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A Uniform Two-Variable Analytic Approximation for the Modified Bessel Function $I_\nu(x)$ in the Interval $-1/2 \leq \nu \leq 1/4$

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Abstract

A global, uniform two-variable analytic approximation for the modified Bessel function $I_\nu(x)$ is presented, valid for all real x and for orders $-1/2 \leq \nu \leq 1/4$. The approximation is constructed using a two-variable multipoint quasi-rational approximation (MPQA) approach, in which the argument x and the order ν are treated simultaneously as independent variables. The method consistently incorporates the power-series expansion at small arguments and the asymptotic behavior at large arguments, leading to an explicit analytic representation that preserves the correct limiting behaviors. The resulting approximation remains suitable for analytical differentiation and integration, while all parameters are obtained from linear equations, avoiding numerical fitting procedures. A numerical analysis over the entire domain considered shows excellent agreement with the exact function. The largest relative error observed is $\varepsilon_r = 0.0213$, occurring at $\nu = -0.34$ and $x = 2.56$. These results indicate that the proposed approximation provides an accurate and efficient analytic representation of $I_\nu(x)$ throughout the investigated domain.

Keywords: multipoint quasi-rational approximation (MPQA); Bessel functions; analytic approximations

MSC: 33C10

1. Introduction

The modified Bessel functions $I_\nu(x)$ play a fundamental role in a wide variety of scientific and engineering applications, particularly in problems involving cylindrical symmetry and exponential-type solutions [1–5]. They naturally arise in heat conduction and diffusion processes in cylindrical geometries [6], electromagnetic wave propagation in waveguides and optical fibers [7], mechanical stress and vibration analysis of cylindrical structures [8], and fluid dynamics [9]. In condensed matter physics and materials science, $I_\nu(x)$ appear in models of electronic and photonic transport [10], superconductivity, thermodynamic partition functions, and nanoscale systems such as nanowires and nanotubes [11–13]. Furthermore, they are also encountered in stochastic processes, statistical mechanics [8–10], medical imaging [14], and field-theoretical models [15], highlighting their broad relevance across multiple disciplines.

Several analytical approximations have been developed for Bessel and modified Bessel functions. Classical approaches include rational minimax approximations [16,17], Chebyshev polynomial expansions [18], and uniform asymptotic expansions for large arguments or orders [19]. These methods often achieve high accuracy within specific



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parameter ranges, but may require piecewise representations, numerical optimization procedures, tabulated coefficients, or separate treatments for different orders. In contrast, the present work seeks a single analytic approximation that depends simultaneously on the argument x and the order ν , while preserving both the power-series behavior near the origin and the asymptotic behavior for large arguments without numerical fitting.

This work presents a new technique for obtaining accurate analytical approximations in the variables x and ν , valid for real values of x and for $-1/2 \leq \nu \leq 1/4$. The method, based on the two-variable multipoint quasi-rational approximation (MPQA) technique [20], combines power-series and asymptotic expansions to construct an explicit analytic approximation that smoothly bridges both regimes. The resulting approximation preserves the relevant symmetries of the problem and remains suitable for analytical differentiation, integration, and computational applications.

A further distinction between the present MPQA approach and standard Padé approximations [21,22] lies in the incorporation of asymptotic information into the approximation structure itself. Classical Padé approximants are constructed exclusively from rational functions and are designed to reproduce a finite number of terms of a local power-series expansion. In contrast, the present MPQA formulation combines quasi-rational functions with elementary functions such as $\sinh x$ and $\cosh x$, allowing the exponential asymptotic behavior of $I_\nu(x)$ to be incorporated directly into the approximation. Consequently, both the local behavior near the origin and the asymptotic behavior at infinity are imposed simultaneously during the construction of the approximation. Since the asymptotic expansions employed are derived for real variables, the resulting approximations are also intended for real values of x . For this reason, the present approach should not be regarded as a direct analytic continuation to complex or imaginary arguments. For instance, attempting to obtain $I_\nu(x)$ from the approximation for $J_\nu(x)$ reported in Refs. [23,24] through analytic continuation generally leads to significant difficulties, including the appearance of spurious poles and defect-like structures.

The remainder of this paper is organized as follows. Section 2 presents the theoretical framework and the construction of the approximation. Section 3 contains the analytical results together with the corresponding accuracy analysis. Finally, Section 4 summarizes the main conclusions.

2. Theoretical Analysis

This section presents the construction of the proposed two-variable MPQA approximation for the modified Bessel function $I_\nu(x)$. The procedure is based on matching the power-series expansion valid for small arguments with the corresponding asymptotic expansion for large arguments. The objective is to obtain an explicit analytic approximation that accurately reproduces both limiting behaviors while maintaining a simple analytical structure.

The power series of the modified Bessel functions [5] are

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k}, \quad (1)$$

which can also be written as

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \left(1 + a_1 x^2 + a_2 x^4 + a_3 x^6 + \dots\right), \quad (2)$$

wherein

$$a_1 = \frac{1}{4(\nu+1)}, \quad (3)$$

$$a_2 = \frac{1}{32(\nu+1)(\nu+2)}, \quad (4)$$

$$a_3 = \frac{1}{384(\nu+1)(\nu+2)(\nu+3)}, \quad (5)$$

and the terms shown here are those to be used in this work.

