

Article

Strongly Exposed Points of Orlicz Sequence Spaces Equipped with the p -Amemiya Norm

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Abstract

Using some new techniques, criteria for strongly exposed points of Orlicz sequence spaces generated by arbitrary Orlicz function and equipped with the p -Amemiya ($1 < p < \infty$) norm are obtained. Next, strongly exposed property of these spaces is deduced. The obtained results broaden the knowledge about these notions in Orlicz sequence spaces equipped with the p -Amemiya norm.

Keywords: Orlicz spaces; p -Amemiya norm; strongly exposed points

MSC: 46E30; 46E20; 46B45; 46B42; 46A80

1. Introduction

Let X be a real Banach space, and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X , respectively. By X^* we denote the dual space of X . For any $u \in S(X)$ we denote by $\text{Grad}(u)$ the set of all supporting functionals at x , i.e.,

$$\text{Grad}(u) = \{f \in S(X^*) : f(u) = \|u\|\}.$$

Exposed points were first defined by Straszewicz in 1935 in the case of finite-dimensional spaces (see [1]). The concept of strongly exposed points were introduced by Lindenstrauss in 1963 (see [2]). Both exposed points and strongly exposed points are basic concepts in the geometric theory of Banach spaces. There is a connection of strongly exposed points and the Radon-Nikodym property. Phelps showed in 1974 that a Banach space X has a Radon-Nikodym property if every non-empty closed, bounded convex subset is contained in a closed convex hull of its strongly exposed points (see [3]). Criteria for exposed points in classical Orlicz spaces equipped the Orlicz norm, Luxemburg nom and p -Amemiya norms were presented in [4–8]. Criteria for strongly exposed points in classical Orlicz spaces and Musielak-Orlicz spaces equipped the Orlicz norm and Luxemburg nom were given in [9–13]. The p -Amemiya norms are generalization of the Orlicz norm and Luxemburg norm. Up to now, no characterization of strong exposed points and strongly exposed property of Orlicz sequence spaces equipped with the p -Amemiya norm was known. The main aim of the paper is to complete this gap.

Definition 1 ([1]). A point $u \in S(X)$ is said to be an exposed point if there exists $f \in \text{Grad}(u)$, such that for all $v \in B(X) \setminus \{u\}$, $f(v) < f(u) = 1$. Then f is called an exposed functional of u .



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Definition 2 ([2]). A point $u \in S(X)$ is called a strongly exposed point if there exists $f \in \text{Grad}(u)$, such that for any $\{u_n\}_{n=1}^\infty \subset B(X)$, $f(u_n) \rightarrow f(u) = 1$ implies $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. Then f is called a strongly exposed functional of u .

Obviously, a strongly exposed point is an exposed point. If every $x \in S(X)$ is an (strongly) exposed point, then X has the (strongly) exposed property. In a strictly convex Banach space all points of its unit sphere are exposed, while in a locally uniformly convex Banach space all points of its unit sphere are strongly exposed (see [5,9]).

In the sequel, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

The triple $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ stands for the counting measure space and $l^0(\mu) = l^0$ denotes the space of all real sequences $u = (u(i))$. A Banach space X is called a Köthe sequence space if it is a subspace of l^0 such that

- (i) if $u \in l^0, v \in X$ and $|u(i)| \leq |v(i)|$ for $i \in \mathbb{N}$, then $u \in X$ and $\|u\| \leq \|v\|$,
- (ii) there exists $u \in X$ such that $\text{supp } u = \mathbb{N}$, where $\text{supp } u = \{i \in \mathbb{N} : u(i) \neq 0\}$.

For any map $\Phi : \mathbb{R} \rightarrow [0, \infty]$, define

$$a_\Phi = \sup\{x \geq 0 : \Phi(x) = 0\}, \quad b_\Phi = \sup\{x > 0 : \Phi(x) < \infty\}.$$

Note that if Φ is even on $\mathbb{R}$, $a_\Phi = 0$ means that Φ vanishes only at zero while $b_\Phi = \infty$ means that Φ takes only finite values.

A map $\Phi : \mathbb{R} \rightarrow [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero, Φ is even and convex on the interval $(-b_\Phi, b_\Phi)$ and Φ is left continuous at b_Φ , i.e., $\lim_{x \rightarrow b_\Phi^-} \Phi(x) = \Phi(b_\Phi)$.

For every Orlicz function Φ , we define the complementary function, in the sense of Young, $\Psi : \mathbb{R} \rightarrow [0, \infty]$ by the formula

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}.$$

Let $\Phi'_+(x)$ and $\Phi'_-(x)$ ($\Psi'_+(y)$ and $\Psi'_-(y)$) stand for the right and left derivatives of Φ (Ψ) at $x \in \mathbb{R}$ with $0 \leq x < b_\Phi$ (at $y \in \mathbb{R}$ with $0 \leq y < b_\Psi$), respectively. Here we define $\Phi'_+(b_\Phi) = \infty$ and $\Phi'_-(x) = \Phi'_+(x) = \infty$ for all $x > b_\Phi$, $\Psi'_+(b_\Psi) = \infty$ and $\Psi'_-(y) = \Psi'_+(y) = \infty$ for all $y > b_\Psi$.

For every $x, y \in \mathbb{R}$, we have the following Young Inequality:

$$|xy| \leq \Phi(x) + \Psi(y) \tag{1}$$

which reduces to an equality when $y \in \partial\Phi(x)$ if x is given, or when $x \in \partial\Psi(y)$ if y is given, and

$$\partial\Phi(x) = \begin{cases} [\Phi'_-(x), \Phi'_+(x)], & \text{for } -b_\Phi < x < b_\Phi, \\ [\Phi'_-(x), +\infty), & \text{for } x = b_\Phi \text{ and } \Phi'_-(b_\Phi) < \infty, \\ (-\infty, \Phi'_+(x)], & \text{for } x = -b_\Phi \text{ and } \Phi'_+(-b_\Phi) > -\infty. \end{cases}$$

Given any Orlicz function Φ , we define on l^0 the convex modular of u by

$$I_\Phi(u) = \sum_{i=1}^\infty \Phi(u(i)), \text{ for any } u = (u(i)) \in l^0.$$

The Orlicz sequence space l_Φ generated by an Orlicz function Φ is a linear space of real sequences defined by

$$l_{\Phi} = \{u \in l^0 : I_{\Phi}(cu) < \infty \text{ for some } c > 0 \text{ depending on } u\}.$$

The space h_{Φ} is the subspace of all order continuous elements of l_{Φ} defined by

$$h_{\Phi} = \{u \in l^0 : I_{\Phi}(cu) < \infty \text{ for any } c > 0\}.$$

The Orlicz sequence space l_{Φ} is a Köthe sequence space when it is equipped with the Orlicz norm

$$\|u\|^o = \sup_{I_{\Psi}(v) \leq 1} \sum_{i=1}^{\infty} u(i)v(i)$$

or the Luxemburg norm

$$\|u\|_{\Phi} = \inf \left\{ \lambda > 0 : I_{\Phi} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

At first sight the Orlicz and Luxemburg norms seem far from similar. In fact, in many cases the geometric properties of Orlicz spaces under each of these norms differ from each other. But in the fifties, I. Amemiya (see [14]) considered a norm defined by the formula

$$\|u\|^A = \inf_{k>0} \frac{1}{k} (1 + I_{\Phi}(ku)).$$

In 2000, Hudzik and Maligranda proved the Amemiya norm and the Orlicz norm coincide for any Orlicz function Φ , i.e., $\|\cdot\|^o = \|\cdot\|^A$ (see [15]). Moreover, it is not difficult to verify that the Luxemburg norm can be expressed by an Amemiya-like formula (see [16,17]), namely

$$\|u\|_{\Phi} = \inf_{k>0} \frac{1}{k} \max\{1, I_{\Phi}(ku)\}.$$

The only difference between the two Amemiya formulas is the function under the infimum operation (we will call it the outer function): for all $t \geq 0$, $s(t) = 1 + t$ (for the Orlicz norm) and $s(t) = \max\{1, t\}$ (for the Luxemburg norm). In [15], Hudzik and Maligranda proposed to investigate the Amemiya formula-norms generated by the outer functions of the type

$$s_p(t) = \begin{cases} (1 + t^p)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ \max\{1, t\}, & \text{for } p = \infty, \end{cases}$$

and for any $u \in l_{\Phi}$,

$$\|u\|_{\Phi,p} = \inf_{k>0} \frac{1}{k} s_p(I_{\Phi}(ku)) \quad (1 \leq p \leq \infty).$$

We obtain a family of topologically equivalent norms (called the p -Amemiya norm), indexed by $1 \leq p \leq \infty$ and satisfying the inequalities

$$\|u\|_{\Phi} = \|u\|_{\Phi,\infty} \leq \|u\|_{\Phi,p} \leq \|u\|_{\Phi,p'} \leq \|u\|_{\Phi,1} = \|u\|^o \leq 2^{\frac{1}{p}} \|u\|_{\Phi,p},$$

for all $1 \leq p' \leq p \leq \infty$.

