

Article

The New Exponentiated Half Logistic-Generalized-Topp-Leone Family: Theory, Estimation, and Applications in Reliability Engineering

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Abstract

This work presents a new family of distributions (FoDs) called the exponentiated half logistic-generalized-Topp-Leone-G (EHL-GEN-TL-G) family. This family can be expressed as an infinite linear combination of exponentiated-G densities, which facilitates the derivation of its important statistical properties. The shapes of the density and hazard rate functions were investigated for special cases. The model parameters were estimated using six different methods, with the maximum likelihood technique emerging as the best approach. The consistency of the parameter estimates was then validated through Monte Carlo simulations. The exponentiated half logistic-generalized-Topp-Leone-Weibull (EHL-GEN-TL-W) distribution, a sub-model of the EHL-GEN-TL-G family, was applied to three sets of engineering failure time data. The results indicated that, based on in-sample goodness-of-fit criteria, the EHL-GEN-TL-W model provided the best fit among the several established models considered. Additionally, the EHL-GEN-TL-W regression model was developed, and its practical utility in modeling failure data was demonstrated.

Keywords: generalized-topp-leone-G; exponentiated half logistic-G; estimation; simulations; failure time; regression

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1. Introduction

Due to the shortcomings of traditional probability distributions in accurately representing empirical data, generalized distributions have emerged. These generalized distributions include additional parameters that increase their flexibility, allowing them to better model skewed data and control characteristics such as skewness, kurtosis, and tail behavior. Recent advancements in this domain have led to the development of several generalized families of distributions, including the extended Topp-Leone-G [1], the inverted Topp-Leone family [2], the Topp-Leone Odd Burr III-G [3], the type II exponentiated half logistic [4], and the Topp-Leone Kumaraswamy Marshall-Olkin generated family [5]. Other notable generalizations include the Fréchet Topp Leone-G [6], the type II Topp-Leone Dagum [7], the odd Weibull-Topp-Leone-G power series [8], the complementary geometric Topp-Leone-G [9], Type II Topp-Leone-G [10], Type II generalized Topp-Leone-G [11],

the new power Topp–Leone-G [12], Type II power Topp–Leone [13], Topp–Leone Odd Fréchet [14], Sine Topp–Leone-G [15], and the sine unit exponentiated half-logistic [16].

These advances have expanded the ability of researchers to describe and model real data in multiple fields, including engineering, health, economics, finance, and biological studies.

Ref. [17] pioneered the exponentiated half logistic-G (EHL-G) FoDs with cumulative distribution function (cdf)

$$F(z; \alpha, \delta, \nu) = \left[\frac{1 - \bar{G}^\delta(z; \nu)}{1 + \bar{G}^\delta(z; \nu)} \right]^\alpha, \quad (1)$$

and probability density function (pdf)

$$f(z; \alpha, \delta, \nu) = 2\alpha\delta \frac{\bar{G}^{\delta-1}g(z; \nu)[1 - \bar{G}^\delta(z; \nu)]^{\alpha-1}}{[1 + \bar{G}^\delta(z; \nu)]^{\alpha+1}}, \quad (2)$$

respectively, for $z, \alpha, \delta > 0$, where $g(z; \nu)$ and $G(z; \nu)$ are the pdf and cdf of the base-line distribution depending on parameter vector ν . We shall set $\delta = 1$ to avoid over-parameterization.

In this work, we develop a new family of FoDs called the exponentiated half logistic-generalized-Topp–Leone-G (EHL-GEN-TL-G). The primary motivation for this new family is twofold. First, many existing generalized families of distributions (for example, the EHL-G family, the standard TL-G family, and their various extensions) suffer from limited capacity to simultaneously model the full range of hazard rate shapes—particularly complex non-monotonic patterns such as bathtub followed by inverted bathtub—while also accommodating extreme skewness and heavy tails in a single, parsimonious framework. Second, the introduction of additional shape parameters aims to provide a superior fit for the highly skewed and heavy-tailed data frequently encountered in lifetime and failure time analyses. By integrating the characteristics of its parent distributions, the EHL-GEN-TL-G family achieves enhanced versatility and flexibility, making it exceptionally well-suited for modeling complex real-world data.

The research extends beyond theoretical development by presenting practical applications of the EHL-GEN-TL-G distribution through one of its special cases, the EHL-GEN-TL-Weibull, thus bridging the gap between theoretical concepts and real-world utility. In general, this study aims to offer an enhanced family of distributions capable of accommodating various data behaviors while remaining effectively applicable in practical scenarios.

The remainder of this paper is organized as follows: Section 2 presents the generalized family of distributions, including sub-families, special cases, density expansion, and statistical properties. Section 3 provides a discussion of parameter estimation. Section 4 details the findings of the simulations conducted. The development of the EHL-GEN-TL-W regression model is detailed in Section 5. To further demonstrate the applicability of the proposed family of distributions, Section 6 offers several data examples. Finally, Section 7 concludes the paper with summary remarks.

2. The New EHL-GEN-TL-G FoDs and Its Properties

Ref. [18] pioneered the Topp–Leone generalized FoDs. The cdf of the generalized-Topp–Leone-G (GEN-TL-G) FoDs is

$$F(z; b, \beta, \nu) = 1 - \left[1 - \left(1 - \bar{G}^2(z; \nu) \right)^b \right]^\beta, \quad (3)$$

and the pdf is

$$f(z; b, \beta, \nu) = 2b\beta g(z; \nu) \left[1 - \left(1 - \bar{G}^2(z; \nu) \right)^b \right]^{\beta-1} \left(1 - \bar{G}^2(z; \nu) \right)^{b-1} \bar{G}(z; \nu) \quad (4)$$

for $b, \beta, z > 0$, where $\bar{G}(z; \nu) = 1 - G(z; \nu)$, and ν is a parameter vector from the baseline distribution $G(\cdot)$. We take $\Theta_b(z; \nu) = 1 - \left(1 - \bar{G}^2(z; \nu) \right)^b$.

Now, utilizing the baseline cdf in Equation (1) to be the GEN-TL-G cdf yields a new FoD called EHL-GEN-TL-G FoD with cdf, reliability function, and pdf given as

$$F(z; \alpha, b, \beta, \nu) = \left[\frac{1 - (\Theta_b(z; \nu))^\beta}{1 + (\Theta_b(z; \nu))^\beta} \right]^\alpha, \quad (5)$$

$$R(z; \alpha, b, \beta, \nu) = 1 - \left[\frac{1 - (\Theta_b(z; \nu))^\beta}{1 + (\Theta_b(z; \nu))^\beta} \right]^\alpha \quad (6)$$

and

$$\begin{aligned} f(z; \alpha, b, \beta, \nu) &= 4\alpha b\beta g(z; \nu) \bar{G}(z; \nu) \left(1 - \bar{G}^2(z; \nu) \right)^{b-1} \\ &\times \frac{[\Theta_b(z; \nu)]^{\beta-1} \left[1 - (\Theta_b(z; \nu))^\beta \right]^{\alpha-1}}{\left[1 + (\Theta_b(z; \nu))^\beta \right]^{\alpha+1}}, \end{aligned} \quad (7)$$

for $\alpha, b, \beta > 0$. The hazard rate function (hrf) of the EHL-GEN-TL-G FoDs is

$$\begin{aligned} h(z; \alpha, b, \beta, \nu) &= 4\alpha b\beta g(z; \nu) \bar{G}(z; \nu) \left(1 - \bar{G}^2(z; \nu) \right)^{b-1} \frac{[\Theta_b(z; \nu)]^{\beta-1} \left[1 - (\Theta_b(z; \nu))^\beta \right]^{\alpha-1}}{\left[1 + (\Theta_b(z; \nu))^\beta \right]^{\alpha+1}} \\ &\times \left(1 - \left[\frac{1 - (\Theta_b(z; \nu))^\beta}{1 + (\Theta_b(z; \nu))^\beta} \right]^\alpha \right)^{-1}. \end{aligned} \quad (8)$$

2.1. EHL-GEN-TL-G Sub-Families

Table 1 presents various sub-families associated with the EHL-GEN-TL-G FoDs, accompanied by their corresponding nomenclature. Note that the compositional function $\Theta_b(z; \nu) = 1 - \left(1 - \bar{G}^2(z; \nu) \right)^b$.

2.2. Particular Cases

This subsection presents some particular cases of EHL-GEN-TL- FoDs by specifying $g(z; \nu)$ and $G(z; \nu)$ in Equations (5) and (7). The Lindley and Weibull distributions serve as baseline distributions in this article.

The shape parameters α , b , and β are crucial to enhancing the flexibility of the EHL-GEN-TL-G family. The parameter α mainly governs the tail weight and is a key to generating reversed-J shapes. The parameter b strongly influences the left tail and skewness, making it effective in creating bathtub-shaped hazard rates. Meanwhile, β controls the kurtosis and steepness of the density and is vital for capturing inverted bathtub hazard shapes. Their synergistic effect enables the family to model a wide spectrum of density shapes, including symmetric, skewed, reversed-J, and bimodal, as well as all fundamental hazard rate patterns, as visually confirmed in the accompanying figures.

Table 1. Sub-families of EHL-GEN-TL-G FoDs.

α	b	β	Resultant Distribution	Distribution Nomenclature
1	-	-	$F(z; b, \beta, \nu) = \left[\frac{1 - (\Theta_b(z; \nu))^\beta}{1 + (\Theta_b(z; \nu))^\beta} \right]$	Half-logistic-Generalized-Topp-Leone-G
-	-	1	$F(z; \alpha, b, \nu) = \left[\frac{[1 - \tilde{G}^2(z; \nu)]^b}{2 - [1 - \tilde{G}^2(z; \nu)]^b} \right]^\alpha$	Exponentiated Half-Logistic-Generalized-Topp-Leone-G
-	1	-	$F(z; \alpha, \beta, \nu) = \left[\frac{1 - (\tilde{G}^2(z; \nu))^\beta}{1 + (\tilde{G}^2(z; \nu))^\beta} \right]^\alpha$	Exponentiated Half-Logistic-Topp-Leone-G (EHL-TL-G)
1	1	-	$F(z; \beta, \nu) = \left[\frac{1 - (\tilde{G}^2(z; \nu))^\beta}{1 + (\tilde{G}^2(z; \nu))^\beta} \right]$	Half-Logistic-Topp-Leone-G (HL-TL-G)
-	1	1	$F(z; \alpha, \nu) = \left[\frac{1 - \tilde{G}^2(z; \nu)}{2 - \tilde{G}^2(z; \nu)} \right]^\alpha$	Exponentiated Half-Logistic-Topp-Leone-G
1	-	1	$F(z; b, \nu) = \left[\frac{[1 - \tilde{G}^2(z; \nu)]^b}{2 - [1 - \tilde{G}^2(z; \nu)]^b} \right]$	Half-Logistic-Generalized-Topp-Leone-G
1	1	1	$F(z; \nu) = \frac{1 - \tilde{G}^2(z; \nu)}{1 + \tilde{G}^2(z; \nu)}$	Half-Logistic-Topp-Leone-G (HL-TL-G)

Table 2 offers an empirical guide to the density shapes exhibited by the EHL-GEN-TL-G family under selected parameter settings. The suggested links between parameter regions and shape characteristics are based on numerical exploration and visual inspection, not formal mathematical derivations. These classifications should therefore be interpreted as heuristic observations rather than strict theoretical criteria.

Table 2. Shape regimes of the EHL-GEN-TL-G family with corresponding parameter conditions and baseline requirements.

Shape	Typical (α, b, β)	Baseline Condition
Symmetric	$\alpha \approx 1, b \approx 1$	Baseline near-symmetric (Weibull $d \approx 3.6$)
Positive skew	$\alpha < 1, b < 1$	Any
Negative skew	$\alpha > 1, b > 1$	Any
Reversed-J	$\alpha \leq 1, b \leq 1$	$g(0) > 0$ (Weibull $d \geq 1$)
Bimodal	$\alpha > 1, \beta > 1, b < 1$	Baseline unimodal (Weibull $d > 1$)

2.2.1. Exponentiated Half Logistic-Generalized-Topp-Leone-Lindley (EHL-GEN-TL-LIN) Distribution

Taking into account the Lindley distribution with the cdf and pdf given by $G(z; c) = 1 - (1 + \frac{cz}{1+c}) \exp(-cz)$ and $g(z; c) = \frac{c^2}{1+c} (1+z) \exp(-cz)$ as the baseline distribution, for $z, c > 0$, we have the EHL-GEN-TL-LIN distribution with cdf

$$F(z; \alpha, b, \beta, c) = \left[\frac{1 - \left(1 - (1 - L^2(z; c))^b\right)^\beta}{1 + \left(1 - (1 - L^2(z; c))^b\right)^\beta} \right]^\alpha,$$

and pdf

$$f(z; \alpha, b, \beta, c) = 4\alpha b \beta \frac{c^2}{1+c} (1+z) \exp(-cz) L(z; c) \left(1 - L^2(z; c)\right)^{b-1} \times \frac{\left[1 - (1 - L^2(z; c))^b\right]^{\beta-1} \left[1 - \left(1 - (1 - L^2(z; c))^b\right)^\beta\right]^{\alpha-1}}{\left[1 + \left(1 - (1 - L^2(z; c))^b\right)^\beta\right]^{\alpha+1}},$$

for $\alpha, b, \beta, c, z > 0$, where $L(z; c) = \left(1 + \frac{cz}{1+c}\right) \exp(-cz)$.

The density plots in Figure 1a illustrate that the EHL-GEN-TL-LIN distribution can accommodate data that are positively skewed, negatively skewed, nearly symmetric, and reversed-J. The hrf plot of the EHL-GEN-TL-LIN distribution in Figure 1b effectively captures both monotonic and complex hazard rate patterns.

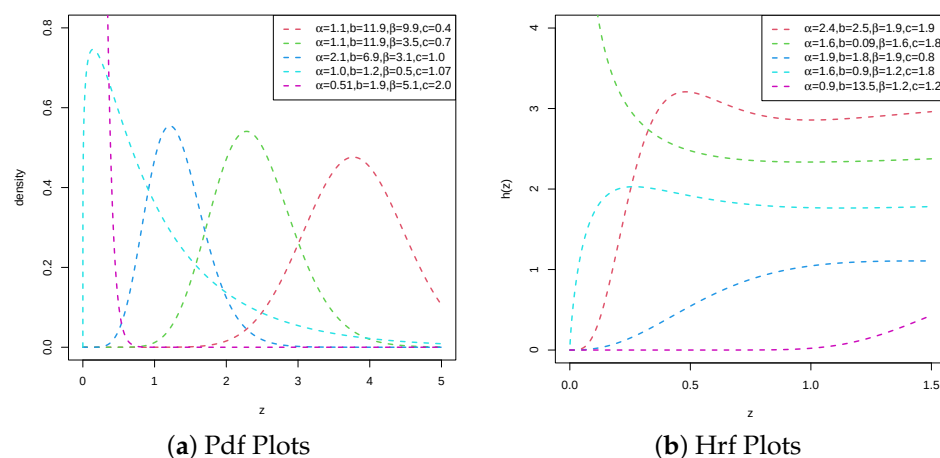


Figure 1. Plots of pdf and hrf for EHL-GEN-TL-LIN distribution.

