

Article

Approximate Synchronization of Memristive Hopfield Neural Networks

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Abstract

Asymptotic synchronization is one of the essential differences between artificial neural networks and biologically inspired neural networks due to mismatches from the dynamical update of weight parameters and heterogeneous activations. In this paper, a new concept of approximate synchronization is proposed and investigated for Hopfield neural networks coupled with nonlinear memristors. It is proved that global solution dynamics are robustly dissipative and a sharp ultimate bound is acquired. Through *a priori* uniform estimates on the interneuron differencing equations, it is rigorously and analytically shown that approximate synchronization to any prescribed small gap at an exponential convergence rate of the memristive Hopfield neural networks occurs if an explicitly computable threshold condition is satisfied by the interneuron coupling strength parameter. The main result is also extended to Hopfield neural networks with Hebbian learning rules for a broad range of applications in unsupervised learning. The contribution of this approximate synchronization framework and the analytic methodology in this work advance the exploration of asymptotic dynamics for more AI mathematical models.

Keywords: Hopfield neural network; approximate synchronization; memristive coupling; Hebbian weight dynamics; synchronization threshold condition

MSC: 34D06; 34D45; 37N25; 68T07; 92B20

1. Introduction

Artificial neural networks are used to somehow emulate the dynamic behavior and functions of biological neuron networks and nervous systems through a variety of mathematical models and computational algorithms [1–4]. The Hopfield neural network (HNN), initially proposed in 1982 [5], is a recursive or recurrent artificial neural network [6,7]. HNN models coupled with memristor-based synapses have been actively studied with rich and important results and applications achieved in recent decades [8–11].

Memristor is a physics concept that relates the electric charge of a circuit and the electromagnetic flux expressed by the time integral of the voltage. It was initiated by Chua [12]. It is recognized [13–15] that a memristor is a promising and ideal element in spiking neural networks to mimic the learning and memorizing functions of the human brain. Many researches have been reported in recent years on memristive biological neural network synchronization [2,16–18], image encryption [9,19], nanotechnology [20], and quantum computers [21]. In recent years wide-ranging memristor-enhanced research and new applications have emerged, such as the robust polynomial hyper-chaotic map on multi-channel



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image encryption [22], neuromorphic computing [23], and the artificial synapse regulation of signal transmission speed in neuromorphic networks [24].

Synchronization dynamics is an important topic on cognitive functions and information processing for biological neural networks, chaotic systems, and deep learning [25–27]. For biological neural network models in terms of Hindmarsh–Rose equations [17], FitzHugh–Nagumo equations [18], and general reaction–diffusion equations [28] with memristive synapses or cellular models with boundary feedback [29], complete synchronization (meaning the pairwise interneuron gaps for all solution trajectories converge to zero) can be triggered if the network coupling strength satisfies a threshold condition because, for these models, all the nodes in a network are represented by a system of qualitatively identical differential or difference equations.

In contrast, for general artificial neural networks used in machine learning and various AI applications, such complete synchronization cannot be achieved because of the mismatched weight dynamics, as well as the heterogeneous activation functions among different neuron nodes in a network, which are inherently characterized. The mismatching component equations of neuron nodes in artificial neural network models will inevitably cause nonzero asymptotic neuron state gaps and possibly somewhat incomplete synchronization of the network.

It is essentially open to establish an analytic theory on approximate synchronization for typical ODE models of artificial neural networks in view of the currently sporadic research reported on particular reaction–diffusion neural networks of FHN neurons; cf. [30] and references therein.

In this paper, we shall quantitatively define approximate synchronization for artificial neural network models and investigate a novel model of memristive Hopfield neural networks (mHNN) featuring nonlinear memductance-potential coupling and weak synapses among all neuron nodes. We shall analytically prove a main result that approximate synchronization of the mHNN occurs if the coefficient of cross-network coupling strength satisfies an explicitly computable threshold condition. The extension main result for memristive HNNs with Hebbian learning rules is also achieved.

Consider a memristive Hopfield neural network composed of m neuron nodes, denoted by $\mathcal{NW} = \{\mathcal{N}_i : i = 1, 2, \dots, m\}$, where $m \geq 2$ is a positive integer. Each neuron $\mathcal{N}_i, 1 \leq i \leq m$, and a memristor characterized by its memductance variable ρ are presented by the following two differential equations:

$$\begin{aligned} \frac{du_i}{dt} &= -a_i u_i + \sum_{j=1}^m w_{ij} f_j(u_j) + k \varphi_i(\rho) u_i + J_i - P u_i \sum_{j=1}^m \Gamma(u_j), \quad 1 \leq i \leq m, \\ \frac{d\rho}{dt} &= \sum_{i=1}^m \gamma_i u_i - b\rho, \quad t > 0. \end{aligned} \quad (1)$$

The coefficient $P > 0$ is called the network coupling strength and the interneuron synaptic function

$$\Gamma(s) = \frac{1}{1 + \exp[-r(s - V)]}, \quad s \in \mathbb{R}, \quad (2)$$

is a sigmoidal function, in which the parameter $r > 0$ shapes the slope of $\Gamma(s)$ and the parameter $V \in \mathbb{R}$ is a switch for neuron bursting. The nonlinear sigmoidal coupling $P u_i \sum_{j=1}^m \Gamma(u_j)$ across the network was originated from the Hodgkin–Huxley neuron network model, which provides computational efficiency in artificial synapses and can be called weak coupling [28] versus the convenient linear neuron coupling.

The state variable $u_i(t)$ simulates the electrical potential of the i th neuron $\mathcal{N}_i$, for $1 \leq i \leq m$. The initial states of the ODE system (1) will be denoted by

$$u_i^0 = u_i(0), \quad \rho^0 = \rho(0), \quad 1 \leq i \leq m. \quad (3)$$

We denote the initial values in (3) by $g^0 = \text{col}(u_1^0, u_2^0, \dots, u_m^0, \rho^0) \in \mathbb{R}^{m+1}$. The memristive feature of this model is reflected by the memductance-potential coupling term $k\varphi_i(\rho)u_i$.

For this mHNN model, we notice that the interaction weights $\{w_{ij} : 1 \leq i, j \leq m\}$ and the activation functions $\{f_j(u_j) : 1 \leq j \leq m\}$ can be different [7], while the memristor window functions $\varphi_i(\rho)$ for different neuron nodes $\mathcal{N}_i$ can also be heterogeneous [15]. It implies that complete synchronization for such artificial neural networks in general will not be possible and approximate synchronization can be the aim to expect and pursue.