The first paper constituting some basic and crucial results allowing further research and containing the complete characterization of rotundity and extreme points in Orlicz spaces equipped with the p -Amemiya norms ($1 \leq p \leq \infty$) was written by Cui et al. (see [17]). After that, an intensive development of research connected with Orlicz spaces equipped with the p -Amemiya norm have taken place. Necessary and sufficient conditions for non-squareness properties, smoothness, midpoint local uniform rotundity, local uniform rotundity, etc., were presented in [18–23]. These results broaden the knowledge about the geometry of these spaces.

In order to simplify notation, the Orlicz sequence spaces equipped with p -Amemiya norms are denoted by $l_{\Phi,p} = (l_{\Phi}, \|\cdot\|_{\Phi,p})$, $h_{\Phi,p} = (h_{\Phi}, \|\cdot\|_{\Phi,p})$.

An interval $[a, b]$ is called a structural affine interval of Φ , or simply a SAI of Φ , provided that Φ is affine on $[a, b]$ and it is not affine on either $[a - \varepsilon, b]$ or $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let $([a_i, b_i])_i$ be all SAIs of Φ . We call $SC_{\Phi} = \mathbb{R} \setminus (\cup_{i=1}^{\infty} (a_i, b_i))$ the set of strictly convex points of Φ . Denote

$$A' = \bigcup \{a_i : \text{SAI}(a_i, b_i) \text{ with } \Phi'_-(a_i) = \Phi'_+(a_i)\},$$

$$B' = \bigcup \{b_i : \text{SAI}(a_i, b_i) \text{ with } \Phi'_-(b_i) = \Phi'_+(b_i)\},$$

$$A = \bigcup \{a_i : \text{SAI}(a_i, b_i) \text{ with } \Phi'_-(a_i) < \Phi'_+(a_i)\},$$

$$B = \bigcup \{b_i : \text{SAI}(a_i, b_i) \text{ with } \Phi'_-(b_i) < \Phi'_+(b_i)\}.$$

We say an Orlicz function Φ satisfies the Δ_2 -condition at zero ($\Phi \in \Delta_2(0)$, for short) if there are positive constants k and $0 < x_0 < \infty$ with $0 < \Phi(x_0) < \infty$ such that $\Phi(2x) \leq k\Phi(x)$ for $|x| \leq x_0$.

For any Orlicz function Φ , $\Phi \in \Delta_2(0)$, we also write that its complementary function Ψ satisfies the ∇_2 -condition at zero, i.e., $\Psi \in \nabla_2(0)$. If $\Phi \in \Delta_2(0)$ and $\Psi \in \Delta_2(0)$, then we write $\Phi \in \Delta_2(0) \cap \nabla_2(0)$. It is not difficult to prove that $l_{\Phi} = h_{\Phi} \Leftrightarrow \Phi \in \Delta_2(0) \cap \nabla_2(0)$ (see [24]).

By $\text{sgn}x$ we will denote the sign function defined on $\mathbb{R}$. This function $\text{sgn}x$ is equal to 1 for $x > 0$, -1 for $x < 0$ and 0 for $x = 0$.

Further details about Orlicz spaces, we refer to [24–27]. Some fundamental topological and geometrical properties of Orlicz spaces equipped with p -Amemiya norms have been investigated in [17,22].

2. Auxiliary Results

In this section we will present a few auxiliary definitions and results that will be used in the main part of the paper.

Let $1 \leq p \leq \infty$, for any $u \in l_{\Phi,p}$, $i < j$, we denote

$$[u]_j^i = (0, \dots, 0, u(i+1), \dots, u(j), 0, \dots),$$

$$[u]_j = (u(1), u(2), \dots, u(j), 0, \dots).$$

By Theorem 1.43 in [24] and equivalence of p -Amemiya ($1 \leq p \leq \infty$) norms, we have

$$\lim_{n \rightarrow \infty} \|u - [u]_n\|_{\Phi,p} = \theta_{\Phi}(u) = \inf \left\{ \lambda > 0, I_{\Phi} \left(\frac{u}{\lambda} \right) < \infty \right\}.$$

For any $u \in l_{\Phi,p}$, define

$$k_p^*(u) = \inf \{k \geq 0 : I_{\Phi}^{p-1}(ku) \cdot I_{\Psi}(\Phi'_+(ku)) \geq 1\} \text{ (with } \inf \emptyset = \infty),$$

$$k_p^{**}(u) = \sup \{k \geq 0 : I_{\Phi}^{p-1}(ku) \cdot I_{\Psi}(\Phi'_+(ku)) \leq 1\}.$$

It is evident that $k_p^*(u) \leq k_p^{**}(u)$ for every $u \in l_{\Phi,p} \setminus \{0\}$. Set $K_p(u) = \{0 < k < \infty : k_p^*(u) \leq k \leq k_p^{**}(u)\}$.

Lemma 1 ([17]). *For any $1 \leq p \leq \infty$ the set of those k 's at which the p -Amemiya norm of an element $u \in l_{\Phi,p} \setminus \{0\}$ is attained is equal to the interval $K_p(u)$.*

Lemma 2 ([17]). Let $1 < p < \infty$, the interval of norm attainable of all $u \in l_{\Phi,p} \setminus \{0\}$ is nonempty, closed and bounded (in fact, equal to $[k_p^*(u), k_p^{**}(u)]$) if and only if Φ is not linear on $[0, \infty)$.

Lemma 3 ([22]). Let $1 \leq p < \infty$. For every $u \in l_{\Phi,p} \setminus \{0\}$, if $K_p(u) = \emptyset$, then $\|u\|_{\Phi,p} = b_{\Psi} \sum_{i=1}^{\infty} |u(i)|$ where $b_{\Psi} = \lim_{x \rightarrow \infty} \Phi'_+(x) = \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x}$.

Lemma 4 ([19]). Let $1 \leq p \leq \infty$. If $u \in B(l_{\Phi,p}) \setminus \{0\}$, then $I_{\Phi}(u) \leq \|u\|_{\Phi,p}$.

We say that $\varphi \in (l_{\Phi,p})^*$ is a singular functional, or simply, $\varphi \in F$, if $\varphi(h_{\Phi}) = \{0\}$.

Lemma 5 ([22]). Let Φ, Ψ be Orlicz functions complementary to each other in the sense of Young that take finite values only and let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If the p -Amemiya norm is k^* -finite, then $f \in (l_{\Phi,p})^* \setminus \{0\}$ has the unique decomposition $f = v + \varphi$ where $v \in l_{\Psi,q}$, $\varphi \in F$, and

$$\|f\| = \|f\|_{\Psi,q} = \inf_{l > 0} \frac{1}{l} s_q(I_{\Psi}(lv) + l\|\varphi\|).$$

Lemma 6 ([22]). Let $1 < p, q < \infty$, $u \in l_{\Phi,p} \setminus \{0\}$ and $K_p(u) \neq \emptyset$. $v \in S(l_{\Psi,q})$ is a supporting functional of u if and only if:

- (i) $v(i) = \frac{w(i)}{\|w\|_{\Psi,q}} \cdot \text{sgn}(u(i))$, for some w satisfying $\Phi'_-(k|u(i)|) \leq w(i) \leq \Phi'_+(k|u(i)|)$, $i \in \mathbb{N}$, $k \in K_p(u)$,
- (ii) $I_{\Phi}^{p-1}(ku) \cdot I_{\Psi}(w) = 1$.

Lemma 7 ([20]). Let $1 < p < \infty$, $l_{\Phi,p}$ is strictly convex if and only if $a_{\Phi} = 0$ and Φ is strictly convex on the interval $[0, \pi_{\Phi,p}(\frac{1}{2})]$ where

$$\pi_{\Phi,p}\left(\frac{1}{2}\right) = \sup \left\{ t > 0 : (\Phi(t))^{p-1} \cdot \Psi(\Phi'_+(t)) \leq \frac{1}{2^p} \right\}.$$

Lemma 8 ([23]). Let $1 < p < \infty$, $l_{\Phi,p}$ is locally uniformly convex if and only if

- (i) $\Phi \in \Delta_2(0)$ and $\Psi \in \Delta_2(0)$ (i.e., $\Phi \in \Delta_2(0) \cap \nabla_2(0)$).
- (ii) Φ is strictly convex on the interval $\left[0, \pi_{\Phi,p}(\frac{1}{2})\right]$.