Figure 2 displays 3D plots that illustrate the skewness and peakedness features of the EHL-GEN-TL-LIN distribution. These plots demonstrate that fixing various parameter combinations yields differing levels of skewness and kurtosis, thereby underscoring the distribution's sensitivity to parameter adjustments and showcasing the model's inherent versatility.

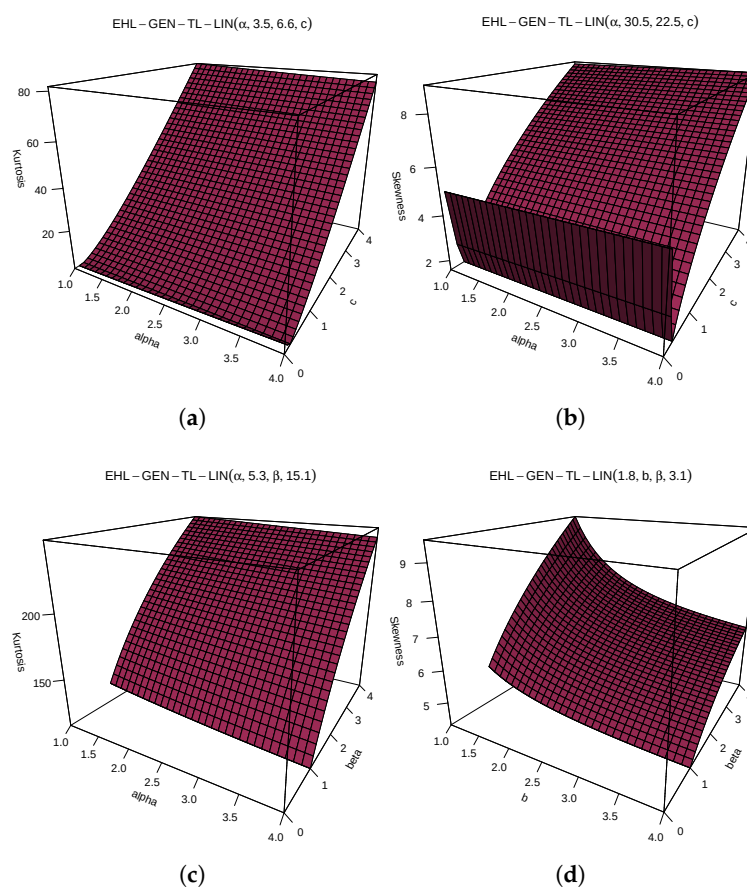


Figure 2. EHL-GEN-TL-LIN kurtosis and skewness plots for selected parameter values. (a) Kurtosis (α, c); (b) Skewness (α, c); (c) Kurtosis (α, β); (d) Skewness (b, β).

2.2.2. Exponentiated Half Logistic-Generalized-Topp-Leone-Weibull (EHL-GEN-TL-W) Distribution

The Weibull distribution has cdf $G(z;d) = 1 - \exp(-z^d)$ and pdf $g(z;d) = dz^{d-1} \exp(-z^d)$, for $z, d > 0$. If the Weibull distribution is applied to the EHL-GEN-TL-G distribution as the baseline, we obtain the EHL-GEN-TL-W distribution with cdf

$$F(z; \alpha, b, \beta, d) = \left(\frac{1 - [K_W(z; b, d)]^\beta}{1 + [K_W(z; b, d)]^\beta} \right)^\alpha$$

and pdf

$$f(z; \alpha, b, \beta, d) = 4\alpha b \beta d z^{d-1} \exp(-2z^d) \left(1 - [\exp(-z^d)]^2 \right)^{b-1} \times \frac{[K_W(z; b, d)]^{\beta-1} [1 - [K_W(z; b, d)]^\beta]^{\alpha-1}}{[1 + [K_W(z; b, d)]^\beta]^{\alpha+1}}$$

for $\alpha, b, \beta, d, z > 0$, where $K_W(z; b, d) = 1 - \left(1 - [\exp(-z^d)]^2 \right)^b$.

The density plots in Figure 3a demonstrate the adaptability of the EHL-GEN-TL-W distribution in accommodating diverse datasets, including those that are positively skewed, negatively skewed, nearly symmetric, and reversed-J-shaped. Similarly, the hrf plots in Figure 3b demonstrate the effectiveness of the EHL-GEN-TL-W distribution in capturing patterns such as decreasing, increasing, bathtub-shaped, inverted bathtub-shaped, and more complex geometries involving combinations of these shapes.

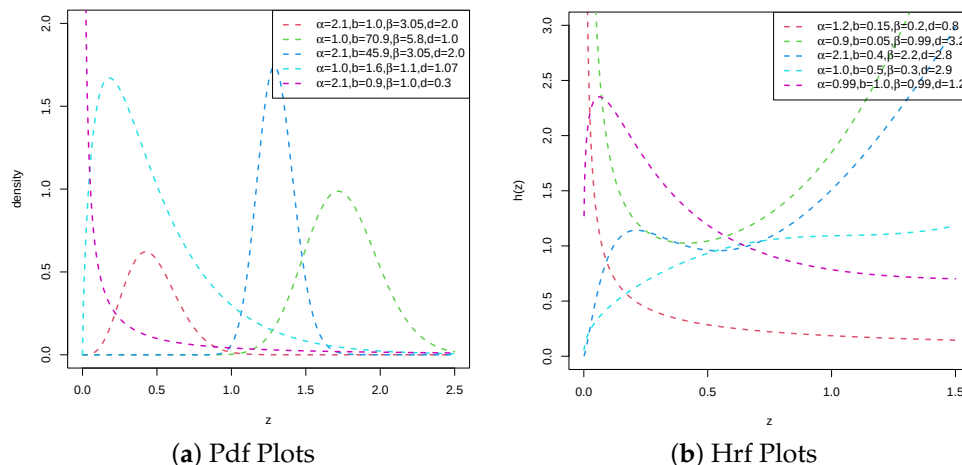


Figure 3. Plots showing the pdf and hrf for the EHL-GEN-TL-W distribution.

In Figure 4, a set of 3D graphs illustrates the variability in the skewness and kurtosis characteristics of the EHL-GEN-TL-W distribution. The plots reveal that fixing varying combinations of parameters yields distinct levels of skewness and kurtosis.

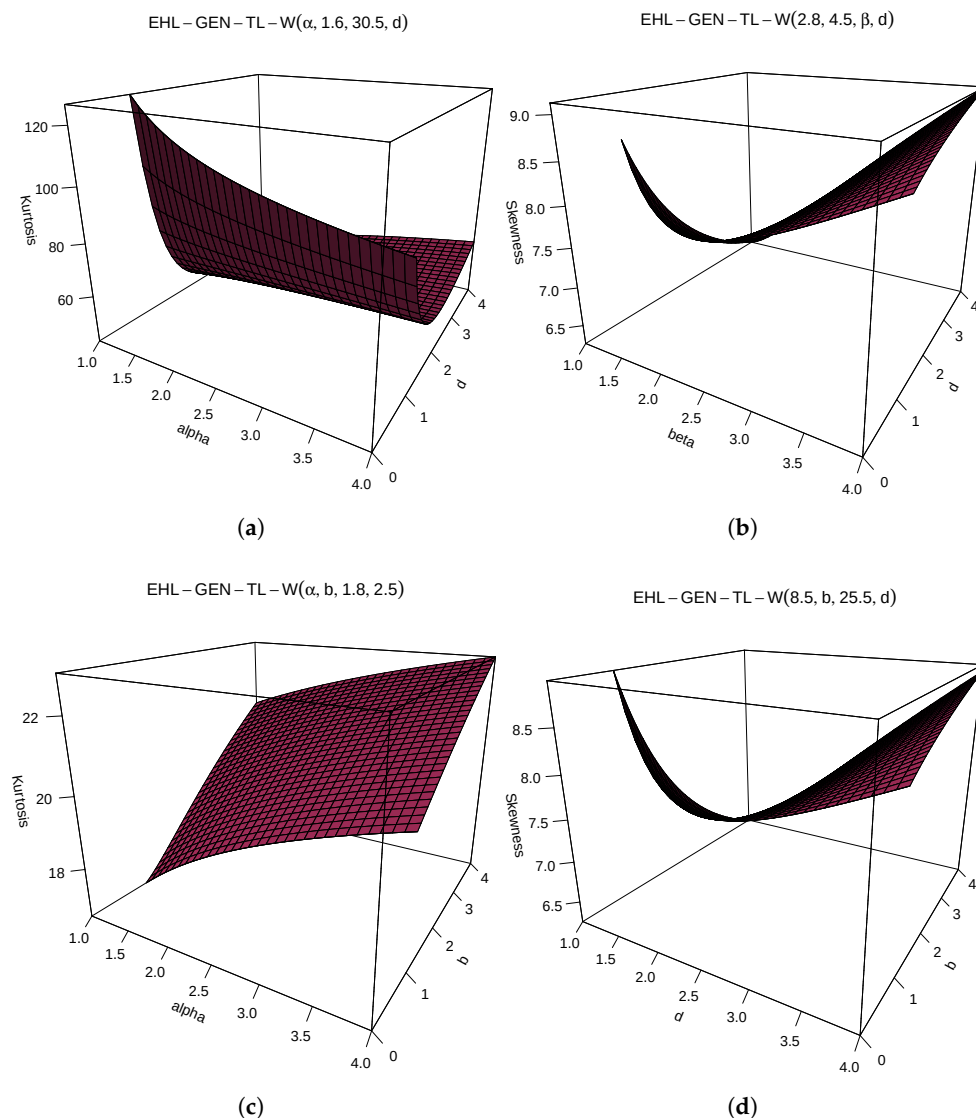


Figure 4. Kurtosis and skewness plots for the EHL-GEN-TL-W distribution. (a) Kurtosis (α, d); (b) Skewness (β, d); (c) Kurtosis (α, b); (d) Skewness (b, d).

2.3. Quantile Function

The quantile function (QF) is a highly versatile statistical concept with various applications, including probability estimation, percentile calculation, Monte Carlo simulations, and quantile regression. In the context of analysis of failure mechanisms, the QF is a valuable tool for estimating failure times, calculating reliability, and determining tolerance limits.

The QF of the EHL-GEN-TL-G FoDs is determined by inverting the cdf, represented as $F(x; \alpha, b, \beta, \nu) = q$, where $q \in [0, 1]$. Following the detailed derivations provided in Supplementary Material, the EHL-GEN-TL-G FoDs' QF is

$$Q_Z(q) = G^{-1} \left(1 - \sqrt{1 - \left[1 - \left(\frac{1 - q^{1/\alpha}}{1 + q^{1/\alpha}} \right)^{1/\beta} \right]^{1/b}} \right). \quad (9)$$

Quantile values of the EHL-GEN-TL-G FoDs are consequently found by solving Equation (9) using numerical methods in R software for a specified baseline distribution $G(\cdot)$.

2.4. Density Function Series Expansion

The pdf of the EHL-GEN-TL-G FoDs is

$$f(z; \alpha, b, \beta, \nu) = 4\alpha b \beta g(z; \nu) \bar{G}(z; \nu) \left[1 - \left(1 - \bar{G}^2(z; \nu) \right)^b \right]^{\beta-1} \left(1 - \bar{G}^2(z; \nu) \right)^{b-1} \\ \times \frac{\left[1 - \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^\beta \right]^{\alpha-1}}{\left[1 + \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^\beta \right]^{\alpha+1}}. \quad (10)$$

Applying the generalized binomial expansions $(1 - z)^k = \sum_{h=0}^{\infty} (-1)^h \binom{k}{h} z^h$ and $(1 + z)^{-k} = \sum_{h=0}^{\infty} (-1)^h \binom{k+h-1}{h} z^h$ to $|z| < 1$, we have

$$\left[1 - \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^\beta \right]^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^{\beta i},$$

$$\left[1 + \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^\beta \right]^{-(\alpha+1)} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j}{j} \left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^{\beta j},$$

$$\left(1 - [1 - \bar{G}^2(z; \nu)]^b \right)^{\beta(i+j+1)-1} = \sum_{k=0}^{\infty} (-1)^k \binom{\beta(i+j+1)-1}{k} \left(1 - \bar{G}^2(z; \nu) \right)^{bk},$$

and

$$\left(1 - \bar{G}^2(z; \nu) \right)^{b(k+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b(k+1)-1}{m} \bar{G}^{2m}(z; \nu).$$

Now with $\bar{G}^{2m+1}(z; \nu) = (1 - G(z; \nu))^{2m+1}$, and the generalized binomial expansion of

$$(1 - G(z; \nu))^{2m+1} = \sum_{p=0}^{\infty} (-1)^p \binom{2m+1}{p} G^p(z; \nu),$$

we write

$$f(z; \alpha, b, \beta, \nu) = 4\alpha b \beta \sum_{i,j,k,m,p=0}^{\infty} (-1)^{i+j+k+m+p} \binom{\alpha-1}{i} \binom{\alpha+j}{j} \binom{\beta(i+j+1)-1}{k} \\ \times \binom{b(k+1)-1}{m} \binom{2m+1}{p} \binom{p+1}{p+1} g(z; \nu) G^p(z; \nu) \\ = \sum_{p=0}^{\infty} Y_{p+1} g_{p+1}(z; \nu), \quad (11)$$

where $g_{p+1}(z; \nu) = (p+1)G^p(z; \nu)g(z; \nu)$ is the exponentiated-G (Expon-G) distribution with power parameter $(p+1)$, and

$$Y_{p+1} = 4\alpha b \beta \sum_{i,j,k,m=0}^{\infty} (-1)^{i+j+k+m+p} \binom{\alpha-1}{i} \binom{\alpha+j}{j} \binom{\beta(i+j+1)-1}{k} \\ \times \binom{b(k+1)-1}{m} \binom{2m+1}{p} \left(\frac{1}{p+1} \right) \quad (12)$$

is the linear component. Consequently, the EHL-GEN-TL-G FODs can be expressed as an unbounded linear combination of Expon-G densities, allowing for the direct derivation of various crucial statistical properties.