The corresponding asymptotic expansions of these functions are expressed as [5]

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left[1 - \frac{\mu-1}{8x} \pm \mathcal{O}(1/x^2) \right], \quad (6)$$

where

$$\mu = 4\nu^2. \quad (7)$$

In order to simplify the notation, the following constants and variables will be introduced:

$$a(\nu) = 2^\nu \Gamma(\nu+1), \quad (8)$$

$$\beta(\nu) = \frac{2\nu-1}{4}, \quad (9)$$

$$\gamma(\nu) = |1 - 2(\nu - [\nu])|, \quad (10)$$

wherein $[\nu]$ denotes the integer-part function obtained by truncation toward zero. For instance, $[-1/4] = 0$ and $[1/4] = 0$. It is also well-known that

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{\cosh x}{\sqrt{|x|}}, \quad (11)$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{\sinh x}{\sqrt{|x|}}, \quad (12)$$

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi}} \left(\frac{\sinh x}{\sqrt{|x|^3}} - \frac{\cosh x}{\sqrt{|x|}} \right). \quad (13)$$

The bridge approximation connecting power and asymptotic expansions has been constructed as

$$\tilde{I}_\nu(x) = \frac{x^{\nu-1} \left[(p_0 + \tilde{p}_0 \sqrt{1 + \gamma^4 x^2} + p_2 x^2) \sinh x + (p_1 + p_3 \sqrt{1 + \gamma^4 x^2}) x \cosh x \right]}{a(1 + \gamma^4 x^2)^\beta (q_0 + x^2)}. \quad (14)$$

To explain how this structure is reached, it is interesting to pay attention to the following points:

1. Inspection of the power-series expansion shows that only even powers of x appear. Therefore, the approximation should be constructed in such a way that its Taylor expansion preserves this property.
2. The exponential behavior of the asymptotic expansion, together with the parity requirements of the power series, motivates the use of the functions $\sinh x$ and $\cosh x$. For large values of x , both functions behave asymptotically as $e^x/2$, thereby reproducing the exponential growth of $I_\nu(x)$. Moreover, the combination of $\sinh x$ and $x \cosh x$ generates only even powers of x after multiplication by the prefactor $x^{\nu-1}$, which is consistent with the structure of the power series. The factor $\sqrt{1 + \gamma^4 x^2}$ is introduced because it contributes only even powers in its Taylor expansion while scaling linearly with x for large arguments, thus providing additional flexibility to simultaneously satisfy the power-series and asymptotic constraints. The asymptotic

power $x^{-1/2}$ is obtained because the term p_2x^2 is compensated by the denominator factor $(q_0 + x^2)$, while the prefactor $x^{\nu-1}$ combines with the factor $(1 + \gamma^4x^2)^\beta$ to yield $\nu - 1 - 2\beta = -1/2$. In the case of the cosh x contribution, the additional power of x is similarly compensated by the denominator structure.

3. A further requirement is that the approximation must reproduce exactly the known analytical expressions for $I_{-1/2}(x)$, $I_{1/2}(x)$, and $I_{3/2}(x)$ once the parameters are determined. To achieve this, the factor $(1 + \gamma^4x^2)^\beta$ must reduce to unity for these particular values of ν . This is accomplished through the definition of $\gamma(\nu)$, which satisfies $\gamma = 2$ for $\nu = -1/2$ and $\nu = 1/2$, and $\gamma = 0$ for $\nu = 3/2$.

The next step is first to determine the parameters p_0 , $\tilde{p}_0$, p_1 , p_2 , p_3 , and q_0 , using the power series and the asymptotic expansions. It is important to point out that, in order to avoid complicated problems, the parameters should be determined by linear equations, and in this way non-linear equations between parameters should be avoided. In order to accomplish this, before matching the corresponding terms, we must multiply the series $I_\nu(x)$ by the factors $(q_0 + x^2)(1 + \gamma^4x^2)^\beta$.

To determine the six parameters of the approximation, the power-series expansions of both $I_\nu(x)$ and $\tilde{I}_\nu(x)$ are matched up to order x^6 . Accordingly, the Taylor expansions of $\sinh x$, $\cosh x$, and $\sqrt{1 + \gamma^4x^2}$ are truncated consistently at this order. This procedure provides four matching conditions associated with the coefficients of 1 , x^2 , x^4 , and x^6 . Together with the two additional conditions obtained from the asymptotic expansion, a system of six linear equations is obtained for the six unknown parameters. Thus, for $I_\nu(x)$, the following expansions are obtained:

$$\begin{aligned} & ax^{-\nu}(1 + \gamma^4x^2)^\beta(q_0 + x^2)I_\nu(x) \\ &= q_0 + \left(1 + q_0\beta\gamma^4 + q_0a_1\right)x^2 + \left(\beta\gamma^4 + \frac{1}{2}q_0\tilde{\beta}\gamma^8 + a_1(1 + q_0\beta\gamma^4) + a_2q_0\right)x^4 \\ &+ \left(\frac{q_0B\gamma^{12}}{6} + \frac{a_1}{2}\tilde{\beta}\gamma^8(4\nu + q_0 + 4)\right. \\ &\left.+ a_2\beta\gamma^4(8\nu + q_0 + 16) + a_3(12\nu + q_0 + 36)\right)x^6 + \mathcal{O}(x^8), \end{aligned} \quad (15)$$

