G}(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{i_1(t)\} + \mathcal{L}\{i_2(t)\} - \mathcal{L}\{i_3(t)\} \quad (54)$$

where $i_1(t)$, $i_2(t)$, and $i_3(t)$ are given by Equations (47), (51), and (52), respectively.

To find the Laplace transform of each of the three functions (Equations (47), (51) and (52)), we use the series representations of Kummer's and KdF functions (Equations (9) and (12)), the known Laplace transform of a monomial (Equation (6)), the relations between Pochhammer symbols and binomial coefficients (Equations (14)–(17)), as well as the definitions of binomial and infinite geometric series (Equations (30) and (31)).

The Laplace transform of $i_1(t)$ is:

$$\begin{aligned}
 \mathcal{L}\{i_1(t)\} &= \mathcal{L}\left\{\frac{t^{q-1/2}}{\Gamma(q+1/2)} M\left(1/2, q+1/2, -At\right)\right\} = \\
 &= \frac{1}{\Gamma(q+1/2)} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)} (-A)^n}{n! (q+1/2)^{(n)}} \mathcal{L}\{t^{q-1/2+n}\} = \\
 &= \frac{1}{\Gamma(q+1/2)} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)} (-A)^n}{n! (q+1/2)^{(n)}} \frac{\Gamma(q+1/2+n)}{s^{q+1/2+n}} = \\
 &= \frac{1}{s^{q+1/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(-\frac{A}{s}\right)^n = \frac{1}{s^q \sqrt{s+A}}
 \end{aligned} \tag{55}$$

The Laplace transform of $i_2(t)$ is:

$$\begin{aligned}
 \mathcal{L}\{i_2(t)\} &= \mathcal{L}\left\{\frac{2B^2 t^{q+1/2}}{\sqrt{\pi}} F_{1,1}^{1,1}\left(\begin{matrix} 3/2 \\ q+3/2 \end{matrix}; \begin{matrix} 1/2 \\ 3/2 \end{matrix}; -B^2 t, (B^2 - A)t\right)\right\} = \\
 &= \frac{2B^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{\Gamma(q+3/2)} \frac{(3/2)^{(n+k)}}{(q+3/2)^{(n+k)}} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \frac{(-B^2)^n}{n!} \frac{(B^2 - A)^k}{k!} \mathcal{L}\{t^{n+k+q+1/2}\} = \\
 &= \frac{2B^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{1} \frac{(3/2)^{(n+k)}}{1} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \frac{(-B^2)^n}{n!} \frac{(B^2 - A)^k}{k!} \frac{1}{s^{n+k+q+1/2}} = \\
 &= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n)} (3/2+n)^{(k)}}{n! k!} \frac{(1/2)^{(n)}}{(3/2)^{(n)}} \left(\frac{-B^2}{s}\right)^n \left(\frac{B^2 - A}{s}\right)^k = \\
 &= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s}\right)^n \sum_{k=0}^{\infty} \frac{(3/2+n)^{(k)}}{k!} \left(\frac{B^2 - A}{s}\right)^k = \\
 &= \frac{B^2}{s^{q+3/2}} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s}\right)^n \left(1 - \frac{B^2 - A}{s}\right)^{-3/2-n} = \\
 &= \frac{B^2}{s^{q+3/2}} \left(1 - \frac{B^2 - A}{s}\right)^{-3/2} \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!} \left(\frac{-B^2}{s - B^2 + A}\right)^n = \\
 &= \frac{B^2}{s^q} \frac{1}{(s - B^2 + A)^{3/2}} \left(1 - \frac{-B^2}{s - B^2 + A}\right) = \\
 &= \frac{B^2}{s^q (s - B^2 + A) \sqrt{s+A}}
 \end{aligned} \tag{56}$$

The Laplace transform of $i_3(t)$ is:

$$\begin{aligned}
 \mathcal{L}\{i_3(t)\} &= \mathcal{L}\left\{\frac{B t^q}{\Gamma(q+1)} M\left(1; q+1; (B^2 - A)t\right)\right\} = \\
 &= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{n! (B^2 - A)^n}{n! (q+1)^{(n)}} \mathcal{L}\{t^{q+n}\} = \\
 &= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{(B^2 - A)^n}{(q+1)^{(n)}} \frac{\Gamma(q+n+1)}{s^{q+n+1}} = \\
 &= \frac{B}{\Gamma(q+1)} \sum_{n=0}^{\infty} \frac{(B^2 - A)^n}{(q+1)^{(n)}} \frac{\Gamma(q+1)(q+1)^{(n)}}{s^{q+n+1}} = \\
 &= \frac{B}{s^{q+1}} \sum_{n=0}^{\infty} \left(\frac{B^2 - A}{s}\right)^n = \frac{B}{s^{q+1}} \frac{1}{1 - \frac{B^2 - A}{s}} = \\
 &= \frac{B}{s^q (s - B^2 + A)}
 \end{aligned} \tag{57}$$