Lemma 9 ([22]). Let $1 < p < \infty$, $\Phi \in \Delta_2(0)$ and $u \in l_{\Phi,p} \setminus \{0\}$. For any $i \in \mathbb{N}$, if $u_n(i) \rightarrow u(i)$ (as $n \rightarrow \infty$) and

$$\lim_{i_0 \rightarrow \infty} \sup_n \sum_{i > i_0} \Phi(u_n(i)) = 0,$$

then $\|u_n - u\|_{\Phi,p} \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 10 ([28]). If $s_p(x) = (1 + x^p)^{\frac{1}{p}}$, $s_q(y) = (1 + y^q)^{\frac{1}{q}}$, for any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and the Hölder equality $x + y = s_p(x)s_q(y) = (1 + x^p)^{\frac{1}{p}}(1 + y^q)^{\frac{1}{q}}$ for all $0 < x, y < \infty$ holds true, it is necessary and sufficient that $x^{\frac{1}{q}}y^{\frac{1}{p}} = 1$ (i.e., $x^{p-1}y = 1$ or $xy^{q-1} = 1$).

Lemma 11 ([22]). (Minkowski Inequality) Let $1 \leq q < \infty$. For any sequences $\{\xi_k\}, \{\eta_k\} \subset \mathbb{R}$, we have

- (i) $(\sum_k |\xi_k + \eta_k|^q)^{\frac{1}{q}} \leq (\sum_k |\xi_k|^q)^{\frac{1}{q}} + (\sum_k |\eta_k|^q)^{\frac{1}{q}}$.
- (ii) $(1 + (x + y)^q)^{\frac{1}{q}} \leq (1 + x^q)^{\frac{1}{q}} + y$ for all $x, y \geq 0$.

Lemma 12 ([8]). Let $1 < p < \infty$, $u \in S(l_{\Phi,p})$ and $K_p(u) = \emptyset$. u is an exposed point if and only if $\text{card}(\text{supp}(u)) = 1$.

Lemma 13 ([8]). Let $1 < p < \infty$, $u \in S(l_{\Phi,p})$ and $K_p(u) \neq \emptyset$. u is an exposed point of $B(l_{\Phi,p})$ if and only if:

- (i) $K_p(u)$ is a singleton set, i.e., $K_p(u) = \{k_u\}$.
- (ii) $\text{card}(\text{supp}(u)) = 1$ or $\text{card}\{i \in \mathbb{N} : k_u u(i) \in (\mathbb{R} \setminus SC_{\Phi})\} \leq 1$.
- (iii) $\text{card}\{i \in \mathbb{N} : k_u u(i) \in A'\} \cdot \text{card}\{i \in \mathbb{N} : k_u u(i) \in B'\} = 0$.
- (iv) If $I_{\Phi}^{p-1}(k_u u) \cdot I_{\Psi}(\Phi'_-(k_u u)) = 1$, then

$$\text{card}\{i \in \mathbb{N} : k_u u(i) \in A'\} \cdot \text{card}\{i \in \mathbb{N} : k_u u(i) \in B\} = 0.$$

If $I_{\Phi}^{p-1}(k_u u) \cdot I_{\Psi}(\Phi'_+(k_u u)) = 1$, and $\theta_{\Phi}(k_u u) < 1$, then

$$\text{card}\{i \in \mathbb{N} : k_u u(i) \in A\} \cdot \text{card}\{i \in \mathbb{N} : k_u u(i) \in B'\} = 0.$$

3. Main Results

Theorem 1. Let $1 < p < \infty$, $u \in S(l_{\Phi,p})$ and $K_p(u) = \emptyset$. u is a strongly exposed point if and only if $\text{card}(\text{supp}(u)) = 1$.

Proof. Necessity. Since a strongly exposed point is an exposed point, by Lemma 12, $\text{card}(\text{supp}(u)) = 1$ is necessary.

Sufficiency. Without loss of generality, we assume $u = (u(1), 0, \dots)$ and $u(1) > 0$. By $K_p(u) = \emptyset$ and Lemma 3, $\|u\|_{\Phi,p} = b_{\Psi} u(1)$ where $b_{\Psi} = \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x}$. Thus $v = (b_{\Psi}, 0, \dots)$ is the supporting functional of u . If there is $\{u_n\}_{n=1}^{\infty} \subset B(l_{\Phi,p})$ that satisfies

$$\langle v, u_n \rangle = b_{\Psi} u_n(1) \rightarrow b_{\Psi} u(1),$$

then $u_n(1) \rightarrow u(1)$ as $n \rightarrow \infty$.

We will consider the following two cases separately.

(I) There are infinite numbers of n such that $K_p(u_n) = \emptyset$. Passing to a subsequence of $\{u_n\}$, if necessary, we may assume that for any $n \in \mathbb{N}$, $K_p(u_n) = \emptyset$. By Lemma 3, we have

$$\begin{aligned} 1 \geq \|u_n\|_{\Phi,p} &= b_{\Psi} \sum_{i=1}^{\infty} |u_n(i)| \\ &= b_{\Psi} |u_n(1)| + b_{\Psi} \sum_{i=2}^{\infty} |u_n(i)| \\ &\rightarrow 1 + b_{\Psi} \sum_{i=2}^{\infty} |u_n(i)| \end{aligned}$$

as $n \rightarrow \infty$. Therefore, $\sum_{i=2}^{\infty} |u_n(i)| \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$\begin{aligned} \|u - u_n\|_{\Phi,p} &= \inf_{k>0} \frac{1}{k} (1 + I_{\Phi}^p(k(u - u_n)))^{\frac{1}{p}} \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{k} (1 + I_{\Phi}^p(k(u - u_n)))^{\frac{1}{p}} \\ &= \lim_{k \rightarrow \infty} \left(\frac{\Phi(k(u(1) - u_n(1)))}{k} + \sum_{i=2}^{\infty} \frac{\Phi(k u_n(i))}{k} \right) \\ &= \lim_{k \rightarrow \infty} \left(\frac{\Phi(k(u(1) - u_n(1)))}{k |u(1) - u_n(1)|} |u(1) - u_n(1)| + \sum_{i=2}^{\infty} \frac{\Phi(k u_n(i))}{k |u_n(i)|} |u_n(i)| \right) \\ &= b_{\Psi} |u(1) - u_n(1)| + b_{\Psi} \sum_{i=2}^{\infty} |u_n(i)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus, u is a strongly exposed point of $S(l_{\Phi,p})$.

(II) There are infinite numbers of n such that $K_p(u_n) \neq \emptyset$. Passing to a subsequence of $\{u_n\}$, if necessary, we may assume that for any $n \in \mathbb{N}$, $K_p(u_n) \neq \emptyset$.

Take $k_n \in K_p(u_n)$. We will show that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Otherwise, without loss of generality, we may assume that $k_n \rightarrow k_0$ as $n \rightarrow \infty$. By the definition of $\|u\|_{\Phi,p}$, we get

$$\begin{aligned} \|u\|_{\Phi,p} &< \frac{1}{k_0} (1 + I_{\Phi}^p(k_0 u))^{\frac{1}{p}} \\ &= \frac{1}{k_0} (1 + (\Phi(k_0 u(1)))^p)^{\frac{1}{p}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{k_n} (1 + (\Phi(k_n u_n(1)))^p)^{\frac{1}{p}} \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{k_n} (1 + I_{\Phi}^p(k_n u_n))^{\frac{1}{p}} \\ &= \lim_{n \rightarrow \infty} \|u_n\|_{\Phi,p} \leq 1 = \|u\|_{\Phi,p} \end{aligned}$$

a contradiction.

Since,

$$\begin{aligned} 1 &\geq \|u_n\|_{\Phi,p} = \frac{1}{k_n} (1 + I_{\Phi}^p(k_n u_n))^{\frac{1}{p}} \\ &= \frac{1}{k_n} \left(1 + \left(\Phi(k_n u_n(1)) + \sum_{i \geq 2} \Phi(k_n u_n(i)) \right)^p \right)^{\frac{1}{p}} \\ &\geq \frac{1}{k_n} (1 + (\Phi(k_n u_n(1)))^p)^{\frac{1}{p}} \geq \|[u_n]_1\|_{\Phi,p} \rightarrow \|u\|_{\Phi,p} = 1. \end{aligned}$$

Therefore, $\sum_{i=2}^{\infty} \Phi(k_n u_n(i)) \rightarrow 0$ as $n \rightarrow \infty$. Thus, we obtain

$$\|u_n - [u_n]_1\|_{\Phi,p} \leq \frac{1}{k_n} \left(1 + \left(\sum_{i \geq 1} \Phi(k_n u_n(i)) \right)^p \right)^{\frac{1}{p}} \rightarrow 0$$

as $n \rightarrow \infty$. Hence,

$$\|u - u_n\|_{\Phi,p} \leq \|u - [u_n]_1\|_{\Phi,p} + \|u_n - [u_n]_1\|_{\Phi,p} \rightarrow 0$$

as $n \rightarrow \infty$. Thus, u is a strongly exposed point of $S(l_{\Phi,p})$. $\square$

Theorem 2. Let $1 < q < \infty$, $v \in S(l_{\Psi,q})$ and $\theta_{\Psi}(v) < 1$. If there is $\{u_n\}_{n=1}^{\infty} \subset B(l_{\Phi,p})$ such that $\langle v, u_n \rangle \rightarrow 1$ as $n \rightarrow \infty$, then

$$\lim_{i_0 \rightarrow \infty} \sup_n \sum_{i > i_0} \Phi(u_n(i)) = 0. \quad (2)$$

Proof. By $v \in S(l_{\Psi,q})$, $\{u_n\}_{n=1}^{\infty} \subset B(l_{\Phi,p})$ and $\langle v, u_n \rangle \rightarrow 1$ as $n \rightarrow \infty$, we get $\|u_n\|_{\Phi,p} \rightarrow 1$ as $n \rightarrow \infty$. If there is $i_0 \nearrow \infty$, such that $\sum_{i > i_0} \Phi(u_n(i)) \geq \delta_0$. We will consider the following two cases separately.