where $\tilde{\beta} = \beta(\beta - 1)$ and $B = \beta(\beta - 1)(\beta - 2)$. Similarly, for $\tilde{I}_\nu(x)$ the expansions are

$$\begin{aligned} & \left(p_0 + \tilde{p}_0\sqrt{1 + \gamma^4x^2} + p_2x^2\right)\frac{\sinh x}{x} \\ &= p_0 + \tilde{p}_0 + \frac{1}{6}\left(6p_2 + p_0 + \tilde{p}_0(3\gamma^4 + 1)\right)x^2 \\ &+ \frac{1}{120}\left(p_0 + \tilde{p}_0 + 20p_2 + 10\tilde{p}_0\gamma^4 - 15\tilde{p}_0\gamma^8\right)x^4 \\ &+ \frac{1}{5040}\left(42p_2 + p_0 + \tilde{p}_0(21\gamma^4 - 105\gamma^8 + 315\gamma^{12} + 1)\right)x^6 + \mathcal{O}(x^8) \end{aligned} \quad (16)$$

$$\begin{aligned} & \left(p_1 + p_3\sqrt{1 + \gamma^4x^2}\right)\cosh x \\ &= p_1 + p_3 + \frac{1}{2}\left(p_1 + p_3(\gamma^4 + 1)\right)x^2 + \frac{1}{24}\left(p_1 + p_3 + 6p_3\gamma^4 - 3p_3\gamma^8\right)x^4 \\ &+ \frac{1}{720}\left(p_1 + p_3\left(15\gamma^4(1 + 3\gamma^4(\gamma^4 - 1)) + 1\right)\right)x^6 + \mathcal{O}(x^8) \end{aligned} \quad (17)$$

In this way the following four equations for the parameters are obtained

$$q_0 = p_0 + \tilde{p}_0 + p_1 + p_3, \quad (18)$$

$$1 + q_0\beta\gamma^4 + q_0a_1 = \frac{1}{6} \left(p_0 + \tilde{p}_0 + 3(p_1 + 2p_2 + p_3) + 3(p_0 + p_3)\gamma^4 \right), \quad (19)$$

$$120 \left(\beta\gamma^4 + \frac{1}{2}q_0\tilde{\beta}\gamma^8 + (1 + q_0\beta\gamma^4)a_1 + q_0a_2 \right) \\ = p_0 + \tilde{p}_0 + 5(p_1 + 4p_2 + p_3) + 10(\tilde{p}_0 + 3p_3)\gamma^4 - 15(\tilde{p}_0 + p_3)\gamma^8, \quad (20)$$

$$\frac{q_0B\gamma^{12}}{6} + \tilde{\beta}\gamma^8(4\nu + q_0 + 4)a_1 + \beta\gamma^4(8\nu + q_0 + 16)a_2 + (12\nu + q_0 + 36)a_3 \\ = \frac{1}{5040} \left(p_0 + \tilde{p}_0 + 7(p_1 + 6p_2 + p_3) + 21(p_0 + 5p_3)\gamma^4 - 105(\tilde{p}_0 + 3p_3)\gamma^8 + 315(\tilde{p}_0 + p_3)\gamma^{12} \right), \quad (21)$$

and from the asymptotic expansions [see Appendix A for details]:

$$p_2 + p_3\gamma^2 = \sqrt{\frac{2}{\pi}} a \gamma^{4\beta}, \quad (22)$$

$$p_1 + \tilde{p}_0\gamma^2 = -\sqrt{\frac{2}{\pi}} a \gamma^{4\beta} \left(\frac{\mu - 1}{8} \right). \quad (23)$$

Now six equations are obtained for the six parameters $p_0, \tilde{p}_0, p_1, p_2, p_3$, and q_0 . The resulting linear system is well posed provided that the determinant of its coefficient matrix is nonzero. An analysis of this determinant reveals the existence of isolated singular values of ν at which the coefficient matrix becomes singular and the corresponding parameters develop pole-like divergences. Since the first singular value occurs at approximately $\nu \approx 0.26374 > 1/4$, the present work restricts its attention to the interval $[-1/2 \leq \nu \leq 1/4]$, where the coefficient matrix remains nonsingular and the approximation parameters are uniquely defined. Throughout this interval, the resulting approximation is regular and can be applied without encountering singular parameter values.

3. Results

This section presents the explicit analytical expressions obtained for the parameters of the proposed approximation, together with a numerical assessment of its accuracy. Particular attention is devoted to the dependence of the parameters on the order ν , the behavior of the approximation across the domain of interest, and the resulting relative errors.