2.5. Moments and Incomplete Moments

Moments enable the exploration and quantification of central tendency, dispersion, skewness, kurtosis, and other higher-order distributional attributes of data. Suppose Y_{p+1} represents an Expon-G distributed random variable with a power parameter $(p + 1)$; then the r^{th} moment of the EHL-GEN-TL-G FoDs can be expressed as

$$E(Z^r) = \sum_{p=0}^{\infty} Y_{p+1} E(Y_{p+1}^r),$$

where $E(Y_{p+1}^r)$ represents the r^{th} moment of Y_{p+1} , and Y_{p+1} is defined in Equation (12). The r^{th} incomplete moment can be derived as

$$I_Z(t) = \int_0^t z^r f(z) dz = \sum_{p=0}^{\infty} Y_{p+1} I_{p+1}(t; r, \nu),$$

where $I_{p+1}(t; r, \nu) = \int_0^t z^r g_{p+1}(z; \nu) dz$ defines the incomplete moment of Y_{p+1} . Additionally, the moment generating function (MGF) of Z is

$$M_Z(t) = \sum_{p=0}^{\infty} Y_{p+1} E(e^{tY_{p+1}}),$$

with $E(e^{tY_{p+1}})$ representing the *mgf* of Y_{p+1} , and Y_{p+1} is specified in Equation (12).

Mastery of incomplete moment-based techniques is essential for deepening understanding of distributional analysis and the multifaceted nature of inequality. Incomplete moments are essential for the construction and interpretation of Bonferroni and Lorenz curves, which are indispensable tools for analyzing income inequality and have a wide range of applications in economics, finance, and social sciences.

2.6. Entropy

Entropy is crucial in failure mechanisms because it quantifies uncertainty and provides insight into the randomness within a system, aiding in risk assessment. In addition, it helps in predictive modeling and in optimizing designs to reduce failure probabilities. Rényi entropy, introduced by [19], for a random variable Z that follows the EHL-GEN-TL-G FoDs is expressed as follows:

$$\begin{aligned} I_R(\xi) &= \frac{\log \left[\int_{-\infty}^{\infty} f_{EHL-GEN-TL-G}^{\pi}(z; \alpha, b, \beta, \nu) dz \right]}{1 - \xi} \\ &= \frac{\log \left[\sum_{j=0}^{\infty} \varphi_j e^{[(1-\xi)I_{REG}]} \right]}{1 - \xi}, \xi \neq 1, \xi > 0, \end{aligned}$$

where

$$\begin{aligned} \varphi_j &= (4\alpha b \beta)^{\xi} \sum_{e,f,h,i=0}^{\infty} (-1)^{e+f+h+i+p} \binom{\xi(\alpha-1)}{e} \binom{\xi(\alpha+1)+f-1}{f} \\ &\times \binom{\beta(e+f+\xi)-e}{h} \binom{bh+\xi(b-1)}{i} \binom{2i+\xi}{j} \left(\frac{1}{\frac{j}{\xi}+1} \right), \end{aligned}$$

and $I_{REG} = \left[\int_0^{\infty} \left(\left(\frac{j}{\xi} + 1 \right) [g(z; \nu) G^{\frac{j}{\xi}}(z; \nu)] \right)^{\xi} dz \right]$ is the Rényi entropy of the Expon-G distribution with power parameter $(\frac{j}{\xi} + 1)$. Consequently, the Rényi entropy of the EHL-

GEN-TL-G FoDs can be directly derived from the Rényi entropy of the Expon-G distribution. Refer to Supplementary Material for detailed derivations.

2.7. Order Statistics

Order statistics are essential analytical tools that find widespread application in system failure analysis. They facilitate understanding of component lifetimes and system reliability, empowering engineers to optimize design, identify critical failure points, and improve predictive maintenance. Let $Z_1, Z_2, \dots, Z_n$ denote independent and identically distributed random variables that follow the EHL-GEN-TL-G FoDs. The pdf of the i^{th} order statistic derived from the EHL-GEN-TL-G FoDs is

$$f_{i:n}(z) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{r=0}^{\infty} \binom{n-i}{j} \omega_{r+1} g_{r+1}(z; \nu),$$

where the function $B(\cdot, \cdot)$ is the beta function, $g_{r+1}(z; \nu) = (r+1)g(z; \nu)G^r(z; \nu)$ is the Expon-G distribution with the power parameter $(r+1)$, and

$$\begin{aligned} \omega_{r+1} &= 4ab\beta \sum_{l,m,p,q=0}^{\infty} (-1)^{l+m+p+q+r} \binom{\alpha(j+i)-1}{l} \binom{\alpha(j+i+1)+m-1}{m} \\ &\times \binom{\beta(1+l+m)-1}{p} \binom{b(p+1)-1}{q} \binom{2q+1}{r} \left(\frac{1}{r+1}\right). \end{aligned}$$

Based on this analysis, we can infer that the pdf of the i^{th} order statistic for the EHL-GEN-TL-G FoDs can be represented as an infinite linear combination of the Expon-G densities. For further elaboration on the derivations, please consult Supplementary Material.

3. Estimation Methods

In this section, various estimation methods are employed to estimate the unknown parameters of the EHL-GEN-TL-G FoDs. These methods encompass maximum likelihood (ML), Anderson-Darling (AD), right-tail AD (RTAD), ordinary least squares (OLS), weighted least squares (WLS), and maximum product of spacings (MPS) (See [20–23]).

3.1. Maximum Likelihood

Given a random sample of size n , where each observation Z_i adheres to the EHL-GEN-TL-G FoDs distribution, and considering $\Delta = (\alpha, b, \beta, \nu)^T$ as the parameter vector, the log-likelihood function, $\ell(\Delta)$, is

$$\begin{aligned} \ell(\Delta) &= n \ln(4ab\beta) + \sum_{i=1}^n \ln[g(z_i; \nu)] + \sum_{i=1}^n \ln[\bar{G}(z_i; \nu)] + (\beta-1) \sum_{i=1}^n \ln[\Theta_b(z_i; \nu)] \\ &+ (b-1) \sum_{i=1}^n \ln\left(1 - \bar{G}^2(z_i; \nu)\right) + (\alpha-1) \sum_{i=1}^n \ln\left[1 - (\Theta_b(z_i; \nu))^{\beta}\right] \\ &- (\alpha+1) \sum_{i=1}^n \ln\left[1 + (\Theta_b(z_i; \nu))^{\beta}\right]. \end{aligned} \quad (13)$$

The numerical solution of the nonlinear equations $\left[\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \nu_k}\right]^T = \mathbf{0}$ using iterative methods in R software Version 4.5.2. for a specific baseline cdf $G(x; \nu)$ produces the ML estimates of a, b, β , and ν_k , respectively. Refer to Supplementary Material for individual components of the score vector.

It is important to note that, to obtain confidence intervals for the parameters $(\alpha, b, \beta, \nu_k)$ and perform hypothesis tests based on Equation (13), it is necessary to have access to the observed information matrix, which can be calculated as follows:

$$J(\Delta) = \begin{bmatrix} J_{aa}(\Delta) & J_{ab}(\Delta) & J_{a\beta}(\Delta) & J_{av}(\Delta) \\ J_{ba}(\Delta) & J_{bb}(\Delta) & J_{b\beta}(\Delta) & J_{bv}(\Delta) \\ J_{\beta a}(\Delta) & J_{\beta b}(\Delta) & J_{\beta\beta}(\Delta) & J_{\beta v}(\Delta) \\ J_{va}(\Delta) & J_{vb}(\Delta) & J_{v\beta}(\Delta) & J_{vv}(\Delta) \end{bmatrix}, \quad (14)$$

where $J_{i,j}(\Delta) = -\frac{\partial^2 \ell_n(\Delta)}{\partial \Delta_i \partial \Delta_j}$, $i, j = \alpha, b, \beta, \nu$, and ν is a vector of components k of the base-line parameters. Under the usual regularity assumptions (see [24]), as the sample size n approaches infinity, the estimated parameter vector $\hat{\Delta}$ follows an asymptotic normal distribution; that is, $\hat{\Delta} \sim N_{3+\rho}(\Delta, I^{-1}(\Delta))$. Here, $I(\Delta)$ denotes the anticipated information matrix. Importantly, this asymptotic behavior persists even when the expected information matrix $I(\Delta)$ is substituted with the observed information matrix $J(\hat{\Delta})$ computed at the estimated parameter vector $\hat{\Delta}$.

3.2. Minimum Distance Methods

The underlying principle of the minimum distance estimation methods is to find the parameter estimates that result in the best fit between the estimated cdf and the observed empirical distribution function. This is achieved by minimizing a selected goodness-of-fit statistic, which quantifies the discrepancy between the two distributions. In this subsection, we shall consider two such techniques, AD and RTAD.

AD and RTAD Methods

The AD and RTAD estimates of the EHL-GEN-TL-G FoDs vector of parameters are, respectively, found by minimizing the functions

$$AD(\Delta) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\log(F(z_{(i)}|\alpha, b, \beta, \nu)) + \log(S(z_{(i)}|\alpha, b, \beta, \nu)) \right],$$

and

$$RTAD(\Delta) = \frac{n}{2} - 2 \sum_{i=1}^n F(z_{(i)}; \alpha, b, \beta, \nu) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(z_{(n-i+1)}; \alpha, b, \beta, \nu)$$

with respect to α, b, β , and ν_k , respectively, where $F(\cdot)$ is defined by Equation (5). $S(\cdot) = 1 - F(\cdot)$ is evaluated in ordered observations, often in reverse order to emphasize the right tail.

3.3. OLS and WLS Methods

Let $z_{1:n} \leq z_{2:n} \leq z_{3:n} \leq \dots \leq z_{n:n}$ be the order statistics of a random sample of size n drawn from the EHL-GEN-TL-G FoDs. The OLS and WLS estimates of the EHL-GEN-TL-G FoDs vector of parameters are, respectively, found by minimizing the following functions:

$$OLS(\Delta) = \sum_{i=1}^n \left(F(z_{(i)}|\alpha, b, \beta, \nu) - \frac{i}{n+1} \right)^2,$$

and

$$WLS(\Delta) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F(z_{(i)}|\alpha, b, \beta, \nu) - \frac{i}{n+1} \right)^2.$$

with respect to α, b, β, ν_k .

3.4. MPS Method

The MPS method ([25,26]), serves as an alternative to the traditional ML estimation approach. Their research showcased that MPS estimates exhibit favorable statistical characteristics, demonstrating efficiency comparable to ML estimates. Within the framework of the EHL-GEN-TL-G FoDs, the MPS parameter vector is determined by maximizing the function

$$MPS(\Delta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \left[F(z_{(i)}; (\alpha, b, \beta, \nu)) - F(z_{(i-1)}; (\alpha, b, \beta, \nu)) \right],$$

with respect to α, b, β, ν_k .

4. Simulations

This section examines the consistency of the six parameter estimation techniques through Monte Carlo simulations conducted on the EHL-GEN-TL-W distribution. The estimation techniques considered are ML, AD, RTAD, OLS, WLS, and MPS. Tables 3 and 4 show the average bias (AvBIAS) and the root mean square error (RMSE) for each estimation technique for various parameter values. The AvBIAS and RMSE for an estimated parameter, for instance $\hat{\beta}$, are calculated as follows: $AvBIAS(\hat{\beta}) = \frac{\sum_{i=1}^N \hat{\beta}_i}{N} - \beta$ and $RMSE(\hat{\beta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\beta}_i - \beta)^2}{N}}$, respectively.

Table 3. Simulation results for various estimation methods with parameter values $\alpha = 0.4, b = 0.4, \beta = 0.3, d = 1.5$.