(I) $K_p(u_n) \neq \emptyset$ ($n = 1, 2, \dots$).

Take $k_n \in K_p(u_n)$, $l \in K_q(v)$. By $\{u_n\}_{n=1}^{\infty} \subset B(l_{\Phi,p})$, $v \in S(l_{\Psi,q})$, we get $k_n \geq 1$, $l \geq 1$. Since $\theta_{\Psi}(v) < 1$, there exists $\alpha > 0$ such that $I_{\Psi}(l(1 + \alpha)v) < \infty$. For $0 < \varepsilon < \frac{\alpha \delta_0}{2}$, take $j_0 \in \mathbb{N}$ large enough satisfying $\sum_{i > j_0} \Psi(l(1 + \alpha)v(i)) < \varepsilon$. Thus,

$$\begin{aligned}
1 &\leftarrow \frac{1}{k_n l} \langle k_n u_n, l v \rangle \\
&= \frac{1}{k_n l} \left(\sum_{i=1}^{j_0} k_n u_n(i) l v(i) + \frac{1}{1+\alpha} \sum_{i>j_0} k_n u_n(i) l (1+\alpha) v(i) \right) \\
&\leq \frac{1}{k_n l} \left(\sum_{i=1}^{j_0} \Phi(k_n u_n(i)) + \sum_{i=1}^{j_0} \Psi(l v(i)) \right. \\
&\quad \left. + \frac{1}{1+\alpha} \left(\sum_{i>j_0} \Phi(k_n u_n(i)) + \sum_{i>j_0} \Psi(l (1+\alpha) v(i)) \right) \right) \\
&\leq \frac{1}{k_n l} \left(I_\Phi(k_n u_n) + I_\Psi(l v) - \frac{\alpha}{1+\alpha} \sum_{i>j_0} \Phi(k_n u_n(i)) + \frac{1}{1+\alpha} \sum_{i>j_0} \Psi(l (1+\alpha) v(i)) \right) \\
&\leq \frac{1}{k_n} s_p(I_\Phi(k_n u_n)) \frac{1}{l} s_q(I_\Psi(l v)) - \frac{1}{k_n l} \frac{\alpha}{1+\alpha} k_n \delta_0 + \frac{1}{k_n l} \frac{\varepsilon}{1+\alpha} \\
&\rightarrow 1 - \frac{1}{k_n l} \frac{\alpha}{1+\alpha} k_n \delta_0 + \frac{1}{k_n l} \frac{\varepsilon}{1+\alpha} \\
&< 1 - \frac{1}{l} \frac{\alpha \delta_0}{1+\alpha} \left(1 - \frac{1}{2k_n} \right) < 1 - \frac{\alpha \delta_0}{2l(1+\alpha)}
\end{aligned}$$

as $n \rightarrow \infty$, this is a contradiction.

(II) $K_p(u_n) = \emptyset$ ($n = 1, 2, \dots$).

By Lemma 3, We have $\|u_n\|_{\Phi,p} = b_\Psi \sum_{i=1}^\infty |u_n(i)|$. By $\theta_\Psi(v) < 1$, there is $\lambda > 0$ and $i_0 \in \mathbb{N}$ such that

$$\sum_{i>i_0}^\infty \Psi\left(\frac{v(i)}{1-\lambda}\right) < \infty.$$

So, for any $i > i_0$, $\frac{v(i)}{1-\lambda} < b_\Psi$. From

$$\Phi(u_n(i)) = \int_0^{|u_n(i)|} \Phi'_+(t) dt \leq \int_0^{|u_n(i)|} b_\Psi dt = b_\Psi |u_n(i)|.$$

We obtain $\sum_{i>i_n} b_\Psi |u_n(i)| \geq \sum_{i>i_n} \Phi(u_n(i)) \geq \delta_0$ for $i_n > i_0$. Thus,

$$\begin{aligned}
1 \leftarrow \langle v, u_n \rangle &= \sum_{i=1}^{i_n} v(i) u_n(i) + (1-\lambda) \sum_{i \geq i_n} \frac{v(i)}{1-\lambda} u_n(i) \\
&\leq b_\Psi \sum_{i=1}^{i_n} |u_n(i)| + (1-\lambda) \sum_{i \geq i_n} b_\Psi |u_n(i)| \\
&= b_\Psi \sum_{i=1}^\infty |u_n(i)| - \lambda \sum_{i \geq i_n} b_\Psi |u_n(i)| \\
&\leq \|u_n\|_{\Phi,p} - \lambda \delta_0 \rightarrow 1 - \lambda \delta_0
\end{aligned}$$

as $n \rightarrow \infty$, this is a contradiction. $\square$

Theorem 3. Let $1 < p < \infty$, if $\Phi \notin \Delta_2(0)$ and the p -Amemiya norm $\|\cdot\|_{\Phi,p}$ is k_p^* -finite, then $S(l_{\Phi,p})$ has no strongly exposed point.

Proof. Let $u \in S(l_{\Phi,p})$. Since $k_p^*(u) < \infty$, take $k \in K_p(u)$, then $\theta_\Phi(ku) \leq 1$. We will consider the following two cases separately.

(I) $\theta_\Phi(ku) = 1$.

For $\varepsilon > 0$, $j \in \mathbb{N}$, by the definition of $\theta_\Phi(ku)$, we have $\sum_{i=j}^\infty \Phi\left(\frac{ku(i)}{1-\varepsilon}\right) = \infty$. Take $0 = n_0 < n_1 < n_2 < \dots$, such that

$$I_\Phi\left(\frac{k[u]_{n_j}^{n_{j-1}}}{1-\frac{1}{j}}\right) = \sum_{i=n_{j-1}+1}^{n_j} \Phi\left(\frac{ku(i)}{1-\frac{1}{j}}\right) > 1 \quad (j = 2, 3, \dots).$$

By the definition of $\|\cdot\|_{\Phi,\infty}$, we obtain

$$\|[u]_{n_j}^{n_{j-1}}\|_{\Phi,\infty} \geq \frac{1}{k}\left(1 - \frac{1}{j}\right).$$

Set $u_j = u - [u]_{n_j}^{n_{j-1}}$, then $u_j \in B(l_{\Phi,p})$. Since $\Phi \notin \Delta_2(0)$, the supporting functional of u must be in $l_{\Psi,q} \oplus F$. Suppose $f = (v + \varphi) \in \text{Grad}(u)$ where $v \in l_{\Psi,q}$, $\varphi \in F$. We have

$$\begin{aligned} 1 \geq f(u_j) &= \langle v, u - [u]_{n_j}^{n_{j-1}} \rangle + \varphi(u - [u]_{n_j}^{n_{j-1}}) \\ &\geq \sum_{i=1}^{n_{j-1}} v(i)u(i) + \varphi(u) \\ &\rightarrow \langle v, u \rangle + \varphi(u) = f(u) = 1 \end{aligned}$$

as $j \rightarrow \infty$. However,

$$\|u - u_j\|_{\Phi,p} = \|[u]_{n_j}^{n_{j-1}}\|_{\Phi,p} \geq \|[u]_{n_j}^{n_{j-1}}\|_{\Phi,\infty} \geq \frac{1}{k}\left(1 - \frac{1}{j}\right) \rightarrow \frac{1}{k}$$

as $j \rightarrow \infty$. This shows that u is not a strongly exposed point of $S(l_{\Phi,p})$.

(II) $\theta_\Phi(ku) < 1$.

