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On New Classes of Stretch Minkowskian Product Finsler Manifolds

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Abstract

Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds. A Minkowskian product Finsler manifold is defined to be the product manifold $M_1 \times M_2$, which is endowed with a Finsler metric F. This metric F is constructed by taking the square root of a product function f, which itself operates on the squares of the original metrics F_1 and F_2. This paper focuses on new classes of stretch Minkowskian product Finsler manifolds. We prove that the Minkowskian product Finsler manifold (M, F) is a $\widetilde{\mathbf{B}}$-manifold (resp. $\widetilde{\mathbf{B}}$-stretch manifold, $\mathbf{H}$-stretch manifold) if and only if (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-manifold (resp. $\widetilde{\mathbf{B}}$-stretch manifold, $\mathbf{H}$-stretch manifold). Thus an effective method for constructing special Finsler manifolds mentioned above is given.

Keywords: Finsler manifold; Minkowskian product; $\widetilde{\mathbf{B}}$-manifold; $\widetilde{\mathbf{B}}$-stretch manifold; $\mathbf{H}$-stretch manifold

MSC: 53C60; 53B40

1. Introduction

In Finsler geometry, there exist several significant non-Riemannian geometric quantities, such as $\mathbf{B}$ curvature (i.e., Berwald curvature), $\widetilde{\mathbf{B}}$-curvature, stretch $\widetilde{\mathbf{B}}$-curvature, Cartan torsion, stretch curvature and the Landsberg curvature etc., all of these quantities vanish for Riemannian manifolds. One of the fundamental problems in Finsler geometry is studying Finsler manifolds with special curvature properties. Therefore, these quantities mentioned above merit extra attention.

Let (M, F) be a Finsler manifold. The third-order derivatives of F^2 is called the Cartan torsion. The Cartan torsion provides a quantitative measure of the deviation of a Finsler metric from a Riemannian metric on a manifold [1]. The rate of change in the Cartan torsion along Finslerian geodesics is called the Landsberg curvature [2]. In 1926, Berwald [3] defined the concept of stretch curvature as a natural extension of the Landsberg curvature.

In 1927, Berwald [4,5] first defined $\mathbf{B}$ curvature, whose curvature coefficients are third-order differentials of the geodesic coefficients induced by the Finsler metric. Shen [6] introduced another non-Riemannian quantity from the $\mathbf{B}$ curvature through horizontal covariant derivative along Finslerian geodesics, which is called $\widetilde{\mathbf{B}}$-curvature. Abbas and Kozma [7] defined a new non-Riemannian geometric quantity called the stretch $\widetilde{\mathbf{B}}$-curvature using the $\widetilde{\mathbf{B}}$-curvature, with the relationship between the two being analogous to that between the Landsberg curvature and the stretch curvature. A Finsler manifold is said to be $\widetilde{\mathbf{B}}$-stretch manifold if its stretch $\widetilde{\mathbf{B}}$-curvature vanishes. Recently, Abbas and Kozma [8]

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proved that every generalized Douglas manifold with vanishing stretch $\widetilde{\mathbf{B}}$-curvature is a Douglas manifold under the condition that the mean Berwald curvature is horizontally constant along geodesics of F.

The trace of **B** curvature is called **E** curvature (i.e., mean Berwald curvature) [6]. Akbar-Zadeh [9,10] originally defined **H** curvature as a horizontal covariant derivative of **E** curvature along geodesics, and revealed the connection between **H** curvature and the flag curvature. **H** curvature is a positively homogeneous scalar function of degree zero on the slit tangent bundle [11]. Abbas and Kozma got the non-Riemannian quantity, which is called stretch **H**-curvature by taking a trace of stretch $\widetilde{\mathbf{B}}$-curvature. A Finsler manifold is said to be **H**-stretch manifold if its stretch **H**-curvature vanishes. In 2023, Abbas and Kozma [8] showed that if a Finsler manifold is a Douglas Finsler manifold, then the Finsler metric F is an **H**-stretch metric if and only if it is a $\widetilde{\mathbf{B}}$-metric.

The Minkowskian product plays a key role in constructing Finsler manifolds that possess special curvature properties in Finsler geometry. In 1982, Okada [12] not only first introduced the concept of the Minkowskian product of Finsler manifolds but also successfully determined their geodesics. In 2023, He, Li, et al. [13] proved that a Minkowskian product Finsler manifold is a Berwald (resp. weakly Berwald, Landsberg, or weakly Landsberg) manifold if and only if the component manifolds of it are both Berwald (resp. weakly Berwald, Landsberg, weakly Landsberg) manifolds; thus, they gave a new way to construct the special Finsler manifolds mentioned above. Later, Li, He, et al. [14] characterized dually flat and projectively flat Minkowskian product Finsler manifolds. Tian, He, et al. [15] gave a characterization by differential equations for a Minkowskian product Finsler manifold has scalar flag curvature. Zhang, Lu and Han [16] obtained the necessary and sufficient conditions for a Minkowskian product Finsler manifold to be a Douglas manifold or Wely manifold.

Motivated by the above research, we consider the following questions: If (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-manifold ($\widetilde{\mathbf{B}}$-stretch manifold or **H**-stretch manifold), then whether the Minkowskian product of Finsler manifolds (M_1, F_1) and (M_2, F_2) is also a $\widetilde{\mathbf{B}}$-manifold ($\widetilde{\mathbf{B}}$-stretch manifold or **H**-stretch manifold). The novelty of this paper lies in, based on the aforementioned issues, exploring an effective method for constructing $\widetilde{\mathbf{B}}$-manifold ($\widetilde{\mathbf{B}}$-stretch manifold or **H**-stretch manifold) using the Minkowskian product.

