

Article

A Structure-Preserving Scheme for the Space-Fractional Klein-Gordon-Schrödinger System with the Invariant Energy Quadratization Method

Wenye Jiang ¹, Yu Li ¹  and Yan Fan ^{2,*} 

¹ School of Science, Northeast Forestry University, Harbin 150040, China; 2023113175@nefu.edu.cn (W.J.); liy@nefu.edu.cn (Y.L.)

² Xingzhi College, Zhejiang Normal University, Jinhua 321004, China

* Correspondence: fanyan@zjnu.edu.cn

Abstract

This paper investigates the conservation of mass and energy in the space-fractional Klein-Gordon-Schrödinger system with fractional Laplacian operators. Firstly, the invariant energy quadratization method is applied to transform the original system into an equivalent form. For spatial discretization, Fourier spectral methods are employed, yielding a semi-discrete scheme. Subsequently, an invariant energy quadratization Runge-Kutta approach is used for temporal discretization, resulting in a fully discrete scheme. Owing to its diagonally implicit structure, the proposed scheme is both highly accurate and efficient while preserving mass and energy exactly. Numerical experiments are conducted to verify the accuracy and conservation properties of the method.

Keywords: space-fractional Klein-Gordon-Schrödinger equation; Fourier spectral method; invariant energy quadratization method; diagonally implicit Runge-Kutta method; mass and energy conservation

MSC: 65M08; 65M12

1. Introduction

The Klein-Gordon-Schrödinger (KGS) equation is an important model in relativistic quantum mechanics and quantum field theory [1]. It describes the interaction between a conservative complex nucleon field and a neutral meson field via Yukawa coupling in quantum field theory. The nonlinear Schrödinger equation, one of the most significant equations in quantum mechanics, describes various physical phenomena such as optics. By coupling the Schrödinger equation with the Klein-Gordon equation, a relativistic version of the Schrödinger equation, the well-known KGS system is obtained. The KGS equation plays a significant role in quantum optics, particularly in biomedical imaging applications. These equations describe the dynamics of quantum systems, which can be used to detect and process individual X-ray quanta. This capability is crucial in biomedical applications where high-resolution imaging is essential for diagnosing and treating medical conditions. Beyond its fundamental role in quantum field theory, the KGS system has recently emerged as a valuable model for describing nonlinear wave phenomena in optical media, particularly in the context of soliton dynamics within materials exhibiting nonlocal responses.

Leveraging the nonlocal nature of fractional derivatives to describe memory and hereditary properties of materials, Laskin extended the Feynman path integral to the



Academic Editors: Branislav M. Randelović, Branislav Vlahovic and Roman Dmytryshyn

Received: 21 December 2025

Revised: 5 February 2026

Accepted: 26 February 2026

Published: 1 March 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\) license](https://creativecommons.org/licenses/by/4.0/).

Lévy path [2,3], establishing a Schrödinger model that incorporates a nonlocal Laplacian operator [4,5]. Recently, fractional KGS equations have introduced novel perspectives in a wide range of scientific and engineering fields [6–8], gaining considerable attention due to their improved accuracy over integer-order models in depicting quantum state evolution, making them essential in quantum mechanics [9]. Huang et al. [10] investigated the coupled fractional KGS system, focusing on the low-regularity local or global well-posedness of its Cauchy problem and numerical solutions, respectively.

In this paper, we consider the following coupled space-fractional KGS system with fractional Laplacian operators:

$$iu_t - \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}}u + \gamma_1 u\phi + 2\gamma_2 |u|^2 u\phi = 0, \quad (1)$$

$$\phi_{tt} + (-\Delta)^{\frac{\beta}{2}}\phi + m^2\phi - \gamma_1 |u|^2 - \gamma_2 |u|^4 = 0, \quad (2)$$

$$u(x, 0) = u_0(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad x \in \Omega, \quad (3)$$

where $i = \sqrt{-1}$, $1 < \alpha, \beta \leq 2$, $x \in \Omega \subset \mathbb{R}$, $t \in (0, T]$, and $u(x, t)$ and $\phi(x, t)$ are complex-valued and real-valued unknown functions, respectively. Here, u_t denotes the first partial derivative of u with respect to t , while ϕ_{tt} denotes the second partial derivative of ϕ with respect to t . The fractional Laplacian operator is defined as [11]

$$(-\Delta)^{\frac{s}{2}}u(x, t) = \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(u(\xi, t))),$$

where $\mathcal{F}$ represents the Fourier transform and $1 < s \leq 2$. If $\alpha = 2$ and $\beta = 2$, the system reduces to the standard KGS equation.

Felmer et al. [12] demonstrated that, under suitable regularity conditions, the solution of the Schrödinger equation with the fractional Laplacian exhibits polynomial decay to zero as $|x| \rightarrow \infty$. Consequently, for computational purposes, Equations (1)–(3) can be efficiently confined to a finite computational domain $[-L/2, L/2]$ by imposing periodic boundary conditions

$$u\left(x - \frac{L}{2}, t\right) = u\left(x + \frac{L}{2}, t\right), \quad \phi\left(x - \frac{L}{2}, t\right) = \phi\left(x + \frac{L}{2}, t\right),$$

with L chosen sufficiently large. Under periodic boundary conditions, the Fourier transform of the fractional Laplacian can be expressed as [11]

$$(-\Delta)^{\frac{\alpha}{2}}u = \sum_{k \in \mathbb{Z}} |k|^\alpha \hat{u}_k e^{ik(x+L/2)}, \quad \hat{u}_k = \frac{1}{2\pi} \int_{\Omega} u(x, t) e^{-ik(x+L/2)} dx.$$

An important property of the fractional KGS system is that it preserves the mass of the nucleon field,

$$M(t) = \|u(\cdot, t)\|^2 = \|u_0\|^2 = M(0),$$

and the total energy,

$$E(t) = \|\phi_t\|^2 + \left\|(-\Delta)^{\frac{\beta}{4}}\phi\right\|^2 + m^2\|\phi\|^2 + \left\|(-\Delta)^{\frac{\alpha}{4}}u\right\|^2 - 2\gamma_1(\phi, |u|^2) - 2\gamma_2(\phi, |u|^4) = E(0),$$

where $(\cdot, \cdot)$ denotes the inner product. Since analytical methods are only available for obtaining certain special solutions, and explicit analytical solutions for the coupled fractional KGS equations are generally difficult to obtain, constructing effective numerical methods becomes crucial.

Fourier spectral methods have demonstrated remarkable efficiency in solving classical KGS systems and space-fractional Schrödinger equations. As shown by [13], when

discretizing fractional Laplacian operators spatially, Fourier spectral methods achieve spectral convergence up to machine accuracy regardless of the order of the fractional Laplacian. Moreover, under periodic boundary conditions, the spatial semi-discrete scheme obtained via Fourier spectral methods preserves the specific conservation laws inherent in the original differential equation [11].

The core idea of the Invariant Energy Quadratization (IEQ) method is to introduce auxiliary variables based on the original energy functional, thereby transforming the original equation with high-order energy terms into an equivalent coupled system. This approach enables the preservation of the energy conservation law for polynomial structures of any order. Since its initial proposal by Yang et al. [14], several conservative or unconditionally energy-stable methods for various equations have been developed using the IEQ method [15–17].

Recently, Zhang et al. proposed a class of high-order energy-conserving diagonally implicit Runge-Kutta (DIRK) methods for nonlinear Hamiltonian ordinary differential equations [18]. They first used the IEQ method to transform the nonlinear Hamiltonian ordinary differential equations into an equivalent form with quadratic energy, then they discretized the reformulated equations using DIRK schemes. The resulting algorithms preserve the modified quadratic energy conservation law and achieve high-order accuracy. Subsequently, for the nonlinear Schrödinger equation, a DIRK scheme was proposed in [19] that exhibits high-order convergence while preserving both mass and energy.

Compared with existing literature, the main contributions of this paper are as follows: we systematically apply the IEQ-Fourier-DIRK framework to the fractional-order KGS system, which enhances both the accuracy and stability of numerical solutions, and the rigorous proof of mass and energy conservation for arbitrary fractional-order parameters. Numerical experiments validate the high-order accuracy and reliability in preserving structural properties over long-time simulations of the proposed method.

The structure of this paper is as follows. In Section 2, we first apply the IEQ method to transform Systems (1)–(3) into an equivalent coupled system, then we use Fourier spectral methods to discretize the coupled system, obtaining a semi-discrete scheme with high spatial accuracy. In Section 3, we adopt the DIRK method for temporal discretization and verify the conservation of energy and mass. Section 4 presents an error analysis and its proof. Section 5 provides several numerical examples to evaluate the performance of the proposed algorithm. Finally, Section 6 concludes this paper with a summary and future perspectives.