As a setting of the problem, we make the following assumption: The scalar functions $f_i(s)$ and $\varphi_i(s)$, $1 \leq i \leq m$, are locally Lipschitz continuous functions and satisfy the following conditions:

$$a_i > k, \quad |f_i(s)| \leq \beta \quad \text{and} \quad \varphi_i(s) = 1 - \eta_i s^2, \quad \text{for } s \in \mathbb{R}, \quad 1 \leq i \leq m. \quad (4)$$

Assume that the parameters a_i, b, β, η_i and the memristive coupling strength k can be any given positive constants, while weight coefficients w_{ik} , input/bias constants J_i and coefficients γ_i can be any given real numbers.

Note that the window functions $k\varphi_i(\rho)$ set in (4) are adaptable to deal with other types of memristors such as a linear, hyperbolic-tangent, or titanium dioxide memristor [15] whose window function is $\hat{\varphi}(s) = cs(1 - s)$.

Definition 1. An artificial neural network with a mathematical model of differential equations in a state space Z such as the aforementioned mHNN $\mathcal{NW}$ is said to be approximately synchronizable if there exists a threshold condition on the network coupling parameter $\mathcal{P}$ such that for any small prescribed gap $\varepsilon > 0$ there is a threshold value $\mathcal{P}^*(\varepsilon) > 0$ to trigger an approximate convergence with the network synchronous degree

$$\deg_s(\mathcal{NW}) = \sup_{g^0 \in Z} \left\{ \max_{1 \leq i < j \leq m} \left\{ \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| \right\} \right\} < \varepsilon \quad (5)$$

once $\mathcal{P} > \mathcal{P}^*(\varepsilon)$, where $u_i(t)$ is the state variable of the i th neuron node, $1 \leq i \leq m$. If the limsup convergence in (5) admits a uniform exponential rate, then it is called approximate exponential synchronization.

The rest of the paper is organized as follows. In Section 2, we start with proving the global existence and uniqueness in time of the solutions to the initial value problem of the evolutionary equation formulated from the mHNN model (1)–(3). Then we analyze dissipative dynamics of the solution semiflow and prove the existence of an absorbing set in the state space. In Section 3, we shall rigorously prove the main theorem on approximate exponential synchronization of this memristive Hopfield neural network through the approach of sharp uniform estimates of the pairwise neuron state gaps governed by the interneuron differencing equations. In Section 4, the main result is generalized to memristive Hopfield-Hebbian neural networks, which are used in a broad range of unsupervised AI learning applications. In Section 5 Conclusions, we will summarize the key contributions of this work and outlook for potential extension research and challenges.

2. Dissipative Dynamics and Global Attractor

The mathematical model (1)–(3) of the memristive Hopfield neural network $\mathcal{NW}$ can be formulated into an initial value problem of the evolutionary equation:

$$\begin{aligned} \frac{dg}{dt} &= Ag + F(g), \quad t > 0, \\ g(0) &= g^0 \in \mathbb{R}^{m+1}. \end{aligned} \quad (6)$$

In (6) the column vector function $g(t) = \text{col}(u_1(t), u_2(t), \dots, u_m(t), \rho(t))$ and the initial state is

$$g(0) = g^0 = \text{col}(u_1^0, u_2^0, \dots, u_m^0, \rho^0).$$

The norm $\|g(t)\|$ in the state space $\mathbb{R}^{m+1}$ for (6) is defined to be $\|g(t)\|^2 = \sum_{i=1}^m |u_i(t)|^2 + |\rho(t)|^2$. In (6) the diagonal square matrix

$$A = \text{diag}(-a_1, -a_2, \dots, -a_m, -b) \quad (7)$$

and the nonlinear vector

$$F(g) = \begin{pmatrix} \sum_{j=1}^m w_{1j} f_j(u_j) + k \varphi_1(\rho) u_1 + I_1 - P u_1 \sum_{j=1}^m \Gamma(u_j) \\ \sum_{j=1}^m w_{2j} f_j(u_j) + k \varphi_2(\rho) u_2 + I_2 - P u_2 \sum_{j=1}^m \Gamma(u_j) \\ \vdots \\ \sum_{j=1}^m w_{mj} f_j(u_j) + k \varphi_m(\rho) u_m + I_m - P u_m \sum_{j=1}^m \Gamma(u_j) \\ \sum_{i=1}^m \gamma_i u_i \end{pmatrix} \quad (8)$$

is a locally Lipschitz continuous vector function.

As preliminaries [31,32], a general dynamical system only for time $t \geq 0$ can be called semiflow. For a semiflow denoted by $\{S(t)\}_{t \geq 0}$ on a Banach space $\mathcal{X}$, a bounded subset $B^* \subset \mathcal{X}$ is called an absorbing set if for any given bounded set B in the space $\mathcal{X}$ there is a finite time $T_B \geq 0$ such that $S(t)B \subset B^*$ for all $t > T_B$. A semiflow is called dissipative if there exists an absorbing set in the state space $\mathcal{X}$, which also implies the existence of a global attractor if $\mathcal{X}$ is finite dimensional.

The Young's inequality is often used in differential or integral inequality management. It states that for any positive numbers x and y , if $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1, q > 1$, then

$$xy \leq \frac{1}{p} \varepsilon x^p + \frac{1}{q} C(\varepsilon, p) y^q \leq \varepsilon x^p + C(\varepsilon, p) y^q, \quad (9)$$

where $C(\varepsilon, p) = \varepsilon^{-q/p}$ and the constant $\varepsilon > 0$ can be arbitrarily given.

We start with the existence and uniqueness of solutions of the initial value problem (6) in the state space.

Theorem 1. For any given initial state $g^0 \in \mathbb{R}^{m+1}$, there exists a unique global solution $g(t; g^0)$ in time, $t \in [0, \infty)$, for the initial value problem (6) of the memristive Hopfield neural network $\mathcal{NW}$ presented by the model (1) with the Assumption (4).

Proof. Since the nonlinear vector function $F(g)$ in the semilinear Equation (6) is locally Lipschitz continuous, there exists a unique local solution $g(t; g^0)$ in time for the initial value problem (6). For study of asymptotic dynamics, we shall consider the continuation of the solution existence for time $t > 0$ by the approach of scaling group estimation.