The remainder of this paper is structured as follows: In Section 2, we briefly review the basic concepts and notations required for this paper's research. In Section 3, we establish the formulas for both the $\widetilde{\mathbf{B}}$-curvature and the stretch $\widetilde{\mathbf{B}}$-curvature of a Minkowskian product Finsler manifold. Furthermore, we characterize the necessary and sufficient conditions for such a manifold to be a $\widetilde{\mathbf{B}}$-manifold or a $\widetilde{\mathbf{B}}$-stretch manifold. In Section 4, we shall prove that a Minkowskian product Finsler manifold is a **H**-stretch manifold if and only if the component manifolds of it are both **H**-stretch manifolds.

2. Preliminary

In this section, we recall some basic concepts and notations in Finsler geometry which we need.

Let M be an n-dimensional Finsler manifold. Let TM denote the tangent bundle of M, and the complement of the zero section in TM is denoted by $\widetilde{M}$. Let $x = (x^1, \ldots, x^n)$ be the local coordinates on M, then the induced local coordinates on TM are given by $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$.

Definition 1 ([17]). *A Finsler metric F on a manifold M is a function $F : TM \to \mathbb{R}^+$ satisfying the following properties:*

(i) $G = F^2$ *is smooth on* $\widetilde{M} = TM \setminus \{0\}$.

(ii) $F(x,y) > 0$ *for any* $(x,y) \in \widetilde{M}$.
(iii) $F(x,\lambda y) = |\lambda| F(x,y)$ *for any* $(x,y) \in TM$ *and* $\lambda \in \mathbb{R}$.
(iv) *the Hessian matrix* $(G_{\alpha\beta}) = (\frac{\partial^2 G}{\partial y^\alpha \partial y^\beta})$ *is positive definite on* $\widetilde{M}$.

In this paper, we denote $(G^{\alpha\beta})$ the inverse matrix of $(G_{\alpha\beta})$.

Given a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^\alpha \frac{\partial}{\partial x^\alpha} - 2\mathbb{G}^\alpha(x,y)\frac{\partial}{\partial y^\alpha}$, where

$$\mathbb{G}^\alpha := \frac{1}{2} G^{\alpha\beta}\Big\{\frac{\partial^2 G}{\partial x^\gamma \partial y^\beta} y^\gamma - \frac{\partial G}{\partial x^\beta}\Big\}, y \in T_x M.$$

The $\mathbf{G}$ is called the spray associated to (M,F) [6].

In Finsler geometry, the $\mathbf{B}$ curvature is an important non-Riemannian geometric quantity.

The $\mathbf{B}$ curvature $\mathbf{B} : T_xM \otimes T_xM \otimes T_xM \to T_xM$ of (M,F) is defined by $\mathbf{B} := B^\alpha_{\beta\gamma\eta} dx^\beta \otimes dx^\gamma \otimes dx^\eta \otimes \frac{\partial}{\partial x^\alpha}|_x$, where

$$B^\alpha_{\beta\gamma\eta} := \frac{\partial^3 \mathbb{G}^\alpha}{\partial y^\beta \partial y^\gamma \partial y^\eta}. \tag{1}$$

A Finsler manifold (M,F) is called a Berwald manifold if and only if $\mathbf{B} = 0$ [6]. It is well known that all tangent spaces of a Berwald manifold are linearly isometric to each other [18].

The $\mathbf{E}$ curvature $\mathbf{E} : T_xM \otimes T_xM \to \mathbb{R}$ of (M,F) is defined by $\mathbf{E} := E_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where

$$E_{\alpha\beta} = \frac{1}{2} B^\gamma_{\alpha\beta\gamma}. \tag{2}$$

A Finsler manifold (M,F) is called weakly Berwald manifold if and only if $\mathbf{E} = 0$ [6].

Berwald connection is an important connection in Finsler geometry [19]. The Berwald connection $\hat{D} : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*\tilde{M} \otimes \mathcal{V})$ was first introduced by Berwald [17], and systemically studied in [20]. Its connection 1-forms can be expressed as

$$\check{\omega}^\alpha_\beta = \check{\Gamma}^\alpha_{\beta;\gamma}\, dx^\gamma,$$

where

$$\check{\Gamma}^\alpha_{\beta;\gamma} = \frac{\partial N^\alpha_\beta}{\partial y^\gamma};\ N^\alpha_\beta := \frac{\partial \mathbb{G}^\alpha}{\partial y^\beta}, \tag{3}$$

$\check{\Gamma}^\alpha_{\beta;\gamma}$ is the non-linear connection coefficient of the Berwald connection [21]. In this paper, we use " | " to denote the horizontal covariant derivation of the geometric quantity related to Finsler geometry about the Berwald connection of (M,F).

The $\widetilde{\mathbf{B}}$ curvature $\widetilde{\mathbf{B}} : T_xM \otimes T_xM \otimes T_xM \to T_xM$ of (M,F) is defined by $\widetilde{\mathbf{B}} := \widetilde{B}^\alpha_{\beta\gamma\eta} dx^\beta \otimes dx^\gamma \otimes dx^\eta \otimes \frac{\partial}{\partial x^\alpha}|_x$, where [6]

$$\widetilde{B}^\alpha_{\beta\gamma\eta} := B^\alpha_{\beta\gamma\eta|\lambda} y^\lambda, \tag{4}$$

and

$$B^\alpha_{\beta\gamma\eta|\mu} = \frac{\delta B^\alpha_{\beta\gamma\eta}}{\delta x^\mu} + B^\lambda_{\beta\gamma\eta}\check{\Gamma}^\alpha_{\lambda;\mu} - B^\alpha_{\lambda\gamma\eta}\check{\Gamma}^\lambda_{\beta;\mu} - B^\alpha_{\beta\lambda\eta}\check{\Gamma}^\lambda_{\gamma;\mu} - B^\alpha_{\beta\gamma\lambda}\check{\Gamma}^\lambda_{\eta;\mu}, \tag{5}$$

while

$$\frac{\delta}{\delta x^\alpha} := \frac{\partial}{\partial x^\alpha} - N^\beta_\alpha \frac{\partial}{\partial y^\beta}. \tag{6}$$

A Finsler manifold (M, F) is called $\widetilde{\mathbf{B}}$-manifold if and only if $\widetilde{\mathbf{B}} = 0$.