Finally, combining Equations (54)–(57), we recover the original complex function $\tilde{G}(s)$:

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \frac{1}{s^q \sqrt{s+A}} - \frac{B}{s^q (s-B^2+A)} + \frac{B^2}{s^q (s-B^2+A) \sqrt{s+A}} = \\ &= \frac{1}{s^q} \frac{s-B^2+A-B\sqrt{s+A}+B^2}{(s-B^2+A)\sqrt{s+A}} = \\ &= \frac{1}{s^q} \frac{\sqrt{s+A}(\sqrt{s+A}-B)}{(\sqrt{s+A}-B)(\sqrt{s+A}+B)\sqrt{s+A}} = \\ &= \frac{1}{s^q} \frac{1}{\sqrt{s+A}+B} = \tilde{G}(s)\end{aligned}\quad (58)$$

Equation (58) completes the proof of Theorem 1 in the Laplace transform direction ($\Leftarrow$). $\square$

This proof confirms that the derived inverse Laplace transform in Equation (53) is consistent and exact.

5. Analysis and Discussion

Theorem 1 serves as a unifying result from which numerous previously known inverse Laplace transforms follow as special cases.

In this section, we demonstrate that many well-known analytical solutions reported in the literature are contained in the general solution derived in this work as special cases. Table 1 summarizes several special cases that are embedded in the complex function considered in Equation (1). For each case, the corresponding inverse Laplace transform is provided together with its known analytical form from the literature [25].

Starting from the general solution Equation (33), it is shown that in all these specific parameter regimes, the obtained expression naturally reduces to the classical analytical results. It is worth noting that the derivation presented here cannot be obtained through symbolic computation or AI-assisted algebraic tools alone, as it relies on structural recognition and non-trivial transformations within the theory of special functions.

Table 1. Known solutions of the inverse Laplace transform for special cases of combined shifted quasi-rational functions with square-root radicals and power-law decay terms [25].

Special Case Number	Parameters	$\tilde{G}(s)$	$g(t)$
1	$q \in (0, +\infty), A = 0, B = 0$	$\frac{1}{s^{q+1/2}}$	$\frac{t^{q-1/2}}{\Gamma(q+1/2)}$
2	$q = 1/2, A \neq 0, B = 0$	$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{s+A}}$	$e^{-At/2} I_0\left(\frac{At}{2}\right)$
3	$q = 1, A \neq 0, B = 0$	$\frac{1}{s} \frac{1}{\sqrt{s+A}}$	$\frac{1}{\sqrt{A}} \operatorname{erf}(\sqrt{At})$
4	$q = 3/2, A \neq 0, B = 0$	$\frac{1}{s^{3/2}} \frac{1}{\sqrt{s+A}}$	$te^{-At/2} (I_0(At/2) + I_1(At/2))$
5	$q = 1/2, A = 0, B \neq 0$	$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{s+B}}$	$e^{B^2 t} \operatorname{erfc}(B\sqrt{t})$
6	$q = 1, A = 0, B \neq 0$	$\frac{1}{s} \frac{1}{\sqrt{s+B}}$	$\frac{1}{B} \left(1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t})\right)$
7	$q = 1, A = 0, B \neq 0$	$\frac{1}{s} \frac{1}{\sqrt{s+B}}$	$\frac{1}{B} \left(1 - E_{1/2}(-B\sqrt{t})\right)$

Lemma 1.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{q+1/2}}\right\} = \frac{t^{q-1/2}}{\Gamma(q+1/2)} M(1/2, q+1/2, 0) = \frac{t^{q-1/2}}{\Gamma(q+1/2)}, \quad q \in (0, +\infty)$$

Proof. By substituting the identity given in Equation (19) ($M(1/2, q + 1/2, 0)$) into Equation (33), the following relation is obtained:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{q+1/2}}\right\} = \frac{t^{q-1/2}}{\Gamma(q+1/2)}M(1/2, q+1/2, 0) = \frac{t^{q-1/2}}{\Gamma(q+1/2)}$$

Hence, the Lemma is proven. $\square$

Lemma 2.