Multiply the u_i -equation in (1) by $C_1 u_i(t)$ for $t \in [0, T_{max})$, which is the maximal existence interval of that solution. Then sum them altogether for $1 \leq i \leq m$. By the Assumption (4) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{i=1}^m C_1 u_i^2(t) &\leq \frac{1}{2} \frac{d}{dt} \sum_{i=1}^m C_1 u_i^2(t) + C_1 \sum_{i=1}^m \sum_{j=1}^m \frac{P u_i^2}{1 + \exp[-r(u_j - V)]} \\ &= \sum_{i=1}^m C_1 \left[-a_i u_i^2(t) + \sum_{j=1}^m w_{ij} f_j(u_j(t)) u_i(t) + k \varphi_i(\rho(t)) u_i^2(t) + J_i u_i(t) \right] \\ &\leq \sum_{i=1}^m C_1 \left[-a u_i^2(t) + m W \beta |u_i(t)| + k (1 - \eta_i \rho^2(t)) u_i^2(t) + J |u_i(t)| \right] \\ &\leq \sum_{i=1}^m C_1 \left[-(a - k) u_i^2(t) + (m W \beta + J) |u_i(t)| \right] \\ &\leq \sum_{i=1}^m C_1 \left[-\frac{1}{2} (a - k) u_i^2(t) + \frac{(m W \beta + J)^2}{2(a - k)} \right], \quad \text{for } t \in [0, T_{max}), \end{aligned} \quad (10)$$

where $a = \min\{a_i : 1 \leq i \leq m\} > k$, the double summation term on the right-hand side of the first inequality is nonnegative, $-k \eta_i \rho^2(t) u_i^2(t) \leq 0$, and a square completion is used in the last step in (10). The scaling constant $C_1 > 0$ is yet to be specified. Here the new positive parameters are $a = \min\{a_i : 1 \leq i \leq m\}$, $W = \max\{|w_{ij}| : 1 \leq i, j \leq m\}$, and $J = \max\{|J_i| : 1 \leq i \leq m\}$.

Then multiply the ρ equation in (1) with $\rho(t)$, for $t \in [0, T_{max})$. By the Young's inequality (9), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \rho^2(t) &= \sum_{i=1}^m \gamma_i u_i(t) \rho(t) - b \rho^2(t) \\ &\leq \sum_{i=1}^m \left[\frac{m \gamma_i^2}{2b} u_i^2(t) + \frac{b}{2m} \rho^2(t) \right] - b \rho^2(t) = \sum_{i=1}^m \frac{m \gamma_i^2}{2b} u_i^2(t) - \frac{b}{2} \rho^2(t), \quad t \in [0, T_{max}). \end{aligned} \quad (11)$$

Add together the above two inequalities (10) and (11). We come up with

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\sum_{i=1}^m C_1 u_i^2(t) + \rho^2(t) \right) \\ &\leq \sum_{i=1}^m C_1 \left[-\frac{1}{2} (a - k) u_i^2(t) + \frac{(m W \beta + J)^2}{2(a - k)} \right] + \sum_{i=1}^m \frac{m \gamma_i^2}{2b} u_i^2(t) - \frac{b}{2} \rho^2(t) \\ &= -\frac{1}{2} \sum_{i=1}^m \left[C_1 (a - k) - \frac{m \gamma_i^2}{b} \right] u_i^2(t) - \frac{b}{2} \rho^2(t) + \frac{C_1 m (m W \beta + J)^2}{2(a - k)}, \end{aligned} \quad (12)$$

for $t \in [0, T_{max})$, which is the maximal existence interval of a local solution $g(t; g^0)$ of this initial value problem (6) and T_{max} may depend on the initial state g^0 .

Now we can choose the scaling constant to be

$$C_1 = \frac{1}{(a - k)} \left(\frac{m \gamma_i^2}{b} + b \right) \quad \text{so that} \quad C_1 (a - k) - \frac{m \gamma_i^2}{b} = b. \quad (13)$$

With this choice, from (12) we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{i=1}^m C_1 u_i^2(t) + \rho^2(t) \right) + b \left(\sum_{i=1}^m u_i^2(t) + \rho^2(t) \right) \\ & \leq \frac{1}{a-k} \left(\frac{m\gamma_i^2}{b} + b \right) \frac{m(mW\beta + J)^2}{a-k}, \quad \text{for } t \in [0, T_{\max}). \end{aligned} \quad (14)$$

Denote the positive constant on the right-hand side of (14) by

$$C_2 = \left(\frac{m\gamma_i^2}{b} + b \right) \frac{m(mW\beta + J)^2}{(a-k)^2}. \quad (15)$$

Then (14) gives rise to a Gronwall-type differential inequality:

$$\frac{d}{dt} \left(\sum_{i=1}^m C_1 u_i^2(t) + \rho^2(t) \right) + \mu \left(\sum_{i=1}^m C_1 u_i^2(t) + \rho^2(t) \right) \leq C_2, \quad \text{for } t \in [0, T_{\max}), \quad (16)$$

where

$$\mu = b \min \left\{ \frac{1}{C_1}, 1 \right\}. \quad (17)$$

We can solve this Gronwall differential inequality (16) to get the following bounding estimate of all the solutions of the initial value problem (6) and the memristive Hopfield neural network model (1),

$$\left(\sum_{i=1}^m C_1 u_i^2(t) + \rho^2(t) \right) \leq e^{-\mu t} \left(\sum_{i=1}^m C_1 u_i^2(0) + \rho^2(0) \right) + \frac{C_2}{\mu},$$

so that

$$\|g(t; g^0)\|^2 = \sum_{i=1}^m |u_i(t)|^2 + |\rho(t)|^2 \leq \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} e^{-\mu t} \|g^0\|^2 + \frac{C_2}{\mu \min\{C_1, 1\}}, \quad t \in [0, \infty). \quad (18)$$

It also shows that the existence interval $[0, T_{\max}) = [0, \infty)$ for every solution and any initial state $g^0 \in \mathbb{R}^{m+1}$, because the solution $g(t; g^0)$ will never blow up at any finite time. Therefore, under the Assumption (4), for any given initial state there exists a unique global solution in time $t \in [0, \infty)$ for the mHNN model (1). $\square$

Based on the global existence and uniqueness of solutions shown in Theorem 1, the solution semiflow $\{S(t) : g^0 \rightarrow g(t; g^0)\}_{t \geq 0}$ on the state space $\mathbb{R}^{m+1}$ is called the mHNN semiflow. The next theorem shows that $\{S(t)\}_{t \geq 0}$ is a dissipative dynamical system and there exists a global attractor in the state space.

Theorem 2. *There exists a bounded absorbing set for the mHNN semiflow $\{S(t)\}_{t \geq 0}$ in the state space $\mathbb{R}^{m+1}$, which is the bounded ball*

$$B^* = \{g \in \mathbb{R}^{m+1} : \|g\|^2 \leq Q\}. \quad (19)$$

Here the uniform constant bound

$$Q = 1 + \frac{C_2}{\mu \min\{C_1, 1\}} \quad (20)$$

is independent of any initial state. The constants C_1 and C_2 are given in (13) and (15). Moreover, there exists a global attractor $\mathcal{A}$ for this mHNN semiflow.