The stretch $\widetilde{\mathbf{B}}$-curvature $\boldsymbol{\mathcal{K}} : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to T_xM$ of (M, F) is defined by $\boldsymbol{\mathcal{K}} := \mathcal{K}^\alpha_{\beta\gamma\eta\lambda} dx^\beta \otimes dx^\gamma \otimes dx^\eta \otimes dx^\lambda \otimes \frac{\partial}{\partial x^\alpha}|_x$, where [7]

$$\mathcal{K}^\alpha_{\beta\gamma\eta\lambda} := 2(\widetilde{B}^\alpha_{\beta\gamma\eta|\lambda} - \widetilde{B}^\alpha_{\beta\gamma\lambda|\eta}), \tag{7}$$

and

$$\widetilde{B}^\alpha_{\beta\gamma\eta|\mu} = \frac{\delta \widetilde{B}^\alpha_{\beta\gamma\eta}}{\delta x^\mu} + \widetilde{B}^\lambda_{\beta\gamma\eta} \check{\Gamma}^\alpha_{\lambda;\mu} - \widetilde{B}^\alpha_{\lambda\gamma\eta} \check{\Gamma}^\lambda_{\beta;\mu} - \widetilde{B}^\alpha_{\beta\lambda\eta} \check{\Gamma}^\lambda_{\gamma;\mu} - \widetilde{B}^\alpha_{\beta\gamma\lambda} \check{\Gamma}^\lambda_{\eta;\mu}. \tag{8}$$

A Finsler manifold (M, F) is said to be $\widetilde{\mathbf{B}}$-stretch manifold if and only if $\boldsymbol{\mathcal{K}} = 0$.

According to Equations (1), (4) and (7), we have the following inclusion relations:

$$\{\text{Berwald manifold}\} \subset \left\{\widetilde{\mathbf{B}}\text{-manifold}\right\} \subset \left\{\widetilde{\mathbf{B}}\text{-stretch manifold}\right\}.$$

H curvature is another important non-Riemannian geometric quantity in Finsler geometry. The **H** curvature $\mathbf{H} : T_xM \otimes T_xM \to \mathbb{R}$ of (M, F) is defined by $\mathbf{H} := H_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where

$$H_{\alpha\beta} := E_{\alpha\beta|\gamma} y^\gamma. \tag{9}$$

A Finsler manifold (M, F) is called **H**-manifold if and only if $\mathbf{H} = 0$ [9].

The stretch **H**-curvature $\boldsymbol{\kappa} : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ of (M, F) is defined by $\boldsymbol{\kappa} := \kappa_{\alpha\beta\gamma} dx^\alpha \otimes dx^\beta \otimes dx^\gamma$, where [7]

$$\kappa_{\alpha\beta\gamma} := 2(H_{\alpha\beta|\gamma} - H_{\alpha\gamma|\beta}), \tag{10}$$

and

$$H_{\alpha\beta|\gamma} = \frac{\delta H_{\alpha\beta}}{\delta x^\gamma} - H_{\lambda\beta} \check{\Gamma}^\lambda_{\alpha;\gamma} - H_{\alpha\lambda} \check{\Gamma}^\lambda_{\beta;\gamma}. \tag{11}$$

A Finsler manifold (M, F) is called **H**-stretch manifold if and only if $\boldsymbol{\kappa} = 0$.

According to Equations (2), (9) and (10), the inclusion relations are as follows:

$$\{\text{weakly Berwald manifold}\} \subset \{\mathbf{H}\text{-manifold}\} \subset \{\mathbf{H}\text{-stretch manifold}\}.$$

Let M_1 and M_2 be m-dimensional and n-dimensional smooth manifolds, respectively, then the dimension of the product manifold $M = M_1 \times M_2$ is $m + n$.

Let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the natural projection maps, for any $x_1 = (x^1, \cdots, x^m) \in M_1$, $x_2 = (x^{m+1}, \cdots, x^{m+n}) \in M_2$, $x = (x_1, x_2) \in M$, we have $\pi_1(x) = x_1$ and $\pi_2(x) = x_2$.

Let TM_1, TM_2 and TM be the tangent bundles of M_1, M_2 and M, respectively. Denote $d\pi_1 : TM \to TM_1$ and $d\pi_2 : TM \to TM_2$ be the tangent maps induced by π_1 and π_2, respectively. Note that $d\pi_1(x, y) = (x_1, y_1) = (x^1, \cdots, x^m, y^1, \cdots, y^m)$ and $d\pi_2(x, y) = (x_2, y_2) = (x^{m+1}, \cdots, x^{m+n}, y^{m+1}, \cdots, y^{m+n})$. We have natural isomorphism $TM \cong TM_1 \oplus TM_2$. Denote $\widetilde{M_1} = TM_1 \setminus \{0\}$, $\widetilde{M_2} = TM_2 \setminus \{0\}$ and $\widetilde{M} = \widetilde{M_1} \times \widetilde{M_2} \subset TM \setminus \{0\}$.

In the following, we employ the Einstein summation convention and specify the following index conventions: lowercase Greek indices $1 \leq \alpha, \beta, \gamma \cdots \leq m+n$; lowercase Latin indices $1 \leq i, j, k \cdots \leq m$; and primed lowercase Latin indices $m+1 \leq i', j', k' \cdots \leq m+n$. Geometric quantities related to F_1 or F_2 are distinguished by placing the superscripts 1 or 2 directly above the quantities, respectively.