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\frac{1}{\sqrt{s+A}}\right\} = M(1/2, 1, -At) = e^{-At/2}I_0(At/2)$$

Proof. The proof employs Kummer's transformation Equation (11) together with the relation between Kummer's function and the modified Bessel function given in Equation (21):

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\frac{1}{\sqrt{s+A}}\right\} &= M(1/2, 1, -At) = e^{-At}M(1/2, 1, At) = \\ &= e^{-At}e^{At/2}I_0(At/2) = e^{-At/2}I_0(At/2)\end{aligned}$$

Thus, Lemma 2 is thus proven. $\square$

Lemma 3.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s+A}}\right\} = \frac{t^{1/2}}{\Gamma(3/2)}M(1/2, 3/2, -At) = \frac{1}{\sqrt{A}}\operatorname{erf}(\sqrt{At})$$

Proof. Starting from Equation (33) and replacing the obtained Kummer function $M(1/2, 3/2, -At)$ using Equation (23), the following identity follows:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s+A}}\right\} &= \frac{t^{1/2}}{\Gamma(3/2)}M(1/2, 3/2, -At) = \frac{t^{1/2}}{(1/2)\Gamma(1/2)}\frac{\sqrt{\pi}}{2(\sqrt{At})}\operatorname{erf}(\sqrt{At}) = \\ &= \frac{1}{\sqrt{A}}\operatorname{erf}(\sqrt{At})\end{aligned}$$

Lemma 3 is thereby proven. $\square$

Lemma 4.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\frac{1}{\sqrt{s+A}}\right\} = tM(1/2, 2, -At) = te^{-At/2}(I_0(At/2) + I_1(At/2))$$

Proof. We use two contiguous relations for the confluent hypergeometric function (Equations (18) and (19)), along with the connections between Kummer and modified Bessel functions (Equations (21) and (22)).

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\frac{1}{\sqrt{s+A}}\right\} &= tM(1/2, 2, -At) = te^{-At}M(3/2, 2, At) = \\ &= te^{-At}\left[M(1/2, 1, At) + \frac{At}{4}M(3/2, 3, At)\right] = \\ &= te^{-At/2}(I_0(At/2) + I_1(At/2))\end{aligned}$$

Hence, Lemma 4 is proven. $\square$

Lemma 5.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}(\sqrt{s}+B)}\right\}=M(1/2, 1, 0)+B^2tF_{1;1}^{1;1}\left(\frac{3/2}{2};\frac{1/2}{3/2};-B^2t, B^2t\right)-\frac{B\sqrt{t}}{\Gamma(3/2)}M(1, 3/2, B^2t)=e^{B^2t}\operatorname{erfc}(B\sqrt{t})$$

Proof. We apply Equation (20), Kummer's transformation (Equation (11)), the relation between Kummer and error functions (Equation (23)), and the reduction identity for the KdF function (Equation (26), proven in Appendix B).

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}(\sqrt{s}+B)}\right\}&=M(1/2, 1, 0)+B^2tF_{1;1}^{1;1}\left(\frac{3/2}{2};\frac{1/2}{3/2};-B^2t, B^2t\right)-\\&\quad -\frac{B\sqrt{t}}{\Gamma(3/2)}M(1, 3/2, B^2t)=\\&=1+B^2t\left(\frac{e^{B^2t}-1}{B^2t}\right)-\frac{B\sqrt{t}}{\Gamma(3/2)}\Gamma(3/2)\frac{\operatorname{erf}(B\sqrt{t})}{B\sqrt{t}}e^{B^2t}=\\&=1+(e^{B^2t}-1)-e^{B^2t}\operatorname{erf}(B\sqrt{t})=e^{B^2t}(1-\operatorname{erf}(B\sqrt{t}))=\\&=e^{B^2t}\operatorname{erfc}(B\sqrt{t})\end{aligned}$$

This completes the proof of Lemma 5. $\square$

Lemma 6.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s}+B}\right\}=\frac{\sqrt{t}}{\Gamma(3/2)}M(1/2, 3/2, 0)+\frac{B^2t^{3/2}}{\Gamma(5/2)}F_{1;1}^{1;1}\left(\frac{3/2}{5/2};\frac{1/2}{3/2};-B^2t, B^2t\right)-\frac{Bt}{\Gamma(2)}M(1, 2, B^2t)=\frac{1}{B}(1-e^{B^2t}\operatorname{erfc}(B\sqrt{t}))$$

Proof. We use Equations (20) and (25) (linking Kummer and exponential function), and the reduction formulas for KdF given by Equation (27) (proven in Appendix B).