Take $w \in l_{\Phi,p}$ such that $I_\Phi(w) < \infty$ and $\theta_\Phi(ku - w) \neq 0$ (indeed, if $\theta_\Phi(ku) = 0$, we take $w \in S(l_{\Phi,p})$ with $\theta_\Phi(w) \neq 0$, if $\theta_\Phi(ku) \neq 0$, we take $w = 0$). Since $\Phi \notin \Delta_2(0)$, there exists $\varphi_0 \in S(F)$ such that $\varphi_0(ku - w) \neq 0$. Set

$$u_n = \left(u(1), \dots, u(n), \frac{1}{k}w(n+1), \frac{1}{k}w(n+2), \dots\right).$$

By Minkowski Inequality, we have

$$\begin{aligned} \|u_n\|_{\Phi,p} &\leq \frac{1}{k}(1 + I_\Phi^p(ku_n))^{\frac{1}{p}} \\ &= \frac{1}{k}\left(1 + \left(\sum_{i=1}^n \Phi(ku(i)) + \sum_{i=n+1}^\infty \Phi(w(i))\right)^p\right)^{\frac{1}{p}} \\ &\leq \frac{1}{k}\left(1 + \left(\sum_{i=1}^n \Phi(ku(i))\right)^p\right)^{\frac{1}{p}} + \frac{1}{k}\sum_{i=n+1}^\infty \Phi(w(i)) \\ &\leq \frac{1}{k}(1 + I_\Phi^p(ku))^{\frac{1}{p}} + \frac{1}{k}\sum_{i=n+1}^\infty \Phi(w(i)) \\ &\rightarrow \|u\|_{\Phi,p} \end{aligned}$$

as $n \rightarrow \infty$ where $\sum_{i=n+1}^\infty \Phi(w(i)) \rightarrow 0$ is deduced from $I_\Phi(w) < \infty$. By $\theta_\Phi(ku) < 1$ and Lemma 8 in [22], we have $\text{Grad}(u) \subset l_{\Psi,q}$, take $f = v \in \text{Grad}(u)$, then

$$\begin{aligned} 1 \geq f(u_n) &= \langle v, u_n \rangle \\ &= \sum_{i=1}^n u(i)v(i) + \frac{1}{k} \sum_{i=n+1}^{\infty} w(i)v(i) \\ &\rightarrow \langle v, u \rangle = f(u) = 1 \end{aligned}$$

as $n \rightarrow \infty$. But

$$\|u - u_n\|_{\Phi,p} \geq |\varphi_0(u - u_n)| = \left| \varphi_0\left(u - \frac{1}{k}w\right) \right| > 0.$$

So, u is not a strongly exposed point of $S(l_{\Phi,p})$. $\square$

Theorem 4. Let $1 < p < \infty$, $u \in S(l_{\Phi,p})$ and $K_p(u) \neq \emptyset$. u is a strongly exposed point of $B(l_{\Phi,p})$ if and only if

(i) u is an exposed point.

(ii) $\Phi \in \Delta_2(0)$.

(iii) u has a supporting functional $v \in l_{\Psi,q}$ with $\theta_{\Psi}(v) < 1$.

Proof. Necessity. Since a strongly exposed point is an exposed point, (i) is trivial. By Theorem 3, $\Phi \in \Delta_2(0)$ is necessary. We only need to prove (iii). If (iii) does not hold, then for any supporting functional $v \in l_{\Psi,q}$, $\theta_{\Psi}(v) = 1$, i.e., for and any $\tau > 0$, $I_{\Psi}((1 + \tau)v) = \infty$. By Theorem 1.43 in [24] and equivalence of p -Amemiya norms, we have

$$\lim_{n \rightarrow \infty} \|v - [v]_n\|_{\Psi,q} = \theta_{\Psi}(v) = 1.$$

Since $\|v - [v]_n\|_{\Psi,q} = \sup_{\|u\|_{\Phi,p}=1} \langle u, v - [v]_n \rangle$, there exists $\{u_n\}_{n=1}^{\infty} \subset S(l_{\Phi,p})$ satisfying

$$\langle v - [v]_n, u_n \rangle = \sum_{i=n+1}^{\infty} v(i)u_n(i) \geq \|v - [v]_n\|_{\Psi,q} - \frac{1}{n}.$$

Without loss of generality, let

$$u_n = (0, \dots, 0, u_n(n+1), u_n(n+2), \dots).$$

Put

$$z_n = \frac{1}{2}(u(1), \dots, u(n), u_n(n+1), u_n(n+2), \dots) = \frac{[u]_n + u_n}{2}.$$

Then

$$\begin{aligned} 1 &\geq \frac{1}{2}(\|[u]_n\|_{\Phi,p} + \|u_n\|_{\Phi,p}) \\ &\geq \|z_n\|_{\Phi,p} \geq \sum_{i=1}^{\infty} z_n(i)v(i) \\ &= \frac{1}{2} \left(\sum_{i=1}^n u(i)v(i) + \sum_{i=n+1}^{\infty} u_n(i)v(i) \right) \\ &\geq \frac{1}{2} \left(\sum_{i=1}^n u(i)v(i) + \|v - [v]_n\|_{\Psi,q} - \frac{1}{n} \right) \rightarrow 1 \end{aligned}$$

as $n \rightarrow \infty$. But $\|z_n - u\|_{\Phi,p} \geq \frac{1}{2}\|[u]_n\|_{\Phi,p} \rightarrow \frac{1}{2} > 0$ as $n \rightarrow \infty$. We obtain that u is not a strongly exposed point of $S(l_{\Phi,p})$.

Sufficiency. Now, we prove the sufficiency for $\text{card}(\text{supp}(u)) = 1$.

Assume $u = (u(1), 0, \dots)$ and $u(1) > 0$. Since $u \in h_{\Phi,p}$, we have $\text{Grad}(u) \subset l_{\Psi,q}$. Suppose $v = (v(1), 0, \dots) \in \text{Grad}(u)$, then $\|u\|_{\Phi,p} = u(1)v(1) = 1$. If there exists $\{u_n\}_{n=1}^\infty \subset B(l_{\Phi,p})$, satisfying $\langle u_n, v \rangle \rightarrow \langle u, v \rangle$, then $u_n(1) \rightarrow u(1)$ as $n \rightarrow \infty$. Since $K_p(u) \neq \emptyset$, by Lemma 2, $K_p(u_n) \neq \emptyset$, take $k_n \in K_p(u_n)$, we have

$$\begin{aligned}\|u_n\|_{\Phi,p} &= \frac{1}{k_n} (1 + I_\Phi^p(k_n u_n))^{\frac{1}{p}} \\ &\geq \frac{1}{k_n} (1 + (\Phi(k_n u_n(1)))^p)^{\frac{1}{p}} \\ &\geq \|u_n\|_1 \rightarrow \|u\|_{\Phi,p} = 1\end{aligned}$$

as $n \rightarrow \infty$. Hence, $\sum_{i=2}^\infty \Phi(k_n u_n(i)) \rightarrow 0$ as $n \rightarrow \infty$, which implies $\sum_{i=2}^\infty \Phi(u_n(i)) \rightarrow 0$ as $n \rightarrow \infty$. By $\Phi \in \Delta_2(0)$ and Theorem 1.39 in [24], we have

$$\|u_n - [u_n]_1\|_{\Phi,\infty} \rightarrow 0$$

as $n \rightarrow \infty$. By equivalence of p -Amemiya norms,

$$\|u_n - [u_n]_1\|_{\Phi,p} \rightarrow 0$$

as $n \rightarrow \infty$. So,

$$\|u - u_n\|_{\Phi,p} \leq \|u - [u_n]_1\|_{\Phi,p} + \|u_n - [u_n]_1\|_{\Phi,p} \rightarrow 0$$

as $n \rightarrow \infty$. Thus, u is a strongly exposed point of $S(l_{\Phi,p})$.

Next, we will prove the sufficiency for $\text{card}(\text{supp}(u)) > 1$.

Suppose $\{u_n\}_{n=1}^\infty \subset B(l_{\Phi,p})$, $\langle u_n, v \rangle \rightarrow \langle u, v \rangle = 1$. By Theorem 2,

$$\lim_{i_0 \rightarrow \infty} \sup_n \sum_{i > i_0} \Phi(u_n(i)) = 0 \quad (3)$$

holds. Combine $\Phi \in \Delta_2(0)$ and Lemma 9, to prove $\|u_n - u\|_{\Phi,p} \rightarrow 0$ as $n \rightarrow \infty$, we only need to prove, for any $i \in \mathbb{N}$, $u_n(i) \rightarrow u(i)$ as $n \rightarrow \infty$. Take $k_n \in K_p(u_n)$. We can prove, for any $i \in \mathbb{N}$, $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$ where $k_u \in K_p(u)$.