Let $f : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ be a continuous function such that [12]

(a) $f(s,t) = 0$ if and only if $(s,t) = (0,0)$;
(b) $f(\lambda s, \lambda t) = \lambda f(s,t)$ for any $\lambda \in [0,+\infty)$;
(c) f is smooth on $(0,+\infty) \times (0,+\infty)$;
(d) $\frac{\partial f}{\partial s} \neq 0$, $\frac{\partial f}{\partial t} \neq 0$ for any $(s,t) \in (0,+\infty) \times (0,+\infty)$;
(e) $\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} - 2f\frac{\partial^2 f}{\partial s \partial t} \neq 0$ for any $(s,t) \in (0,+\infty) \times (0,+\infty)$;

Definition 2 ([12]). *Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds, and f be continuous function satisfying (a)–(e). Denote $S = {F_1}^2$, $T = {F_2}^2$, the Minkowskian product of Finsler manifold (M_1, F_1) and (M_2, F_2) with respect to the product function f is the product manifold $M = M_1 \times M_2$ endowed with the Finsler metric $F : \widetilde{M} \to R^+$ defined by*

$$F(x,y) = \sqrt{f(S(x^i, y^i), T(x^{i'}, y^{i'}))}, \tag{12}$$

where $(x,y) \in \widetilde{M}$, $(x^i, y^i) \in \widetilde{M_1}$, $(x^{i'}, y^{i'}) \in \widetilde{M_2}$ with $x = (x^i, x^{i'})$, $y = (y^i, y^{i'})$. Clearly, (M,F) is a Finsler manifold. (M,F) is called Minkowskian product Finsler manifold for short. (M_1, F_1) and (M_2, F_2) are called the component manifolds of (M,F).

It should be noted that a Finsler metric is smooth on the slit tangent bundle, but not necessarily on the whole TM_i unless it is Riemannian for $i = 1, 2$. Consequently, F is defined on $\widetilde{M}$, rather than on $T(M_1 \times M_2) \setminus \{0\}$, or on $\widetilde{M_1} \times TM_2$, or on $TM_1 \times \widetilde{M_2}$. It is clear that the function F, as given by Equation (12), is a Finsler metric on $\widetilde{M}$.

3. $\widetilde{\mathbf{B}}$-Stretch Minkowskian Product Manifold

In this section, we shall study $\widetilde{\mathbf{B}}$-stretch Minkowskian product manifold. We firstly recall the following lemmas.

Lemma 1 ([13]). *Let (M,F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the coefficients of spray associated to (M,F) are given by*

$$\mathbb{G}^i = \overset{1}{\mathbb{G}}{}^i, \tag{13}$$

$$\mathbb{G}^{i'} = \overset{2}{\mathbb{G}}{}^{i'}. \tag{14}$$

Lemma 2 ([13]). *Let (M,F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the $\mathbf{B}$ curvature coefficients of (M,F) are given by*

$$B^i_{jkl} = \overset{1}{B}{}^i_{jkl}, \tag{15}$$

$$B^{i'}_{j'k'l'} = \overset{2}{B}{}^{i'}_{j'k'l'},$$

$$B^i_{j'kl} = B^i_{jk'l} = B^i_{jkl'} = B^i_{j'k'l} = B^i_{j'kl'} = B^i_{jk'l'} = B^i_{j'k'l'} = 0, \tag{16}$$

$$B^{i'}_{j'kl} = B^{i'}_{jk'l} = B^{i'}_{jkl'} = B^{i'}_{j'k'l} = B^{i'}_{j'kl'} = B^{i'}_{jk'l'} = B^{i'}_{jkl} = 0. \tag{17}$$

Lemma 3 ([13]). *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the Berwald connection coefficients associated to (M, F) are given by*

$$\check{\Gamma}^i_{j;l} = \overset{1}{\check{\Gamma}}{}^i_{j;l}, \quad \check{\Gamma}^{i'}_{j';l'} = \overset{2}{\check{\Gamma}}{}^{i'}_{j';l'}, \tag{18}$$

$$\check{\Gamma}^i_{j';l} = \check{\Gamma}^i_{j;l'} = \check{\Gamma}^i_{j';l'} = \check{\Gamma}^{i'}_{j';l} = \check{\Gamma}^{i'}_{j;l'} = \check{\Gamma}^{i'}_{j;l} = 0. \tag{19}$$

Proposition 1. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). $B^\alpha_{\beta\gamma\eta|\mu}$ are coefficients of the horizontal covariant derivatives of Berwald curvature. Then*

$$B^i_{jkl|s} = \overset{1}{B}{}^i_{jkl|s}, \tag{20}$$

$$B^{i'}_{j'k'l'|s'} = \overset{2}{B}{}^{i'}_{j'k'l'|s'}, \tag{21}$$

$$B^i_{j'kl|s} = B^i_{jk'l|s} = B^i_{jkl'|s} = B^i_{jkl|s'} = 0, \tag{22}$$

$$B^i_{j'k'l|s} = B^i_{j'kl'|s} = B^i_{j'kl|s'} = B^i_{jk'l'|s} = B^i_{jk'l|s'} = B^i_{jkl'|s'} = 0, \tag{23}$$

$$B^i_{j'k'l'|s} = B^i_{j'k'l|s'} = B^i_{j'kl'|s'} = B^i_{jk'l'|s'} = B^i_{j'k'l'|s'} = B^{i'}_{jkl|s} = 0, \tag{24}$$

$$B^{i'}_{j'kl|s} = B^{i'}_{jk'l|s} = B^{i'}_{jkl'|s} = B^{i'}_{jkl|s'} = B^{i'}_{j'k'l|s} = B^{i'}_{j'kl'|s} = B^{i'}_{j'kl|s'} = 0,$$

$$B^{i'}_{jk'l'|s} = B^{i'}_{jk'l|s'} = B^{i'}_{jkl'|s'} = B^{i'}_{j'k'l'|s} = B^{i'}_{j'k'l|s'} = B^{i'}_{j'kl'|s'} = B^{i'}_{jk'l'|s'} = 0.$$

Proof. Step 1: we first provide the explicit expression of $B^\alpha_{\beta\gamma\eta|\mu}$.