2. Equivalent System Construction and Semi-Discrete Conservative Scheme

By introducing an auxiliary variable $r = |u|^2$, the coupled space-fractional KGS Equations (1)–(3) can be transformed into the following equivalent system:

$$u_t = -\frac{i}{2}(-\Delta)^{\frac{\alpha}{2}}u + i\gamma_1 u\phi + 2i\gamma_2 ru\phi, \quad (4)$$

$$\phi_t = v, \quad (5)$$

$$v_t = -(-\Delta)^{\frac{\beta}{2}}\phi - m^2\phi + \gamma_1 r + \gamma_2 r^2, \quad (6)$$

$$r_t = u\bar{u}_t + \bar{u}u_t, \quad (7)$$

where $\bar{u}$ denotes the complex conjugate of u . This transformation results in a quadratic energy functional in the variables u, ϕ, r , which facilitates a concise and systematic proof of the discrete energy conservation laws.

Theorem 1. The equivalent system (4)–(7) satisfies the following mass and modified energy conservation laws:

$$M(t) = \|u(x, t)\|^2 = M(0), \quad (8)$$

$$E(t) = \|v\|^2 + \left\|(-\Delta)^{\frac{\beta}{4}}\phi\right\|^2 + m^2\|\phi\|^2 + \left\|(-\Delta)^{\frac{\alpha}{4}}u\right\|^2 - 2\gamma_1(\phi, r) - 2\gamma_2(\phi, r^2) = E(0). \quad (9)$$

Proof. Multiplying Equation (4) by $i\bar{u}$ yields

$$i\bar{u}u_t = \frac{1}{2}\bar{u}(-\Delta)^{\frac{\alpha}{2}}u - \bar{u}\gamma_1u\phi - 2\bar{u}\gamma_2ru\phi.$$

Taking the imaginary part and integrating over space gives

$$(\bar{u}, u_t) = \frac{\partial}{\partial t}\|u\|^2 = 0,$$

which proves the mass conservation law (8). Multiplying Equation (4) by $-2i\bar{u}_t$ gives

$$\begin{aligned} -2i\bar{u}_tu_t &= \bar{u}_t(-\Delta)^{\frac{\alpha}{2}}u + 2\gamma_1\bar{u}_tu\phi + 4\gamma_2r\bar{u}_tu\phi \\ &= \bar{u}_t(-\Delta)^{\frac{\alpha}{4}} \cdot (-\Delta)^{\frac{\alpha}{4}}u + (-\Delta)^{\frac{\alpha}{4}}\bar{u}_t(-\Delta)^{\frac{\alpha}{4}} \cdot u \\ &\quad - (-\Delta)^{\frac{\alpha}{4}}\bar{u}_t \cdot (-\Delta)^{\frac{\alpha}{4}}u + 2\gamma_1\bar{u}_tu\phi + 4\gamma_2r\bar{u}_tu\phi \\ &= (-\Delta)^{\frac{\alpha}{4}}\left(\bar{u}_t(-\Delta)^{\frac{\alpha}{4}}u\right) + 2\gamma_1\bar{u}_tu\phi + 4\gamma_2r\bar{u}_tu\phi, \end{aligned} \quad (10)$$

Taking the complex conjugate of Equation (4) and multiplying by $-2iu_t$ gives

$$\begin{aligned} -2iu_t\bar{u}_t &= u_t(-\Delta)^{\frac{\alpha}{2}}\bar{u} + 2\gamma_1u_t\bar{u}\phi + 4\gamma_2ru_t\bar{u}\phi \\ &= u_t(-\Delta)^{\frac{\alpha}{4}} \cdot (-\Delta)^{\frac{\alpha}{4}}\bar{u} + (-\Delta)^{\frac{\alpha}{4}}u_t(-\Delta)^{\frac{\alpha}{4}} \cdot \bar{u} \\ &\quad - (-\Delta)^{\frac{\alpha}{4}}u_t \cdot (-\Delta)^{\frac{\alpha}{4}}\bar{u} + 2\gamma_1u_t\bar{u}\phi + 4\gamma_2ru_t\bar{u}\phi \\ &= (-\Delta)^{\frac{\alpha}{4}}\left(u_t(-\Delta)^{\frac{\alpha}{4}}\bar{u}\right) + 2\gamma_1u_t\bar{u}\phi + 4\gamma_2ru_t\bar{u}\phi. \end{aligned} \quad (11)$$

Adding the real parts of (10) and (11), and noting that $\text{Re}(iu_t\bar{u}_t) = 0$ gives

$$(-\Delta)^{\frac{\alpha}{4}}\left(\bar{u}_t(-\Delta)^{\frac{\alpha}{4}}u + u_t(-\Delta)^{\frac{\alpha}{4}}\bar{u}\right) - \frac{\partial}{\partial t}\left\|(-\Delta)^{\frac{\alpha}{4}}u\right\|^2 + 2\gamma_1\phi r_t + 4\gamma_2\phi r r_t = 0.$$

Integrating over space and eliminating the divergence term

$$(-\Delta)^{\frac{\alpha}{4}}(\bar{u}_t(-\Delta)^{\frac{\alpha}{4}}u + u_t(-\Delta)^{\frac{\alpha}{4}}\bar{u}),$$

we get

$$\frac{\partial}{\partial t}\left\|(-\Delta)^{\frac{\alpha}{4}}u\right\|^2 - 2\gamma_1(\phi, r_t) - 2\gamma_2(\phi, 2rr_t) = 0. \quad (12)$$

Taking the inner product of Equation (5) with $2v_t$ and Equation (6) with $2\phi_t$, respectively, gives

$$(\phi_t, 2v_t) = (v, 2v_t) = \frac{\partial}{\partial t}\|v\|^2, \quad (13)$$

$$\begin{aligned} (v_t, 2\phi_t) &= -\left((-\Delta)^{\frac{\beta}{2}}\phi, 2\phi_t\right) - (m^2\phi, 2\phi_t) + (\gamma_1r, 2\phi_t) + (\gamma_2r^2, 2\phi_t) \\ &= \frac{\partial}{\partial t}\left(-\left\|(-\Delta)^{\frac{\beta}{4}}\phi\right\|^2 - m^2\|\phi\|^2\right) + 2\gamma_1(\phi_t, r) + 2\gamma_2(\phi_t, r^2). \end{aligned} \quad (14)$$

Adding Equations (12)–(14) yields

$$\begin{aligned} \frac{\partial}{\partial t} \left(\left\| (-\Delta)^{\frac{\alpha}{4}} u \right\|^2 + \left\| (-\Delta)^{\frac{\beta}{4}} \phi \right\|^2 + \|v\|^2 \right) - 2\gamma_1(\phi, r_t) - 2\gamma_2(\phi, 2rr_t) \\ - 2\gamma_1(\phi_t, r) - 2\gamma_2(\phi_t, r^2) = 0, \end{aligned}$$

which simplifies to

$$\frac{\partial}{\partial t} \left(\left\| (-\Delta)^{\frac{\alpha}{4}} u \right\|^2 + \left\| (-\Delta)^{\frac{\beta}{4}} \phi \right\|^2 + \|v\|^2 - 2\gamma_1(\phi, r) - 2\gamma_2(\phi, r^2) \right) = 0.$$

Thus, the energy conservation law (9) is proved. $\square$

2.1. Fourier Spectral Approximation of Spatial Derivatives

For a given positive even integer N , the spatial domain $\Omega_h = [-L/2, L/2]$ is uniformly partitioned with step size $h = L/N$. The grid points are $x_k = kL/N$ for $k = 0, 1, \dots, N-1$. Define the inner product and norms as

$$(u, v) = \int_{\Omega} uv dx,$$

$$\|u\| = \sqrt{(u, u)},$$

$$\|u\|_{L^p} = \left(\int_{\Omega} |u|^p dx \right)^{1/p}, 1 \leq p < \infty.$$

Define the interpolation space $\mathcal{J}_N$ as

$$\mathcal{J}_N = \left\{ u(x) \mid u(x) = \sum_{|k| \leq N/2} \hat{u}_k e^{-ik(x + \frac{L}{2})}, N/2 \in \mathbb{Z}^+ \right\}.$$

Let $I_N : L^2(\Omega) \rightarrow \mathcal{J}_N$ be the interpolation operator:

$$I_N u(x) = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} \hat{u}_l e^{il(x + \frac{L}{2})}, \hat{u}_l = \frac{1}{2\pi} \int_{\Omega} u(x) e^{-il(x + \frac{L}{2})} dx = \frac{1}{NC_l} \sum_{k=0}^{N-1} u(x_k) e^{-il(x_k + \frac{L}{2})},$$

where $C_l = 1$ for $|l| < N/2$ and $C_l = 2$ for $|l| = N/2$. Then, $I_N u(x_k) = u(x_k)$ for $k = 0, 1, \dots, N-1$. Applying the fractional Laplacian to $I_N u(x_j)$ gives

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} I_N u(x_j) &= \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} |l|^{\alpha} \left(\frac{1}{NC_l} \sum_{k=0}^{N-1} u(x_k) e^{-il(x_k + \frac{L}{2})} \right) e^{il(x_j + \frac{L}{2})} \\ &= \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{NC_l} \sum_{k=0}^{N-1} u(x_k) |l|^{\alpha} e^{il(x_j - x_k)} = (D_2^{\alpha} \mathbf{u})_j. \end{aligned}$$

where $\mathbf{u} = (u(x_0), \dots, u(x_{N-1}))^T$.