Proof. This is the consequence of the global uniform estimate (18) shown above, which implies that, since $e^{-\mu t} \rightarrow 0$ as $t \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} \|g(t; g^0)\|^2 = \limsup_{t \rightarrow \infty} \left[\sum_{i=1}^m |u_i(t)|^2 + |\rho(t)|^2 \right] < Q \quad (21)$$

for all the solutions of (6) with any initial state $g^0 \in \mathbb{R}^{m+1}$. Moreover, (18) shows that for any given bounded set $B = \{g \in \mathbb{R}^{m+1} : \|g\|^2 \leq L\}$ in the state space, there exists a finite time

$$T_B = \frac{1}{\mu} \log^+ \left(L \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \right) \geq 0 \quad (22)$$

such that all the solution trajectories started at the initial time $t = 0$ from inside the set B will permanently enter the bounded ball B^* shown in (19) for all $t > T_B$. Therefore, the bounded ball B^* is an absorbing set for the semiflow $\{S(t)\}_{t \geq 0}$ so that this mHNN semiflow is dissipative dynamical system.

Finally, since the state space $\mathbb{R}^{m+1}$ is a finite-dimensional Hilbert space, the closed bounded absorbing set B^* is a compact set. According to Theorem 1.1 in [31], there exists a global attractor

$$\mathcal{A} = \Omega(B^*) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)B^*},$$

which is the ω -limit set of the absorbing set for this mHNN semiflow. $\square$

3. Approximate Synchronization of Memristive HNN

In this section we shall prove the main result on approximate synchronization of the memristive Hopfield neural network $\mathcal{NW}$ described by the model (1). We define the interneuron gap function between any two neuron nodes $\mathcal{N}_i$ and $\mathcal{N}_j$ to be

$$U_{ij}(t) = u_i(t) - u_j(t), \quad 1 \leq i \neq j \leq m.$$

The differencing equation for $U_{ij}(t)$ is

$$\begin{aligned} \frac{dU_{ij}}{dt} = & - (a_i U_{ij} + (a_i - a_j) u_j) + \sum_{\ell=1}^m (w_{i\ell} - w_{j\ell}) f_{\ell}(u_{\ell}) \\ & + k (\varphi_i(\rho) u_i - \varphi_j(\rho) u_j) + (J_i - J_j) - P U_{ij} \sum_{\ell=1}^m \Gamma(u_{\ell}). \end{aligned} \quad (23)$$

The secondary parameters involved to analyze the interneuron differencing Equation (23) are denoted by

$$\begin{aligned} a^* &= \max\{|a_i - a_j| : 1 \leq i, j \leq m\}, \quad W^* = \max\{|w_{i\ell} - w_{j\ell}| : 1 \leq i, j, \ell \leq m\}, \\ \eta^* &= \max\{|\eta_i - \eta_j| : 1 \leq i, j \leq m\}, \quad J^* = \max\{|J_i - J_j| : 1 \leq i, j \leq m\}. \end{aligned}$$

Now we present and prove the main result of this paper.

Theorem 3. *The memristive Hopfield neural network $\mathcal{NW}$ presented by the model (1) with Assumption (4) is approximate exponentially synchronizable. For any small prescribed gap $\varepsilon > 0$, there exists a threshold*

$$P^*(\varepsilon) = \frac{1}{m\varepsilon} \left(mW^* \beta + a^* Q^{1/2} + k\eta^* Q^{3/2} + J^* \right) \left[1 + \exp\{r(\sqrt{Q} + |V|)\} \right], \quad (24)$$

where Q is shown in (20), such that if the interneuron coupling strength $P > P^*(\varepsilon)$ then the neural network $\mathcal{NW}$ is approximately synchronized,

$$\deg_s(\mathcal{NW}) = \sup_{g^0 \in \mathbb{R}^{m+1}} \left\{ \max_{1 \leq i < j \leq m} \left\{ \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| \right\} \right\} < \varepsilon \quad (25)$$

at a uniform exponential convergence rate

$$\mu(P) = (a - k) + \frac{mP}{1 + \exp\{r(\sqrt{Q} + |V|)\}} > 0. \quad (26)$$

Proof. In this proof we shall simply denote by $U(t) = U_{ij}(t)$ for a pair of indices (i, j) .

Step 1. For any given $1 \leq i \neq j \leq m$, multiply the Equation (23) with $U_{ij}(t) = U(t)$ and by Assumption (4) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} U^2(t) &= -a_i U^2(t) - (a_i - a_j) u_j(t) U(t) + \sum_{\ell=1}^m (w_{i\ell} - w_{j\ell}) f_{\ell}(u_{\ell}) U(t) \\ &\quad + k(\varphi_i(\rho) - \varphi_j(\rho)) u_i(t) U(t) + k\varphi_j(\rho)(u_i(t) - u_j(t)) U(t) \\ &\quad + (J_i - J_j) U(t) - P U^2(t) \sum_{\ell=1}^m \Gamma(u_{\ell}(t)) \\ &\leq -a U^2(t) + a^* |u_j(t) U(t)| + m W^* \beta |U(t)| + k[(1 - \eta_i \rho^2) - (1 - \eta_j \rho^2)] u_i(t) U(t) \\ &\quad + k(1 - \eta_j \rho^2(t))(u_i(t) - u_j(t)) U(t) + J^* |U(t)| - P U^2(t) \sum_{\ell=1}^m \Gamma(u_{\ell}(t)) \\ &= -(a - k) U^2(t) + (m W^* \beta + J^*) |U(t)| + a^* |u_j(t)| |U(t)| + k(\eta_j - \eta_i) \rho^2(t) u_i(t) U(t) \\ &\quad - k \eta_j \rho^2(t) U^2(t) - P U^2(t) \sum_{\ell=1}^m \Gamma(u_{\ell}(t)) \\ &\leq -(a - k) U^2(t) + (m W^* \beta + J^*) |U(t)| + a^* |u_j(t)| |U(t)| + k \eta^* \rho^2(t) |u_i(t)| |U(t)| \\ &\quad - P U^2(t) \sum_{\ell=1}^m \Gamma(u_{\ell}(t)), \quad \text{for } t > 0. \end{aligned} \quad (27)$$