$$(I) I_\Phi^{p-1}(k_u u) \cdot I_\Psi(\Phi'_-(k_u u)) = 1.$$

By Lemma 6, $v = \frac{\Phi'_-(k_u u)}{\|\Phi'_-(k_u u)\|_{\Psi,q}}$ is the unique supporting functional of u . Since $I_\Phi^{p-1}(k_u u) \cdot I_\Psi(\Phi'_-(k_u u)) = 1$, we get $1 \in K_q(\Phi'_-(k_u u))$. By the Hölder Inequality ([28]) and the Young Inequality, we have

$$\begin{aligned}0 &\leftarrow \frac{1}{k_n} (1 + I_\Phi^p(k_n u_n))^{\frac{1}{p}} - \left\langle \frac{\Phi'_-(k_u u)}{\|\Phi'_-(k_u u)\|_{\Psi,q}}, u_n \right\rangle \\ &= \frac{(1 + I_\Phi^p(k_n u_n))^{\frac{1}{p}} \cdot \|\Phi'_-(k_u u)\|_{\Psi,q} - \langle \Phi'_-(k_u u), k_n u_n \rangle}{k_n \|\Phi'_-(k_u u)\|_{\Psi,q}} \\ &= \frac{(1 + I_\Phi^p(k_n u_n))^{\frac{1}{p}} \cdot (1 + I_\Psi^q(\Phi'_-(k_u u)))^{\frac{1}{q}} - \sum_{i=1}^\infty k_n u_n(i) \Phi'_-(k_u u(i))}{k_n \|\Phi'_-(k_u u)\|_{\Psi,q}} \\ &\geq \frac{I_\Phi(k_n u_n) + I_\Psi(\Phi'_-(k_u u)) - \sum_{i=1}^\infty k_n u_n(i) \Phi'_-(k_u u(i))}{k_n \|\Phi'_-(k_u u)\|_{\Psi,q}} \\ &\geq \frac{I_\Phi(k_n u_n) + I_\Psi(\Phi'_-(k_u u)) - I_\Phi(k_n u_n) - I_\Psi(\Phi'_-(k_u u))}{k_n \|\Phi'_-(k_u u)\|_{\Psi,q}} \\ &= 0\end{aligned}$$

as $n \rightarrow \infty$. Thus, for any $i \in \mathbb{N}$,

$$\Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \rightarrow 0 \quad (4)$$

as $n \rightarrow \infty$. And

$$(1 + I_\Phi^p(k_n u_n))^{\frac{1}{p}} \cdot (1 + I_\Psi^q(\Phi'_-(k_u u)))^{\frac{1}{q}} - I_\Phi(k_n u_n) - I_\Psi(\Phi'_-(k_u u)) \rightarrow 0$$

as $n \rightarrow \infty$, which means

$$I_\Phi^{p-1}(k_n u_n) \cdot I_\Psi(\Phi'_-(k_u u)) \rightarrow 1 = I_\Phi^{p-1}(k_u u) \cdot I_\Psi(\Phi'_-(k_u u))$$

as $n \rightarrow \infty$, i.e.,

$$I_\Phi(k_n u_n) \rightarrow I_\Phi(k_u u) \quad (5)$$

as $n \rightarrow \infty$.

We will discuss the following cases.

Case-1 If $k_u u(i) \in SC_\Phi$, then $k_n u_n(i) \leq k_u u(i) - \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ & \geq \text{the area of figure ABC} > 0. \end{aligned}$$

(see Figure 1). This contradicts to (4). So $k_n u_n(i) > k_u u(i) - \delta$ for n large enough and any $\delta > 0$.

Moreover, $k_n u_n(i) \geq k_u u(i) + \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ & \geq \text{the area of figure ADE} > 0. \end{aligned}$$

(see Figure 1). This contradicts to (4). So $k_n u_n(i) < k_u u(i) + \delta$ for n large enough and any $\delta > 0$. Thus, $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$.

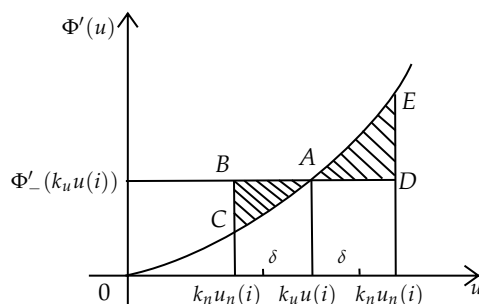


Figure 1. $k_u u(i) \in SC_\Phi$

Case-2 If $k_u u(i) = a \in A$, then $k_n u_n(i) \leq a - \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ & = \Phi(k_n u_n(i)) + \Psi(\Phi'_-(a)) - k_n u_n(i) \cdot \Phi'_-(a) \\ & \geq \text{the area of the figure ABC} > 0. \end{aligned}$$

(see Figure 2). This contradicts to (4). So $k_n u_n(i) > a - \delta$ for n large enough and any $\delta > 0$.

Moreover, $k_n u_n(i) \geq a + \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(a)) - k_n u_n(i) \cdot \Phi'_-(a) \\ & \geq \text{the area of the rectangle ADEF} \\ & \geq (\Phi'_+(a) - \Phi'_-(a))\delta > 0. \end{aligned}$$

(see Figure 2). This contradicts to (4). So $k_n u_n(i) < a + \delta$ for n large enough and any $\delta > 0$. Thus, $k_n u_n(i) \rightarrow a$ as $n \rightarrow \infty$.

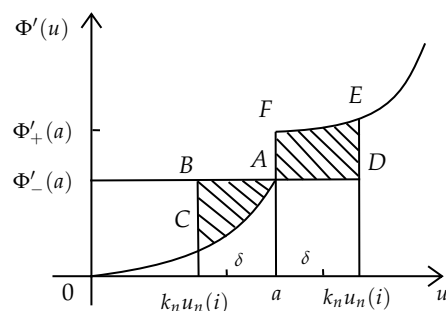


Figure 2. $k_u u(i) = a \in A$

Case-3 If $k_u u(i) = b \in B$, then $k_n u_n(i) \geq b + \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ &= \Phi(k_n u_n(i)) + \Psi(\Phi'_-(b)) - k_n u_n(i) \cdot \Phi'_-(b) \\ &\geq \text{the area of the figure ABCD} \\ &\geq (\Phi'_+(b) - \Phi'_-(b))\delta > 0. \end{aligned}$$

(see Figure 3). This contradicts to (4). So $k_n u_n(i) < b + \delta$ for n large enough and any $\delta > 0$. Moreover, $k_n u_n(i) \leq b - \delta$ for any $\delta \in (0, b - a)$ or $(0, b - a')$ implies

$$\Phi(k_n u_n(i)) + \Psi(\Phi'_-(b)) - k_n u_n(i) \cdot \Phi'_-(b) = 0.$$

(see Figure 3). Thus, $k_n u_n(i) \leq b$ as $n \rightarrow \infty$.

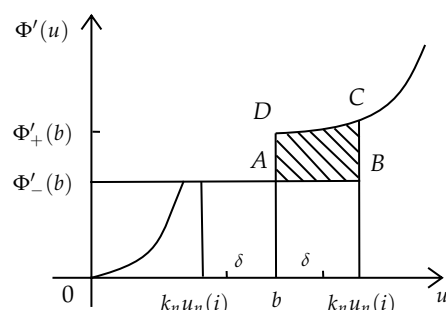


Figure 3. $k_u u(i) = b \in B$

Case-4 If $k_u u(i) = a' \in A'$, then $k_n u_n(i) \leq a' - \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ &= \Phi(k_n u_n(i)) + \Psi(\Phi'_-(a')) - k_n u_n(i) \cdot \Phi'_-(a') \\ &\geq \text{the area of the figure ABC} > 0. \end{aligned}$$

(see Figure 4). This contradicts to (4). So $k_n u_n(i) > a' - \delta$ for n large enough and any $\delta > 0$. Moreover, $k_n u_n(i) \geq a' + \delta$ for any $\delta \in (0, b' - a')$ or $(0, b - a')$ implies

$$\Phi(k_n u_n(i)) + \Psi(\Phi'_-(a')) - k_n u_n(i) \cdot \Phi'_-(a') = 0.$$

(see Figure 4). Thus, $k_n u_n(i) \geq a'$ as $n \rightarrow \infty$.

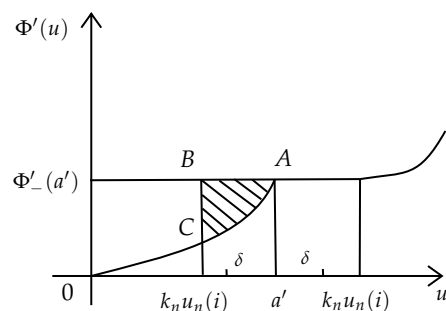


Figure 4. $k_u u(i) = a' \in A'$

Case-5 If $k_u u(i) = b' \in B'$, then $k_n u_n(i) \geq b' + \delta$ for any $\delta > 0$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(\Phi'_-(k_u u(i))) - k_n u_n(i) \cdot \Phi'_-(k_u u(i)) \\ &= \Phi(k_n u_n(i)) + \Psi(\Phi'_-(b')) - k_n u_n(i) \cdot \Phi'_-(b') \\ &\geq \text{the area of the figure ABC} > 0. \end{aligned}$$

(see Figure 5). This contradicts to (4). So $k_n u_n(i) < b' + \delta$ for n large enough and any $\delta > 0$.