The Fourier spectral discretization provides high-order accuracy for approximating fractional Laplacian operators in space. Using the interpolation operator I_N and the differentiation matrix D_2^{α} , fractional derivatives in the semi-discrete system can be computed efficiently via the Fast Fourier Transform (FFT) algorithm. The spatial discretization forms the foundation for constructing a conservative semi-discrete scheme in the next subsection.

2.2. Conservative Fourier Spectral Scheme

Applying the Fourier spectral method to Equations (4)–(7) yields the following semi-discrete scheme with high spatial accuracy:

$$\mathbf{u}_t = -\frac{i}{2}D_2^\alpha \mathbf{u} + i\gamma_1 \mathbf{u} \cdot \boldsymbol{\phi} + 2i\gamma_2 \mathbf{r} \cdot \mathbf{u} \cdot \boldsymbol{\phi}, \quad (15)$$

$$\boldsymbol{\phi}_t = \mathbf{v}, \quad (16)$$

$$\mathbf{v}_t = -D_2^\beta \boldsymbol{\phi} - m^2 \boldsymbol{\phi} + \gamma_1 \mathbf{r} + \gamma_2 \mathbf{r} \cdot \mathbf{r}, \quad (17)$$

$$\mathbf{r}_t = \mathbf{u} \cdot \bar{\mathbf{u}}_t + \bar{\mathbf{u}} \cdot \mathbf{u}_t, \quad (18)$$

where $\mathbf{u} = (u_0, u_1, \dots, u_{N-1})^T$, $\boldsymbol{\phi} = (\phi_0, \dots, \phi_{N-1})^T$, $\mathbf{v} = (v_0, \dots, v_{N-1})^T$, $\mathbf{r} = (r_0, \dots, r_{N-1})^T$, and $\cdot$ denotes pointwise multiplication of vectors.

Theorem 2. The scheme in (15)–(18) satisfies the semi-discrete mass and energy conservation laws:

$$\tilde{M}(t) = \|\mathbf{u}\|^2 = \tilde{M}(0), \quad (19)$$

$$\tilde{E}(t) = \left\| D_2^{\frac{\alpha}{2}} \mathbf{u} \right\|^2 + \left\| D_2^{\frac{\beta}{2}} \boldsymbol{\phi} \right\|^2 + m^2 \|\boldsymbol{\phi}\|^2 + \|\mathbf{v}\|^2 - 2\gamma_1(\boldsymbol{\phi}, \mathbf{r}) - 2\gamma_2(\boldsymbol{\phi}, \mathbf{r} \cdot \mathbf{r}) = \tilde{E}(0). \quad (20)$$

Proof. Taking the inner product of Equation (15) with $i\bar{\mathbf{u}}$ and extracting the imaginary part gives

$$\Im(\mathbf{u}_t, i\bar{\mathbf{u}}) = \Im\left(-\frac{i}{2}D_2^\alpha \mathbf{u}, i\bar{\mathbf{u}}\right) + \Im(i\gamma_1 \mathbf{u} \cdot \boldsymbol{\phi}, i\bar{\mathbf{u}}) + \Im(2i\gamma_2 \mathbf{r} \cdot \mathbf{u} \cdot \boldsymbol{\phi}, i\bar{\mathbf{u}}),$$

where $\Im(\cdot)$ denotes the imaginary part. Noting that each term on the right-hand side has zero imaginary part, we obtain

$$\Im(\mathbf{u}_t, i\bar{\mathbf{u}}) = \frac{\partial}{\partial t} \|\mathbf{u}\|^2 = 0,$$

then

$$\frac{\partial}{\partial t} \|\mathbf{u}\|^2 = 0.$$

hence $\frac{\partial}{\partial t} \|\mathbf{u}\|^2 = 0$, proving the mass conservation law (19).

Taking the inner product of Equation (15) with $2i\bar{\mathbf{u}}_t$ and extracting the real part gives

$$\begin{aligned} \Re(\mathbf{u}_t, 2i\bar{\mathbf{u}}_t) &= \Re\left(-\frac{i}{2}D_2^\alpha \mathbf{u}, 2i\bar{\mathbf{u}}_t\right) + \Re(i\gamma_1 \mathbf{u} \cdot \boldsymbol{\phi}, 2i\bar{\mathbf{u}}_t) + \Re(2i\gamma_2 \mathbf{r} \cdot \mathbf{u} \cdot \boldsymbol{\phi}, 2i\bar{\mathbf{u}}_t) \\ &= \left(D_2^{\frac{\alpha}{2}} \mathbf{u}, \frac{\partial}{\partial t} D_2^{\frac{\alpha}{2}} \bar{\mathbf{u}}\right) - 2\gamma_1(\boldsymbol{\phi}, \mathbf{u} \cdot \bar{\mathbf{u}}_t) - 2\gamma_2(\boldsymbol{\phi}, 2\mathbf{r} \cdot (\mathbf{u} \cdot \bar{\mathbf{u}}_t)), \end{aligned} \quad (21)$$

where $\Re(\cdot)$ denotes the real part and $D_2^\alpha = (D_2^{\frac{\alpha}{2}})^2$.

Taking the complex conjugate of Equation (15), multiplying by $2i\mathbf{u}_t$, and extracting the real part gives

$$\begin{aligned} \Re(\bar{\mathbf{u}}_t, 2i\mathbf{u}_t) &= \Re\left(-\frac{i}{2}D_2^\alpha \bar{\mathbf{u}}, 2i\mathbf{u}_t\right) + \Re(i\gamma_1 \bar{\mathbf{u}} \cdot \boldsymbol{\phi}, 2i\mathbf{u}_t) + \Re(2i\gamma_2 \mathbf{r} \cdot \bar{\mathbf{u}} \cdot \boldsymbol{\phi}, 2i\mathbf{u}_t) \\ &= \left(D_2^{\frac{\alpha}{2}} \bar{\mathbf{u}}, \frac{\partial}{\partial t} D_2^{\frac{\alpha}{2}} \mathbf{u}\right) - 2\gamma_1(\boldsymbol{\phi}, \bar{\mathbf{u}} \cdot \mathbf{u}_t) - 2\gamma_2(\boldsymbol{\phi}, 2\mathbf{r} \cdot (\bar{\mathbf{u}} \cdot \mathbf{u}_t)), \end{aligned} \quad (22)$$

Adding (21) and (22), and noting that $\Re(\bar{u}_t, 2iu_t) = \Re(u_t, 2i\bar{u}_t) = 0$, we obtain

$$\begin{aligned} & \left(D_2^{\frac{\alpha}{2}} u, \frac{\partial}{\partial t} D_2^{\frac{\alpha}{2}} \bar{u} \right) + \left(D_2^{\frac{\alpha}{2}} \bar{u}, \frac{\partial}{\partial t} D_2^{\frac{\alpha}{2}} u \right) - 2\gamma_1(\phi, u \cdot \bar{u}_t) - 2\gamma_1(\phi, \bar{u} \cdot u_t) \\ & \quad - 2\gamma_2(\phi, 2r \cdot (u \cdot \bar{u}_t)) - 2\gamma_2(\phi, 2r \cdot (\bar{u} \cdot u_t)) \\ & = \frac{\partial}{\partial t} \left\| D_2^{\frac{\alpha}{2}} u \right\|^2 - 2\gamma_1((\phi, u \cdot \bar{u}_t) + (\phi, \bar{u} \cdot u_t)) - 2\gamma_2(\phi, 2r \cdot (u \cdot \bar{u}_t + \bar{u} \cdot u_t)) = 0, \end{aligned} \quad (23)$$

Taking the inner product of Equation (18) with 2ϕ gives

$$2(\phi, r_t) = 2((\phi, u \cdot \bar{u}_t) + (\phi, \bar{u} \cdot u_t)). \quad (24)$$

Taking the inner product of Equation (16) with $2v_t$ and Equation (17) with $2\phi_t$ yields

$$(\phi_t, 2v_t) = (v, 2v_t) = \frac{\partial}{\partial t} \|v\|^2, \quad (25)$$

$$\begin{aligned} (v_t, 2\phi_t) & = -\left(D_2^{\beta} \phi, 2\phi_t \right) - \left(m^2 \phi, 2\phi_t \right) + (\gamma_1 r, 2\phi_t) + (\gamma_2 r \cdot r, 2\phi_t) \\ & = \frac{\partial}{\partial t} \left(-\left\| D_2^{\frac{\beta}{2}} \phi \right\|^2 - m^2 \|\phi\|^2 \right) + 2\gamma_1(\phi_t, r) + 2\gamma_2(\phi_t, r \cdot r), \end{aligned} \quad (26)$$

where $D_2^{\beta} = (D_2^{\frac{\beta}{2}})^2$. Adding Equations (23)–(26), we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\left\| D_2^{\frac{\alpha}{2}} u \right\|^2 + \left\| D_2^{\frac{\beta}{2}} \phi \right\|^2 + m^2 \|\phi\|^2 + \|v\|^2 \right) - 2\gamma_1((\phi, u \cdot \bar{u}_t) + (\phi, \bar{u} \cdot u_t)) - \\ & \quad 2\gamma_2(\phi, 2r \cdot (u \cdot \bar{u}_t + \bar{u} \cdot u_t)) - 2\gamma_1(\phi_t, r) - 2\gamma_2(\phi_t, r \cdot r) = 0, \end{aligned}$$

Using (24), the above formula simplifies to

$$\frac{\partial}{\partial t} \left(\left\| D_2^{\frac{\alpha}{2}} u \right\|^2 + \left\| D_2^{\frac{\beta}{2}} \phi \right\|^2 + m^2 \|\phi\|^2 + \|v\|^2 \right) - \frac{\partial}{\partial t} (2\gamma_1(\phi, r)) - \frac{\partial}{\partial t} (2\gamma_2(\phi, r \cdot r)) = 0,$$

which complete the proof of the energy conservation law (20). $\square$

3. A Fully Discrete Scheme for the Space-Fractional KGS System

We employ the DIRK method for temporal discretization of the semi-discrete scheme (15)–(18).