By substituting Equations (3) and (6) into Equation (5), we have

$$B^\alpha_{\beta\gamma\eta|\mu} = \frac{\partial B^\alpha_{\beta\gamma\eta}}{\partial x^\mu} - \frac{\partial \mathbb{G}^\lambda}{\partial y^\mu}\frac{\partial B^\alpha_{\beta\gamma\eta}}{\partial y^\lambda} + B^\lambda_{\beta\gamma\eta}\check{\Gamma}^\alpha_{\lambda;\mu} - B^\alpha_{\lambda\gamma\eta}\check{\Gamma}^\lambda_{\beta;\mu} - B^\alpha_{\beta\lambda\eta}\check{\Gamma}^\lambda_{\gamma;\mu} - B^\alpha_{\beta\gamma\lambda}\check{\Gamma}^\lambda_{\eta;\mu}. \tag{25}$$

Step 2: we now proceed to calculate $B^\alpha_{\beta\gamma\eta|\mu}$ on the Minkowskian product Finsler manifold. The key technique lies in expressing it using the corresponding geometric quantities on its component manifolds. Note that $B^\alpha_{\beta\gamma\eta|\mu}$ has 32 distinct component expressions according to the range of the indices. In the concrete computation, we will substitute Equations (15)–(17) from Lemma 2 into Equation (25).

Putting $\alpha = i$, $\beta = j$, $\gamma = k$, $\eta = l$, $\mu = s$ in Equation (25), and plugging Equations (13)–(19) into it, yields

$$\begin{aligned}
B^i_{jkl|s} &= \frac{\partial B^i_{jkl}}{\partial x^s} - \frac{\partial \mathbb{G}^\lambda}{\partial y^s}\frac{\partial B^i_{jkl}}{\partial y^\lambda} + B^\lambda_{jkl}\check{\Gamma}^i_{\lambda;s} - B^i_{\lambda kl}\check{\Gamma}^\lambda_{j;s} - B^i_{j\lambda l}\check{\Gamma}^\lambda_{k;s} - B^i_{jk\lambda}\check{\Gamma}^\lambda_{l;s} \\
&= \frac{\partial B^i_{jkl}}{\partial x^s} - \frac{\partial \mathbb{G}^t}{\partial y^s}\frac{\partial B^i_{jkl}}{\partial y^t} - \frac{\partial \mathbb{G}^{t'}}{\partial y^s}\frac{\partial B^i_{jkl}}{\partial y^{t'}} + B^t_{jkl}\check{\Gamma}^i_{t;s} + B^{t'}_{jkl}\check{\Gamma}^i_{t';s} - B^i_{tkl}\check{\Gamma}^t_{j;s} \\
&\quad - B^i_{t'kl}\check{\Gamma}^{t'}_{j;s} - B^i_{jtl}\check{\Gamma}^t_{k;s} - B^i_{jt'l}\check{\Gamma}^{t'}_{k;s} - B^i_{jkt}\check{\Gamma}^t_{l;s} - B^i_{jkt'}\check{\Gamma}^{t'}_{l;s} \\
&= \frac{\partial \overset{1}{B}{}^i_{jkl}}{\partial x^s} - \frac{\partial \overset{1}{\mathbb{G}}{}^t}{\partial y^s}\frac{\partial \overset{1}{B}{}^i_{jkl}}{\partial y^t} + \overset{1}{B}{}^t_{jkl}\overset{1}{\check{\Gamma}}{}^i_{t;s} - \overset{1}{B}{}^i_{tkl}\overset{1}{\check{\Gamma}}{}^t_{j;s} - \overset{1}{B}{}^i_{jtl}\overset{1}{\check{\Gamma}}{}^t_{k;s} - \overset{1}{B}{}^i_{jkt}\overset{1}{\check{\Gamma}}{}^t_{l;s} \\
&= \overset{1}{B}{}^i_{jkl|s}.
\end{aligned}$$

By a similar argument, the other equalities of the Proposition 1 are obtained. □

Proposition 2. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the $\widetilde{\mathbf{B}}$ curvature coefficients of (M, F) are given by*

$$\widetilde{B}^i_{jkl} = \overset{1}{\widetilde{B}}{}^i_{jkl}, \tag{26}$$

$$\begin{aligned}
&\widetilde{B}^{i'}_{j'k'l'} = \overset{2}{\widetilde{B}}{}^{i'}_{j'k'l'},\\
&\widetilde{B}^i_{j'k'l} = \widetilde{B}^i_{j'kl'} = \widetilde{B}^i_{jk'l'} = \widetilde{B}^{i'}_{j'k'l} = \widetilde{B}^{i'}_{j'kl'} = \widetilde{B}^{i'}_{jk'l'} = \widetilde{B}^i_{j'k'l'} = 0.\\
&\widetilde{B}^{i'}_{jkl} = \widetilde{B}^i_{j'kl} = \widetilde{B}^i_{jk'l} = \widetilde{B}^i_{jkl'} = \widetilde{B}^{i'}_{j'kl} = \widetilde{B}^{i'}_{jk'l} = \widetilde{B}^{i'}_{jkl'} = 0,
\end{aligned} \tag{27}$$

Proof. By putting $\alpha = i, \beta = j, \gamma = k, \eta = l$ in Equation (4), and plugging Equations (20) and (22) into it, we can get

$$\begin{aligned}
\widetilde{B}^i_{jkl} &= B^i_{jkl|\lambda} y^\lambda\\
&= B^i_{jkl|t} y^t + B^i_{jkl|t'} y^{t'}\\
&= \overset{1}{B}{}^i_{jkl|t}\, y^t\\
&= \overset{1}{\widetilde{B}}{}^i_{jkl}\,.
\end{aligned}$$

Similar calculations give the rest of the equalities of Proposition 2. □

Theorem 1. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2), then (M, F) is a $\widetilde{\mathbf{B}}$-manifold if and only if (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-manifold.*

Proof. According to the definition of $\widetilde{\mathbf{B}}$-manifold, (M, F) is $\widetilde{\mathbf{B}}$-manifold if and only if

$$\widetilde{B}^\alpha_{\beta\gamma\eta} = 0. \tag{28}$$

Based on Proposition 2, Equation (28) is equivalent to

$$\overset{1}{\widetilde{B}}{}^i_{jkl} = \overset{2}{\widetilde{B}}{}^{i'}_{j'k'l'} = 0,$$

which mean that (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-manifold. □

Remark 1. *Any Berwald manifold must be a $\widetilde{\mathbf{B}}$-manifold, whereas the reverse implication does not necessarily hold. Consequently, Theorem 1 extends the result of Theorem 4.2 in [12], further enriching the research achievements concerning Berwald manifolds in Finsler geometry.*

Using Proposition 2, formulas (3), (6) and (8), and similar to the proof method of Proposition 1, the following proposition can be obtained.