Step 2. We treat the third term and the fourth term on the rightmost side of the differential inequality (27). According to Theorem 2 and the finite-time bounding estimate (21) and (22), we have

$$\limsup_{t \rightarrow \infty} \left[\sum_{i=1}^m |u_i(t)|^2 + |\rho(t)|^2 \right] < Q,$$

where $Q > 0$ is a uniform constant given in (20). Thus for any given initial state $g^0 \in \mathbb{R}^{m+1}$ there exists a finite time $T(g^0) \geq 0$, which may depend on g^0 , such that the solution $g(t; g^0) = (u_1(t), \dots, u_m(t), \rho(t))$ is ultimately bounded:

$$\sum_{i=1}^m |u_i(t)|^2 + |\rho(t)|^2 < Q, \quad \text{for } t > T(g^0). \quad (28)$$

Therefore, we have

$$a^* |u_j(t)| < a^* Q^{1/2} \quad \text{and} \quad \rho^2(t) |u_i(t)| < Q^{3/2}, \quad \text{for } t > T(g^0), \quad 1 \leq i, j \leq m. \quad (29)$$

Substitute (29) in two terms $a^*|u_j(t)||U(t)|$ and $k\eta^*\rho^2(t)|u_i(t)||U(t)|$ on the rightmost side of (27). We see that, for any given initial state g^0 , the solution $U(t) = U_{ij}(t)$ of the differencing Equation (23) satisfies the following differential inequality:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} U^2(t) + PU^2(t) \sum_{\ell=1}^m \Gamma(u_\ell(t)) \\ & \leq -(a-k)U^2(t) + (mW^*\beta + J^*)|U(t)| + (a^*Q^{1/2} + k\eta^*Q^{3/2})|U(t)| \\ & = -(a-k)U^2(t) + (mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)|U(t)|, \text{ for } t > T(g^0). \end{aligned} \quad (30)$$

Next we handle the neuron coupling term $PU^2(t) \sum_{\ell=1}^m \Gamma(u_\ell(t))$ in (30), which can be called the weak interneuron coupling. Since (28) implies that $|u_\ell(t)| \leq \sqrt{Q}$ for $t > T(g^0)$ and for all $1 \leq \ell \leq m$, we have

$$\sum_{\ell=1}^m \Gamma(u_\ell(t)) = \sum_{\ell=1}^m \frac{1}{1 + \exp[-r(u_\ell(t) - V)]} \geq \frac{m}{1 + \exp\{r(\sqrt{Q} + |V|)\}} \quad (31)$$

for $t > T(g^0)$. Substitute (31) in the inequality (30). Then we get

$$\begin{aligned} & \frac{d}{dt} U^2(t) + 2 \left[(a-k) + \frac{mP}{1 + \exp\{r(\sqrt{Q} + |V|)\}} \right] U^2(t) \\ & \leq 2(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)|U(t)|, \text{ for } t > T(g^0). \end{aligned} \quad (32)$$

Step 3. Finally we have to tackle the nonlinear differential Gronwall inequality (32). Denote by

$$\mu = (a-k) + \frac{mP}{1 + \exp\{r(\sqrt{Q} + |V|)\}}. \quad (33)$$

We see that (32) yields the following differential inequality

$$\begin{aligned} & \frac{d}{dt} U^2(t) + 2\mu U^2(t) \leq 2(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)|U(t)| \\ & \leq \mu U^2(t) + \frac{1}{\mu} (mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)^2, \quad t > T(g^0). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{d}{dt} U^2(t) + \mu U^2(t) \leq \frac{(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)^2}{(a-k) + \frac{mP}{1 + \exp\{r(\sqrt{Q} + |V|)\}}} \\ & = \frac{(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)^2 (1 + \exp\{r(\sqrt{Q} + |V|)\})}{(a-k)(1 + \exp\{r(\sqrt{Q} + |V|)\}) + mP}, \quad t > T(g^0). \end{aligned} \quad (34)$$

Given any small prescribed gap $\varepsilon > 0$, if the threshold condition $P > P^*(\varepsilon)$ is satisfied, then the gap function $U(t) = U_{ij}(t) = u_i(t) - u_j(t)$ for any two neurons $\mathcal{N}_i$ and $\mathcal{N}_j$ satisfies

$$\begin{aligned} U^2(t) &\leq e^{-\mu(t-T(g^0))} U^2(T(g^0)) \\ &\quad + \frac{1}{\mu} \left[\frac{(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)^2(1 + \exp\{r(\sqrt{Q} + |V|)\})}{(a-k)(1 + \exp\{r(\sqrt{Q} + |V|)\}) + mP} \right] \\ &= e^{-\mu(t-T(g^0))} U^2(T(g^0)) \\ &\quad + \left[\frac{(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)(1 + \exp\{r(\sqrt{Q} + |V|)\})}{(a-k)(1 + \exp\{r(\sqrt{Q} + |V|)\}) + mP} \right]^2, \end{aligned} \quad (35)$$

for $t > T(g^0)$, $1 \leq i \neq j \leq m$, and for any initial state g^0 .

For any prescribed gap $\varepsilon > 0$, if the threshold condition $P > P^*(\varepsilon)$ shown by (24) is satisfied, then the inequality (35) shows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| &= \limsup_{t \rightarrow \infty} |U(t)| \\ &\leq \frac{(mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)(1 + \exp\{r(\sqrt{Q} + |V|)\})}{(a-k)(1 + \exp\{r(\sqrt{Q} + |V|)\}) + mP} \\ &< \frac{1}{mP^*(\varepsilon)} (mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*)(1 + \exp\{r(\sqrt{Q} + |V|)\}) = \varepsilon \end{aligned} \quad (36)$$

for any $1 \leq i \neq j \leq m$ and for any initial state g^0 . (36) means that we have shown

$$\deg_s(\mathcal{NW}) = \sup_{g^0 \in \mathbb{Z}} \left\{ \max_{1 \leq i < j \leq m} \left\{ \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| \right\} \right\} < \varepsilon$$

if the interneuron coupling strength P exceeds the threshold $P^*(\varepsilon)$. Thus the statement (25) and the exponential convergence rate μ in (26) are proved. According to the Definition (5), the proof of this theorem is completed. $\square$

Corollary 1. If the weak network coupling $-Pu_i \sum_{j=1}^m \Gamma(u_j)$ in the model (1) is replaced by the linear network coupling $-P \sum_{j=1}^m (u_i - u_j)$ or called strong network coupling, then the approximate synchronization stated in Theorem 3 remains valid with the simpler threshold

$$P^*(\varepsilon) = \frac{1}{m\varepsilon} (mW^*\beta + a^*Q^{1/2} + k\eta^*Q^{3/2} + J^*) \quad (37)$$

and the exponential convergence rate is $\mu = a - k + P$.