Let b_i, a_{ij} ($i, j = 1, \dots, s$) be real numbers with $a_{ij} = b_j$, $a_{ii} = b_i/2$, and $c_i = \sum_{j=1}^s a_{ij}$. For a positive even integer P , define the time step $\tau = T/P$. Let $t_n = n\tau$, and denote by u^n, ϕ^n, v^n, r^n the numerical approximations to u, ϕ, v, r at time t_n . For convenience, define

$$\begin{aligned} \tilde{f}(u, \phi, v, r) & = u_t = -\frac{i}{2} D_2^{\alpha} u + i\gamma_1 u \cdot \phi + 2i\gamma_2 r \cdot u \cdot \phi, \\ \tilde{g}(u, \phi, v, r) & = \phi_t = v, \\ \tilde{h}(u, \phi, v, r) & = v_t = -D_2^{\beta} \phi - m^2 \phi + \gamma_1 r + \gamma_2 r \cdot r, \\ \tilde{w}(u, \phi, v, r) & = r_t = u \cdot \bar{u}_t + \bar{u} \cdot u_t. \end{aligned}$$

The intermediate values of the s -stage Runge-Kutta method are:

$$\begin{cases} u_i = u^n + \tau \sum_{j=1}^s a_{ij} \tilde{f}_j, \\ \phi_i = \phi^n + \tau \sum_{j=1}^s a_{ij} \tilde{g}_j, \\ v_i = v^n + \tau \sum_{j=1}^s a_{ij} \tilde{h}_j, \\ r_i = r^n + \tau \sum_{j=1}^s a_{ij} \tilde{w}_j, \end{cases} \quad (27)$$

where $\tilde{f}_j = \tilde{f}(u_i, \phi_i, v_i, r_i)$, and $\tilde{g}_j, \tilde{h}_j, \tilde{w}_j$ are defined similarly. The numerical solution at t_{n+1} is then updated by:

$$\begin{cases} u^{n+1} = u^n + \tau \sum_{i=1}^s b_i \tilde{f}_i, \\ \phi^{n+1} = \phi^n + \tau \sum_{i=1}^s b_i \tilde{g}_i, \\ v^{n+1} = v^n + \tau \sum_{i=1}^s b_i \tilde{h}_i, \\ r^{n+1} = r^n + \tau \sum_{i=1}^s b_i \tilde{w}_i. \end{cases} \quad (28)$$

The discrete mass and modified energy are defined as:

$$M^n = \|u^n\|^2, \quad (29)$$

$$E^n = \left\| D_2^{\frac{\alpha}{2}} u^n \right\|^2 + \left\| D_2^{\frac{\beta}{2}} \phi^n \right\|^2 + m^2 \|\phi^n\|^2 + \|v^n\|^2 - 2\gamma_1(\phi^n, r^n) - 2\gamma_2(\phi^n, r^{2n}). \quad (30)$$

To ensure the fully discrete scheme satisfies discrete mass and energy conservation, we require DIRK coefficients satisfying $y_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j = 0$ for all $i, j = 1, \dots, s$. The Butcher tableaux of DIRK methods can be represented as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & b_1/2 & & \\ c_2 & b_1 & b_2/2 & \\ \vdots & \vdots & \vdots & \ddots \\ c_s & b_1 & b_2 & \cdots & b_s/2 \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

The corresponding coefficients for various stages and orders of different DIRK methods are listed in [20,21] and presented below.

(Stage, order)	b_1 b_4	b_2 b_5	b_3 b_6
(1,2)	1		
(2,2)	0.5	0.5	
(3,3)	1.351207	1.351207	−1.702414
(4,4)	2.70309412 1.8606818856	−0.53652708	2.37893931
(5,4)	−2.150611289942181 1.452223059167718	1.452223059167718 −2.150611289942181	2.3967764615489258
(6,5)	0.5080048194000274 0.5685658926458251	1.360107162294827 −1.4598520495864393	2.0192933591817224 −1.9961191839359627

Theorem 3. The scheme in (27) and (28) possesses the discrete mass and energy conservation laws

$$M^{n+1} = M^n, \quad E^{n+1} = E^n, \quad 0 \leq n \leq N-1, \quad (31)$$

if and only if the coefficient matrix $Y = (y_{ij})$ with elements $y_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j$ ($i, j = 1, \dots, s$) is zero.

Proof. Differentiating Equations (19) and (20) with respect to t gives

$$(\tilde{f}, u) = 0, \quad (32)$$

$$\begin{aligned} & \left(D_2^{\frac{\alpha}{2}} \tilde{f}, D_2^{\frac{\alpha}{2}} u \right) + \left(D_2^{\frac{\beta}{2}} \tilde{g}, D_2^{\frac{\beta}{2}} \phi \right) + m^2 (\tilde{g}, \phi) + (\tilde{h}, v) - \gamma_1 (\phi, \tilde{w}) - \gamma_1 (\tilde{g}, r) \\ & - \gamma_2 (\phi, 2r \cdot \tilde{w}) - \gamma_2 (\tilde{g}, r \cdot r) = 0. \end{aligned} \quad (33)$$

We begin with the energy conservation. Multiplying the first equation in (28) by $D_2^{\frac{\alpha}{2}}$ gives

$$D_2^{\frac{\alpha}{2}} u^{n+1} = D_2^{\frac{\alpha}{2}} u^n + \tau \sum_{i=1}^s b_i D_2^{\frac{\alpha}{2}} \tilde{f}_i. \quad (34)$$

Taking the inner product of Equations (34) with itself and combining with the first expression in Equation (27), we have

$$\begin{aligned} \|D_2^{\frac{\alpha}{2}} u^{n+1}\|^2 &= \|D_2^{\frac{\alpha}{2}} u^n\|^2 + \tau \sum_{i=1}^s (b_i D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} u^n) + \tau \sum_{j=1}^s (b_j D_2^{\frac{\alpha}{2}} \tilde{f}_j, D_2^{\frac{\alpha}{2}} u^n) \\ &+ \tau^2 \sum_{i,j=1}^s (b_i D_2^{\frac{\alpha}{2}} \tilde{f}_i, b_j D_2^{\frac{\alpha}{2}} \tilde{f}_j) \\ &= \|D_2^{\frac{\alpha}{2}} u^n\|^2 + \tau \sum_{i=1}^s \left(b_i D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} \left(u_i - \tau \sum_{j=1}^s a_{ij} \tilde{f}_j \right) \right) \\ &+ \tau \sum_{j=1}^s \left(b_j D_2^{\frac{\alpha}{2}} \tilde{f}_j, D_2^{\frac{\alpha}{2}} \left(u_j - \tau \sum_{i=1}^s a_{ji} \tilde{f}_i \right) \right) + \tau^2 \sum_{i,j=1}^s (b_i D_2^{\frac{\alpha}{2}} \tilde{f}_i, b_j D_2^{\frac{\alpha}{2}} \tilde{f}_j) \\ &= \|D_2^{\frac{\alpha}{2}} u^n\|^2 - \tau^2 \sum_{i,j=1}^s (b_i a_{ij} + b_j a_{ji} - b_i b_j) (D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} \tilde{f}_j) \\ &+ \tau \sum_{i=1}^s 2b_i (D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} u_i). \end{aligned} \quad (35)$$