n	Parameters	ML		AD		RTAD		OLS		WLS		MPS	
		RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS
20	α	1.0204 ₍₂₎	0.5046 ₍₂₎	1.1982 ₍₃₎	0.5692 ₍₄₎	1.0092 ₍₁₎	0.4903 ₍₁₎	1.3041 ₍₆₎	0.6592 ₍₆₎	1.2376 ₍₄₎	0.5903 ₍₅₎	1.0304 ₍₅₎	0.5384 ₍₃₎
	b	1.3779 ₍₁₎	0.8414 ₍₁₎	1.6321 ₍₆₎	0.9426 ₍₆₎	1.3821 ₍₂₎	0.8429 ₍₂₎	1.4213 ₍₅₎	0.8865 ₍₄₎	1.3863 ₍₃₎	0.8635 ₍₃₎	1.4025 ₍₄₎	0.9213 ₍₅₎
	β	0.9508 ₍₁₎	0.4717 ₍₁₎	0.9641 ₍₃₎	0.4812 ₍₂₎	0.9621 ₍₂₎	0.4821 ₍₃₎	0.9743 ₍₅₎	0.4832 ₍₄₎	0.9923 ₍₆₎	0.4977 ₍₆₎	0.9734 ₍₄₎	0.4876 ₍₅₎
	d	1.2938 ₍₂₎	0.4199 ₍₂₎	1.3021 ₍₄₎	0.4291 ₍₄₎	1.1993 ₍₁₎	0.3921 ₍₁₎	1.3512 ₍₆₎	0.5314 ₍₆₎	1.3217 ₍₅₎	0.5143 ₍₅₎	1.2957 ₍₃₎	0.4201 ₍₃₎
	Σ ranks		12		32		15		42		37		32
40	α	0.8551 ₍₂₎	0.3591 ₍₃₎	0.9929 ₍₆₎	0.3498 ₍₁₎	0.9034 ₍₄₎	0.5238 ₍₄₎	0.9074 ₍₄₎	0.4032 ₍₅₎	0.8532 ₍₁₎	0.3542 ₍₂₎	0.9355 ₍₅₎	0.4127 ₍₆₎
	b	1.0189 ₍₂₎	0.5342 ₍₁₎	1.2146 ₍₅₎	0.6389 ₍₅₎	1.0231 ₍₃₎	0.5441 ₍₂₎	1.2367 ₍₆₎	0.5539 ₍₃₎	1.0155 ₍₁₎	0.6001 ₍₄₎	1.2034 ₍₄₎	0.7269 ₍₆₎
	β	0.7444 ₍₁₎	0.3058 ₍₁₎	0.7526 ₍₂₎	0.3102 ₍₂₎	0.7621 ₍₃₎	0.3103 ₍₃₎	0.8058 ₍₅₎	0.4112 ₍₅₎	0.9032 ₍₆₎	0.5326 ₍₆₎	0.7638 ₍₄₎	0.3126 ₍₄₎
	d	1.1921 ₍₂₎	0.3796 ₍₁₎	1.2221 ₍₄₎	0.3841 ₍₂₎	1.1865 ₍₁₎	0.3921 ₍₃₎	1.2731 ₍₆₎	0.4136 ₍₅₎	1.2608 ₍₅₎	0.4241 ₍₆₎	1.1938 ₍₃₎	0.4013 ₍₄₎
	Σ ranks		13		27		22		39		31		36
100	α	0.6128 ₍₁₎	0.1914 ₍₁₎	0.7321 ₍₄₎	0.2141 ₍₅₎	0.6213 ₍₂₎	0.1983 ₍₂₎	0.7412 ₍₅₎	0.2113 ₍₄₎	0.6849 ₍₃₎	0.2003 ₍₃₎	0.8217 ₍₆₎	0.2264 ₍₆₎
	b	0.7593 ₍₁₎	0.2782 ₍₁₎	0.8469 ₍₆₎	0.3431 ₍₆₎	0.7921 ₍₂₎	0.2843 ₍₂₎	0.8416 ₍₅₎	0.3421 ₍₅₎	0.8026 ₍₃₎	0.2863 ₍₃₎	0.8249 ₍₄₎	0.3216 ₍₄₎
	β	0.5357 ₍₁₎	0.1654 ₍₁₎	0.5426 ₍₃₎	0.1728 ₍₄₎	0.5462 ₍₅₎	0.1799 ₍₅₎	0.5429 ₍₄₎	0.1721 ₍₃₎	0.5945 ₍₆₎	0.1811 ₍₆₎	0.5421 ₍₂₎	0.1719 ₍₂₎
	d	1.0641 ₍₃₎	0.2005 ₍₃₎	1.0763 ₍₆₎	0.2114 _(4,5)	1.0746 ₍₅₎	0.2147 ₍₆₎	1.0763 ₍₄₎	0.2114 _(4,5)	1.0635 ₍₂₎	0.2003 ₍₂₎	0.9923 ₍₁₎	0.1992 ₍₁₎
	Σ ranks		12		38.5		29		34.5		32		26
300	α	0.2229 ₍₂₎	0.0381 ₍₃₎	0.3260 ₍₆₎	0.0458 ₍₆₎	0.2341 ₍₄₎	0.0421 ₍₅₎	0.2243 ₍₃₎	0.0364 ₍₂₎	0.2148 ₍₁₎	0.0321 ₍₁₎	0.2514 ₍₅₎	0.0403 ₍₄₎
	b	0.5460 ₍₃₎	0.1256 ₍₃₎	0.5341 ₍₁₎	0.1162 ₍₁₎	0.5490 ₍₄₎	0.1286 ₍₄₎	0.5926 ₍₅₎	0.1368 ₍₅₎	0.5392 ₍₂₎	0.1212 ₍₂₎	0.6239 ₍₆₎	0.1423 ₍₆₎
	β	0.2887 ₍₁₎	0.0462 ₍₁₎	0.3221 ₍₄₎	0.0501 ₍₃₎	0.2931 ₍₂₎	0.0521 ₍₃₎	0.3141 ₍₃₎	0.0593 ₍₄₎	0.3247 ₍₅₎	0.0731 ₍₆₎	0.3296 ₍₆₎	0.0632 ₍₅₎
	d	0.7830 ₍₁₎	0.1152 ₍₁₎	0.7923 ₍₃₎	0.1205 ₍₃₎	0.7926 ₍₄₎	0.1237 ₍₄₎	0.8219 ₍₆₎	0.1314 ₍₆₎	0.7859 ₍₂₎	0.1187 ₍₂₎	0.7930 ₍₅₎	0.1246 ₍₅₎
	Σ ranks		15		27		30		34		21		42
900	α	0.1775 ₍₃₎	0.0154 ₍₃₎	0.1821 ₍₅₎	0.0213 ₍₆₎	0.1628 ₍₁₎	0.0142 _(1,5)	0.1699 ₍₂₎	0.0142 _(1,5)	0.1782 ₍₄₎	0.0169 ₍₄₎	0.1824 ₍₆₎	0.0175 ₍₅₎
	b	0.3156 ₍₂₎	0.0391 ₍₂₎	0.2993 ₍₁₎	0.0316 ₍₁₎	0.3421 _(5,5)	0.0469 _(5,5)	0.3292 ₍₄₎	0.0415 ₍₄₎	0.3184 ₍₃₎	0.0401 ₍₃₎	0.3421 _(5,5)	0.0469 _(5,5)
	β	0.1477 ₍₁₎	0.0104 ₍₂₎	0.1463 ₍₁₎	0.0109 ₍₃₎	0.1521 ₍₃₎	0.0164 ₍₅₎	0.1491 ₍₅₎	0.0116 ₍₂₎	0.1503 ₍₂₎	0.0142 ₍₄₎	0.1739 ₍₆₎	0.0341 ₍₆₎
	d	0.0530 ₍₂₎	0.0141 ₍₂₎	0.0821 ₍₆₎	0.0186 ₍₆₎	0.0631 ₍₅₎	0.0158 ₍₄₎	0.0598 ₍₃₎	0.0162 ₍₅₎	0.0305 ₍₁₎	0.0095 ₍₁₎	0.0613 ₍₄₎	0.0151 ₍₃₎
	Σ ranks		17		29		30.5		26.5		22		41

Data were generated using the inverse transform method based on the quantile function (Equation (9)), with numerical computation in R via the uniroot function. All parameters were constrained to be positive without upper bounds. For maximum likelihood estimation, starting values were set to 0.5 for all parameters; for other methods, true parameter values were used as starting points to ensure fair comparison. The nlm function in R was employed with a maximum of 500 iterations, achieving convergence in over 98% of replications; non-convergent runs were discarded and re-simulated, and

multiple starting values (0.3, 0.5, 0.7) were tested to avoid local optima, with the highest log-likelihood solution selected.

Three thousand ($N = 3000$) replications with sample sizes $n = 20, 40, 100, 300$, and 900 and initial parameter values $(\alpha, b, \beta, d) = (0.4, 0.4, 0.3, 1.5)$ and $(\alpha, b, \beta, d) = (0.2, 1.2, 1.0, 1.2)$ were utilized in the simulation experiments.

The ranking of each estimation method is based on the sum of the individual ranks of AvBIAS and RMSE for all four parameters (α, b, β, d) . For each sample size n and each parameter configuration, we assign rank 1 to the method with the smallest AvBIAS (or RMSE) for a given parameter, rank 2 to the second smallest, and so on. These ranks are summed across all parameters and both criteria to obtain a total rank sum for that method. The process is repeated across different sample sizes and parameter configurations. The final overall rank (Table 4) is obtained by summing these sums of rank in all scenarios; the method with the smallest cumulative rank sum receives the best final rank (1) and the largest receives the worst final rank (6).

Table 4. Simulation results for various estimation methods with parameter values $\alpha = 0.2$, $b = 1.2$, $\beta = 1.0$, $d = 1.2$.

n	Parameters	ML		AD		RTAD		OLS		WLS		MPS	
		RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS	RMSE	AvBIAS
20	α	0.9220 ₍₂₎	0.4354 ₍₂₎	1.0272 ₍₄₎	0.4632 ₍₃₎	0.9784 ₍₃₎	0.5592 ₍₅₎	1.3412 ₍₅₎	0.5234 ₍₄₎	0.8922 ₍₁₎	0.4233 ₍₁₎	2.0103 ₍₆₎	0.9592 ₍₆₎
	b	1.9637 ₍₆₎	0.5890 ₍₆₎	1.9402 ₍₅₎	0.5793 ₍₅₎	1.9321 ₍₄₎	0.5428 ₍₃₎	1.9235 ₍₃₎	0.5346 ₍₂₎	1.0358 ₍₁₎	0.4401 ₍₁₎	1.8752 ₍₂₎	0.5764 ₍₄₎
	β	1.8843 ₍₄₎	0.9361 ₍₄₎	1.8824 ₍₃₎	0.9324 ₍₃₎	1.9024 ₍₆₎	0.9436 ₍₆₎	1.8521 ₍₁₎	0.8921 ₍₁₎	1.8621 ₍₂₎	0.9021 ₍₂₎	1.9004 ₍₅₎	0.9421 ₍₅₎
	d	0.8988 ₍₂₎	0.1094 ₍₂₎	0.9001 ₍₃₎	0.1104 ₍₃₎	0.9104 ₍₄₎	0.1184 ₍₄₎	0.9121 ₍₅₎	0.1196 ₍₅₎	0.8841 ₍₁₎	0.0934 ₍₁₎	0.9126 ₍₆₎	0.1199 ₍₆₎
	Σ ranks	28		29		35		26		10		40	
40	α	0.6477 ₍₁₎	0.2946 ₍₁₎	0.7506 ₍₃₎	0.3686 ₍₃₎	0.7632 ₍₄₎	0.3808 ₍₅₎	0.7721 ₍₅₎	0.3721 ₍₄₎	0.6521 ₍₂₎	0.3051 ₍₂₎	1.5249 ₍₆₎	0.6359 ₍₆₎
	b	1.7675 ₍₂₎	0.4700 ₍₂₎	1.8346 ₍₆₎	0.4836 ₍₆₎	1.8299 _(4,5)	0.4803 _(4,5)	1.8299 _(4,5)	0.4803 _(4,5)	0.9947 ₍₁₎	0.3924 ₍₁₎	1.7734 ₍₃₎	0.4726 ₍₃₎
	β	1.1580 ₍₂₎	0.4728 ₍₂₎	1.1603 ₍₃₎	0.4828 ₍₃₎	1.1704 ₍₅₎	0.4903 ₍₅₎	1.1621 ₍₄₎	0.3828 ₍₄₎	1.1821 ₍₆₎	0.5121 ₍₆₎	1.1528 ₍₁₎	0.4704 ₍₁₎
	d	0.7709 ₍₁₎	0.1004 ₍₁₎	0.7923 _(2,5)	0.1099 _(2,5)	0.8034 ₍₅₎	0.1103 ₍₅₎	0.7923 _(2,5)	0.1099 _(2,5)	0.7834 ₍₄₎	0.1027 ₍₄₎	0.8155 ₍₆₎	0.1128 ₍₆₎
	Σ ranks	12		29		37		32		26		32	
100	α	0.4985 ₍₂₎	0.1919 ₍₂₎	0.5331 ₍₄₎	0.2141 ₍₄₎	0.6121 ₍₅₎	0.2697 ₍₅₎	0.4921 ₍₁₎	0.1901 ₍₁₎	0.5037 ₍₃₎	0.2001 ₍₃₎	1.0249 ₍₆₎	0.4359 ₍₆₎
	b	1.6197 ₍₃₎	0.3293 ₍₃₎	1.6265 ₍₄₎	0.3301 ₍₄₎	1.6137 ₍₂₎	0.3258 ₍₂₎	1.6401 ₍₅₎	0.3421 ₍₅₎	0.8921 ₍₁₎	0.1427 ₍₁₎	1.6454 ₍₆₎	0.3468 ₍₆₎
	β	0.8190 ₍₂₎	0.1129 ₍₂₎	0.8055 ₍₁₎	0.1101 ₍₁₎	0.8203 ₍₃₎	0.1203 ₍₃₎	0.8235 ₍₄₎	0.1248 ₍₄₎	0.8941 ₍₆₎	0.1428 ₍₆₎	0.8331 ₍₅₎	0.1369 ₍₅₎
	d	0.6442 ₍₁₎	0.0900 ₍₁₎	0.6623 ₍₄₎	0.0997 ₍₄₎	0.6587 ₍₃₎	0.0921 ₍₃₎	0.6625 ₍₅₎	0.0998 ₍₅₎	0.6536 ₍₂₎	0.0901 ₍₂₎	0.6664 ₍₆₎	0.1053 ₍₆₎
	Σ ranks	16		26		26		30		24		46	
300	α	0.2510 ₍₁₎	0.0669 ₍₁₎	0.2739 ₍₄₎	0.0835 ₍₄₎	0.3604 ₍₅₎	0.1336 ₍₅₎	0.2642 ₍₃₎	0.0800 ₍₃₎	0.2633 ₍₂₎	0.0742 ₍₂₎	0.8516 ₍₆₎	0.2539 ₍₆₎
	b	0.3400 ₍₁₎	0.2470 ₍₁₎	0.3427 ₍₃₎	0.2481 ₍₃₎	0.3401 ₍₂₎	0.2392 ₍₂₎	0.3465 ₍₄₎	0.2501 ₍₄₎	0.3907 ₍₆₎	0.2713 ₍₆₎	0.3651 ₍₅₎	0.2603 ₍₅₎
	β	0.6343 ₍₄₎	0.0322 ₍₄₎	0.6421 ₍₅₎	0.0395 ₍₅₎	0.6321 ₍₃₎	0.0299 ₍₃₎	0.6246 ₍₂₎	0.0214 ₍₄₎	0.6121 ₍₁₎	0.0123 ₍₁₎	0.6467 ₍₆₎	0.0401 ₍₆₎
	d	0.2789 ₍₁₎	0.0437 ₍₁₎	0.3022 ₍₅₎	0.0654 ₍₅₎	0.2994 ₍₄₎	0.0536 ₍₄₎	0.2801 ₍₂₎	0.0502 ₍₂₎	0.2831 ₍₃₎	0.0504 ₍₃₎	0.3155 ₍₆₎	0.0651 ₍₆₎
	Σ ranks	17		34		28		22		24		46	
900	α	0.1227 ₍₂₎	0.0184 ₍₂₎	0.1321 ₍₄₎	0.0211 ₍₄₎	0.1383 ₍₅₎	0.0436 ₍₅₎	0.1299 ₍₃₎	0.0201 ₍₃₎	0.1164 ₍₁₎	0.0106 ₍₁₎	0.1986 ₍₆₎	0.1067 ₍₆₎
	b	0.0697 ₍₄₎	0.0505 ₍₄₎	0.0583 ₍₃₎	0.0491 ₍₃₎	0.0428 ₍₂₎	0.0427 ₍₂₎	0.0736 ₍₅₎	0.0525 ₍₅₎	0.0401 ₍₁₎	0.0321 ₍₁₎	0.0816 ₍₆₎	0.0699 ₍₆₎
	β	0.3146 ₍₄₎	0.0293 ₍₄₎	0.3293 ₍₆₎	0.0311 ₍₆₎	0.3027 ₍₂₎	0.0212 ₍₂₎	0.3114 ₍₃₎	0.0263 ₍₃₎	0.2934 ₍₁₎	0.0143 ₍₁₎	0.3251 ₍₅₎	0.0304 ₍₅₎
	d	0.0087 ₍₁₎	0.0140 ₍₁₎	0.0115 ₍₅₎	0.0199 ₍₅₎	0.0106 ₍₄₎	0.0187 ₍₄₎	0.0103 _(2,5)	0.0184 _(2,5)	0.0103 _(2,5)	0.0057 _(2,5)	0.0205 ₍₆₎	0.0231 ₍₆₎
	Σ ranks	22		36		26		26		11		46	

Tables 3 and 4 include a row labeled ‘ Σ ranks,’ which represents the partial sum of the ranks. The subscript (in parentheses) next to each estimate indicates its rank for a specific metric. For example, in Table 4, the RMSE of $\hat{\beta}$ obtained using the RTAD method is shown as 1.9024₍₆₎ for $n = 20$. This means that the RTAD method is sixth among the other estimators in terms of the RMSE of $\hat{\beta}$. Furthermore, the AvBIAS of $\hat{\alpha}$ derived by the ML approach is 0.2946₍₁₎ for $n = 40$, which means the best AvBIAS of $\hat{\alpha}$ compared with the other estimates. Table 5 presents the partial and final ranks assigned to the estimation techniques used to estimate the EHL-GEN-TL-W distribution for different values of the model parameter. Figures 5 and 6 show line graphs showing a gradual decrease in RMSEs for the estimated parameters with increasing sample size for each of the estimation methods.