Moreover, $k_n u_n(i) \leq b' - \delta$ for any $\delta \in (0, b' - a')$ or $(0, b' - a)$ implies

$$\Phi(k_n u_n(i)) + \Psi(\Phi'_-(b')) - k_n u_n(i) \cdot \Phi'_-(b') = 0.$$

(see Figure 5). Thus, $k_n u_n(i) \leq b'$ as $n \rightarrow \infty$.

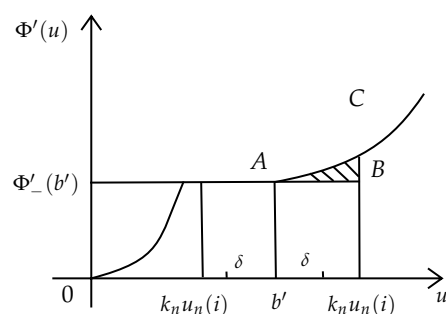


Figure 5. $k_u u(i) = b' \in B'$

If $k_u u(i) \in A'$, by u is an exposed point and Lemma 13, we get

$$\{i \in \mathbb{N} : k_u u(i) \in B'\} = \{i \in \mathbb{N} : k_u u(i) \in B\} = \emptyset.$$

Set $I_0 = \{i \in \mathbb{N} : k_u u(i) \in A'\}$. By case-1, case-2 and case-4, we have

$$\begin{aligned} k_n u_n(i) &\rightarrow k_u u(i) & (i \in \mathbb{N} \setminus I_0) & \text{ as } n \rightarrow \infty, \\ k_n u_n(i) &\geq k_u u(i) & (i \in I_0) & \text{ as } n \rightarrow \infty. \end{aligned}$$

In fact, for any $i \in I_0$, we can claim that $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$. Otherwise, there exists $i_0 \in I_0$, $\varepsilon_0 > 0$, $n_0 \in \mathbb{N}$ such that $k_n u_n(i_0) > k_u u(i_0) + \varepsilon_0$ for all $n > n_0$. Since Φ is affine on $[a', b]$ or $[a', b']$: $\Phi(u) = cu + d$ where $c, d \in \mathbb{R}$ for $u \in [a', b]$ or $[a', b']$. For all $n > n_0$, we have

$$\begin{aligned} \Phi(k_n u_n(i_0)) &= ck_n u_n(i_0) + d \\ &> c(k_u u(i_0) + \varepsilon_0) + d \\ &= ck_u u(i_0) + d + c\varepsilon_0 \\ &= \Phi(k_u u(i_0)) + c\varepsilon_0. \end{aligned}$$

Then,

$$\begin{aligned} I_{\Phi}(k_n u_n) &= \sum_{i \neq i_0} \Phi(k_n u_n(i)) + \Phi(k_n u_n(i_0)) \\ &> \sum_{i \neq i_0} \Phi(k_n u_n(i)) + (\Phi(k_u u(i_0)) + c\varepsilon_0) \\ &\geq I_{\Phi}(k_u u) + c\varepsilon_0 \end{aligned}$$

as $n \rightarrow \infty$, this contradicts to (5). Hence, for any $i \in \mathbb{N}$, $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$.

If $k_u u(i) \in B \cup B'$, by u is an exposed point and Lemma 13, we get

$$\{i \in \mathbb{N} : k_u u(i) \in A'\} = \emptyset.$$

Set $I_1 = \{i \in \mathbb{N} : k_u u(i) \in B \cup B'\}$. By case-1, case-2, case-3 and case-5, we have

$$\begin{aligned} k_n u_n(i) &\rightarrow k_u u(i) \quad (i \in \mathbb{N} \setminus I_1) \quad \text{as } n \rightarrow \infty, \\ k_n u_n(i) &\leq k_u u(i) \quad (i \in I_1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Use the analogous way as above, we obtain, for any $i \in \mathbb{N}$, $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$.

Thus, for any $i \in \mathbb{N}$, we have $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$. By $\|u_n\|_{\Phi,p} \rightarrow 1$, $\|u\|_{\Phi,p} = 1$, we have $(1 + I_{\Phi}^p(k_n u_n))^{\frac{1}{p}} \rightarrow k_n$ and $(1 + I_{\Phi}^p(k_u u))^{\frac{1}{p}} = k_u$. From (5), we get $k_n \rightarrow k_u$ as $n \rightarrow \infty$. Hence, for any $i \in \mathbb{N}$, $u_n(i) \rightarrow u(i)$ as $n \rightarrow \infty$. By Lemma 9, u is a strongly exposed point of $S(l_{\Phi,p})$.

$$(II) I_{\Phi}^{p-1}(k_u u) I_{\Psi}(\Phi'_+(k_u u)) = 1.$$

By Lemma 6, $v = \frac{\Phi'_+(k_u u)}{\|\Phi'_+(k_u u)\|_{\Psi,q}}$ is the unique supporting functional of u . Use the analogous way as case (I), we have, for any $i \in \mathbb{N}$, $u_n(i) \rightarrow u(i)$ as $n \rightarrow \infty$, and u is a strongly exposed point of $S(l_{\Phi,p})$.

$$(III) I_{\Phi}^{p-1}(k_u u) I_{\Psi}(\Phi'_-(k_u u)) < 1 < I_{\Phi}^{p-1}(k_u u) I_{\Psi}(\Phi'_+(k_u u)).$$

By $\Phi \in \Delta_2(0)$, we have $(l_{\Phi,p})^* = (h_{\Phi,p})^* = l_{\Psi,q}$. Thus, there exists $v \in l_{\Psi,q}$ such that $\langle u, v \rangle = \|u\|_{\Phi,p} \cdot \|v\|_{\Psi,q}$, i.e., $\frac{v}{\|v\|_{\Psi,q}} \in \text{Grad}(u)$, and $\theta(\frac{v}{\|v\|_{\Psi,q}}) < 1$. By Lemma 6, $\Phi'_-(k_u u(i)) \leq v(i) \leq \Phi'_+(k_u u(i))$ for any $i \in \mathbb{N}$, and $I_{\Phi}^{p-1}(k_u u) \cdot I_{\Psi}(v) = 1$.

Set $J_0 = \{i \in \mathbb{N} : \Phi'_-(k_u u(i)) < \Phi'_+(k_u u(i))\}$. Then

$$v(i) = \begin{cases} \Phi'_-(k_u u(i)), & i \notin J_0, \\ \Phi'_-(k_u u(i)) + \varepsilon_i, & i \in J_0, \end{cases} \quad (6)$$

where for any $i \in J_0$, $\Phi'_-(k_u u(i)) + \varepsilon_i \leq \Phi'_+(k_u u(i))$. By $I_{\Phi}^{p-1}(k_u u) \cdot I_{\Psi}(v) = 1$, we obtain $1 \in K_q(v)$. For any $\{u_n\}_{n=1}^{\infty} \subset B(l_{\Phi,p})$ with $\langle \frac{v}{\|v\|_{\Psi,q}}, u_n \rangle \rightarrow 1$ as $n \rightarrow \infty$. Take $k_n \in K_p(u_n)$. By the Young Inequality and the Hölder Inequality, we have

$$\begin{aligned} \left\langle \frac{v}{\|v\|_{\Psi,q}}, u_n \right\rangle &= \frac{1}{k_n \|v\|_{\Psi,q}} \langle v, k_n u_n \rangle \leq \frac{1}{k_n \|v\|_{\Psi,q}} (I_{\Psi}(v) + I_{\Phi}(k_n u_n)) \\ &\leq \frac{1}{k_n} s_p(I_{\Phi}(k_n u_n)) \cdot \frac{1}{\|v\|_{\Psi,q}} s_q(I_{\Psi}(v)) \rightarrow 1. \end{aligned}$$

as $n \rightarrow \infty$. Thus, for any $i \in \mathbb{N}$,

$$\Phi(k_n u_n(i)) + \Psi(v(i)) - k_n u_n(i) \cdot v(i) \rightarrow 0, \quad (7)$$

as $n \rightarrow \infty$. And

$$(1 + I_{\Phi}^p(k_n u_n))^{\frac{1}{p}} \cdot (1 + I_{\Psi}^q(v))^{\frac{1}{q}} - I_{\Phi}(k_n u_n) - I_{\Psi}(v) \rightarrow 0,$$

as $n \rightarrow \infty$, which means

$$I_{\Phi}^{p-1}(k_n u_n) \cdot I_{\Psi}(v) \rightarrow 1 = I_{\Phi}^{p-1}(k_u u) \cdot I_{\Psi}(v),$$

as $n \rightarrow \infty$, i.e.,

$$I_{\Phi}(k_n u_n) \rightarrow I_{\Phi}(k_u u), \quad (8)$$

as $n \rightarrow \infty$.