Proof. The linear network coupling $-P \sum_{j=1}^m (u_i - u_j)$ will only cause the following change of the last term on the right-hand side of the first inequality in (10),

$$\begin{aligned} C_1 \sum_{i=1}^m \sum_{j=1}^m P(u_i - u_j)u_i &= \sum_{1 \leq i < j \leq m} C_1 P[(u_i - u_j)u_i + (u_j - u_i)u_j] \\ &= \sum_{1 \leq i < j \leq m} C_1 P(u_i - u_j)^2 = \frac{1}{2} C_1 P \sum_{i=1}^m \sum_{j=1}^m (u_i - u_j)^2. \end{aligned} \quad (38)$$

The rest of proof can be verified through the same steps by simpler recalculation. $\square$

4. Extension to Memristive Hopfield–Hebbian Neural Networks

In this section we shall extend the study of approximate synchronization to memristive Hopfield neural networks with Hebbian learning rules.

There have been growing interests in recent years to adopt biologically inspired learning rules and algorithms to train artificial neural networks [11], especially for improving the efficiency and accuracy related to the backpropagation process. Hebbian learning rules [33,34] are used to update the training or learning weights in neural networks by dynamical equations and have demonstrated good performances in unsupervised machine learning tasks on certain datasets like MNIST and CIFAR with applications in image recognition and classification [35,36].

Here we consider a memristive Hopfield–Hebbian neural network composed of m neuron nodes, denoted by $\mathcal{NWH}$, and governed by the following mathematical model:

$$\begin{aligned} \frac{du_i}{dt} &= -a_i u_i + \sum_{j=1}^m w_{ij} f_j(u_j) + k_i \psi_i(\rho) u_i + J_i - P \sum_{j=1}^m (u_i - u_j), \quad 1 \leq i \leq m, \\ \frac{dw_{ij}}{dt} &= -c_{ij} w_{ij} + \lambda_{ij} f_i(u_i) f_j(u_j), \quad 1 \leq i, j \leq m, \\ \frac{d\rho}{dt} &= \sum_{i=1}^m \gamma_i u_i - b\rho, \quad t > 0. \end{aligned} \quad (39)$$

The u_i -equations and the ρ equation here in (39) are similar to (1) but the synaptic weights $w_{ij}(t)$ in this model is dynamically updated and governed by the w_{ij} equations, which is called Hebbian learning rules. The case with all coefficients $\lambda_{ij} > 0$ is called Hebbian learning and the case with all $\lambda_{ij} < 0$ is called anti-Hebbian learning. The initial conditions for the system (39) are denoted by $u_i(0) = u_i^0$, $w_{ij}(0) = w_{ij}^0$, $\rho(0) = \rho^0$. It can be denoted as a vector $g^0 = (u_i^0, w_{ij}^0, \rho^0)$. The state space of this memristive Hopfield–Hebbian neural network becomes $\mathbb{R}^{m(1+m)+1}$.

Several remarks can be made in regard to the features about this memristive Hopfield–Hebbian neural network model. The activation functions $\{f_i(x) : 1 \leq i \leq m\}$ in this model (39) can be heterogeneous for different neurons. Typical activation functions [7] are piecewise continuously differentiable and piecewise monotone functions, including Softplus $Sp(x)$, ReLU (x) , GeLU (x) , etc. Since the transmembrane potential for emulated biological neurons are uniformly bounded between -70 mV and $+40$ mV, one can plausibly assume these activation functions to be bounded.

The memristor window functions $\psi_i(\rho)$ in (39) and $\varphi_i(\rho)$ in (1) are typical quadratic window functions with different parameters for different neuron nodes. In general these window functions can also be heterogeneous types for different neuron nodes. Note that the Hebbian type w_{ij} -equations for updating weights carry the key factors of locality, synaptic dissipativity, and cooperative or competitive tendency possessed in biological neuron networks.

We make the following assumptions on this model (39):

$$\begin{aligned} a &= \min\{a_i : 1 \leq i \leq m\} > \frac{1}{2} k \eta^2, \quad c_{ij} > 0, \quad \lambda_{ij} \in \mathbb{R}, \quad 1 \leq i, j \leq m, \\ \eta &= \min\{\eta_i : 1 \leq i \leq m\} > 0, \quad |f_i(s)| \leq \beta, \quad \psi_i(s) = s(\eta_i - s), \quad 1 \leq i \leq m. \end{aligned} \quad (40)$$

The descriptions on the other parameters are the same as in Section 1. We denote the secondary parameters of relevant differences by

$$a^* = \max\{|a_i - a_j| : 1 \leq i, j \leq m\}, \quad W_0^* = \max\{|w_{i\ell}(0) - w_{j\ell}(0)| : 1 \leq i, j, \ell \leq m\},$$

$$\eta^* = \max\{|\eta_i - \eta_j| : 1 \leq i, j \leq m\}, \quad J^* = \max\{|J_i - J_j| : 1 \leq i, j \leq m\}.$$

Here is the extension of the main theorem for the memristive Hopfield–Hebbian neural networks presented by (39).

Theorem 4. *The memristive Hopfield–Hebbian neural network $\mathcal{NWH}$ presented by the model (39) with Assumption (40) is approximate exponentially synchronizable. For any prescribed gap $\varepsilon > 0$, there exists a threshold $\hat{P}^*(\varepsilon) > 0$ of network coupling strength,*

$$\hat{P}^*(\varepsilon) = \frac{1}{m\varepsilon} \left[a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right], \quad (41)$$

where G is shown in (52), such that if the interneuron coupling strength $P > \hat{P}^*(\varepsilon)$, then the neural network $\mathcal{NWH}$ is approximately synchronized,

$$\deg_s(\mathcal{NWH}) = \sup_{g^0 \in \mathbb{Z}} \left\{ \max_{1 \leq i < j \leq m} \left\{ \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| \right\} \right\} < \varepsilon \quad (42)$$

at a uniform exponential convergence rate

$$\zeta(P) = a - \frac{1}{2} k \eta + mP. \quad (43)$$

Proof. The proof will go through two steps. The first step aims to outcropping a uniform ultimate bound for all the solutions of this Hopfield–Hebbian neural network model. The second step will carry out the solution analysis of the interneuron differencing equations for pairwise neuron gap functions to reach the result of approximate exponential synchronization.