Proposition 3. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). $\widetilde{B}^\alpha_{\beta\gamma\eta|\mu}$ are coefficients of the horizontal covariant derivatives of $\widetilde{\mathbf{B}}$. Then*

$$\widetilde{B}^i_{jkl|s} = \overset{1}{\widetilde{B}}{}^i_{jkl|s}, \tag{29}$$

$$\begin{aligned}
&\widetilde{B}^{i'}_{j'k'l'|s'} = \overset{2}{\widetilde{B}}{}^{i'}_{j'k'l'|s'},\\
&\widetilde{B}^i_{j'kl|s} = \widetilde{B}^i_{jk'l|s} = \widetilde{B}^i_{jkl'|s} = \widetilde{B}^i_{jkl|s'} = 0,
\end{aligned}$$

$$\widetilde{B}^i_{j'k'l|s} = \widetilde{B}^i_{j'kl'|s} = \widetilde{B}^i_{j'kl|s'} = \widetilde{B}^i_{jk'l'|s} = \widetilde{B}^i_{jk'l|s'} = \widetilde{B}^i_{jkl'|s'} = 0,$$
$$\widetilde{B}^i_{j'k'l'|s} = \widetilde{B}^i_{j'k'l|s'} = \widetilde{B}^i_{j'kl'|s'} = \widetilde{B}^i_{jk'l'|s'} = \widetilde{B}^i_{j'k'l'|s'} = \widetilde{B}^{i'}_{jkl|s} = 0,$$
$$\widetilde{B}^{i'}_{j'kl|s} = \widetilde{B}^{i'}_{jk'l|s} = \widetilde{B}^{i'}_{jkl'|s} = \widetilde{B}^{i'}_{jkl|s'} = \widetilde{B}^{i'}_{j'k'l|s} = \widetilde{B}^{i'}_{j'kl'|s} = \widetilde{B}^{i'}_{j'kl|s'} = 0,$$
$$\widetilde{B}^{i'}_{jk'l'|s} = \widetilde{B}^{i'}_{jk'l|s'} = \widetilde{B}^{i'}_{jkl'|s'} = \widetilde{B}^{i'}_{j'k'l'|s} = \widetilde{B}^{i'}_{j'k'l|s'} = \widetilde{B}^{i'}_{j'kl'|s'} = \widetilde{B}^{i'}_{jk'l'|s'} = 0.$$

Proposition 4. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the stretch $\widetilde{\mathbf{B}}$-curvature coefficients of (M, F) are given by*

$$\mathcal{K}^i_{jkls} = \overset{1}{\mathcal{K}}{}^i_{jkls},$$
$$\mathcal{K}^{i'}_{j'k'l's'} = \overset{2}{\mathcal{K}}{}^{i'}_{j'k'l's'},$$
$$\mathcal{K}^i_{j'kls} = \mathcal{K}^i_{jk'ls} = \mathcal{K}^i_{jkl's} = \mathcal{K}^i_{jkls'} = 0,$$
$$\mathcal{K}^i_{j'k'ls} = \mathcal{K}^i_{j'kl's} = \mathcal{K}^i_{j'kls'} = \mathcal{K}^i_{jk'l's} = \mathcal{K}^i_{jk'ls'} = \mathcal{K}^i_{jkl's'} = 0,$$
$$\mathcal{K}^i_{j'k'l's} = \mathcal{K}^i_{j'k'ls'} = \mathcal{K}^i_{j'kl's'} = \mathcal{K}^i_{jk'l's'} = \mathcal{K}^i_{j'k'l's'} = \mathcal{K}^{i'}_{jkls} = 0,$$
$$\mathcal{K}^{i'}_{j'kls} = \mathcal{K}^{i'}_{jk'ls} = \mathcal{K}^{i'}_{jkl's} = \mathcal{K}^{i'}_{jkls'} = \mathcal{K}^{i'}_{j'k'ls} = \mathcal{K}^{i'}_{j'kl's} = \mathcal{K}^{i'}_{j'kls'} = 0,$$
$$\mathcal{K}^{i'}_{jk'l's} = \mathcal{K}^{i'}_{jk'ls'} = \mathcal{K}^{i'}_{jkl's'} = \mathcal{K}^{i'}_{j'k'l's} = \mathcal{K}^{i'}_{j'k'ls'} = \mathcal{K}^{i'}_{j'kl's'} = \mathcal{K}^{i'}_{jk'l's'} = 0.$$

Proof. By putting $\alpha = i$, $\beta = j$, $\gamma = k$, $\eta = l$, $\lambda = s$ in Equation (7), and plugging Equation (29) into it, we get

$$\begin{aligned}\mathcal{K}^i_{jkls} &= 2(\widetilde{B}^i_{jkl|s} - \widetilde{B}^i_{jks|l}) \\ &= 2(\overset{1}{\widetilde{B}}{}^i_{jkl|s} - \overset{1}{\widetilde{B}}{}^i_{jks|l}) \\ &= \overset{1}{\mathcal{K}}{}^i_{jkls}\,.\end{aligned}$$

By a similar calculation, the other equations of the Proposition 4 are obtained. □

Theorem 2. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then (M, F) is a $\widetilde{\mathbf{B}}$-stretch manifold if and only if (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-stretch manifold.*

Proof. According to the definition of $\widetilde{B}$-stretch manifold, (M, F) is $\widetilde{\mathbf{B}}$-stretch manifold if and only if

$$\mathcal{K}^\alpha_{\beta\gamma\eta\mu} = 0. \tag{30}$$

Based on Proposition 4, it follows that Equation (30) is equivalent to

$$\overset{1}{\mathcal{K}}{}^i_{jkls} = \overset{2}{\mathcal{K}}{}^{i'}_{j'k'l's'} = 0,$$

which implies that (M_1, F_1) and (M_2, F_2) are both $\widetilde{\mathbf{B}}$-stretch manifold. □