Similarly, we have

$$\begin{aligned} \|D_2^{\frac{\beta}{2}} \phi^{n+1}\|^2 &= \|D_2^{\frac{\beta}{2}} \phi^n\|^2 - \tau^2 \sum_{i,j=1}^s (b_i a_{ij} + b_j a_{ji} - b_i b_j) (D_2^{\frac{\beta}{2}} \tilde{g}_i, D_2^{\frac{\beta}{2}} \tilde{g}_j) \\ &+ \tau \sum_{i=1}^s 2b_i (D_2^{\frac{\beta}{2}} \tilde{g}_i, D_2^{\frac{\beta}{2}} \phi_i), \end{aligned} \quad (36)$$

$$\begin{aligned} m^2 \|\phi^{n+1}\|^2 &= m^2 \|\phi^n\|^2 - \tau^2 \sum_{i,j=1}^s (b_i a_{ij} + b_j a_{ji} - b_i b_j) m^2 (\tilde{g}_i, \tilde{g}_j) \\ &+ \tau \sum_{i=1}^s 2b_i m^2 (\tilde{g}_i, \phi_i), \end{aligned} \quad (37)$$

$$\|v^{n+1}\|^2 = \|v^n\|^2 - \tau^2 \sum_{i,j=1}^s (b_i a_{ij} + b_j a_{ji} - b_i b_j) (\tilde{h}_i, \tilde{h}_j) + \tau \sum_{i=1}^s 2b_i (\tilde{h}_i, v_i), \quad (38)$$

$$2\gamma_1(\phi^{n+1}, r^{n+1}) = 2\gamma_1(\phi^n, r^n) - 2\gamma_1\tau^2 \sum_{i,j=1}^s (b_{ia_{ij}} + b_{ja_{ji}} - b_i b_j) (\tilde{w}_i, \tilde{g}_j) \\ + \tau \sum_{i=1}^s 2b_i (\gamma_1(\phi_i, \tilde{w}_i) + \gamma_1(\tilde{g}_i, r_i)). \quad (39)$$

$$2\gamma_2(\phi^{n+1}, r^{n+1} \cdot r^{n+1}) = 2\gamma_2(\phi^n, r^{2n}) - 2\gamma_2\tau^2 \sum_{i,j=1}^s (b_{ia_{ij}} + b_{ja_{ji}} - b_i b_j) (r_i \cdot \tilde{w}_i, \tilde{g}_j) \\ + \tau \sum_{i=1}^s 2b_i (\gamma_2(\phi_i, 2r_i \cdot \tilde{w}_i) + \gamma_2(\tilde{g}_i, r_i \cdot r_i)). \quad (40)$$

From Equation (33), it yields

$$\left(D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} u_i\right) + \left(D_2^{\frac{\beta}{2}} \tilde{g}_i, D_2^{\frac{\beta}{2}} \phi_i\right) + m^2(\tilde{g}_i, \phi_i) + (\tilde{h}_i, v_i) \\ - \gamma_1(\phi_i, \tilde{w}_i) - \gamma_1(\tilde{g}_i, r_i) - \gamma_2(\phi_i, 2r_i \cdot \tilde{w}_i) - \gamma_2(\tilde{g}_i, r_i \cdot r_i) = 0. \quad (41)$$

Summing Equations (35)–(40) and then substituting Equation (41) into the resulting sum yields

$$E^{n+1} = E^n - \tau^2 \sum_{i,j=1}^s (b_{ia_{ij}} + b_{ja_{ji}} - b_i b_j) \left(\left(D_2^{\frac{\alpha}{2}} \tilde{f}_i, D_2^{\frac{\alpha}{2}} \tilde{f}_j\right) + \left(D_2^{\frac{\beta}{2}} \tilde{g}_i, D_2^{\frac{\beta}{2}} \tilde{g}_j\right) \right) \\ + m^2(\tilde{g}_i, \tilde{g}_j) + (\tilde{h}_i, \tilde{h}_j) - 2\gamma_1(\tilde{w}_i, \tilde{g}_j) - 2\gamma_2(r_i \cdot \tilde{w}_i, \tilde{g}_j).$$

Due to the arbitrariness of the functions u_t, ϕ_t, v_t and r_t , $E^{n+1} = E^n$ holds if and only if $Y = 0$.

Regarding the law of mass conservation, a similar procedure to that outlined in Equation (35) leads to the following result:

$$\|u^{n+1}\|^2 = \|u^n\|^2 - \tau^2 \sum_{i,j=1}^s (b_{ia_{ij}} + b_{ja_{ji}} - b_i b_j) (\tilde{f}_i, \tilde{f}_j) + \tau \sum_{i=1}^s 2b_i (\tilde{f}_i, u_i), \quad (42)$$

Substituting Equation (32) into Equation (40) completes the proof. $\square$

4. Error Analysis

To establish a rigorous foundation for the subsequent theoretical analysis, we introduce the following two assumptions.

Assumption 1. (Regularity of exact solutions). Let $H^m(\Omega)$ be the Sobolev space with $m > \max(\alpha, \beta)$; assume the exact solutions $u, \phi \in C^5([0, T]; H^m(\Omega))$, and there exists a constant $M > 0$ such that

$$\|u\|_{L^\infty(\Omega \times [0, T])}, \|\phi\|_{L^\infty(\Omega \times [0, T])} \leq M.$$

Assumption 2. (Local Lipschitz condition for nonlinear terms). Under periodic boundary conditions and the given boundedness of solutions in Assumption 1, assume the nonlinear terms

$$F_1(u, \phi) = \gamma_1 u \phi + 2\gamma_2 |u|^2 u \phi, \quad F_2(u) = \gamma_1 |u|^2 + \gamma_2 |u|^4,$$

satisfy a local Lipschitz condition: there exists a constant $L = L(M) > 0$ such that for any two pairs (u_1, ϕ_1) and (u_2, ϕ_2) with $\|u_i\|_{L^\infty}, \|\phi_i\|_{L^\infty} \leq M$ ($i = 1, 2$), there are

$$\|F_1(u_1, \phi_1) - F_1(u_2, \phi_2)\| \leq L(\|u_1 - u_2\| + \|\phi_1 - \phi_2\|),$$

$$\|F_2(u_1) - F_2(u_2)\| \leq L\|u_1 - u_2\|.$$

Furthermore, the fractional Laplacian operators $(-\Delta)^{\alpha/2}$ and $(-\Delta)^{\beta/2}$ are self-adjoint and positive definite under periodic boundary conditions, ensuring the well-posedness of the spectral discretization.

Theorem 4. Let $\mathbf{e}_n$ be the global error. For the fully discrete system derived from the semi-discrete spectral system (15)–(18) via a fourth-order Runge-Kutta method, if the exact solutions $u, \phi \in C^5([0, T]; H^m(\Omega))$ with $m > \max(\alpha, \beta)$, then there exist positive constants C_1, C_2 independent of τ and N such that

$$\|\mathbf{e}_n\| \leq C_1 N^{-m} + C_2 \tau^4,$$

where N is the number of Fourier modes and m is the regularity index of the solution.

Proof. Let the projection of the exact solutions $u(x, t)$ and $\phi(x, t)$ onto grid points be $\mathbf{Q}(t) = [U(t)^T, \Phi(t)^T]^T$, where $U(t)$ and $\Phi(t)$ are the discretizations of $u(x_i, t)$ and $\phi(x_i, t)$, respectively.

Spatial error: By the approximation theory of Fourier spectral methods, if $u, \phi \in H^m(\Omega)$, the spatial truncation error satisfies

$$\|(-\Delta)^{\alpha/2} u - D_2^\alpha u_N\| \leq K_1 N^{-m} \|u\|_{H^m}, \quad \|(-\Delta)^{\beta/2} \phi - D_2^\beta \phi_N\| \leq K_1 N^{-m} \|\phi\|_{H^m},$$

where $K_1 > 0$ is a constant, and u_N, ϕ_N denote the spectral projection solutions.

Thus, the solution $\mathbf{Q}_h(t)$ of the semi-discrete system (15)–(18) satisfies

$$\|\mathbf{Q}(t) - \mathbf{Q}_h(t)\| \leq C_1 N^{-m}, \quad \forall t \in [0, T],$$

where $C_1 > 0$ depends on m and T but is independent of N .

Temporal error: Consider the ordinary differential equation system formed by the semi-discrete equations:

$$\mathbf{y}_t = \mathbf{F}(\mathbf{y}), \quad \mathbf{y} = (u, \phi, v, r)^\top,$$

where $\mathbf{F}(\mathbf{y})$ encapsulates the nonlinear coupling via the spectral operators D_2^α, D_2^β and the source terms.

The local truncation error (LTE) of the fourth-order DIRK method (Table 1) is $\mathcal{O}(\tau^5)$ for smooth solutions. Given the exact solution $\mathbf{y}(t) \in C^5([0, T])$, the LTE satisfies

$$\|\mathbf{y}(t_{n+1}) - \Phi(\mathbf{y}(t_n); \tau)\| \leq K\tau^5,$$

where $\Phi(\mathbf{y}(t_n))$ denotes the numerical solution obtained by advancing one time step of size τ from the exact solution $\mathbf{y}(t_n)$ at time t_n using the DIRK method, with $K = \sup_{t \in [0, T]} \|\mathbf{y}^{(5)}(t)\|$, which remains finite under the regularity assumption $u, \phi \in C^5([0, T]; H^m(\Omega))$.