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{\sqrt{s}+B}\right\}&=\frac{\sqrt{t}}{\Gamma(3/2)}M(1/2, 3/2, 0)+\frac{B^2t^{3/2}}{\Gamma(5/2)}F_{1;1}^{1;1}\left(\frac{3/2}{5/2};\frac{1/2}{3/2};-B^2t, B^2t\right)-\\&\quad -\frac{Bt}{\Gamma(2)}M(1, 2, B^2t)=\\&=\frac{B\sqrt{t}}{B\Gamma(3/2)}+\frac{(B\sqrt{t})^3}{B\Gamma(5/2)}\frac{3}{2B^2t}(M(1, 3/2, B^2t)-1)-\frac{B^2te^{B^2t}-1}{B}=\\&=\frac{B\sqrt{t}}{B\Gamma(3/2)}+\frac{B\sqrt{t}}{B\Gamma(3/2)}(e^{B^2t}M(1/2, 3/2, -B^2t)-1)-\frac{1}{B}(e^{B^2t}-1)=\\&=\frac{1}{B}e^{B^2t}\operatorname{erf}(B\sqrt{t})-\frac{1}{B}(e^{B^2t}-1)=\\&=\frac{1}{B}(e^{B^2t}(\operatorname{erf}(B\sqrt{t})-1)+1)=\\&=\frac{1}{B}(1-e^{B^2t}\operatorname{erfc}(B\sqrt{t}))\end{aligned}$$

Lemma 6 is thus established. $\square$

Lemma 7.

$$\mathcal{L}^{-1}\left\{\frac{1}{s(\sqrt{s}+B)}\right\} = \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \frac{1}{B} \left(1 - E_{1/2}(-B\sqrt{t})\right)$$

Proof. Using Lemma 6 and the relation between the ML and error function given by Equation (29), we obtain:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s(\sqrt{s}+B)}\right\} &= \frac{\sqrt{t}}{\Gamma(3/2)} M(1/2, 3/2, 0) + \frac{B^2 t^{3/2}}{\Gamma(5/2)} F_{1;1}^{1;1}\left(\frac{3/2}{5/2}; \frac{1/2}{3/2}; -B^2 t, B^2 t\right) - \\ &\quad - \frac{Bt}{\Gamma(2)} M(1, 2, B^2 t) = \frac{1}{B} \left(1 - e^{B^2 t} \operatorname{erfc}(B\sqrt{t})\right) = \\ &= \frac{1}{B} \left(1 - E_{1/2}(-B\sqrt{t})\right) \end{aligned}$$

Hence, Lemma 7 is proven. $\square$

6. Conclusions

We derived a rigorous closed-form analytical expression for the inverse Laplace transform of a class of shifted quasi-rational spectral functions with square-root radicals. This constitutes the main contribution of the paper, representing a solution not previously reported in the literature. The derived inverse transform is expressed in terms of generalized hypergeometric functions, including Kummer and five-parameter Kampé de Fériet functions, and exhibits a convolution structure that has not been represented analytically before. Reduction formulas for a subclass of the Kampé de Fériet function were obtained as a secondary result, demonstrating how the general solution reduces to known special cases. The closed-form solution is validated analytically by reapplying the Laplace transform, which fully recovers the original spectral function.

This work represents the first step of a modular research framework, which will be extended in future studies to include asymptotic analysis, numerical validation, and systematic applications in photothermal and photoacoustic signal analysis, as motivated by the broader context of coupled diffusion-type processes and inverse procedures highlighted in the Introduction, demonstrating the utility of analytic inverses in both theoretical and experimental investigations.

Author Contributions: All authors contributed to this work: conceptualization, S.G.; methodology, S.G.; validation, S.G., A.D. and K.L.D.; formal analysis, S.G., A.D. and K.L.D.; investigation, S.G., A.D. and K.L.D. writing—original draft preparation, S.G.; writing—review and editing, S.G., A.D. and K.L.D. All authors have read and agreed to the published version of the manuscript.

Funding: This work was financially supported by the Ministry of Science, Technological Development, and Innovation of the Republic of Serbia (Contract No. 451-03-136/2025-03/200017).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors gratefully acknowledge D. Chevizovich for valuable discussions and support in the preparation of this manuscript. During the preparation of this manuscript/study, the author(s) used [CHAT GPT, ver.3] for the purposes of [language editing and language polishing].

The authors have reviewed and edited the output and take full responsibility for the content of this publication.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PT	Photothermal
PA	Photoacoustic
KdF	Kampé de Fériet
ML	Mittag-Leffler

Appendix A. The Function $\operatorname{erf}(x)$ Is Entire Function-Proof

If $f(z)$ is equal to the sum of its Taylor series for all z in complex plane, it is called entire function. It means that the radius of convergence of Taylor series of entire function is infinite. Such functions are exponential function, trigonometric function sine and cosine, etc.