From the previous equations, all parameters can be explicitly determined. As a first step, we determine the parameter q_0 , obtaining

$$q_0 = \frac{N_0}{D_0}, \quad (24)$$

wherein N_0 is given by

$$N_0 = -72\sqrt{\pi} \left(80a_1 - 1120a_2 - 4 + 40(1 - 16a_1 + 168a_2)\gamma^2 \right. \\ - 5(25 + 2520a_2 - 16\beta + 28a_1(8\beta - 11))\gamma^4 \\ + 10(5 + 840a_2 - 64\beta + 672a_1\beta)\gamma^6 + 140(3 + 11\beta - 30a_1(1 + 3\beta) - 4\tilde{\beta})\gamma^8 \\ - 840(1 - 5a_1(1 + 2\beta) - 4\tilde{\beta})\gamma^{10} + 525(1 - 8\beta - 12\tilde{\beta})\gamma^{12} + 4200\beta^2\gamma^{14} \\ \left. + 3\sqrt{2}a\gamma^{2+4\beta} \left(256 + 6720\gamma^6 + 80\gamma^4(\mu - 37) + 525\gamma^{10}(\mu - 25) \right. \right. \\ \left. \left. - 420\gamma^8(\mu - 9) - 16\gamma^2(39 + \mu) \right) \right), \quad (25)$$

and D_0 by

$$\begin{aligned} \frac{D_0}{4\sqrt{\pi}} = & 4 - 72a_1 + 1440a_2 - 20160a_3 - 48(1 - 15a_1 + 240a_2 - 2520a_3)\gamma^2 \\ & + 9(21 - 25200a_3 + 280a_2(11 - 8\beta) - 8\beta + 10a_1(16\beta - 25))\gamma^4 \\ & + 180(840a_3 - 1 + a_1(5 - 64\beta) + 4\beta + 672a_2\beta)\gamma^6 \\ & - 90(7 + 25\beta + 840a_2(1 + 3\beta) + 28a_1(\beta(4\beta - 15) - 3) - 8\tilde{\beta})\gamma^8 \\ & + 90(21 + 10\beta + 840a_2(1 + 2\beta) + 168a_1(4\tilde{\beta} - 1) - 64\tilde{\beta})\gamma^{10} \\ & - 105(15 - 72\beta + 90a_1(2\beta - 1)(1 + 6\beta) - 132\tilde{\beta} + 32B)\gamma^{12} \\ & + 5040(3\beta(5a_1\beta - 1) + 4B)\gamma^{14} + 9450(\beta - 4(\tilde{\beta} + B))\gamma^{16} + 12600(3\tilde{\beta} + 2B)\gamma^{18}. \end{aligned} \quad (26)$$

Within the interval $-1/2 \leq \nu \leq 1/4$, the denominator D_0 remains nonzero, and therefore the parameter q_0 remains finite throughout the entire domain considered. The same regular behavior is observed for the remaining coefficients obtained from the linear system.

The results for the remaining parameters (p_0 , $\tilde{p}_0$, p_1 , p_2 , and p_3) are given in terms of q_0 by the following equations, where we have defined $f = \gamma^4(24\gamma^2 - 45\gamma^4 + 30\gamma^6 - 4)$:

$$\begin{aligned} 8\sqrt{\pi} f p_0 = & 4\sqrt{\pi} \left(24a_1(q_0 - 10) + 2(12 - q_0 - 120a_2q_0) \right. \\ & - 2(60 + 60a_1(q_0 - 6) - 7q_0 - 360a_2q_0)\gamma^2 \\ & - 6(40a_1(3 + q_0(\beta - 1)) + q_0(7 + 120a_2 - 4\beta) + 40(\beta - 1))\gamma^4 \\ & + 15(5q_0 - 12 + 24a_2q_0 - 8(q_0 - 6)\beta + 12a_1(2 - q_0 + 4q_0\beta))\gamma^6 \\ & - 30(24\beta + q_0(3 - 8\beta + 24a_1\beta + 4\tilde{\beta}))\gamma^8 \\ & + 45(2 + 8\beta + q_0(1 + 2a_1 - 4\beta + 8a_1\beta + 8\tilde{\beta}))\gamma^{10} \\ & - 360q_0\tilde{\beta}\gamma^{12} + 90q_0(\beta + 2\tilde{\beta})\gamma^{14} \Big) \\ & + \sqrt{2} a \gamma^{4\beta} \left(64 + \gamma^2(8 + 480\gamma^4 + 24\gamma^2(\mu - 21) + 45\gamma^8(\mu - 9) \right. \\ & \left. - 45\gamma^6(\mu - 1) - 8\mu) \right), \end{aligned} \quad (27)$$