Remark 2. *A $\widetilde{\mathbf{B}}$-manifold is necessarily a $\widetilde{\mathbf{B}}$-stretch manifold, but the converse does not hold in general. Thus, Theorem 2 addresses a more general case than Theorem 1, provides a more effective approach to characterizing special Finsler manifolds.*

Example 1. *Let (M_1, F_1) be a 3-dimensional Euclidean space endowed with a Randers metric $F_1 = \alpha + \beta$, where $\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \beta = y^1$. Let (M_2, F_2) be an unit ball $\mathbb{B}^n (n = 2)$ endowed with a Finsler metric F_2 , where*

$$F_2(x,y) = \frac{\left(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \varepsilon\langle x,y\rangle\right)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}},\ x \in \mathbb{B}^2,\ y \in T\mathbb{B}^2 \setminus \{0\}.$$

Let $F = \sqrt{f}$, where $f = F_1^2 + F_2^2$, then (M, F) is a $\widetilde{\mathbf{B}}$-stretch manifold.

Proof. It follows from [20] that (M_1, F_1) is a Berwald manifold, we can know (M_1, F_1) is a $\widetilde{\mathbf{B}}$-stretch manifold. It has been proven in [6] that (M_2, F_2) is a $\widetilde{\mathbf{B}}$-stretch manifold. It is easy to verify that the product function f satisfies conditions (a)–(e). By Definition 2, we know that (M, F) is a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then, it follows from Theorem 2 that (M, F) is a $\widetilde{\mathbf{B}}$-stretch manifold. □

4. H-Stretch Minkowskian Product Manifold

Tayebi and Koh proved that (M, F) has vanishing main scalar if and only if it has vanishing **H** curvature [22]. In 2025, Tayebi showed that a homogeneous Finsler metric of scalar flag curvature has constant flag curvature if and only if its **H** curvature vanishes [23]. In this section, our primary goal is to study whether the Minkowskian product Finsler manifold is also a **H**-stretch manifold when (M_1, F_1) and (M_2, F_2) are both **H**-stretch manifold, and investigate whether the converse holds as well.

Proposition 5 ([24])**.** *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the* **H** *curvature coefficients of (M, F) are given by*

$$H_{ij} = \overset{1}{H}_{ij}, \tag{31}$$

$$H_{i'j'} = \overset{2}{H}_{i'j'},$$

$$H_{i'j} = H_{ij'} = 0. \tag{32}$$

Proposition 6. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the coefficients of the horizontal covariant derivatives of* **H** *curvature are given by*

$$H_{ij|k} = \overset{1}{H}_{ij|k}, \tag{33}$$

$$H_{i'j'|k'} = \overset{2}{H}_{i'j'|k'}, \tag{34}$$

$$H_{i'j|k} = H_{ij'|k} = H_{ij|k'} = H_{i'j'|k} = H_{i'j|k'} = H_{ij'|k'} = 0. \tag{35}$$

Proof. Substituting Equations (3), (6) into Equation (11), yields

$$H_{\alpha\beta|\gamma} = \frac{\partial H_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial \mathbb{G}^\lambda}{\partial y^\gamma}\frac{\partial H_{\alpha\beta}}{\partial y^\lambda} - H_{\lambda\beta}\check{\Gamma}^\lambda_{\alpha;\gamma} - H_{\alpha\lambda}\check{\Gamma}^\lambda_{\beta;\gamma}. \tag{36}$$

Putting $\alpha = i,\ \beta = j,\ \gamma = k$ in Equation (36), and plugging Equations (13), (14), (18), (19), (31) and (32), into it, we have

$$\begin{aligned} H_{ij|k} &= \frac{\partial H_{ij}}{\partial x^k} - \frac{\partial \mathbb{G}^\lambda}{\partial y^k}\frac{\partial H_{ij}}{\partial y^\lambda} - H_{\lambda j}\check{\Gamma}^\lambda_{i;k} - H_{i\lambda}\check{\Gamma}^\lambda_{j;k} \\ &= \frac{\partial H_{ij}}{\partial x^k} - \frac{\partial \mathbb{G}^t}{\partial y^k}\frac{\partial H_{ij}}{\partial y^t} - \frac{\partial \mathbb{G}^{t'}}{\partial y^k}\frac{\partial H_{ij}}{\partial y^{t'}} - H_{tj}\check{\Gamma}^t_{i;k} - H_{t'j}\check{\Gamma}^{t'}_{i;k} - H_{it}\check{\Gamma}^t_{j;k} - H_{it'}\check{\Gamma}^{t'}_{j;k} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial \overset{1}{H}_{ij}}{\partial x^k} - \frac{\partial \overset{1}{\mathbb{G}}{}^t}{\partial y^k}\frac{\partial \overset{1}{H}_{ij}}{\partial y^t} - \overset{1}{H}_{tj}\,\overset{1}{\check{\Gamma}}{}^t_{i;k} - \overset{1}{H}_{it}\,\overset{1}{\check{\Gamma}}{}^t_{j;k} \\
&= \overset{1}{H}_{ij|k}\,.
\end{aligned}
$$

Similarly, we can obtain Equations (34) and (35). □

Proposition 7. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2). Then the coefficients of stretch* **H***-curvature of (M, F) are given by*

$$
\begin{aligned}
&\kappa_{ijk} = \overset{1}{\kappa}_{ijk}, \\
&\kappa_{i'j'k'} = \overset{2}{\kappa}_{i'j'k'}, \\
&\kappa_{i'jk} = \kappa_{ij'k} = \kappa_{ijk'} = \kappa_{i'j'k} = \kappa_{i'jk'} = \kappa_{ij'k'} = 0.
\end{aligned}
$$