The radius of convergence of infinite series is defined by the following expression [55]. Let

$$(Tf)(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (\text{A1})$$

The radius of convergence of this series is defined by

$$R_{Tf} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|. \quad (\text{A2})$$

Using this definition and Taylor series expansion of error function Equation (48), we obtain

$$R = \lim_{n \rightarrow \infty} \left| \frac{\frac{2(-1)^n}{\sqrt{\pi} n! (2n+1)}}{\frac{2(-1)^{n+1}}{\sqrt{\pi} (n+1)! (2n+3)}} \right|. \quad (\text{A3})$$

After simplification, this yields

$$R = \lim_{n \rightarrow \infty} \frac{(n+1)!(2n+3)}{n!(2n+1)} = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+3)}{2n+1}. \quad (\text{A4})$$

Since

$$\lim_{n \rightarrow \infty} \frac{(n+1)(2n+3)}{2n+1} = +\infty, \quad (\text{A5})$$

we conclude that the radius of convergence is

$$R = +\infty. \quad (\text{A6})$$

Since the radius of convergence of the Taylor series expansion of the $\operatorname{erf}(\cdot)$ function is infinite, this power series can represent the $\operatorname{erf}(\cdot)$ function over its entire domain of definition.

Appendix B. The Proof of Reduction Identities for Specific Five Parameter KdF Function Given by Equations (26) and (27)

In proving the identities given by Equations (26) and (27) (reduction identity for specific five-parameter KdF functions that doesn't depend on fifth parameter and which

arguments are of converse signs), the series representation of the confluent hypergeometric function (Equation (9)) and the representation of the KdF function via double sum (Equation (12)) are used. We also used Kummer's transformation given by Equation (11) and identity given by Equation (25), as well as definition of rising factorial, Equation (13).

As part of argument, we apply diagonal summation, which allows us to rewrite the double sum as a single sum over total index $n = i + j$ [57]. The diagonal summation rule is given by:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j, i+j) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f(n-j, j, n)$$

where $n = i + j$, and each pair $(i, j) \in \mathbb{N}_0$ such that $n = i + j$ is included exactly once.

The Proof of Reduction Identity for KdF Function Given by Equation (26).

$$\begin{aligned} F_{1;1}^{1;1} \left(\begin{matrix} 3/2, 1/2 \\ 2, 3/2 \end{matrix}; -y, y \right) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n+k)} (1/2)^{(n)} (-y)^n (y)^k}{(2)^{(n+k)} (3/2)^{(n)} n! k!} = \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(3/2)^{(m)} (1/2)^{(m-k)} (-1)^{m-k} y^m}{(2)^{(m)} (3/2)^{(m-k)} k! (m-k)!} = \\ &= \sum_{m=0}^{\infty} y^m \frac{(3/2)^{(m)}}{(m+1)!} \sum_{k=0}^m \frac{(1/2)^{(m-k)} (-1)^{m-k}}{(3/2)^{(m-k)} k! (m-k)!} = \\ &= \frac{y^0}{1!} + \frac{y^1}{2!} \frac{3}{2} \left(-\frac{1/2}{3/2} + \frac{1/2}{1/2} \right) + \\ &+ \frac{y^2}{3!} \frac{3}{2} \frac{5}{2} \left(\frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{1!} + \frac{1/2}{1/2} \cdot \frac{1}{2!} \right) + \\ &+ \frac{y^3}{4!} \frac{3}{2} \frac{5}{2} \frac{7}{2} \left(-\frac{1/2}{7/2} \cdot \frac{1}{3!} + \frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{2!} + \frac{1/2}{1/2} \cdot \frac{1}{3!} \right) + \dots = \\ &= \frac{y^0}{1!} + \frac{y^1}{2!} + \frac{y^2}{3!} + \frac{y^3}{4!} + \dots = \\ &= \sum_{j=0}^{\infty} \frac{y^j}{(j+1)!} = \sum_{j=0}^{\infty} \frac{1^{(j)} y^j}{(2)^{(j)} j!} = \\ &= M(1, 2, y) \end{aligned}$$

The Proof of Reduction Identity for KdF Function Given by Equation (27).

$$\begin{aligned} F_{1;1}^{1;1} \left(\begin{matrix} 3/2, 1/2 \\ 5/2, 3/2 \end{matrix}; -y, y \right) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3/2)^{(n+k)} (1/2)^{(n)} (-y)^n (y)^k}{(5/2)^{(n+k)} (3/2)^{(n)} n! k!} = \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{(3/2)^{(m)} (1/2)^{(m-k)} (-1)^{m-k} y^m}{(5/2)^{(m)} (3/2)^{(m-k)} k! (m-k)!} = \\ &= \sum_{m=0}^{\infty} y^m \frac{(3/2)^{(m)}}{(5/2)^{(m)}} \sum_{k=0}^m \frac{(1/2)^{(m-k)} (-1)^{m-k}}{(3/2)^{(m-k)} k! (m-k)!} = \\ &= \sum_{m=0}^{\infty} y^m \frac{3/2}{3/2 + m} \sum_{k=0}^m \frac{(1/2)^{(m-k)} (-1)^{m-k}}{(1/2 + m - k) k! (m-k)!} = \\ &= y^0 \frac{3/2 \cdot 1/2}{3/2 \cdot 1/2} + y^1 \frac{3/2}{5/2} \left(-\frac{1/2}{3/2} + \frac{1/2}{1/2} \right) + \\ &+ y^2 \frac{3/2}{7/2} \left(\frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{1!} + \frac{1/2}{1/2} \cdot \frac{1}{2!} \right) + \end{aligned}$$