$$\begin{aligned} f \tilde{p}_0 = & \frac{1}{2} \left(q_0 - 6 - 6a_1q_0 - 6q_0\beta\gamma^4 + 6a\sqrt{\frac{2}{\pi}}\gamma^{4\beta} \right) \left(5\gamma^2(4 - 6\gamma^2 + 3\gamma^6) - 4 \right) \\ & + 60(2 - 6\gamma^2 + 3\gamma^4) \left(\frac{a\gamma^{4\beta}}{3\sqrt{2\pi}} - \beta\gamma^4 - a_1(1 + q_0\beta\gamma^4) + q_0\left(\frac{1}{120} - a_2 - \frac{\tilde{\beta}}{2}\gamma^8\right) \right) \\ & - \frac{a\gamma^{2+4\beta}(8 - 24\gamma^2 + 15\gamma^6)(\mu - 1)}{4\sqrt{2\pi}}, \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{8\sqrt{\pi}}{\gamma^2} f p_1 = & 4\sqrt{\pi} \left(24a_1(q_0 - 10) + 2(12 - q_0 - 120a_2q_0) \right. \\ & - 2(60 + 60a_1(q_0 - 6) - 7q_0 - 360a_2q_0)\gamma^2 \\ & - 3(q_0(9 + 120a_2 - 8\beta) - 60 + 80\beta + 20a_1(6 - 3q_0 + 4q_0\beta))\gamma^4 \\ & + 120(6 + (6a_1 - 1)q_0)\beta\gamma^6 \\ & - 15(6 + 24\beta + q_0(6a_1 - 1 - 12\beta + 24a_1\beta + 8\tilde{\beta}))\gamma^8 \\ & + 360q_0\tilde{\beta}\gamma^{10} - 90q_0(\beta + 2\tilde{\beta})\gamma^{12} \Big) \\ & - \sqrt{2} a \gamma^{4\beta} \left(480\gamma^4 - 64 + 45\gamma^8(\mu - 1) + 4\gamma^2(\mu - 1) - 45\gamma^6(\mu - 1) \right), \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{8\sqrt{\pi}}{\gamma^4} f p_2 = & 4\sqrt{\pi} \left(24a_1(q_0 - 10) - 2(q_0 - 12 + 120a_2q_0) \right. \\ & - (60 + 60a_1(q_0 - 6) - 7q_0 - 360a_2q_0)\gamma^2 \\ & + 24(q_0 - 10 - 10a_1q_0)\beta\gamma^4 + 15(6 + (6a_1 - 1)q_0)(1 + 4\beta)\gamma^6 \\ & - 120q_0\beta t\gamma^8 + 90q_0(\beta + 2\tilde{\beta})\gamma^{10} \Big) \\ & + \sqrt{2} a \gamma^{4\beta} \left(32 - 360\gamma^4 - 4\gamma^2(\mu - 49) + 15\gamma^6(\mu - 9) \right), \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{8\sqrt{\pi}}{\gamma^2} f p_3 = & 4\sqrt{\pi} \left(2(q_0 + 120a_2q_0 - 12 - 12a_1(q_0 - 10)) \right. \\ & + (60 + 60a_1(q_0 - 6) - 7q_0 - 360a_2q_0)\gamma^2 \\ & + 24(10 + (10a_1 - 1)q_0)\beta\gamma^4 - 15(6 + (6a_1 - 1)q_0)(1 + 4\beta)\gamma^6 \\ & + 120q_0\tilde{\beta}\gamma^8 - 90q_0(\beta + 2\tilde{\beta})\gamma^{10} \Big) \\ & - \sqrt{2} a \gamma^{4\beta} \left(64 + 15\gamma^6(\mu - 25) - 4\gamma^2(\mu - 1) \right). \end{aligned} \quad (31)$$

The relative error is determined using the expression

$$\varepsilon_r(x, \nu) = \left| \frac{I_\nu(x) - \tilde{I}_\nu(x)}{I_\nu(x)} \right|. \quad (32)$$

Since $I_\nu(x)$ grows exponentially as x increases, $\varepsilon_r(x, \nu)$ provides a more meaningful measure of accuracy than the absolute error $\varepsilon_A(x, \nu) = |I_\nu(x) - \tilde{I}_\nu(x)|$. Accordingly, the analysis presented in this work focuses on $\varepsilon_r(x, \nu)$ in what follows.

For each value of ν , the quantity

$$\varepsilon_{r,\max}(\nu) = \max_x [\varepsilon_r(x, \nu)], \quad (33)$$

is computed over the interval of x considered in the numerical analysis. These values are displayed in Figure 1a, while Figure 1b shows the corresponding two-dimensional distribution of $\varepsilon_r(x, \nu)$ over the investigated domain.

The largest relative error observed in the numerical grid is $\varepsilon_r = 0.0213$, occurring at $\nu = -0.34$ and $x = 2.56$. This maximum occurs in the intermediate region where the approximation transitions between the power-series and asymptotic regimes. In this transition region, neither limiting expansion dominates completely, and the approximation must interpolate between both behaviors. Nevertheless, as shown in Figure 1b, the relative error remains below approximately 2.2% throughout the entire domain considered.

In order to further assess the quality of the approximation, Figure 2 presents a direct comparison between the exact function $I_\nu(x)$ and its approximate expression $\tilde{I}_\nu(x)$ for representative values of the order $\nu = -1/4, 0$, and $1/4$. The left panels show the exact and approximated functions, whereas the right panels display the corresponding relative errors. As can be observed, the approximate curves closely reproduce the shape and overall behavior of the exact solutions over the full range of x considered. The relative-error panels make the small deviations visible and confirm that the largest discrepancies occur in the intermediate region of x , where the approximation connects the small- and large-argument regimes.

Table 1 presents representative numerical values of the six parameters entering the approximation. As expected from the analysis of the previous section, all coefficients remain finite throughout the interval considered.

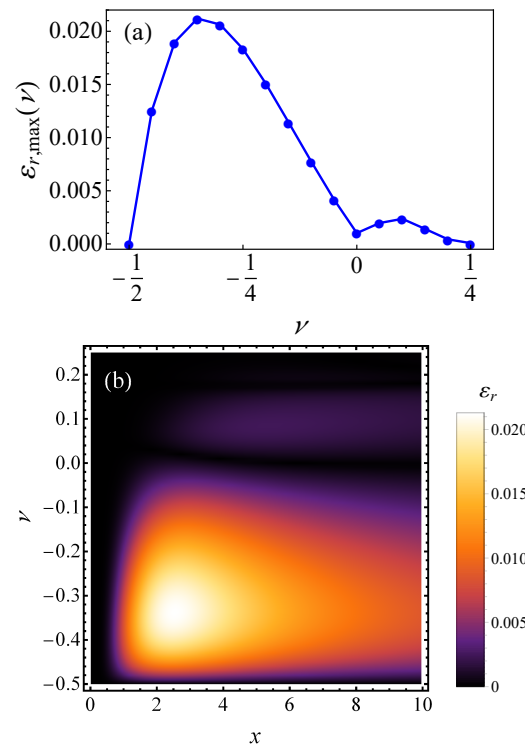


Figure 1. Maximum value of the relative error obtained as function of ν [panel (a)], and color map of the relative error as function of x and ν [panel (b)].