If $i \in J_0$, then $v(i) = \Phi'_-(k_u u(i)) + \varepsilon_i \leq \Phi'_+(k_u u(i))$. For any $\delta > 0$, $k_n u_n(i) \leq k_u u(i) - \delta$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(v(i)) - k_n u_n(i) \cdot v(i) \\ & \geq \text{the area of the figure ABCD} \\ & \geq \varepsilon_i \delta > 0 \end{aligned}$$

(see Figure 6). This contradicts to (7). So $k_n u_n(i) \leq k_u u(i) - \delta$ for n large enough and any $\delta > 0$.

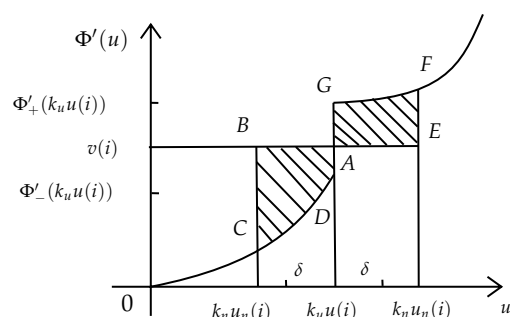


Figure 6. $\Phi'_-(k_u u(i)) < v(i) < \Phi'_+(k_u u(i))$

For any $\delta > 0$, $k_n u_n(i) \geq k_u u(i) + \delta$ implies

$$\begin{aligned} & \Phi(k_n u_n(i)) + \Psi(v(i)) - k_n u_n(i) \cdot v(i) \\ & \geq \text{the area of the figure AEFG} \\ & > (\Phi'_+(k_u u(i)) - v(i))\delta > 0 \end{aligned}$$

(see Figure 6). This contradicts to (7). So $k_n u_n(i) \leq k_u u(i) + \delta$ for n large enough and any $\delta > 0$. Thus, for any $i \in J_0$, $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$. If $i \in \mathbb{N} \setminus J_0$, then $v(i) = \Phi'_-(k_u u(i)) = \Phi'_+(k_u u(i))$. We have $k_n u_n(i) \rightarrow k_u u(i)$ as $n \rightarrow \infty$. Similarly, by (8), we can prove $k_n \rightarrow k_u$ as $n \rightarrow \infty$. Hence, for any $i \in \mathbb{N}$, $u_n(i) \rightarrow u(i)$ as $n \rightarrow \infty$. By Lemma 9, u is a strongly exposed point of $S(l_{\Phi,p})$. $\square$

Theorem 5. Let $1 < p < \infty$, $l_{\Phi,p}$ has the strongly exposed property if and only if

(i) $\Phi \in \Delta_2(0)$ and $\Psi \in \Delta_2(0)$ (i.e., $\Phi \in \Delta_2(0) \cap \nabla_2(0)$).

(ii) Φ is strictly convex on the interval $\left[0, \pi_{\Phi,p}\left(\frac{1}{2}\right)\right]$.

Proof. Sufficiency. By Lemma 8, $l_{\Phi,p}$ is a locally uniform rotundity space. So $l_{\Phi,p}$ has the strongly exposed property.

Necessity. $\Phi \in \Delta_2(0)$ follows from Theorem 4. If condition (ii) fails, by Lemma 7, $l_{\Phi,p}$ is not strictly convex, and therefore it does not have the strongly exposed property either.

So (ii) is necessary. If $\Psi \notin \Delta_2(0)$, for any $n > 2$, there exist $a_i \searrow 0$ such that $a_i \Psi'_+(a_i) < \frac{1}{n^i}$ and $\Psi'_+\left(\left(1 + \frac{1}{i}\right)a_i\right) > n^i \Psi'_+(a_i)$. Choose natural numbers m_i satisfying

$$\frac{1}{n^{i+1}} \leq m_i a_i \Psi'_+(a_i) < \frac{1}{n^i} \quad (i = 1, 2, \dots),$$

and define

$$y = \left(\overbrace{a_1, \dots, a_1}^{m_1}, \overbrace{a_2, \dots, a_2}^{m_2}, \overbrace{a_3, \dots, a_3}^{m_3}, \dots \right).$$

Then we have

$$\langle y, \Psi'_+(y) \rangle = \sum_{i=1}^{\infty} m_i a_i \Psi'_+(a_i) < \sum_{i=1}^{\infty} \frac{1}{n^i} = \frac{1}{n-1}.$$

Whence by $\langle y, \Psi'_+(y) \rangle = I_{\Psi}(y) + I_{\Phi}(\Psi'_+(y))$, we get $I_{\Psi}(y) < \frac{1}{n-1}$ and $I_{\Phi}(\Psi'_+(y)) < \frac{1}{n-1}$. Thus, $I_{\Psi}^{q-1}(y) I_{\Phi}(\Psi'_+(y)) < 1$.

Notice that for any $\lambda > 0$,

$$\begin{aligned} I_{\Phi}(\Psi'_+((1+\lambda)y)) + I_{\Psi}(y) &\geq \langle y, \Psi'_+((1+\lambda)y) \rangle \\ &= \sum_{i=1}^{\infty} m_i a_i \Psi'_+((1+\lambda)a_i) \geq \sum_{i \geq \frac{1}{\lambda}} m_i a_i \Psi'_+\left(\left(1 + \frac{1}{i}\right)a_i\right) \\ &\geq \sum_{i \geq \frac{1}{\lambda}} m_i a_i n^i \Psi'_+(a_i) \geq \sum_{i \geq \frac{1}{\lambda}} \frac{1}{n} = \infty. \end{aligned}$$

So we derive that $I_{\Phi}(\Psi'_+((1+\lambda)y)) = \infty$. We also have that for any $\lambda > 0$,

$$\begin{aligned} I_{\Psi}((1+2\lambda)y) &= \sum_{i=1}^{\infty} m_i \Phi((1+2\lambda)a_i) \\ &\geq \sum_{i=1}^{\infty} m_i \int_{(1+\lambda)a_i}^{(1+2\lambda)a_i} \Psi'_+(s) ds \\ &\geq \sum_{i=1}^{\infty} m_i \lambda a_i \Psi'_+((1+\lambda)a_i) \\ &= \frac{\lambda}{1+\lambda} \sum_{i=1}^{\infty} m_i (1+\lambda)a_i \Psi'_+((1+\lambda)a_i) \\ &= \frac{\lambda}{1+\lambda} \langle (1+\lambda)y, \Psi'_+((1+\lambda)y) \rangle \\ &\geq \frac{\lambda}{1+\lambda} I_{\Phi}(\Psi'_+((1+\lambda)y)) = \infty. \end{aligned}$$

So $I_{\Psi}^{q-1}((1+\lambda)y) \cdot I_{\Phi}(\Psi'_+((1+\lambda)y)) = \infty$, which shows $K_q(y) = \{1\}$, i.e., $\|y\|_{\Psi,q} = 1 + I_{\Psi}(y)$. Set $x = \Psi'_+(y)$, by Lemma 6, $\frac{y}{\|y\|_{\Psi,q}}$ is the unique supporting functional of x . For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{2}{n-1} < \varepsilon$ for all $n > n_0$, then

$$I_{\Psi}\left((1+\varepsilon)\frac{y}{\|y\|_{\Psi,q}}\right) > I_{\Psi}\left(\left(1 + \frac{2}{n-1}\right)\frac{y}{1 + \frac{1}{n-1}}\right) = I_{\Psi}\left(\left(1 + \frac{1}{n}\right)y\right) = \infty.$$

This contradicts to condition (iii) of Theorem 4. So $\Psi \in \Delta_2(0)$ is necessary. $\square$

4. Conclusions

The classical Orlicz and Luxemburg norms generated by an Orlicz function Φ can be defined with the use of the Amemiya formula: $\|u\|_{\Phi}^0 = \inf_{k>0} \frac{1}{k} (1 + I_{\Phi}(ku))$

and $\|u\|_{\Phi} = \inf_{k>0} \frac{1}{k} \{1, I_{\Phi}(ku)\}$ respectively (see [15]). Furthermore, the family of p -Amemiya norms $\|u\|_{\Phi, p} = \inf_{k>0} \frac{1}{k} (1 + I_{\Phi}^p(ku))^{\frac{1}{p}}$ ($1 \leq p \leq \infty$) (under the convention: $(1 + x^{\infty})^{\frac{1}{\infty}} = \lim_{p \rightarrow \infty} (1 + x^p)^{\frac{1}{p}} = \max\{1, x\}$ for all $x \geq 0$) were introduced in the same paper. In last few years, a number of papers about the geometrical properties of Orlicz spaces equipped with the p -Amemiya norms have been published. The criteria for the strongly exposed points and strongly exposed property of Orlicz spaces equipped with the classical norms were presented in [9–11]. However, the criteria for the strongly exposed points and strongly exposed property of Orlicz sequence spaces equipped with the p -Amemiya norms have not been found. In this paper, Theorems 1, 4 and 5 resolves these issues. The obtained results broaden the knowledge about strongly exposed points and strongly exposed property in Orlicz spaces equipped with the p -Amemiya norms.

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