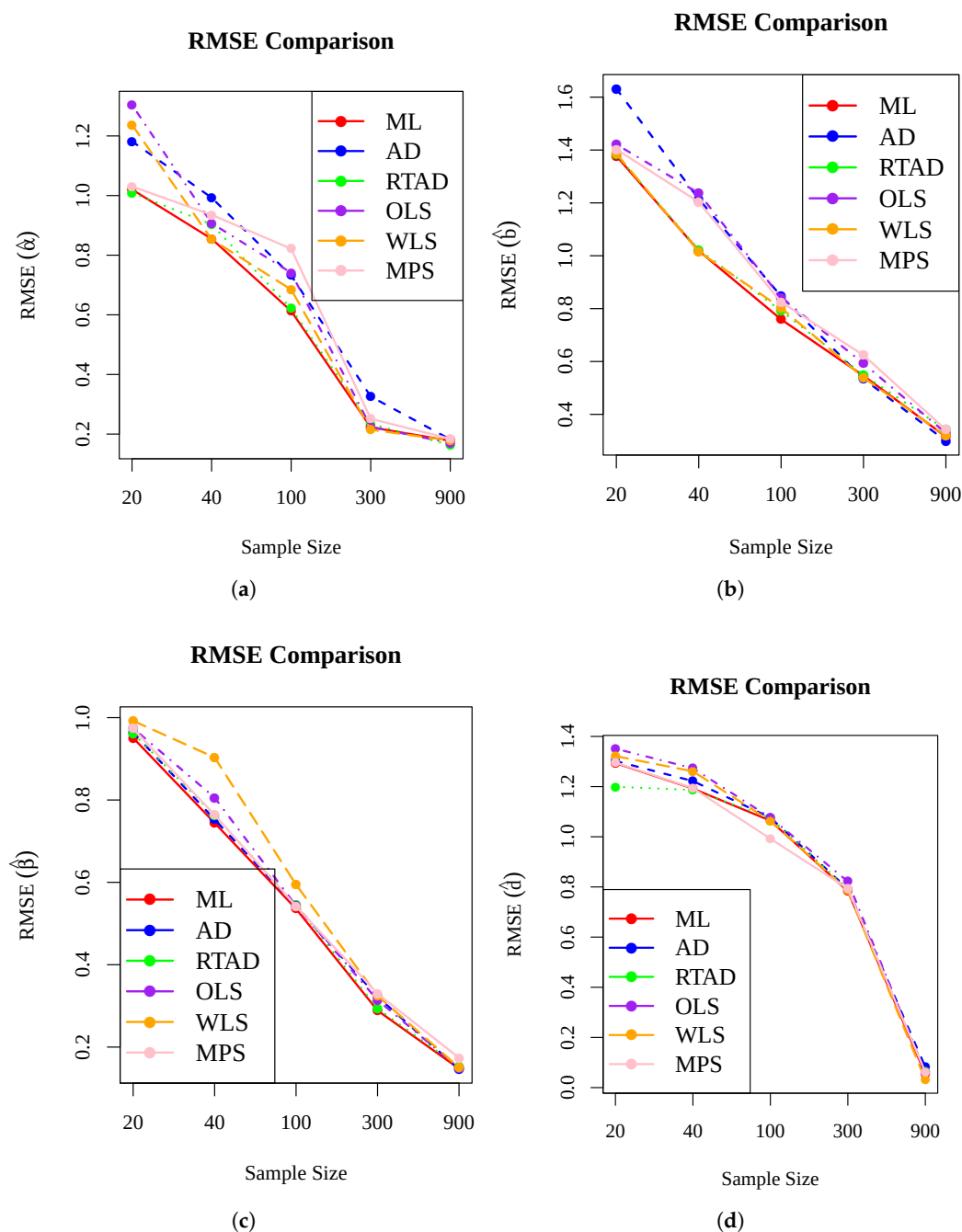


Figure 5. Plots illustrating the reduction in RMSEs for $(\alpha, b, \beta, d) = (0.4, 0.4, 0.3, 1.5)$. (a) RMSE for $\hat{\alpha}$; (b) RMSE for $\hat{b}$; (c) RMSE for $\hat{\beta}$; (d) RMSE for $\hat{d}$

Based on the findings presented in Tables 3 and 4, the following conclusions can be drawn:

- The EHL-GEN-TL-W distribution exhibited stability, as indicated by the modest values of AVBIAS and RMSE for its four parameters.
- RMSE and AvBIAS consistently decreased with increasing sample size in all estimating methods.
- Reliable estimates were achieved across all estimation methods, particularly when larger sample sizes were used.

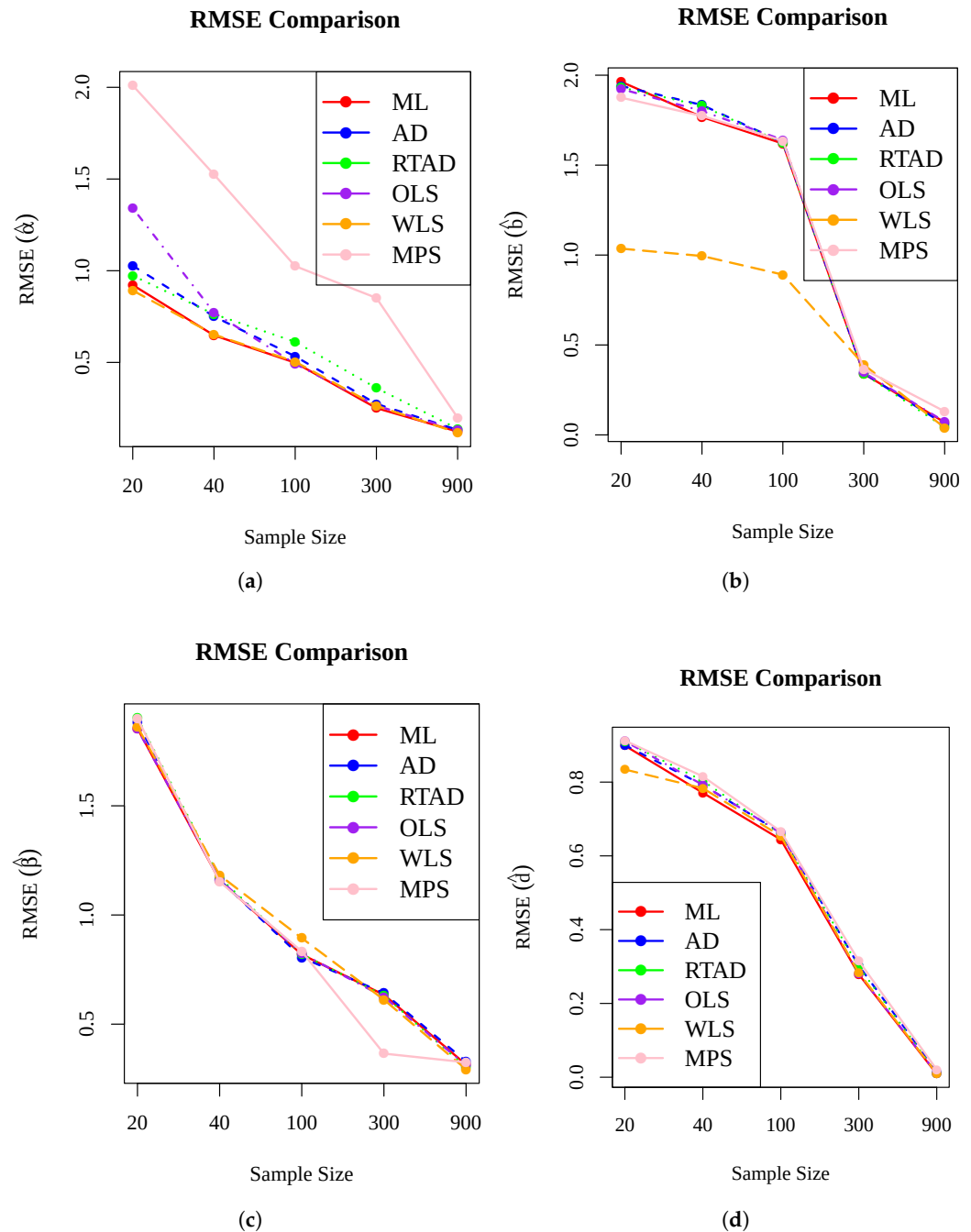


Figure 6. Plots depicting reduction in RMSEs for parameter values $(\alpha, b, \beta, d) = (0.2, 1.2, 1.0, 1.2)$. (a) RMSE for $\hat{\alpha}$; (b) RMSE for $\hat{b}$; (c) RMSE for $\hat{\beta}$; (d) RMSE for $\hat{d}$.

Table 5 demonstrates that ML emerged as the most effective technique to provide accurate estimates of the distribution parameters of EHL-GEN-TL-W. Consequently, we proceeded to estimate the unknown parameters of the EHL-GEN-TL-W distribution using the maximum likelihood estimation technique.

Table 5. Rankings of estimation methods for EHL-GEN-TL-W distribution across different values of the model parameters.

Parameters	n	ML	AD	RTAD	OLS	WLS	MPS
$\alpha = 0.4, b = 0.4, \beta = 0.3, d = 1.5$	20	1	3.5	2	6	5	3.5
	40	1	3	2	6	4	5
	100	1	6	3	5	4	2
	300	1	3	4	5	2	6
	900	1	4	5	2	3	6
$\alpha = 0.2, b = 1.2, \beta = 1.0, d = 1.2$	20	4	3	5	2	1	6
	40	1	3	6	4.5	2	4.5
	100	1	3.5	3.5	5	2	6
	300	1	5	4	2	3	6
	900	2	5	3.5	3.5	1	6
Σ ranks		14	39	38	41	27	51
Final ranking		1	4	3	5	2	6

5. The EHL-GEN-TL-W Regression Model

Conducting a regression analysis on lifetime data involves determining the distribution of a variable Z based on a set of covariates $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_z)^T$. If Z follows the EHL-GEN-TL-W distribution, the random variable $Y = \log(Z)$ follows the $\log$ -EHL-GEN-TL-W distribution. We present a regression model tailored for the EHL-GEN-TL-W distribution, capable of accommodating both censored and uncensored data. To ensure parameter identifiability and parsimony, we shall set the parameter $b = 1$ and introduce a systematic component d ; we can establish a relationship between the parameter d and the covariates using a log-linear link function. The logarithmic link function is defined as $d_j = \exp(\mathbf{u}_j^T \omega)$ for $j = 1, \dots, n$, and $\omega = (\omega_1, \omega_2, \dots, \omega_z)^T$ represents a vector that includes the regression coefficients. The survival function of $Z | u$ then becomes:

$$S(z | u) = 1 - \left(\frac{1 - [\exp(-z^{\exp(\mathbf{u}_j^T \omega)})]^{2\beta}}{1 + [\exp(-z^{\exp(\mathbf{u}_j^T \omega)})]^{2\beta}} \right)^\alpha. \quad (15)$$

The parametric regression model, known as EHL-GEN-TL-W, is denoted by Equation (15). If we consider A and S as two separate groups of individuals, where z_j represents the lifetime of individuals in the set A and S represents the censoring information, the overall log-likelihood function for the parameter vector $\Delta = (\alpha, \beta, \omega^T)^T$ derived from Equation (15) can be expressed in the form $\ell(\Delta) = \sum_{j \in A} \ell_j(\Delta) + \sum_{j \in S} \ell_j^s(\Delta)$, where $\ell_j(\Delta) = \log(f(z_j | u_j))$, $\ell_j^s(\Delta) = \log(S(z_j | u_j))$, and $f(z_j | u_j)$, $S(z_j | u_j)$ represent the density and survival functions of Z , respectively. Let ρ take the value 0 if censoring occurs and 1 if failure is observed. Then, the expression for the log-likelihood function for Δ can be stated as

$$\ell(\Delta) = \sum_{j: \rho_j=1} \log f(z_j | u_j) + \sum_{j: \rho_j=0} \log S(z_j | u_j) \quad (16)$$

Maximizing Equation (16) using R software allows estimation of the maximum likelihood estimator $\hat{\Delta}$ of the vector of unknown parameters. We also consider residuals for the EHL-GEN-TL-W regression model. By plotting the deviation residuals against the index (the numerical identifier assigned to each observation in the data set), one can effectively identify and validate the appropriateness of the fitted model for a typical observation. The

deviance residual, which serves as a measure of the disparity between the observed values and the projected values, can be mathematically defined as

$$r_{D_i} = \text{sign}(r_{M_i}) \left(-2 \left[r_{M_i} + \rho_i \log \left(\frac{\rho_i - r_{M_i}}{\rho_i} \right) \right] \right)^{0.5}$$

which reduces to

$$\sqrt{-2 \log S(z_i)} \quad \text{for censored observations,}$$

and

$$\sqrt{-2[r_{M_i} + \log S(z_i)]} \quad \text{for uncensored observations.}$$

where ρ_i is the censoring indicator (1 for failure, 0 for censored), $S(z_i)$ is the survival function, and r_{M_i} is the martingale residual defined by

$$r_{M_i} = \begin{cases} 1 + \log \left[1 - \left(\frac{1 - \left[\exp(-z \exp(u_j^T \omega)) \right]^{2\beta}}{1 + \left[\exp(-z \exp(u_j^T \omega)) \right]^{2\beta}} \right)^\alpha \right] & , \text{ if } \rho_j = 1, \\ \log \left[1 - \left(\frac{1 - \left[\exp(-z \exp(u_j^T \omega)) \right]^{2\beta}}{1 + \left[\exp(-z \exp(u_j^T \omega)) \right]^{2\beta}} \right)^\alpha \right] & , \text{ if } \rho_j = 0, \end{cases}$$

The $\text{sign}(\cdot)$ assigns the values of +1 when the argument is positive and −1 when the argument is negative.