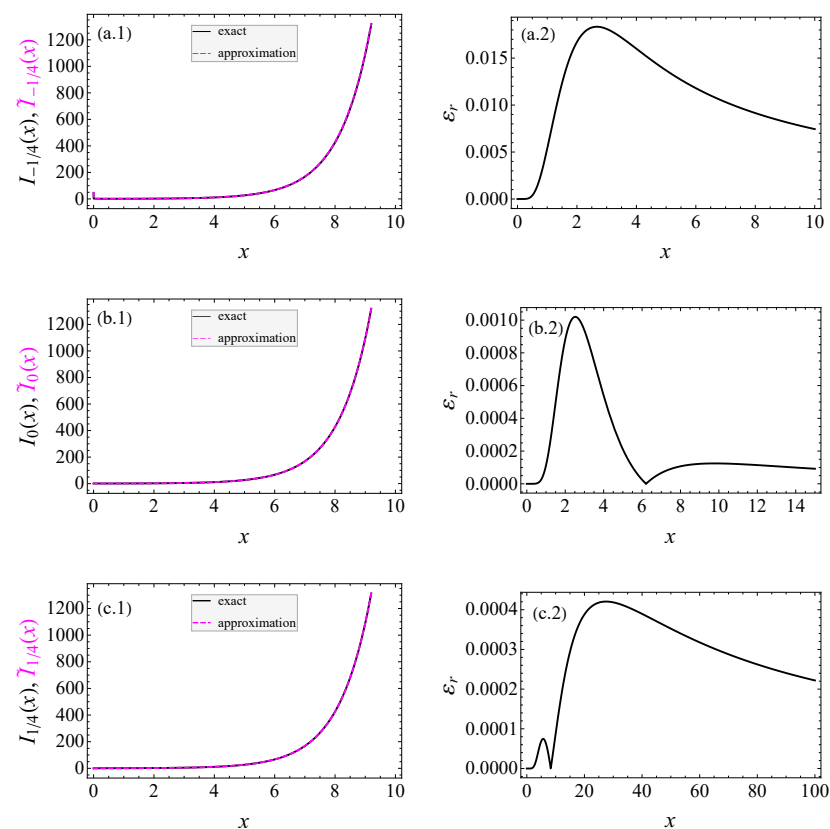


Figure 2. Comparison between the exact and approximated modified Bessel functions for $\nu = -1/4$ [panel (a.1)], $\nu = 0$ [panel (b.1)], and $\nu = 1/4$ [panel (c.1)]. Panels (a.2,b.2,c.2) display the corresponding relative errors.

Table 1. Values of the parameters q_0 , p_0 , $\tilde{p}_0$, p_1 , p_2 , and p_3 for representative values of ν in the interval $-1/2 \leq \nu \leq 1/4$.

	$\nu = -1/4$	$\nu = -1/8$	$\nu = 0$	$\nu = 1/8$	$\nu = 1/4$
q_0	0.183728	0.354446	0.800828	4.01812	22.8017
p_0	0.0238806	0.0901908	0.227587	3.00702	21.2577
$\tilde{p}_0$	0.0618661	0.199792	0.685899	0.599736	−1.23953
p_1	−0.0621194	−0.218746	−0.586163	−0.241327	0.390511
p_2	0.0873106	0.160685	0.324379	0.649589	0.618017
p_3	0.160101	0.28321	0.473505	0.652693	2.39305

4. Conclusions

An analytic approximation for the modified Bessel function $I_\nu(x)$ has been developed for real values of x and orders in the interval $-1/2 \leq \nu \leq 1/4$. The proposed procedure extends the multipoint quasi-rational approximation (MPQA) framework to a genuinely two-variable setting, where both the argument x and the order ν are treated simultaneously. By combining the power-series expansion for small arguments with the asymptotic expansion for large arguments, an explicit analytic approximation is obtained that reproduces the correct limiting behaviors while preserving a simple analytical structure. The resulting approximation is determined entirely from a system of six linear equations for six unknown parameters, thereby avoiding the use of numerical fitting procedures commonly employed in traditional approximations of special functions. The analysis of the coefficient matrix further reveals the existence of isolated singular values of ν , which motivates the restriction of the present study to the interval $-1/2 \leq \nu \leq 1/4$, where the approximation remains regular and uniquely defined. A detailed numerical analysis demonstrates excellent agreement between the exact and approximated functions throughout the investigated domain. The largest relative error observed is $\varepsilon_r = 0.0213$, occurring at $\nu = -0.34$ and $x = 2.56$, while the error remains below approximately 2.2% over the entire domain considered. The largest deviations occur in the transition region between the power-series and asymptotic regimes, where the approximation interpolates between both limiting behaviors. Because of its explicit analytical form, the proposed approximation is suitable for differentiation, integration, and analytical manipulations involving modified Bessel functions. These properties make it a useful alternative for applications requiring simple and accurate analytic representations of $I_\nu(x)$.