Step 1. First we want to find an ultimate bound for all solutions of this Hopfield–Hebbian neural network model. Multiply the w_{ij} -equation in (39) by $w_{ij}(t)$ and use Assumption (40). We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} w_{ij}^2(t) &= -c_{ij} w_{ij}^2(t) + \lambda_{ij} w_{ij}(t) f_i(u_i(t)) f_j(u_j(t)) \\ &\leq -c w_{ij}^2(t) + \lambda \beta^2 w_{ij}(t) \leq -\frac{1}{2} c w_{ij}^2(t) + \frac{1}{2c} \lambda^2 \beta^4, \end{aligned} \quad (44)$$

where $\lambda = \max\{|\lambda_{ij}|\}$ and $c = \min\{c_{ij} : 1 \leq i, j \leq m\} > 0$. It follows that

$$w_{ij}^2(t) \leq e^{-ct} w_{ij}^2(0) + \frac{\lambda^2 \beta^4}{c^2} \leq w_{ij}^2(0) + \frac{\lambda^2 \beta^4}{c^2} \quad t > 0. \quad (45)$$

Sum up the u_i -equations multiplied by $C_3 u_i(t)$ and the ρ -equation multiplied by $\rho(t)$. Without loss of generality, one can take all the initial weights $w_{ij}(0) \in \{0, 1\}$ depending on the connectivity between the two neuron nodes. Note that $\psi_i(\rho) = \rho(\eta_i - \rho) \leq \frac{1}{2} \eta^2 - \frac{1}{2} \rho^2$ and by the treatment of interneuron coupling terms (38), we can conduct the scaled group estimate:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\sum_{i=1}^m C_3 u_i^2 + \rho^2 \right) \leq \frac{1}{2} \frac{d}{dt} \left(\sum_{i=1}^m C_3 u_i^2 + \rho^2 \right) + \frac{1}{2} C_3 P \sum_{i=1}^m \sum_{j=1}^m (u_i - u_j)^2 \\
& = \sum_{i=1}^m C_3 \left[-a_i u_i^2 + \sum_{j=1}^m w_{ij}(t) f_j(u_j) u_i + k \psi_i(\rho) u_i^2 + J_i u_i \right] + \sum_{i=1}^m \gamma_i u_i \rho - b \rho^2 \\
& \leq \sum_{i=1}^m C_3 \left[-a u_i^2(t) + m \left(w_{ij}^2(0) + \frac{\lambda^2 \beta^4}{c^2} \right)^{1/2} \beta |u_i(t)| \right] \\
& \quad + \sum_{i=1}^m C_3 \left[\frac{1}{2} k (\eta^2 - \rho^2(t)) u_i^2(t) + J |u_i(t)| \right] + \sum_{i=1}^m \frac{\gamma_i^2}{2b} u_i^2(t) - \frac{b}{2} \rho^2(t) \\
& \leq - \sum_{i=1}^m C_3 \left(a - \frac{1}{2} k \eta^2 \right) u_i^2(t) + \sum_{i=1}^m \frac{\gamma_i^2}{2b} u_i^2(t) - \frac{b}{2} \rho^2(t) \\
& \quad + \sum_{i=1}^m C_3 \left(J + m \beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} \right) |u_i(t)| \\
& \leq - \frac{1}{2} \sum_{i=1}^m C_3 \left(a - \frac{1}{2} k \eta^2 \right) u_i^2(t) + \frac{1}{2} \left[\sum_{i=1}^m \frac{\gamma_i^2}{b} u_i^2(t) - b \rho^2(t) \right] \\
& \quad + \frac{1}{2} C_3 \left(J + m \beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} \right)^2 \left(a - \frac{1}{2} k \eta^2 \right)^{-1},
\end{aligned} \tag{46}$$

where $C_3 > 0$ is a scaling constant. By Assumption (40), we can choose the positive scaling constant

$$C_3 = \frac{\frac{m \gamma_i^2}{b} + \frac{1}{2} b}{a - \frac{1}{2} k \eta^2} \quad \text{so that} \quad C_3 \left(a - \frac{1}{2} k \eta^2 \right) - \frac{m \gamma_i^2}{b} = \frac{b}{2}. \tag{47}$$

Substitute (47) in (46) to obtain

$$\frac{d}{dt} \left(\sum_{i=1}^m C_3 u_i^2(t) + \rho^2(t) \right) + b \left(\sum_{i=1}^m u_i^2(t) + \rho^2(t) \right) \leq C_4, \quad \text{for } t > 0, \tag{48}$$

where

$$C_4 = \left(\frac{m \gamma_i^2}{b} + \frac{b}{2} \right) \left(J + m \beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} \right)^2 \left(a - \frac{1}{2} k \eta^2 \right)^{-2} \tag{49}$$

is a uniform constant independent of any initial state. The differential inequality (48) can be written as

$$\frac{d}{dt} \left(\sum_{i=1}^m C_3 u_i^2(t) + \rho^2(t) \right) + \sigma \left(\sum_{i=1}^m C_3 u_i^2(t) + \rho^2(t) \right) \leq C_4, \quad \text{for } t > 0, \tag{50}$$

where

$$\sigma = b \min \left\{ \frac{1}{C_3}, 1 \right\}. \tag{51}$$

We solve this differential Gronwall inequality (50) to confirm

$$\min \{C_3, 1\} \left[\sum_{i=1}^m u_i^2(t) + \rho^2(t) \right] \leq \sum_{i=1}^m C_3 u_i^2(t) + \rho^2(t) \leq e^{-\sigma t} \left[\sum_{i=1}^m C_3 u_i^2(0) + \rho^2(0) \right] + \frac{C_4}{\sigma},$$

which shows that

$$\limsup_{t \rightarrow \infty} \left(\sum_{i=1}^m u_i^2(t) + \rho^2(t) \right) < G = 1 + \frac{C_4}{\sigma \min\{C_3, 1\}}. \quad (52)$$

Here the C_3, C_4 and G are uniform positive constants independent of any initial state.