The deviance residuals are plotted against the observation index to help identify poorly fit observations, with horizontal lines added at $y = \pm 2$ to indicate the boundaries of the acceptable range for the residuals [27]. In addition, the $y = \pm 3$ lines help spot potential outliers. These boundaries are common conventions in regression analysis for quickly spotting model fit issues and influential data points.

Ref. [28] proposed a technique to create envelopes around normal probability plots of residuals, known as simulated confidence bands, to visually assess whether residuals are normally distributed—a key assumption for many statistical models. The idea is that, when the model fits well, most of the data points should fall within these envelope ranges and exhibit a random distribution, indicating that the normality assumption is satisfied. The horizontal line at $y = 0$ in a deviance residual versus theoretical quantile plot represents the expected behavior of the residuals under the assumed statistical model, where a good fit of the model would result in the residuals being normally distributed around this line. The envelopes serve as a visual tool to evaluate the appropriateness of the model fit, providing confidence in the model's validity if most residuals fall within the specified envelopes.

6. Applications

The EHL-GEN-TL-W distribution is applied to real data and contrasted with several non-nested models in order to assess its performance. The non-nested models are the odd exponentiated half logistic Burr XII (OEHL-BXII) [29], the Topp-Leone odd Burr III log-logistic (TL-OBIII-LLOG) [3], the exponentiated half logistic odd Weibull Topp-Leone Lomax (EHL-OW-TL-Lx) [30], the exponentiated half logistic odd Burr III-log-logistic Poisson (EHL-OBIII-LLOG) [31], and both the exponentiated half logistic generalized log-logistic Poisson (EHL-GLLOGP) and the exponentiated half logistic generalized Weibull Poisson (EHL-GWP) [32]. Please refer to Supplementary Material for the pdfs of distributions used in the comparisons.

The goodness-of-fit (GoF) statistics employed include the following: $-2 \log$ -likelihood ($-2 \log(L)$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) (visit Supplementary Material for formulae),

Cramér-von Mises (W^*), and Andersen–Darling (AD). These statistics were used to determine the model that best fits a given data set. The selection of the preferred model is accomplished by evaluating various statistical criteria, primarily focusing on the model exhibiting the highest p -value for the Kolmogorov–Smirnov (K–S) statistic and the lowest values for other relevant statistical measures. The closeness to the diagonal line in probability plots is evaluated using the sum of squares (SS), which is given as

$$SS = \sum_{k=1}^n \left[F_{EHL-GEN-TL-W}(z_{(k)}; \hat{\alpha}, \hat{b}, \hat{\beta}, \hat{d}) - \left(\frac{k - 0.375}{n + 0.25} \right) \right]^2, k = 1, 2, 3, \dots, n,$$

where $z_{(k)}$ denotes the k^{th} ordered observed data value. The parameters of the EHL-GEN-TL-W model were estimated using the nonlinear minimization function (**nlm**) in the statistical software R. Standard errors (SEs) (in parentheses) accompany these parameter estimates for each data example. Probability plots with the sum of squares (SS) derived from the plots were also used to evaluate the fit. In addition, fitted densities, empirical cumulative distribution functions (ECDFs), Kaplan–Meier (K–M) survival curves, total time on test (TTT) plots, and hrf plots are presented.

6.1. Windshield Failure Data

The data set comprises the failure times of 84 windshields for a specific aircraft model as discussed in the study conducted by [33]. The data set is presented in Supplementary Material.

The visualizations in Figure 7 show that there are outliers beyond the whiskers, while the violin plot indicates a higher density of data points around the median value. The alignment between the observed data and the EHL-GEN-TL-W model is shown in the fitted densities plot, and the probability plot indicates a good fit of the EHL-GEN-TL-W model, as evidenced by the lowest SS displayed on the plot.

Table 6 presents the ML estimates along with their corresponding SEs (shown in parentheses). The parameters of the EHL-GEN-TL-W distribution for the windshields' failure data can be uniquely identified, as shown in Figure 8. The estimated variance-covariance (vcov) matrix is

$$\begin{bmatrix} 2.86 \times 10^{-6} & -1.86 \times 10^{-6} & -6.91 \times 10^{-5} & 2.28 \times 10^{-4} \\ -1.86 \times 10^{-6} & 1.64 \times 10^{-9} & 5.44 \times 10^{-5} & -1.44 \times 10^{-4} \\ -6.91 \times 10^{-5} & 5.44 \times 10^{-5} & 1.94 \times 10^{-3} & -5.40 \times 10^{-3} \\ 2.28 \times 10^{-4} & -1.44 \times 10^{-4} & -5.40 \times 10^{-3} & 1.82 \times 10^{-2} \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the model parameters are $\alpha \in [11.1400 \pm 3.32 \times 10^{-3}]$, $b \in [7.33 \times 10^{-4} \pm 7.94 \times 10^{-5}]$, $\beta \in [0.2368 \pm 0.0864]$, and $d \in [1.4441 \pm 0.2642]$, respectively.

Table 6. Parameter estimates for windshield failure data.

Distribution	Estimates and SEs			
EHL-GEN-TL-W (α, b, β, d)	11.1400 (1.69×10^{-3})	7.33×10^{-4} (4.05×10^{-5})	0.2368 (0.0441)	1.4441 (0.1348)
OEHL-BXII (α, λ, a, b)	0.2131 (0.0237)	1.20×10^{-4} (2.29×10^{-5})	5.3070 (1.03×10^{-3})	1.2361 (3.83×10^{-3})
EHL-GLLOGP ($\delta, \beta, \lambda, \theta$)	2.2744 (0.6354)	1.1519 (0.3269)	1.8694 (0.4247)	3.20×10^{-8} (0.0144)

Table 6. Cont.

Distribution	Estimates and SEs			
TL-OBIII-LLOG $(\alpha, \beta, b, \lambda)$	0.1870 (0.1290)	6.0491 (1.7338)	0.5805 (0.2009)	7.5948 (5.2389)
EHL-GWP $(\alpha, \beta, \delta, \theta)$	2.1752 (1.3564)	2.31×10^{-3} (9.71×10^{-3})	1.0828 (0.6751)	125.8020 (4.21×10^{-3})
EHL-OW-TL-L $(\alpha, \beta, \delta, \lambda)$	0.3949 (0.2555)	0.0758 (0.0647)	4.4486 (0.9062)	6.5607 (5.7903)
EHL-OBIII-LLOG (a, b, α, c)	2.0401 (0.0287)	0.8621 (0.4553)	2.2729 (1.0807)	0.9981 (0.0586)

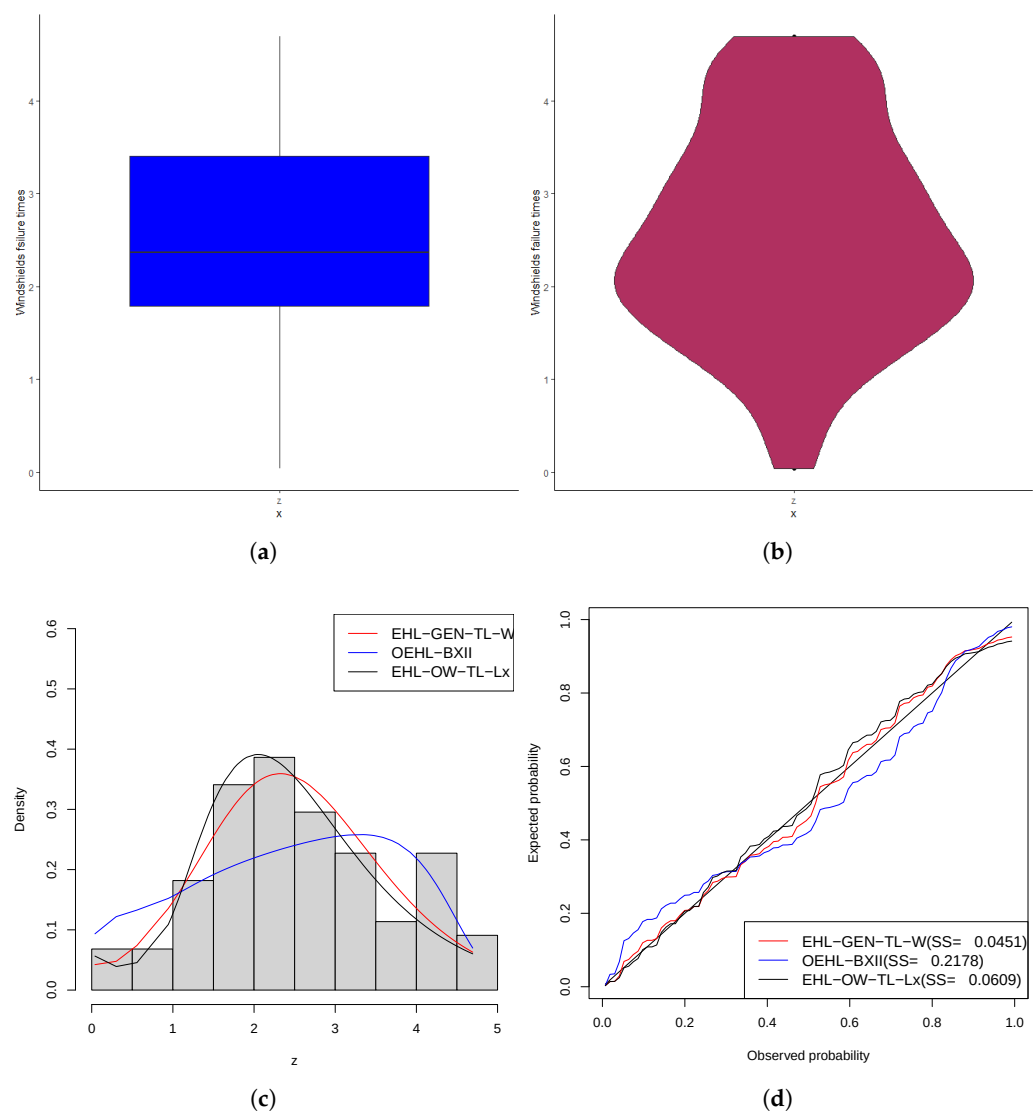


Figure 7. Boxplot, violin plot, fitted density, and probability plot for windshield failure data. (a) Box-Plot; (b) Violin Plot; (c) Fitted Density Plots; (d) Probability Plots

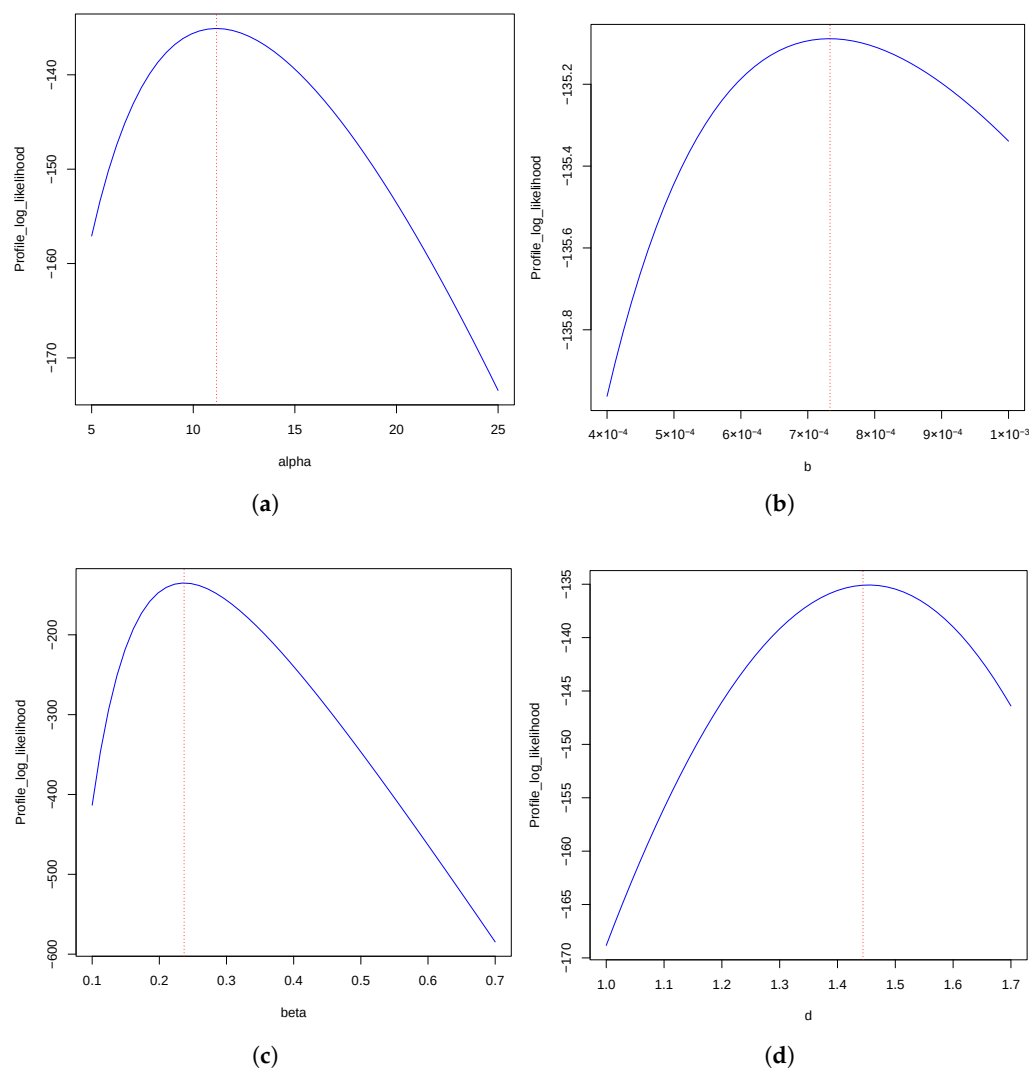


Figure 8. Examining EHL-GEN-TL-W parameter profiles using log-likelihood plots for windshield failure data. (a) Profile Plot for α ; (b) Profile Plot for b ; (c) Profile Plot for β ; (d) Profile Plot for d .