The nonlinear operator $\mathbf{F}(\mathbf{y})$ is Lipschitz continuous in the norm $\|\cdot\|$, i.e.,

$$\|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\| \leq L\|\mathbf{y}_1 - \mathbf{y}_2\|,$$

for some $L > 0$ independent of τ . This implies the DIRK update map satisfies the stability estimate

$$\|\Phi(a; \tau) - \Phi(b; \tau)\| \leq (1 + L\tau)\|a - b\|.$$

Let $\mathbf{y}^n$ denote the numerical solution at t_n , and define the global temporal error $\mathbf{e}_n = \mathbf{y}(t_n) - \mathbf{y}^n$. By the triangle inequality,

$$\|\mathbf{e}_{n+1}\| \leq \underbrace{\|\mathbf{y}(t_{n+1}) - \Phi(\mathbf{y}(t_n); \tau)\|}_{\text{LTE}} + \underbrace{\|\Phi(\mathbf{y}(t_n); \tau) - \Phi(\mathbf{y}^n; \tau)\|}_{\text{Propagated error}}.$$

Combining the LTE bound and Lipschitz stability yields the recurrence

$$\|\mathbf{e}_{n+1}\| \leq (1 + L\tau)\|\mathbf{e}_n\| + K\tau^5.$$

Applying the discrete Grönwall inequality (with $\mathbf{e}_0 = 0$) gives [22]

$$\|\mathbf{e}_n\| \leq \frac{K}{L}(e^{Lt_n} - 1)\tau^4.$$

Since $t_n \leq T$, define $C_2 = \frac{K}{L}(e^{LT} - 1)$ to obtain the global temporal error bound

$$\|\mathbf{Q}_h(t_n) - \mathbf{Q}^n\| \leq C_2\tau^4,$$

where $\mathbf{Q}^n$ is the fully discrete solution vector.

Combined error: Decompose the global error as

$$\mathbf{e}_n = \mathbf{Q}(t_n) - \mathbf{Q}^n = [\mathbf{Q}(t_n) - \mathbf{Q}_h(t_n)] + [\mathbf{Q}_h(t_n) - \mathbf{Q}^n].$$

Applying the triangle inequality and the spatial/temporal bounds yields

$$\|\mathbf{e}_n\| \leq \|\mathbf{Q}(t_n) - \mathbf{Q}_h(t_n)\| + \|\mathbf{Q}_h(t_n) - \mathbf{Q}^n\| \leq C_1N^{-m} + C_2\tau^4.$$

Thus, the fully discrete scheme (27) and (28) is theoretically guaranteed to converge, with the convergence rate jointly determined by the order p of the time discretization method and the spatial spectral accuracy governed by the regularity index m of the solution. $\square$

Remark 1. The error estimate in Theorem 4 implies uniform convergence of the proposed method. Specifically, under the regularity assumptions $u, \phi \in H^m(\Omega)$, the constants C_1, C_2 are independent of N and τ , ensuring uniform convergence in both space and time. This uniform convergence property is crucial for reliable long-time simulations of fractional partial differential equations, as it guarantees numerical accuracy regardless of the fractional orders α and β .

5. Numerical Validation and Optical Soliton Simulations

In this section, we demonstrate the applicability and superiority of the proposed IEQ-Fourier-Dirk scheme by simulating a physically critical phenomenon: the propagation of optical solitons in a nonlinear medium with a fractional nonlocal response.

This example is motivated by recent advances in quantum optics and all-optical signal processing [23], where the accurate modeling of pulse evolution is essential for designing high-performance photonic devices, such as optical switches and logic gates [24,25]. While the propagation in a Kerr medium is classically described by the Nonlinear Schrödinger Equation (NLSE) [1], modern materials like liquid crystals, thermal media, or certain photonic crystals exhibit a nonlocal nonlinear response. In such media, the evolution of the optical field couples strongly with a material response field, which can be effectively modeled by the fractional Klein-Gordon-Schrödinger (KGS) system.

A major challenge in these applications lies in ensuring the long-term numerical stability of the simulation. Conventional numerical methods are frequently susceptible to

cumulative dissipation or dispersion errors, which can gradually distort the soliton shape and its energy over extended propagation times. The proposed scheme maintains the physical fidelity of the soliton dynamics over long-time simulations, which is an essential requirement for applications such as long-haul optical communication. The proposed scheme offers advantages in both accuracy and efficiency for simulating this physically critical phenomenon. The Fourier spectral method leverages the intrinsic definition of the fractional Laplacian in Fourier space, achieving spectral accuracy in approximating nonlocal operators and effectively avoiding the numerical dispersion typically introduced by low-order finite-difference schemes. This ensures that the material's nonlocal memory response is captured with high precision. In addition, the diagonally implicit structure of the temporal integrator permits larger time steps while preserving numerical stability, thereby reducing computational cost relative to fully implicit methods and facilitating efficient parameter exploration in device design. By applying this conservative high-order scheme to the fractional KGS model, the present work aims to provide a reliable numerical tool for investigating how the fractional orders α and β influence soliton interactions and stability, a task that remains analytically intractable yet numerically achievable with the proposed approach.

Consider an optical pulse propagating along the x -direction in a planar waveguide. Let $u(x, t)$ denote the slowly varying envelope of the electric field, and let $\phi(x, t)$ represent the nonlinear refractive index change induced by the light intensity. Following the standard derivation of the Maxwell–Bloch equations for two-level atoms under the slowly varying envelope approximation [24,26], and incorporating fractional Laplacian operators to model nonlocal diffraction and memory effects in the medium [23], one obtains a coupled system for the optical field envelope u and the material response ϕ . After appropriate nondimensionalization and inclusion of higher-order nonlinear terms [25], this system takes the form of the space-fractional KGS equations. In this example, we consider the following space-fractional KGS system:

$$iu_t - \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}}u + 2u\phi = 0, \quad (43)$$

$$\phi_{tt} + (-\Delta)^{\frac{\beta}{2}}\phi + \phi - 2|u|^2 = 0. \quad (44)$$

Here, the fractional orders α and β characterize the degree of nonlocality in the optical diffraction and in the material response, respectively. The parameters γ_1 and γ_2 are nonlinear coupling coefficients (related to the Kerr and higher-order nonlinearities), and m is a scaled resonance frequency of the medium. Compared with the space-fractional KGS Equations (1) and (2), the parameters are set to $\gamma_1 = 2$, $\gamma_2 = 0$, $m = 1$, which are representative of a silica fiber with weak nonlinearity in this example. The system conserves the mass $M(t) = \|u\|^2$, and a modified energy that accounts for the nonlocal interactions,

$$E(t) = \|u_t\|^2 + \|(-\Delta)^{\frac{\beta}{4}}\phi\|^2 + \|\phi\|^2 + \|(-\Delta)^{\frac{\alpha}{4}}u\|^2 - 4(\phi, |u|^2), \quad (45)$$

which remains invariant during propagation.

We simulate the propagation of a fundamental soliton in such a fractional medium. The initial condition is set to the exact solitary wave solution of the local KGS system:

$$u(x, t, v) = \frac{3\sqrt{2}}{4\sqrt{1-v^2}} \operatorname{sech}^2\left(\frac{1}{2\sqrt{1-v^2}}(x - vt - x_0)\right) \cdot \exp\left(i\left(vx + \frac{1-v^2+v^4}{2(1-v^2)}t\right)\right),$$

$$\phi(x, t, v) = \frac{3}{4(1-v^2)} \operatorname{sech}^2\left(\frac{1}{2\sqrt{1-v^2}}(x - vt - x_0)\right),$$

its velocity is taken as $v = 0.8$ and the initial position as $x_0 = -10$, which corresponds to a typical optical pulse in fiber experiments after nondimensionalization:

$$u_0(x) = \frac{3\sqrt{2}}{4\sqrt{1-v^2}} \operatorname{sech}^2\left(\frac{1}{2\sqrt{1-v^2}}(x-x_0)\right) \exp(ivx),$$

$$\phi_0(x) = \frac{3}{4(1-v^2)} \operatorname{sech}^2\left(\frac{1}{2\sqrt{1-v^2}}(x-x_0)\right), \phi_t(x, 0) = 0.$$

The computational domain is $[-40, 40]$ with periodic boundary conditions, discretized with $N = 512$ Fourier modes. The time step is $\tau = 0.0001$, and the simulation runs up to $t = 50$, which corresponds to a propagation distance of about 50 soliton periods.

We first verify the temporal convergence order of the scheme in (27) and (28). The temporal and spatial convergence orders are defined by

$$\operatorname{order}_\tau^{u/\phi} = \ln\left(\frac{\operatorname{Err}(\tau)}{\operatorname{Err}(\tau/2)}\right) / \ln 2, \quad \operatorname{order}_N^{u/\phi} = \ln\left(\frac{\operatorname{Err}(N/2)}{\operatorname{Err}(N)}\right) / \ln 2,$$

where Err denotes the l^2 -norm error. For cases with $\alpha \neq 2$ or $\beta \neq 2$, numerical reference solutions for u and ϕ are computed using a fine spatial grid $N = 512$ and a small time step. Table 1 presents the l^2 -norm errors and corresponding convergence orders at $t = 1$ for different α and β . The results demonstrate fourth-order temporal accuracy.

Table 1. Temporal convergence errors and orders for $N = 512$, $t = 1$.