$$\begin{aligned}
& + y^3 \frac{3/2}{9/2} \left(-\frac{1/2}{7/2} \cdot \frac{1}{3!} + \frac{1/2}{5/2} \cdot \frac{1}{2!} - \frac{1/2}{3/2} \cdot \frac{1}{2!} + \frac{1/2}{1/2} \cdot \frac{1}{3!} \right) + \dots = \\
& = \frac{y^0}{1} + \frac{y^1}{\frac{5}{2}} + \frac{y^2}{\frac{5}{2} \frac{7}{2}} + \frac{y^3}{\frac{5}{2} \frac{7}{2} \frac{9}{2}} + \dots = \\
& = \sum_{j=0}^{\infty} \frac{y^j}{(5/2)^{(j)}} = \frac{3/2}{y} \sum_{j=0}^{\infty} \frac{y^{j+1}}{(3/2) (5/2)^{(j)}} = \\
& = \frac{3/2}{y} \sum_{j=1}^{\infty} \frac{y^j}{(3/2)^{(j)}} = \frac{3/2}{y} \left(\sum_{j=0}^{\infty} \frac{y^j}{(3/2)^{(j)}} - 1 \right) = \\
& = \frac{3/2}{y} \left(\sum_{j=0}^{\infty} \frac{1^{(j)} y^j}{(3/2)^{(j)} j!} - 1 \right) = \frac{3/2}{y} (M(1, 3/2, y) - 1)
\end{aligned}$$

References

1. Lynn, P.A. The Laplace Transform and the z-transform. In *Electronic Signals and Systems*; Macmillan Education: London, UK, 1986; pp. 225–272, ISBN 978-0-333-39164-8.
2. Stojic, M. *Continuous System of Automatic Control*; Naucna Knjiga-Komerc: Belgrade, Serbia, 1991. (In Serbian)
3. Maillet, D.; André, S.; Batsale, J.-C.; Degiovanni, A.; Moyne, C. *Thermal Quadrupoles, Solving the Heat Equation Through Integral Transforms*; John Wiley and Sons: Hoboken, NJ, USA, 2000; ISBN 978-0-471-98320-0.
4. Galovic, S.; Soskic, Z.; Popovic, M. Analysis of Photothermal Response of Thin Solid Films by Analogy with Passive Linear Electric Networks. *Therm. Sci.* **2009**, *13*, 129–142. [CrossRef]
5. Popovic, A.; Šoškić, Z.; Stojanovic, Z.; Cevizovic, D.; Galovic, S. On the applicability of the effective medium approximation to the photoacoustic response of multilayered structures. *Phys. Scr.* **2012**, *26*, 014066. [CrossRef]
6. Galovic, S.; Čukić, M.; Chevizovich, D. Inertial Memory Effects in Molecular Transport Across Nanoporous Membranes. *Membranes* **2025**, *15*, 11. [CrossRef]
7. Caputo, M.; Cametti, C.; Ruggero, V. Time and spatial concentration profile inside a membrane by means of a memory formalism. *Phys. A* **2008**, *387*, 2010–2018. [CrossRef]
8. Anissimov, Y.; Watkinson, A. Modelling skin penetration using the Laplace transform technique. *Ski. Pharmacol. Physiol.* **2013**, *26*, 286–294. [CrossRef] [PubMed]
9. Galovic, S.; Đorđević, K.; Lukić, M.; Chevizovich, D. Vibration phenomena induced by pulsed laser heating of micromechanical cantilever: Influence of laser-pulse temporal shape. *FU Work. Liv. Env. Prot. Special Issue* **2024**, *21*, 465–476.
10. Sun, Y.X.; Fang, D.N.; Soh, A.K. Thermoelastic damping in micro-beam resonators. *Int. J. Solids Struct.* **2006**, *43*, 3213–3229. [CrossRef]
11. Hariharan, S.I.; Sawyer, S.; Dane Quinn, D. A. Laplace transform/potential-theoretic method for acoustic propagation in subsonic flows. *J. Comput. Phys.* **2003**, *185*, 252–270.
12. Metzler, R.; Nonnenmacher, T. F. Fractional diffusion: Exact representations of spectral functions. *J. Phys. A Math. Gen.* **1997**, *30*, 1089–1093.
13. Gogolin, A.O. *Lectures on Complex Integration*; Springer Science and Business Media; Addison Wesley: New York, NY, USA, 2013.
14. Rudin, W. *Real and Complex Analysis*; McGraw-Hill: New Delhi, India, 1987.
15. Luo, S.; Zhao, F. A Semi-Analytical Solution of Inverse Laplace Transform. *J. Math.* **2022**, *2022*, 9129727. [CrossRef]
16. Baumann, G.; Sinc Based Inverse Laplace Transforms, Mittag-Leffler Functions and Their Approximation for Fractional Calculus. *Fractal Fract.* **2021**, *5*, 43. [CrossRef]
17. Trabelsi, K.; H'elie, T.; Matignon, D. On the Numerical Inversion of the Laplace Transform in the Context of Physical Models with Realistic Damping. TR 2009D025. 2009; pp. 1–37. Available online: <http://articles.ircam.fr/textes/Trabelsi09a/index.pdf> (accessed on 12 February 2026).
18. Kamal, R.; Kamran; Rahmat, G.; Ahmadian, A.; Arshad, N. I.; Salahshouret, S. Approximation of linear one dimensional partial differential equations including fractional derivative with non-singular kernel. *Adv. Differ. Equ.* **2021**, *2021*, 317. [CrossRef]
19. Stanimirović, Z.; Stanimirović, I.; Galović, S.; Đorđević, K.; Suljovrujić, E. Haar wavelet operational matrix based numerical inversion of Laplace transform for irrational and transcendental transfer functions. *FU Elec. Energ.* **2023**, *36*, 395–410.
20. Milovanovic, G.V.; Cvetkovic, A.S. Numerical Inversion of the Laplace Transform. *FU Elec. Energ.* **2005**, *20*, 295–310. [CrossRef]
21. Cohen, A.M. *Numerical Methods for Laplace Transform Inversion*; Springer Science+Business Media LLC: New York, NY, USA, 2007.
22. Li, X.; Kamran; Haq, A. U.; Zhang, X. Numerical solution of the linear time fractional Klein-Gordon equation using transform based localized RBF method and quadrature. *AIMS Math.* **2020**, *5*, 5287–5308. [CrossRef]