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Appendix A

In this Appendix we present the derivation of Equations (22) and (23) of the main text.

For large values of x , it is well known that $\cosh x \simeq \sinh x \sim e^x/2$. Furthermore the following expression is used

$$\sqrt{1 + \gamma^4 x^2} \sim \gamma^2 x. \quad (\text{A1})$$

Thus, the following approximations are obtained

$$\left(p_0 + \tilde{p}_0 \sqrt{1 + \gamma^4 x^2} + p_2 x^2 \right) \sinh x \simeq \frac{e^x}{2} \left(\tilde{p}_0 \gamma^2 x + p_2 x^2 \right), \quad (\text{A2})$$

$$\left(p_1 + p_3 \sqrt{1 + \gamma^4 x^2} \right) x \cosh x \simeq \frac{e^x}{2} \left(p_1 x + p_3 \gamma^2 x^2 \right). \quad (\text{A3})$$

On the other hand, the denominator in Equation (14) can be expressed as

$$a(1 + \gamma^4 x^2)^\beta (q_0 + x^2) \simeq a\gamma^{4\beta} x^{2\beta+2}. \quad (\text{A4})$$

At this point it is important to note that for large values of x , the approximated expression for $\tilde{I}_\nu(x)$ is given by

$$\tilde{I}_\nu(x) \simeq \frac{e^x x^{\nu-1}}{2a\gamma^{4\beta} x^{2\beta+2}} \left[(\tilde{p}_0 \gamma^2 + p_1)x + (p_2 + p_3 \gamma^2)x^2 \right]. \quad (\text{A5})$$

Now, since $\beta = (2\nu - 1)/4$ [Equation (9)],

$$x^{\nu-1} x^{-2\beta-2} x = x^{-3/2}, \quad (\text{A6})$$

$$x^{\nu-1} x^{-2\beta-2} x^2 = x^{-1/2}, \quad (\text{A7})$$

and then, by direct comparison with Equation (6), we obtain the expressions given in Equations (22) and (23).

Appendix B

To illustrate the regularity of the parameterization in the interval $-1/2 \leq \nu \leq 1/4$, Figure A1 shows the coefficient $q_0 = N_0/D_0$ as a function of ν . Although q_0 increases rapidly as ν approaches $1/4$, it remains finite throughout the entire interval considered. This behavior is consistent with the fact that the first singularity of the coefficient matrix occurs outside the domain adopted in the present work.

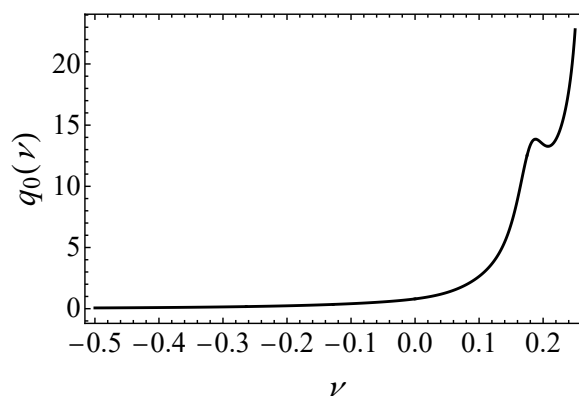


Figure A1. Dependence of the parameter q_0 on the order ν in the interval $-1/2 \leq \nu \leq 1/4$.

Appendix C

In this Appendix we verify that the approximation reproduces the known closed-form expression for $I_{-1/2}(x)$. For $\nu = -1/2$, the parameters take the values $q_0 = p_3 = 1/16$, while $p_0 = \tilde{p}_0 = p_1 = p_2 = 0$. Moreover, $a(-1/2) = \sqrt{\pi/2}$ and $\gamma(-1/2) = 2$. Substituting these values into Equation (14) gives

$$\tilde{I}_{-1/2}(x) = \frac{\frac{x}{16} \sqrt{1 + 16x^2} \cosh x}{\sqrt{\pi/2} x^{3/2} (1 + 16x^2)^{-1/2} \left(x^2 + \frac{1}{16}\right)}, \quad (\text{A8})$$

and then, after performing the corresponding simplifications,

$$\tilde{I}_{-1/2}(x) = I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x. \quad (\text{A9})$$

Therefore, the proposed approximation reproduces exactly the modified Bessel function of order $\nu = -1/2$. This exact reproduction provides an additional consistency check of the proposed approximation.