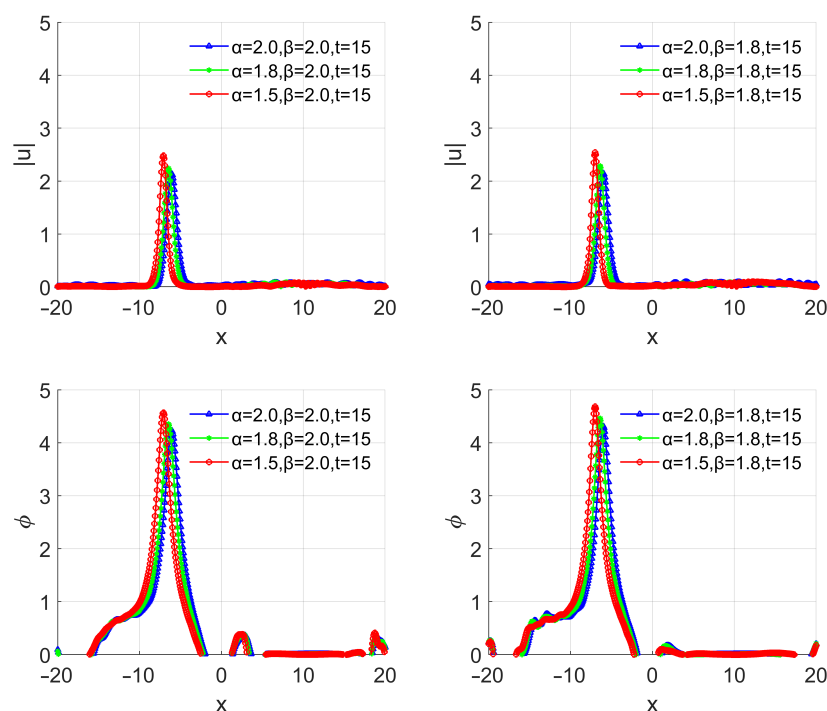
α	β	τ	$\operatorname{Err}^u(\tau)$	$\operatorname{order}_\tau^u$	$\operatorname{Err}^\phi(\tau)$	$\operatorname{order}_\tau^\phi$
2	2	0.005000	4.51×10^{-8}	—	1.74×10^{-8}	—
		0.002500	2.82×10^{-9}	4.00	1.07×10^{-9}	4.02
		0.001250	1.77×10^{-10}	4.04	6.72×10^{-11}	3.99
		0.000625	1.13×10^{-11}	3.92	4.88×10^{-12}	3.92
2	1.8	0.005000	4.87×10^{-8}	—	1.74×10^{-8}	—
		0.002500	3.05×10^{-9}	4.00	1.08×10^{-9}	4.01
		0.001250	1.91×10^{-10}	4.00	6.76×10^{-11}	4.00
		0.000625	1.22×10^{-11}	3.97	4.90×10^{-12}	3.79
1.8	2	0.005000	4.96×10^{-8}	—	2.10×10^{-8}	—
		0.002500	3.11×10^{-9}	4.00	1.29×10^{-9}	4.02
		0.001250	1.95×10^{-10}	4.00	8.06×10^{-11}	4.00
		0.000625	1.24×10^{-11}	3.98	5.66×10^{-12}	3.83
1.8	1.8	0.005000	5.42×10^{-8}	—	2.03×10^{-8}	—
		0.002500	3.40×10^{-9}	4.00	1.25×10^{-9}	4.02
		0.001250	2.13×10^{-10}	4.00	7.85×10^{-11}	4.00
		0.000625	1.34×10^{-11}	3.98	5.54×10^{-12}	3.82

Next, with a fixed time step $\tau = 0.00001$, we examine spatial errors and convergence orders for various α and β values. The results are shown in Table 2, which confirms that numerical errors decrease consistently as the number of grid points N increases.

Figure 1 shows the optical field intensity $|u|$ at $t = 15$ for different values of β while fixing $\alpha = 2$. As β decreases from 2 to 1.5, the soliton propagates more slowly and broadens significantly. This indicates that a stronger nonlocality in the material response (smaller β) leads to increased dispersion and a reduction of the group velocity, a phenomenon reminiscent of slow light in engineered structures. Figure 2 displays the effect of varying α with $\beta = 2$; a similar deceleration and broadening are observed when α is reduced, reflecting the influence of fractional diffraction on pulse spreading.

Table 2. Spatial errors and convergence orders for $\tau = 0.00001$, $t = 1$.

α	β	N	$Err^u(N)$	$order_N^u$	$Err^\phi(N)$	$order_N^\phi$
2	2	16	2.47×10^0	—	2.30×10^0	—
		32	3.21×10^{-1}	2.94	2.89×10^{-1}	2.99
		64	1.12×10^{-3}	8.16	9.85×10^{-4}	8.19
		128	5.64×10^{-7}	10.95	4.91×10^{-7}	10.97
		256	3.21×10^{-8}	4.13	2.89×10^{-8}	4.08
2	1.8	16	2.51×10^0	—	2.35×10^0	—
		32	3.45×10^{-1}	2.86	3.14×10^{-1}	2.90
		64	1.84×10^{-3}	7.55	1.62×10^{-3}	7.60
		128	1.22×10^{-6}	10.56	1.08×10^{-6}	10.55
		256	3.45×10^{-8}	5.14	3.14×10^{-8}	5.10
1.8	2	16	2.55×10^0	—	2.40×10^0	—
		32	3.88×10^{-1}	2.72	3.58×10^{-1}	2.74
		64	3.01×10^{-3}	7.01	2.78×10^{-3}	7.01
		128	2.54×10^{-6}	10.21	2.37×10^{-6}	10.20
		256	3.88×10^{-8}	6.03	3.58×10^{-8}	6.05
1.8	1.8	16	2.58×10^0	—	2.44×10^0	—
		32	4.12×10^{-1}	2.65	3.82×10^{-1}	2.68
		64	4.22×10^{-3}	6.61	3.91×10^{-3}	6.61
		128	4.87×10^{-6}	9.76	4.48×10^{-6}	9.77
		256	4.12×10^{-8}	6.88	3.82×10^{-8}	6.87


Figure 1. Wave profiles of the numerical solution for different α values at $t = 15$.

Crucially, the proposed IEQ-Fourier-DIRK scheme exactly preserves the discrete mass and energy, as demonstrated in Figure 3. Over the entire simulation up to $t = 50$, the relative changes in M^n and E^n remain below 10^{-12} (machine precision). In contrast, a non-conservative scheme would suffer from artificial dissipation, causing the soliton amplitude to decay and the waveform to distort, effects that would accumulate over long distances and corrupt the signal integrity. The conservation properties of our method guarantee that the simulated pulse dynamics faithfully represent the true physical behavior, which is essential for predicting performance in optical communication links.

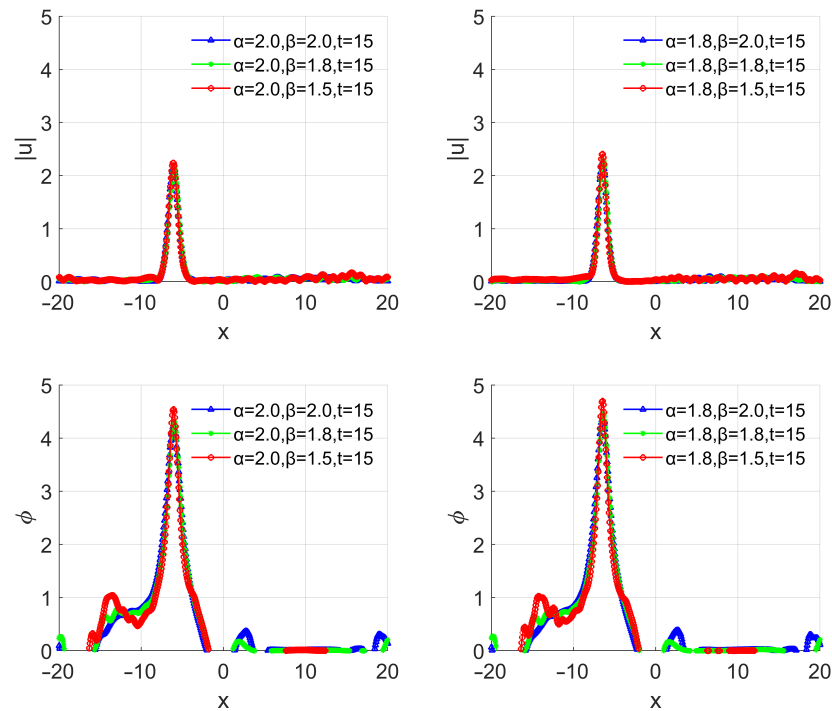


Figure 2. Wave profiles of the numerical solution for different β values at $t = 15$.