23. Liu, X.; Kamran; Yao, Y. Numerical Approximation of Riccati Fractional Differential Equation in the Sense of Caputo-Type Fractional Derivative. *Hindawi J. Math.* **2020**, *2020*, 1274251. [[CrossRef](#)]
24. Oberhettinger, F.; Badii, L. *Tables of Laplace Transforms*; Springer: Berlin/Heidelberg, Germany, 1973.
25. Chen, Y.Q.; Petras, I.; Vinagre, B.M. A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus. 2001. Available online: https://ivopetras.tripod.com/foc_laplace.pdf (accessed on 12 February 2026).
26. Roberts, G.E.; Kaufman, H. *Table of Laplace Transform*; W.B. Saunders Company: Philadelphia, PA, USA; London, UK, 1966.
27. Abramowitz, M.; Stegun, A.I. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Dover Publications: New York, NY, USA, 1965; ISBN-13978-0486612720.
28. Daalhuis, A.B.O. Confluent hypergeometric function. In *NIST Handbook of Mathematical Functions*; Frank, O.W.J., Lozier, D.M., Boisvert, R.F., Clark, C.W., Eds.; Cambridge University Press: Cambridge, UK, 2010; MR2723248. Available online: <https://dlmf.nist.gov/13> (accessed on 12 February 2026).
29. Ito, R.; Kumabe, S.; Nakagawa, A.; Nemoto, Y. Kampé de Fériet hypergeometric functions over finite fields. *Res. Number Theory* **2023**, *9*, 52. [[CrossRef](#)]
30. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications: Theory and Applications*; Springer Monographs in Mathematics; Springer: Berlin/Heidelberg, Germany, 2014.
31. Temme, N.M. Error Functions, Dawson's and Fresnel Integrals. In *NIST Handbook of Mathematical Functions*; Frank, O.W.J., Lozier, D.M.B., Ronald, F.C.W.C., Eds.; Cambridge University Press: Cambridge, UK, 2010; MR 2723248, ISBN 9780521140638.
32. Mandelis, A.; Royce, B.S.H. Time-domain photoacoustic spectroscopy of solids. *J. Appl. Phys.* **1979**, *50*, 4330–4338. [[CrossRef](#)]
33. Rosencwaig, A.; Gersho, A. Theory of the photoacoustic effect with solids. *J. Appl. Phys.* **1976**, *47*, 64–69. [[CrossRef](#)]
34. Bein, B.; Pelzl, J. Theory of signal generation in a photoacoustic cell. *J. Phys. Colloq.* **1983**, *44*, C6-27–C6-34. [[CrossRef](#)]
35. Rousset, G.; Lepoutre, F.; Bertrand, L. Influence of thermoelastic bending on photoacoustic experiments related to measurements of thermal diffusivity of metals. *J. Appl. Phys.* **1983**, *54*, 2383–2391. [[CrossRef](#)]
36. Rojas-Trigos, J.B.; Calderón, A.; Marín, E.A. Practical model for the determination of transport parameters in semiconductors. *J. Mater. Sci.* **2011**, *46*, 7799–7805. [[CrossRef](#)]
37. Mandelis, A. Laser infrared photothermal radiometry of semiconductors: Principles and applications to solid state electronics. *Solid State Electron.* **1998**, *42*, 1–15. [[CrossRef](#)]
38. Todorovic, D.M.; Nikolic, P.M. Carrier transport Contribution on Thermoelastic and Electronic Deformation in Semiconductors. In *Progress in Photothermal and Photoacoustic Science and Technology*; Mandelis, A., Hess, P., Eds.; SPIE Press: Washington, DC, USA, 2000; Chapter 9, Volume 4, p. 272.