Table 7 displays the GoF statistics for the EHL-GEN-TL-W model and its non-nested equi-parameter models. The results show that the EHL-GEN-TL-W model provided the best in-sample fit among the competing models considered. This is corroborated by the smallest values of the GoF statistics associated with the EHL-GEN-TL-W model, alongside the highest p -value observed in the K–S statistics.

The ECDF and K–M survival curves shown in Figure 9 closely align with the windshield failure data. The complementary analysis of the TTT scaled and hrf plots indicates an increasing hazard rate pattern.

Table 7. GoF statistics for windshield failure data.

Distribution	GoF Statistics							
	$-2\log(L)$	AIC	CAIC	BIC	W^*	A^*	K–S	p -Value
EHL-GEN-TL-W	270.1772	278.1772	278.6594	288.0868	0.0482	0.4144	0.0564	0.9421
OEHLBXII	283.4491	291.4491	291.9310	301.3584	0.1868	1.1141	0.0882	0.4997

Table 7. Cont.

Distribution	GoF Statistics							
	$-2\log(L)$	AIC	CAIC	BIC	W^*	A^*	K–S	p -Value
EHL-GLLOGP	319.1587	327.1587	327.6406	337.0681	0.5044	3.4286	0.1536	0.0841
TL-OBIII-LLOG	320.7064	328.7064	329.0097	338.4371	0.5185	3.5307	0.1559	0.0675
EHL-GWP	274.3054	282.3054	282.7874	292.2148	0.0513	0.5378	0.0601	0.8671
EHL-OW-TL-Lx	271.3561	279.3560	279.8379	289.2654	0.0550	0.4819	0.0625	0.8814
EHL-OBIII-LLOG	319.7046	327.7046	328.1866	337.6140	0.5112	3.4635	0.1516	0.0531

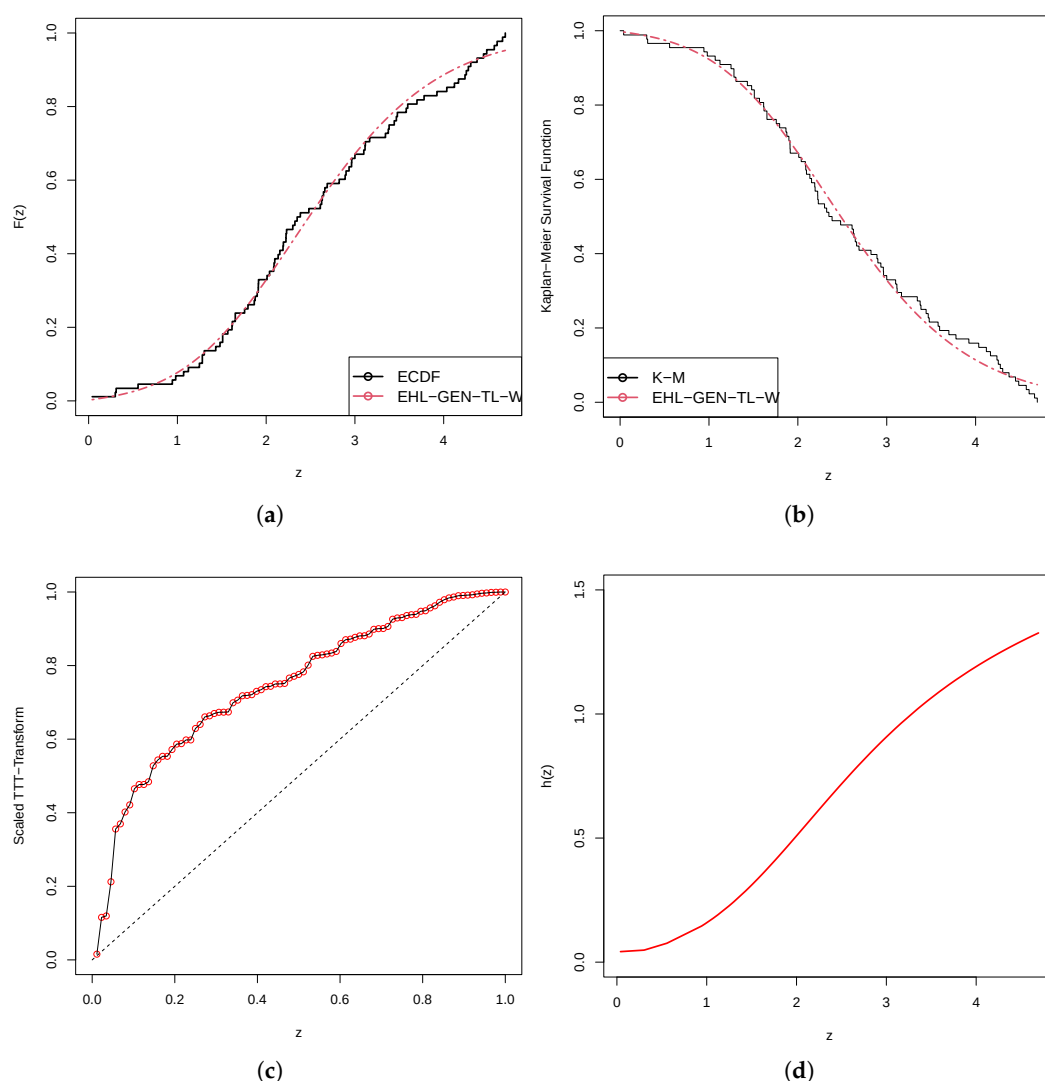


Figure 9. ECDF curve, K–M survival curve, TTT scaled, and hrf plots for windshield failure data. (a) ECDF; (b) K–M Plot; (c) Scaled TTT-Transform Plot; (d) Hrf Plot.

6.2. Kevlar 49/Epoxy Strands Data

The second data set concerns the endurance of Kevlar 49/epoxy strands under sustained pressure at a stress level of 90% until failure occurs. This practical example was originally presented by [34] and subsequently reported by [31]. The data set is included in the Supplementary Material.

The box-and-whisker and violin plots exhibit a wide range of values with a relatively higher density around the median value, as shown in Figure 10. Additionally, the fitted

densities plot and the probability plot indicate that the EHL-GEN-TL-W model provides a reasonable fit relative to the other models, as indicated by the lowest SS displayed on the plot.

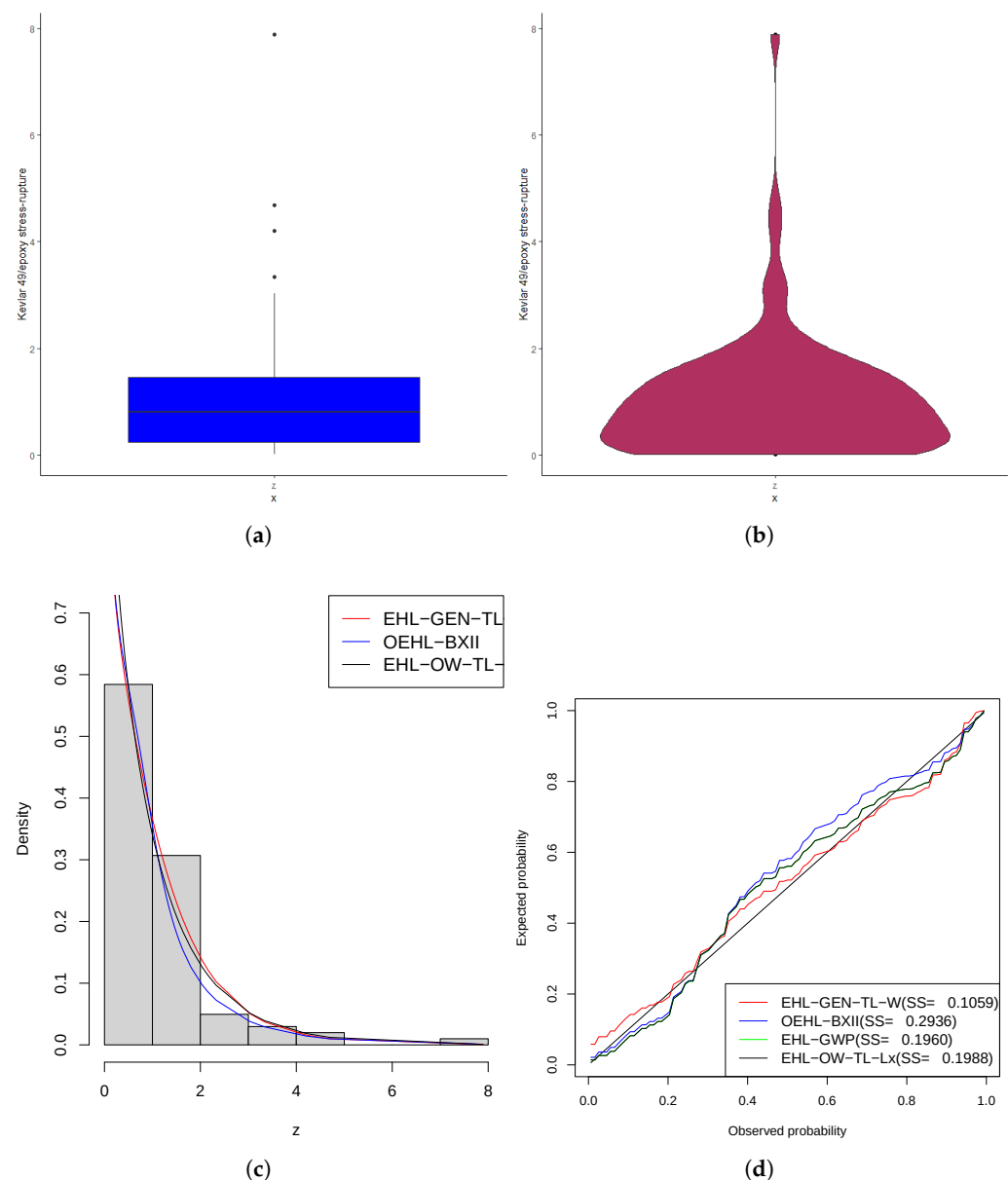


Figure 10. Boxplot, violin plot, fitted density, and probability plot for Kevlar 49/epoxy strands data. (a) Box-Plot; (b) Violin Plot; (c) Fitted Density Plots; (d) Probability Plots.

The ML estimates, along with their SEs (in parentheses), are shown in Table 8. The parameters of the EHL-GEN-TL-W distribution for the Kevlar 49/epoxy data can be uniquely identified, as shown in Figure 11. The estimated var-cov matrix is

$$\begin{bmatrix} 0.0066 & 0.0096 & 0.0039 & -0.0115 \\ 0.0096 & 0.0391 & 0.0138 & -0.0361 \\ 0.0039 & 0.0138 & 0.0063 & -0.0145 \\ -0.0115 & -0.0361 & -0.0145 & 0.0430 \end{bmatrix}.$$

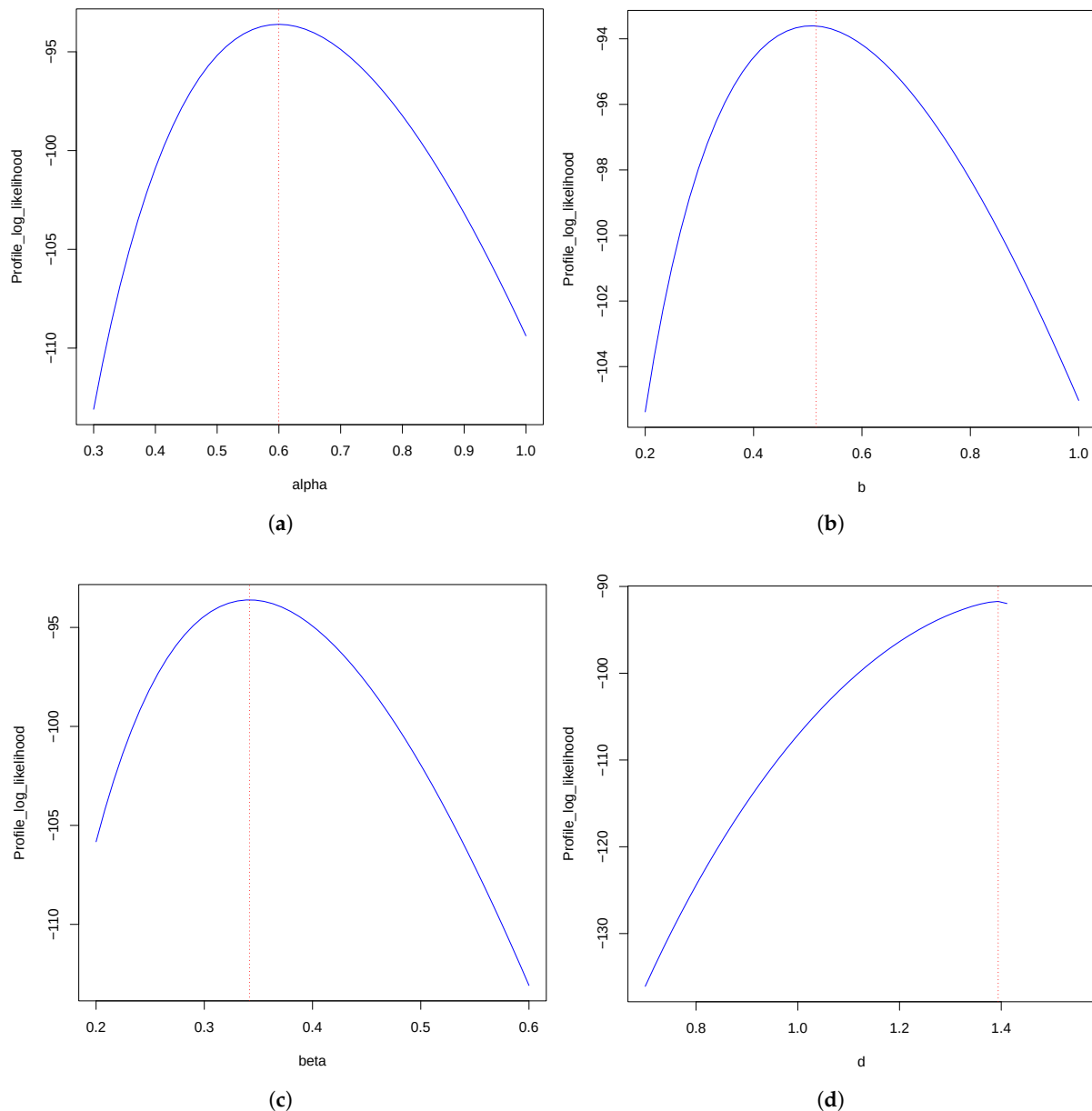


Figure 11. Profile plots illustrating EHL-GEN-TL-W parameters on Kevlar 49/epoxy data. (a) Profile Plot for α ; (b) Profile Plot for b ; (c) Profile Plot for β ; (d) Profile Plot for d .