Step 2. Next we shall carry out the analysis of the dynamic gap functions $V_{ij}(t) = u_i(t) - u_j(t)$ among the neurons, $1 \leq i \neq j \leq m$, which satisfy the following differencing equation:

$$\begin{aligned} \frac{dV_{ij}}{dt} = & - (a_i V_{ij} + (a_i - a_j) u_j) + \sum_{\ell=1}^m (w_{i\ell} - w_{j\ell}) f_{\ell}(u_{\ell}) \\ & + k(\psi_i(\rho) u_i - \psi_j(\rho) u_j) + J_i - J_j - m P V_{ij}, \quad t > 0, \end{aligned} \quad (53)$$

where the last term comes from

$$P \left(\sum_{\ell=1}^m (u_i - u_{\ell}) - \sum_{\ell=1}^m (u_j - u_{\ell}) \right) = P \sum_{\ell=1}^m (u_i - u_j) = m P V_{ij}.$$

For any given $1 \leq i \neq j \leq m$, multiply the V_{ij} -equation by $V_{ij}(t)$. By Assumption (40) and (45) with the setting $w_{ij}(0) \in \{0, 1\}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} V_{ij}^2(t) = & -a_i V_{ij}^2(t) - (a_i - a_j) u_j(t) V_{ij}(t) + \sum_{\ell=1}^m (w_{i\ell}(t) - w_{j\ell}(t)) f_{\ell}(u_{\ell}) V_{ij}(t) \\ & + k(\psi_i(\rho) - \psi_j(\rho)) u_i(t) V_{ij}(t) + k \psi_j(\rho) V_{ij}^2(t) + (J_i - J_j) V_{ij}(t) - m P V_{ij}^2(t) \\ \leq & -a V_{ij}^2(t) + a^* |u_j(t)| |V_{ij}(t)| + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} |V_{ij}(t)| \\ & + k(\eta_i - \eta_j) \rho(t) |u_i(t)| |V_{ij}(t)| + k(\rho(t) \eta_j - \rho^2(t)) V_{ij}^2(t) + J^* |V_{ij}(t)| - m P V_{ij}^2(t) \\ \leq & -a V_{ij}^2(t) + a^* |u_j(t)| |V_{ij}(t)| + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} |V_{ij}(t)| \\ & + k\eta^* |\rho(t)| |u_i(t)| |V_{ij}(t)| + \frac{1}{2} k(\eta - \rho^2(t)) V_{ij}^2(t) + J^* |V_{ij}(t)| - m P V_{ij}^2(t) \\ \leq & - \left(a - \frac{1}{2} k\eta + mP \right) V_{ij}^2(t) \\ & + \left(a^* |u_j(t)| + k\eta^* |\rho(t)| |u_i(t)| + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right) |V_{ij}(t)|, \quad \text{for } t > 0. \end{aligned}$$

According to the asymptotic estimate (52), for any given initial state $g^0 = (u_i^0, w_{ij}^0, \rho^0)$ of the neural network, there is a finite time $T(g^0) > 0$ such that

$$\sum_{i=1}^m u_i^2(t) + \rho^2(t) < G, \quad \text{for } t > T_{g^0}.$$

Therefore, the above differential inequality implies that for $t > T_{g^0}$,

$$\begin{aligned} & \frac{d}{dt} V_{ij}^2(t) + 2 \left(a - \frac{1}{2} k\eta + mP \right) V_{ij}^2(t) \\ & \leq 2 \left(a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right) |V_{ij}(t)| \\ & \leq \left(a - \frac{1}{2} k\eta + mP \right) V_{ij}^2(t) + \frac{\left(a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right)^2}{a - \frac{1}{2} k\eta + mP}. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{d}{dt} V_{ij}^2(t) + \left(a - \frac{1}{2} k\eta + mP \right) V_{ij}^2(t) \\ & \leq \left(a - \frac{1}{2} k\eta + mP \right)^{-1} \left[a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right]^2, \end{aligned} \quad (54)$$

for $t > T_{g^0}$. Denote by

$$\xi = a - \frac{1}{2} k\eta + mP. \quad (55)$$

Integration of the differential Gronwall inequality (54) on the time interval $[T_{g^0}, t]$ shows that for any initial state g^0 and any pair of indices $1 \leq i \neq j \leq m$,

$$V_{ij}^2(t) \leq e^{\xi(t-T_{g^0})} V_{ij}^2(T_{g^0}) + \left[\frac{a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^*}{a - \frac{1}{2} k\eta + mP} \right]^2, \quad t > T_{g^0}. \quad (56)$$

For any prescribed small gap $\varepsilon > 0$, if the threshold condition $P > \hat{P}^*(\varepsilon)$ shown in (41) is satisfied, then from (56) we can conclude that for any $1 \leq i \neq j \leq m$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} |u_i(t) - u_j(t)| = \limsup_{t \rightarrow \infty} |V_{ij}(t)| \\ & \leq \frac{a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^*}{a - \frac{1}{2} k\eta + mP} \\ & < \frac{1}{m\hat{P}^*(\varepsilon)} \left(a^* G^{1/2} + k\eta^* G^{3/2} + 2m\beta \sqrt{1 + \frac{\lambda^2 \beta^4}{c^2}} + J^* \right) = \varepsilon. \end{aligned} \quad (57)$$

Consequently (42) is proved. The exponential convergence rate $\xi(P)$ in (43) follows from the exponential decay term in (56). The proof is completed. $\square$

5. Conclusions

We summarize the contributions in this paper and outlook for further research.

In view that complete synchronization to the zero gap for artificial neural networks is theoretically unrealistic due to multiple factors of mismatched or time-varying synaptic or learning weights and heterogenous activation functions of coupling components, we proposed a new quantitative concept of approximate synchronization for artificial neural networks in this paper.

Two novel models are introduced and investigated. One is the recursive Hopfield neural network with memristor coupling and sigmoidal interneuron coupling. Another is the memristive Hopfield neural network with weights dynamically updated and governed

by Hebbian learning rules, which has good performance in unsupervised data analysis and machine learning. Taking the approach of scaled group estimates of solutions, for both models we analytically proved the existence of the absorbing set with a sharp ultimate bound, which paves the way for the exploration of asymptotic dynamics.

In the main result Theorem 3 and the extension result Theorem 4 of this paper, we established a sufficient threshold condition to be satisfied by a single cross-network coupling strength parameter for achieving approximate exponential synchronization respectively for each model. The merit of this contribution lies in the explicit expression of the threshold and the exponential convergence rate in terms of the original parameters, which is predictable and easily adjustable according to any prescribed tolerance gap.

The theoretical framework and the rigorous analytic methodology in this work can be applied and generalized to many other types of artificial neural networks in unsupervised machine learning with mismatched parameters, heterogeneous activation devices and memristor devices, even for hybrid neural networks [37] and for hypernetworks [38].

In another recent paper [39] by this author, the research strategy of the dissipative approach toward synchronization is applied to the time-fractional memristive Hopfield–Hebbian neural networks and we proved the robust but non-exponential synchronization results. It is conceivable and promising that one can pursue this kind exploration for more time-fractional neural networks and stochastic neural networks.

Mathematical research on synchronization and predictable asymptotic dynamics of artificial neural networks is meaningful in deep learning performance and challenging in AI applications. It remains widely open, especially with respect to tackling the back-propagation operations in multi-layer neural networks, semi-supervised datasets, and computational efficiency and complexity issues.

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