39. Galovic, S.P.; Markushev, D.K.; Markushev, D.D.; Djordjevic, K.L.; Popovic, M.N.; Suljovrujic, E.; Todorovic, D.M. Time-Resolved Photoacoustic Response of Thin Semiconductors Measured with Minimal Volume Cell: Influence of Photoinduced Charge Carriers. *Appl. Sci.* **2025**, *15*, 7290. [[CrossRef](#)]
40. Galovic, S.P.; Stanimirovic, Z.; Stanimirovic, I.; Djordjevic, K.L.; Milicevic, D.; Suljovrujic, E. Time-resolved photoacoustic response of thin solids measured using minimal volume cell. *ICHMT* **20204**, *155*, 107574. [[CrossRef](#)]
41. Buchholz, H. *The Confluent Hypergeometric Function: With Special Emphasis on Its Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013; Volume 15, pp. 34–73.
42. Exton, H. On the reducibility of the Kampé de Fériet function. *J. Comput. Appl. Math.* **1997**, *83*, 119–121. [[CrossRef](#)]
43. Cvijović, Đ.; Miller, A.R. A reduction formula for the Kampé de Fériet function. *Appl. Math. Lett.* **2010**, *23*, 769–771. [[CrossRef](#)]
44. Rathie, A.K.; Pogany, T.K. Novel reductions of Kampé de Fériet function. *Sci. Ser. A Math. Sci.* **2020**, *30*, 27–34.
45. Choi, J.; Milovanovic, G.V.; Rathie, A.K. Generalized Summation Formulas for the Kampé de Fériet Function. *Axioms* **2021**, *10*, 318. [[CrossRef](#)]
46. Available online: <https://mathworld.wolfram.com/KampedeFeretFunction.html> (accessed on 12 February 2026).
47. Available online: <https://mathworld.wolfram.com/FallingFactorial.html> (accessed on 12 February 2026).
48. Available online: <https://mathworld.wolfram.com/Series.html> (accessed on 12 February 2026).
49. Available online: <https://dlmf.nist.gov> (accessed on 12 February 2026).
50. Available online: <https://mathworld.wolfram.com/Mittag-LefflerFunction.html> (accessed on 12 February 2026).
51. Rogosin, S.; Gorenflo, R.; Kilbas, A.A.; Mainardi, F. *Mittag-Leffler Function, Related Topics and Applications*. Springer: Berlin, Germany, 2014; ISBN 978-3-662-43930-2.
52. Biolek, D.; Garrappa, R.; Mainardi, F.; and Popolizio, M. Derivatives of Mittag-Leffler functions: Theory, computation and applications. *Nonlinear Dyn.* **2025**, *113*, 34389–34403. [[CrossRef](#)]
53. Mainardi, F. Why the mittag-leffler function can be considered the queen function of the fractional calculus? *Entropy* **2020**, *22*, 1359. [[CrossRef](#)] [[PubMed](#)]
54. Il'in, V.A.; Sadovnichii, V.A.; Sendov, B.K. *Mathematical Analysis*; Nauka: Moscow, Russia, 1979. (In Russian)
55. Arfken, G.B.; Weber, H.J.; Harris, F.E. *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed.; Academic Press: Cambridge, MA, USA, 2012; ISBN 978-0-12-384654-9.

56. Zill, D.G. *Advanced Engineering Mathematics*, 6th ed.; Jones & Bartlett Learning: Burlington, MA, USA, 2018; ISBN 978-1-284-10590-2.
57. Choi, J. *Notes on Formal Manipulations of Double Series*; CKMS: Seoul, Republic of Korea, 2003; Volume 18, pp. 781–789. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.