References

1. Jackson, J.D. *Classical Electrodynamics*, 3rd ed.; John Wiley and Sons, Inc.: New York, NY, USA, 1998; pp.112–119.
2. Watson, G.N. *A Treatise on the Theory of Bessel Functions*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1966; pp. 85–224.
3. Rothwell, E.J. Exponential Approximation of the Bessel functions $I_0(x)$, $I_1(x)$, $J_0(x)$, $J_1(x)$, $Y_0(x)$ and $H_0(x)$ with applications to electromagnetic scattering, radiation and diffraction [EM Programmer’s Notebooks]. *IEEE Antennas Propag. Mag.* **2009**, *59*, 138–147. [\[CrossRef\]](#)
4. Khosravian-Arab, H.; Dehghan, M.; Eslahchi, M.R. Generalized Bessel functions: Theory and their applications. *Math. Methods Appl. Sci.* **2017**, *40*, 6389–6410. [\[CrossRef\]](#)
5. Abramovitz, M.; Stegun, I.A. *Handbook of Mathematical Functions*, 9th ed.; Dover Books: New York, NY, USA, 1970; pp. 355–379.
6. Parsons, K.; Reichenadter, T.; Vicksman, A.; Segur, H. Explicit Solution for Cylindrical Heat Conduction. *Am. J. Undergrad. Res.* **2016**, *13*, 105–123. [\[CrossRef\]](#)
7. Gómez-Correa, J.E.; Balderas-Mata, S.E.; Coello, V.; Puente, N.P.; Rogel-Salazar, J.; Chávez-Cerda, S. On the physics of propagating Bessel modes in cylindrical waveguides. *Am. J. Phys.* **2017**, *85*, 341–345. [\[CrossRef\]](#)
8. Martinez-Luaces, V.; Ohanian, M. Laplace Transform and Bessel Functions in Solving an Electrochemical Corrosion Problem. *Ann. Math. Phys.* **2025**, *8*, 084–087. [\[CrossRef\]](#)
9. Ridings, K.M.; Hendy, S.C. Nanowire Melting Modes during the Solid–Liquid Phase Transition: Theory and Molecular Dynamics Simulations. *Sci. Rep.* **2022**, *12*, 20052. [\[CrossRef\]](#) [\[PubMed\]](#)
10. Liemert, A.; Kienle, A. Light Diffusion in a Turbid Cylinder. I. Homogeneous Case. *Opt. Express* **2010**, *18*, 9456–9473. [\[CrossRef\]](#) [\[PubMed\]](#)
11. Gupta, S.; Jana, S.P.; Pervin, R.; Gupta, A.K. Gate-tunable crossover between vortex-interaction and pinning dominated regimes in Josephson-coupled lead islands on graphene. *Phys. Rev. B* **2024**, *110*, 024506. [\[CrossRef\]](#)
12. Kalashnikov, D.S.; Seleznev, G.S.; Kudriashov, A.; Babich, I.; Vodolazov, D.Y.; Fominov, Y.V.; Stolyarov, V.S. Diode effect in Shapiro steps in an asymmetric SQUID with a superconducting nanobridge. *Phys. Rev. B* **2025**, *112*, 144504. [\[CrossRef\]](#)
13. Baskov, R.; Weiss, D.K.; Girvin, S.M. Exact amplitudes of parametric processes in driven Josephson circuits. *Phys. Rev. Appl.* **2025**, *24*, 054038. [\[CrossRef\]](#)
14. Kalva, S.K.; Özbek, A.; Reiss, M.; Deán-Ben, X.L.; Razansky, D. Spiral volumetric optoacoustic and ultrasound (SVOPUS) tomography of mice. *Photoacoustics* **2024**, *40*, 100659. [\[CrossRef\]](#) [\[PubMed\]](#)
15. Pini, A. Integrated correlators with a Wilson line in a $\mathcal{N} = 2$ quiver gauge theory at strong coupling. *J. High Energy Phys.* **2025**, *1*, 195. [\[CrossRef\]](#)
16. Blair, J.M.; Edwards, C.A. *Stable Rational Minimax Approximations to the Modified Bessel Functions $I_0(x)$ and $I_1(x)$* ; Atomic Energy of Canada Limited Report AECL-4928; Chalk River Nuclear Laboratories: Renfrew County, ON, Canada, 1974.
17. Cody, W.J. Algorithm 597: Sequence of Modified Bessel Functions of the First Kind. *ACM Trans. Math. Softw.* **1983**, *9*, 242–245. [\[CrossRef\]](#)
18. Luke, Y.L. *Mathematical Functions and Their Approximations*; Academic Press: New York, NY, USA, 1975.
19. Olver, F. *Asymptotics and Special Functions*; Academic Press: New York, NY, USA, 1974.

20. Martin, P.; Castro, E.; Paz, J.L. Multipoint quasi-rational approximations in Quantum Chemistry. In *Chapter 4 in New Developments in Quantum Chemistry*; Paz, J.L., Hernandez, J.A., Eds.; Transworld Research Network: Kerala, India, 2009; Chapter 3, pp. 55–76.
21. Baker, G.A., Jr.; Travers-Morris, P. *Padé Approximants*; Cambridge U.P.: Cambridge, UK, 1996.
22. Peker, H.A. An Introduction to Padé Approximation. In *Current Studies in Basic Sciences, Engineering and Tecnology*; ISRES Publishing: Konya, Turkey, 2021; pp. 143–155.
23. Martin, P.; Ramos-Andrade, J.P. An Analytic Approximation for the Bessel Function $J_\nu(x)$ for $-1/2 < \nu < 3/2$. *Mathematics* **2025**, *13*, 2774. [[CrossRef](#)]
24. Martin, P.; Ramos-Andrade, J.P.; Caro-Pérez, F.; Lastra, F. Accurate Analytical Approximation for the Bessel Function $J_2(x)$. *Math. Comput. Appl.* **2024**, *29*, 63. [[CrossRef](#)]

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