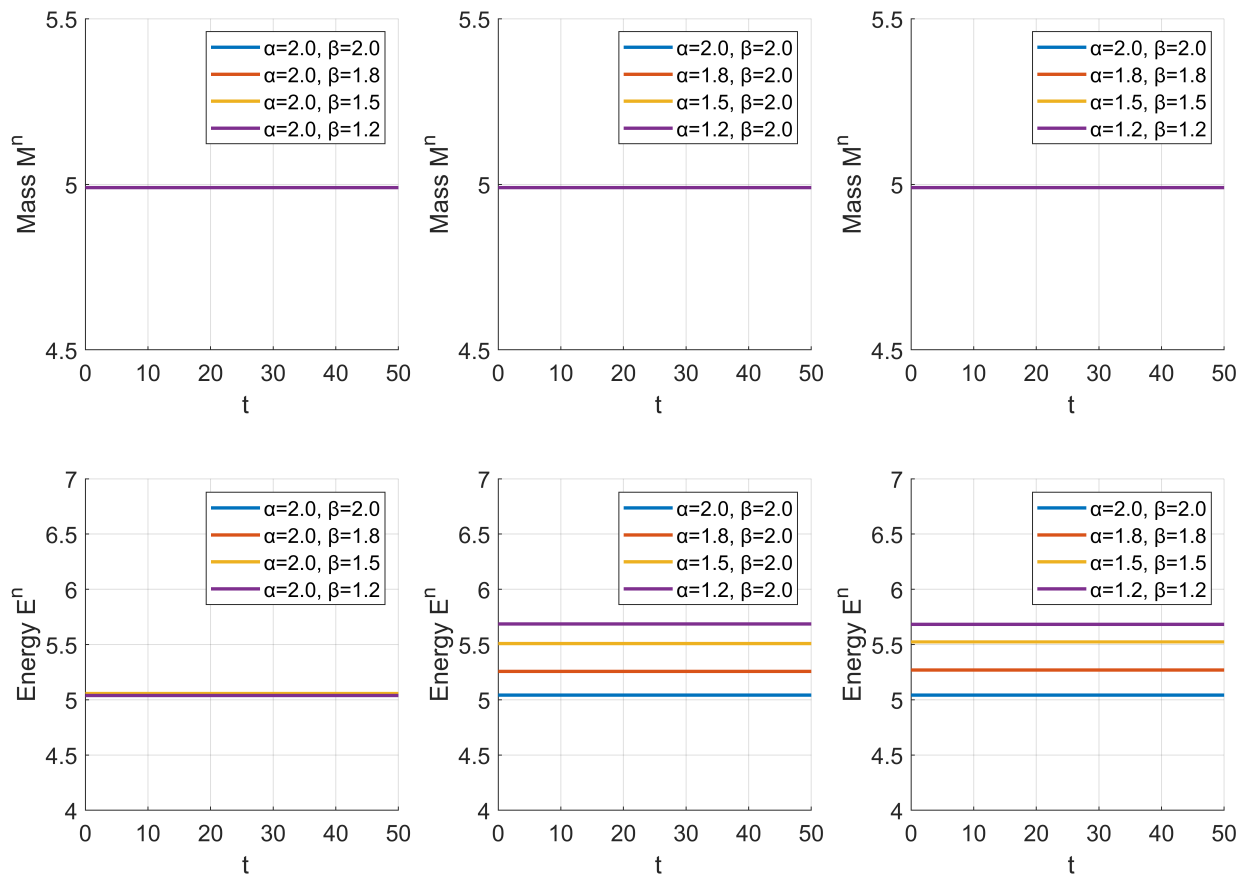


Figure 3. Discrete mass and energy for different α, β values.

6. Conclusions

In this paper, we propose a structure-preserving numerical scheme for solving the space-fractional KGS system. By integrating the IEQ reformulation, Fourier spectral dis-

cretization, and DIRK time stepping, the method achieves high-order accuracy while preserving mass and energy exactly. This work presents the systematic application of the IEQ–Fourier–DIRK framework to fractional KGS systems, accompanied by rigorous proofs of conservation for arbitrary fractional orders.

The proposed scheme is particularly well suited for long-time simulations in applications such as quantum optics and biomedical imaging, where its high accuracy and discrete conservation property offer tangible advantages. The diagonally implicit structure further ensures computational efficiency while maintaining numerical stability. Despite these strengths, the current formulation assumes periodic boundary conditions and uniform spatial grids, which points to natural directions for future work, including adaptive time integration, extension to non-periodic domains, and generalization to higher-dimensional or stochastic fractional partial differential equations.

Author Contributions: Conceptualization, W.J. and Y.F.; methodology, W.J.; software, W.J. and Y.F.; validation, W.J. and Y.L.; formal analysis, W.J.; investigation, W.J.; resources, Y.F.; data curation, W.J. and Y.F.; writing—original draft preparation, W.J.; writing—review and editing, Y.L. and Y.F.; visualization, W.J. and Y.F.; supervision, Y.L.; project administration, Y.F.; funding acquisition, Y.L. and Y.F. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of Heilongjiang Province of China (Grant No. PL2024A002), the National Natural Science Foundation of China (12571525), and the Natural Science Foundation of Zhejiang Province of China (Grant No. LQ21A010007).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Mei, L.; Liu, X.; Jiang, Y. Unconditionally stable explicit exponential methods for the Klein-Gordon-Schrödinger equations. *J. Comput. Phys.* **2025**, *533*, 113993.
2. Laskin, N. Fractional quantum mechanics. *Phys. Rev. E* **2000**, *62*, 3131–3137.
3. Laskin, N. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A* **2000**, *268*, 298–305.
4. Guo, Y.; Fu, Y. Two efficient exponential energy-preserving schemes for the fractional Klein-Gordon-Schrödinger equation. *Math. Comput. Simul.* **2023**, *209*, 169–183.
5. Laskin, N. Fractional Schrödinger equation. *Phys. Rev. E* **2002**, *66*, 056108.
6. Khalil, Q.M.L.; Biala, T.A.; Alzahrani, S.S.; Furati, K.M. Linearly implicit predictor-corrector methods for space-fractional reaction-diffusion equations with non-smooth initial data. *Comput. Math. Appl.* **2018**, *75*, 2629–2657.
7. Kilbas, A.A. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
8. Meerschaert, M.M.; Sikorskii, A. *Stochastic Models for Fractional Calculus*; De Gruyter: Berlin, Germany, 2019.
9. Wu, L.; Ma, Q.; Ding, X. Energy-preserving scheme for the nonlinear fractional Klein-Gordon-Schrödinger equation. *Math. Comput. Simul.* **2021**, *190*, 1110–1129.
10. Li, M.; Huang, C.; Zhao, Y. Fast conservative numerical algorithms for the coupled fractional Klein-Gordon-Schrödinger equation. *Numer. Algorithms* **2020**, *84*, 1081–1119.
11. Wang, J.; Xiao, A. Conservative Fourier spectral method and numerical investigation of space fractional Klein-Gordon-Schrödinger equation. *Appl. Math. Comput.* **2019**, *350*, 348–365.
12. Felmer, P.; Quaas, A.; Tan, J. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. R. Soc. Edinb. Sect. A Math.* **2012**, *142*, 1237–1262.
13. Pindza, E.; Owolabi, K.M. Fourier spectral method for higher order space fractional reaction-diffusion equations. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *40*, 112–128.
14. Yang, X.; Ju, L. Efficient linear schemes with unconditional energy stability for the phase field elastic bending energy model. *Comput. Methods Appl. Mech. Eng.* **2017**, *315*, 691–712.
15. Jiang, C.; Cai, W.; Wang, Y. A linearly implicit and local energy-preserving scheme for the sine-Gordon equation based on the invariant energy quadratization approach. *J. Sci. Comput.* **2019**, *80*, 1629–1655.

16. Gong, Y.; Zhao, J.; Wang, Q. Arbitrarily high-order unconditional energy stable schemes for thermodynamically consistent gradient flow models. *SIAM J. Sci. Comput.* **2020**, *42*, B135–B156.
17. Wang, Y.; Li, Q.; Mei, L. A linear, symmetric and energy-conservative scheme for the space-fractional Klein-Gordon-Schrödinger equations. *Appl. Math. Lett.* **2019**, *95*, 104–113.
18. Zhang, H.; Qian, X.; Song, S. Novel high-order energy-preserving diagonally implicit Runge-Kutta schemes for nonlinear Hamiltonian ODEs. *Appl. Math. Lett.* **2020**, *102*, 106091.
19. Liu, Z.; Zhang, H.; Qian, X.; Song, S. Mass and energy conservative high-order diagonally implicit Runge-Kutta schemes for nonlinear Schrödinger equations. *Appl. Math. Lett.* **2024**, *153*, 109055.
20. Feng, K.; Qin, M. *Symplectic Geometric Algorithms for Hamiltonian Systems*; Springer: Berlin/Heidelberg, Germany, 2010.
21. da Silva, C.C.; Lessig, C. Variational symplectic diagonally implicit Runge-Kutta methods for isospectral systems. *BIT Numer. Math.* **2022**, *62*, 1823–1840.
22. Hairer, E.; Nørsett, S.P.; Wanner, G. *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1993.
23. Longhi, S. Fractional Schrödinger equation in optics. *Opt. Lett.* **2015**, *40*, 1117–1120.
24. Boyd, R.W. *Nonlinear Optics*, 3rd ed.; Academic Press: San Diego, CA, USA, 2008.
25. Kivshar, Y.S.; Agrawal, G.P. *Optical Solitons: From Fibers to Photonic Crystals*; Academic Press: San Diego, CA, USA, 2003.
26. Allen, L.; Eberly, J.H. *Optical Resonance and Two-Level Atoms*; Wiley: New York, NY, USA, 1975.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.