Proof. By putting $\alpha = i,\ \beta = j,\ \gamma = k$ in Equation (10), and plugging Equation (33) into it, we can get

$$
\begin{aligned}
\kappa_{ijk} &= 2(H_{ij|k} - H_{ik|j}) \\
&= 2(\overset{1}{H}_{ij|k} - \overset{1}{H}_{ik|j}) \\
&= \overset{1}{\kappa}_{ijk}\,.
\end{aligned}
$$

By the same reasoning, we can obtain other equalities. □

Theorem 3. *Let (M, F) be a Minkowskian product Finsler manifold of (M_1, F_1) and (M_2, F_2), then (M, F) is a* **H***-stretch manifold if and only if (M_1, F_1) and (M_2, F_2) are both* **H***-stretch manifold.*

Proof. According to the definition of **H**-stretch manifold, (M, F) is **H**-stretch manifold if and only if

$$
\kappa_{\alpha\beta\gamma} = 0. \tag{37}
$$

Based on Proposition 7, we can see that Equation (37) is equivalent to

$$
\overset{1}{\kappa}_{ijk} = \overset{2}{\kappa}_{i'j'k'} = 0,
$$

which mean that (M_1, F_1) and (M_2, F_2) are both **H**-stretch manifold. □

5. Conclusions

The Minkowskian product is a crucial method for constructing special Finsler manifolds. In this paper, we focus on studying new classes of stretch Finsler manifolds and provides an effective method for constructing $\widetilde{\mathbf{B}}$-manifold, $\widetilde{\mathbf{B}}$-stretch manifold and **H**-stretch manifold. In subsequent work, we will further investigate the geometric properties of the classes. Stretch curvatures, such as the relatively isotropic $\widetilde{\mathbf{B}}$-stretch curvature and the relatively isotropic **H**-stretch curvature, with the aim of developing more effective methods for constructing special Finsler manifolds. In future work, we intend to extend these results about the Minkowskian product of two Finsler manifolds to the case of three or more Finsler manifolds, and investigate questions analogous to those addressed in this paper.

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References

1. Cartan, E. Les espaces de Finsler. *Bull. Am. Math. Soc.* **1934**, *40*, 521–522.
2. Tayebi, A.; Nankali, A.; Peyghan, E. Some curvature properties of Cartan spaces with mth root metrics. *J. Lith. Math.* **2014**, *54*, 106–114.
3. Berwald, L. Uber Parallelübertragung in Räumen mit allgemeiner Maßbestimmung. *Jahresber. Dtsch. Math. Ver.* **1926**, *34*, 2013–2020.
4. Berwald, L. Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. *Math. Z.* **1927**, *26*, 176–176.
5. Berwald, L. Über zweidimensionale allgemeine metrische Räume. *J. Die Reine Angew. Math.* **1927**, *156*, 191–122.
6. Shen, Z. *Differental Geometry of Spray and Finsler Spaces*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2013.
7. Abbas, S.A.; Kozma, L. On New Classes of Stretch Finsler Metrics. *J. Finsler Geom. Its Appl.* **2022**, *3*, 86–99.
8. Abbas, S.A.; Kozma, L. On generalized Douglas curvature of Finsler metrics. *Glob. J. Adv. Res. Class. Mod. Geom.* **2023**, *12*, 229–236.
9. Zadeh, H.A. Sur les espaces de Finsler *à* courbures sectionnelles constantes. *Bull. L'Académie R. Belg. Cl. Sci.* **1988**, *74*, 281–322.
10. Najafi, B.; Shen, Z.; Tayebi, A. Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties. *Geom. Dedicata* **2008**, *131*, 87–97.
11. Xia, Q. Some results on the non-riemannian quantity H of a Finsler metric. *Int. J. Math.* **2011**, *22*, 925–936.
12. Okada, T. Minkowskian product of Finsler spaces and Berwald connection. *J. Math. Kyoto Univ.* **1982**, *22*, 323–332.
13. He, Y.; Li, J.; Bian, L.; Zhang, N. On Minkowskian product of Finsler manifolds. *Differ. Geom. Its Appl.* **2023**, *89*, 102019.
14. Li, J.; He, Y.; Zhang, X.; Zhang, N. Dually flat and projectively flat Minkowskian product Finsler metrics. *J. Math. Anal. Appl.* **2023**, *524*, 127093.
15. Tian, C.; He, Y.; Li, S.; Zhang, H. Minkowskian product Finsler metrics with scalar flag curvature. *J. Xinjiang Norm. (Nat. Sci. Ed.)* **2022**, *41*, 51–63.
16. Zhang, N.; Lu, W.; Han, J.; Eryzabk, J. Douglas curvature and Weyl curvature of Minkowskian product Finsler metrics. *J. Xinjiang Norm. (Nat. Sci. Ed.)* **2023**, *42*, 67–74.
17. Abate, M.; Patrizio, G. *Finsler Metrics-A Global Approach: With Applications to Geometric Function Theory*; Springer: Berlin/Heidelberg, Germany, 2006.
18. Ichijyō, Y. Finsler manifolds modeled on a Minkowski space. *J. Math. Kyoto Univ.* **1976**, *16*, 639–652.
19. Zhong, C. On real and complex Berwald connections associated to strongly convex weakly Kähler-Finsler metric. *Differ. Geom. Its Appl.* **2011**, *29*, 338–408.
20. Abate, M. A characterization of the Chern and Berwald connections. *Houst. J. Math.* **1996**, *22*, 701–717.
21. Antonelli, P.L.; Ferraris, A.J.; Spivak, L.A. *Finsler Geometry: An Introduction*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
22. Tayebi, A.; Koh, W.S. On Finsler surfaces with isotropic main scalar. *Mathematics* **2024**, *12*, 2141.

23. Tayebi, A. On E-curvature of homogeneous Finsler manifolds. *Period. Math. Hung.* **2025**, *90*, 140–155.
24. Tian, C.; He, Y.; Lu, W.; Eryzabk, J. On some non-Riemannian curvature of Minkowskian product Finsler metrics. *J. Math. Anal. Appl.* **2024**, *534*, 128070.

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