The 95% asymptotic confidence intervals for the model parameters are $\alpha \in [0.5998 \pm 0.1596]$, $b \in [0.5155 \pm 0.3877]$, $\beta \in [0.3417 \pm 0.1557]$, and $d \in [1.4180 \pm 0.4065]$, respectively. The EHL-GEN-TL-W model provided the lowest AIC, BIC, and K-S statistic values among the competing models, as shown by the GoF statistics and K-S p -values obtained from the analysis of the Kevlar 49/epoxy data.

Table 9 displays the GoF statistics for the models. Among the models considered, the EHL-GEN-TL-W model exhibited the best in-sample fit, with the highest K-S p -value and the lowest AIC and BIC values.

Table 8. Parameter estimates for Kevlar 49/epoxy data.

Distribution		Estimates and SEs		
EHL-GEN-TL-W (α, b, β, d)	0.5998 (0.0814)	0.5155 (0.1978)	0.3417 (0.0795)	1.4180 (0.2074)
OEHL-BXII (α, λ, a, b)	0.1479 (0.0461)	1.1174 (1.7781)	5.1267 (1.5202)	0.1632 (0.1199)
EHL-GLLOGP ($\delta, \beta, \lambda, \theta$)	0.3651 (0.5262)	0.9634 (0.4676)	2.3352 (0.8858)	6.33×10^{-8} (0.0211)
TL-OBIII-LLOG ($\alpha, \beta, b, \lambda$)	0.4646 (0.4014)	3.8593 (1.6443)	0.1893 (0.1029)	2.9737 (2.5691)
EHL-GWP ($\alpha, \beta, \delta, \theta$)	0.6395 (1.7666)	0.0770 (0.7007)	1.4507 (4.0448)	0.0114 (4.51×10^{-3})
EHL-OW-TL-Lx ($\alpha, \beta, \delta, \lambda$)	8.1072 (18.0019)	0.0710 (0.1563)	2.4073 (0.1041)	0.0038 (1.6087)
EHL-OBIII-LLOG (a, b, α, c)	1.6892 (0.0753)	0.6607 (0.3574)	0.5346 (0.2621)	1.3543 (0.0939)

Table 9. GoF statistics for Kevlar 49/epoxy strands data.

GoF Statistics								
Distribution	$-2\log(L)$	AIC	CAIC	BIC	W^*	A^*	K–S	p -Value
EHL-GEN-TL-W	184.5270	192.5270	192.9437	202.9875	0.1170	0.9589	0.0785	0.5631
OEHL-BXII	215.9429	223.9429	224.3596	234.4034	0.1687	0.9856	0.1056	0.2098
EHL-GLLOGP	205.5553	213.5553	213.972	224.0158	0.2067	1.1635	0.1009	0.2552
TL-OBIII-LLOG	210.8527	218.8527	219.2694	229.3132	0.3146	1.7133	0.1343	0.0523
EHL-GWP	205.9439	213.9439	214.3606	224.4044	0.1993	1.1139	0.0907	0.3776
EHL-OW-TL-Lx	205.9826	213.9826	214.3993	224.44314	0.2018	1.1256	0.0915	0.3664
EHL-OBIII-LLOG	204.9657	212.9567	213.3823	223.4262	0.1858	1.0618	0.0976	0.2911

Figure 12 shows that the ECDF and K–M survival curves closely resemble the Kevlar 49/epoxy data. The accompanying analysis of the TTT scaled and hrf plot reveals a bathtub followed by an inverted bathtub hazard rate pattern.

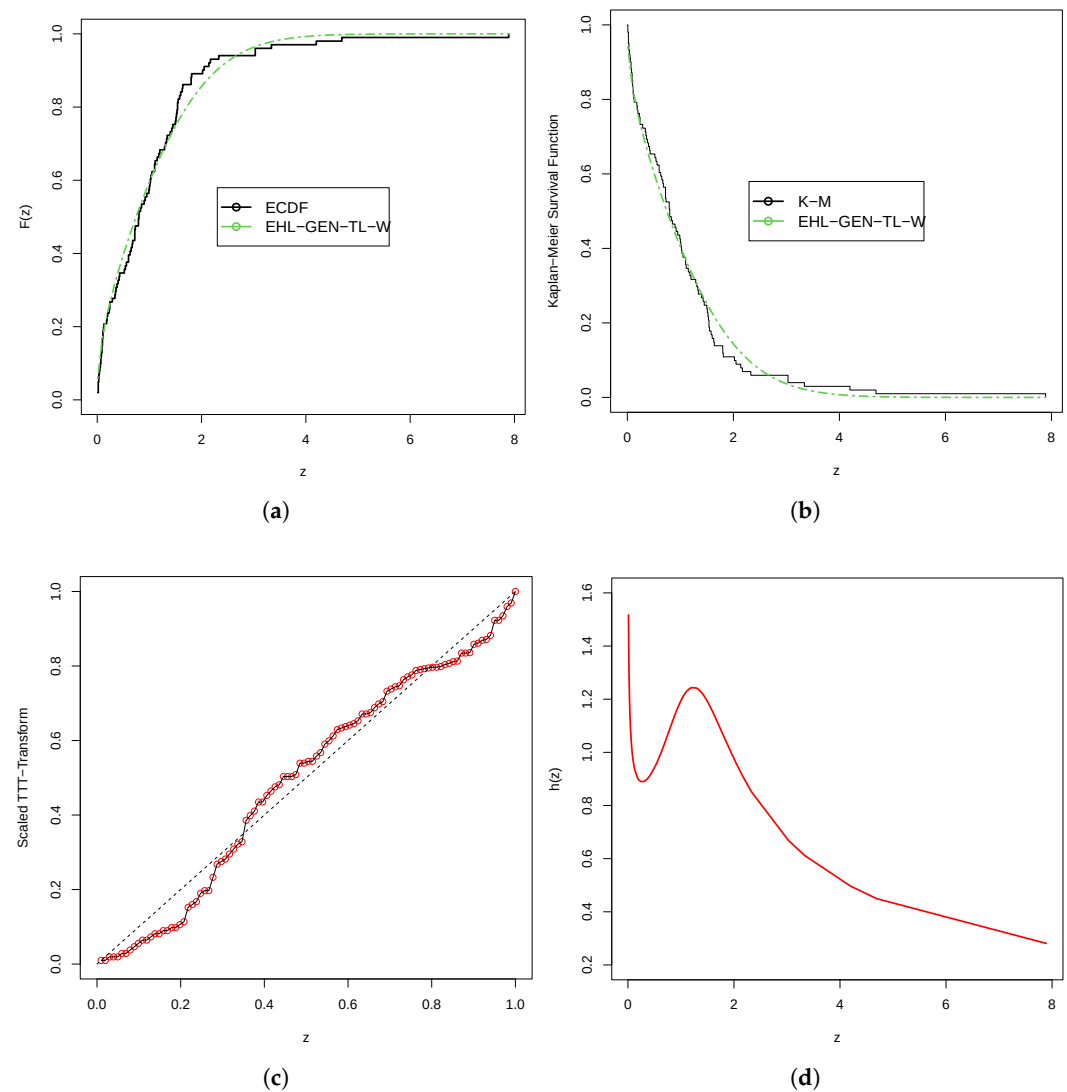


Figure 12. ECDF curve, K–M survival curve, TTT scaled, and hrf plots for Kevlar 49/epoxy data. (a) ECDF; (b) K–M Plot; (c) Scaled TTT-Transform Plot; (d) Hrf Plot.

6.3. Regression Model for Transformer Turns Data

This real data set is sourced from [35] and represents a transformer life test conducted at high voltage, resulting in multiple censored data points. The study reported failures in transformer primary insulation, with 13.33% of observations censored (censored data is shown by a (*)). Data are presented in Table 10.

Table 10. Transformer turns data.

Voltage						Hours				
35.4 kV	40.1	59.4	71.2	166.5	204.7	229.7	308.3	537.9	1002.3 *	1002.3 *
42.4 kV	0.6	13.4	15.2	19.9	25.0	30.2	32.8	44.4	50.2 *	56.2
46.7 kV	3.1	8.3	8.9	9.0	13.6	14.9	16.1	16.9	21.3	48.1 *

The variables considered in this study are z_j =time of failure in hours of the transformer, $j = 1, 2, \dots, 30$, and three voltage levels defined by (35.4 kV, 42.4 kV and 46.7 kV). We consider the following structured regression

$$d_j = \exp(\omega_{10} + \omega_{11}u_{j1} + \omega_{12}u_{j2}),$$

where u_{j1} and u_{j2} are the covariates of $j = 1, 2, 3 \dots, 30$, to maximize the log-likelihood function in Equation (16) in order to obtain the ML estimates of the parameters of the proposed model. We provide the parameter estimates, SEs, and the significance of the ML estimates in Table 11. The findings presented in Table 11 offer convincing empirical support that the only covariate u_1 is significant at 5%, indicating a substantial disparity between the 35.4 kV level and the 46.7kV level.

Table 11. MLEs for regression model fitted to transformer turns data.

Parameters	Estimates	SEs	p-Value
α	0.5201	0.0314	-
β	0.3417	0.0431	-
ω_{10}	4.7123	0.2828	<0.00001
ω_{11}	0.0039	0.0014	0.00035
ω_{12}	−0.0293	0.0349	0.4007

6.4. Goodness-of-Fit

The residual analysis presented in Figure 13 shows that the proposed EHL-GEN-TL-W regression model offers a good fit to the data. Although a small percentage (10%) of observations fit poorly and an outlier is identified, the probability plot shows that all data points fall within the confidence band envelope. In particular, more than 70% of the data points are close to the $y = 0$ line, indicating that the observed values closely align with the expected values of the model. Collectively, these findings suggest that the EHL-GEN-TL-W regression model provides a reasonable fit to this data set, though out-of-sample validation would be needed to assess generalizability.

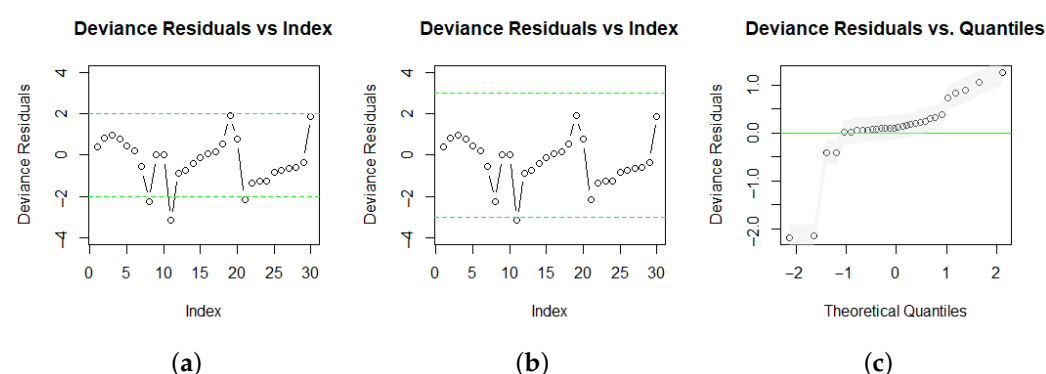


Figure 13. Plots of deviance residuals against index (at (a) $y = \pm 2$ and (b) $y = \pm 3$) and (c) probability plot with envelopes for the deviance component residuals.

7. Summary

In this study, we develop the EHL-GEN-TL-G FoDs and examine their statistical properties as well as their special cases. The study evaluated the consistency of the model parameters using simulation studies that employed six different estimation methods. The findings of the simulation study supported the efficiency of the EHL-GEN-TL-W distribution, which represents a particular case within the broader family of EHL-GEN-TL-G FoDs.

The study investigated the flexibility and use of the EHL-GEN-TL-G FoDs in the engineering field. Specifically, we fit the EHL-GEN-TL-W distribution to three datasets that represent failure time data from engineering systems. These datasets captured the lifetimes of the mechanical components under test, providing a real-world context for the analysis. The EHL-GEN-TL-W distribution outperformed several existing equi-parameter models, including extensions of the Topp-Leone and exponentiated half-logistic distributions, by exhibiting a superior fit to the data. The study also developed and applied the EHL-GEN-TL-W regression model to a specific data set. These findings suggest that EHL-GEN-TL-G FoDs can be a valuable tool for analyzing and modeling diverse types of data across various applications.

We acknowledge that, although the proposed EHL-GEN-TL-W model exhibited the best in-sample goodness-of-fit among the several non-nested competitors considered, its predictive performance was not assessed using cross-validation or out-of-sample criteria. Likelihood ratio tests were also not applicable due to the non-nested nature of the competing models. Consequently, while the proposed model demonstrates strong in-sample fit, future research should incorporate cross-validation or out-of-sample predictive evaluation to more rigorously assess its generalizability.

Furthermore, future research should focus on two main directions. First, predictive accuracy should be assessed using rigorous out-of-sample evaluation techniques, such as leave-one-out cross-validation, expected log predictive density (ELPD), or formal forecasting exercises, to better characterize the generalizability of the proposed model across diverse engineering reliability datasets. Second, Bayesian estimation frameworks offer a promising avenue for the EHL-GEN-TL-G family, leveraging their inherent flexibility and robustness to improve parameter inference and support more informed decision-making under uncertainty. Advancing these research directions will not only deepen the theoretical understanding of the proposed family but also broaden its practical applicability in engineering reliability and lifetime analysis. Ultimately, this may contribute to enhanced design, maintenance, and performance optimization of complex engineering systems.

Supplementary Materials: The following supporting information can be downloaded at: <https://drive.google.com/file/d/1lIr1bA2ktOUscg2uTRTABHIJLnCAR5Pk/view?usp=sharing> (accessed on 10 March 2026).

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