

Article

Octonionic Triality, the Matrix Structure of $\mathfrak{g}_2$, and Principal Bundle Moduli Spaces

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Abstract

We develop a matrix-theoretic framework for the natural embedding of the exceptional Lie algebra $\mathfrak{g}_2 = \text{Der}(\mathbb{O})$ in $\mathfrak{so}(8)$, use it to make constructive a recent existence result on octonionic triality, and derive geometric applications for moduli spaces of principal bundles. Specifically, the derivation condition for $\hat{D} \in \mathfrak{so}(7)$ is reformulated as a homogeneous linear system in the 21 entries of $\hat{D}$, whose solution space is identified with $\mathfrak{g}_2 = \ker \Psi$, where $\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*$ is the Lie derivative with respect to the associative 3-form φ on $\text{Im}(\mathbb{O})$. It is proved that $\text{rank } \Psi = 7$, and an algorithm is given for computing an orthonormal basis of $\mathfrak{g}_2$. The image $\Psi(\hat{A}_\sigma)$ of the triality generator is computed for all triples, yielding six nonzero components and squared norm 12. As geometric applications, the map Ψ is globalized to a morphism of adjoint bundles, giving an intrinsic characterization of the G_2 -reducible locus in $\mathcal{M}(\text{SO}(7))$. The orthogonal decomposition of $\mathfrak{so}(8)$ globalizes to an explicit splitting of the adjoint bundle of any $\text{SO}(8)$ -principal bundle admitting a G_2 -reduction. Finally, $\mathcal{M}(G_2)$ is identified as a connected component of the triality fixed-point locus in $\mathcal{M}(\text{Spin}(8))$, with an explicit description of the tangent and normal spaces in terms of the Lie-algebraic decomposition.

Keywords: octonions; triality; Lie algebra; derivations; exceptional Lie groups

MSC: 17B25; 17A35; 14D20; 22E46

1. Introduction

The octonions $\mathbb{O}$ form the largest normed division algebra over $\mathbb{R}$, unique up to isomorphism by the theorem of Hurwitz [1]. The compact Lie group G_2 , the smallest exceptional Lie group, arises as the automorphism group $G_2 = \text{Aut}(\mathbb{O})$, with Lie algebra $\mathfrak{g}_2 = \text{Der}(\mathbb{O})$ being the space of derivations of $\mathbb{O}$. The relationship between $\mathbb{O}$, G_2 , and the orthogonal group $\text{SO}(8)$ is deepened by the phenomenon of triality, as $\text{Spin}(8)$ is the unique simple Lie group admitting an outer automorphism of order three, intimately linked to the octonionic multiplication. A comprehensive account of these connections can be read in [2].

The exceptional Lie algebra $\mathfrak{g}_2$ was first classified among the simple Lie algebras independently by Killing and Cartan [3]. Its identification with the algebra of derivations of the octonions was established by Cartan and Engel [4]; a modern treatment in the spirit used here appears in [5]. The stabilizer description $G_2 = \text{Stab}_{\text{SO}(7)}(\varphi)$ of the compact form, where $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ is the associative 3-form, goes back to the work of Bonan [6] and was placed in the context of calibrated geometry by Harvey and Lawson [7]. Bryant's



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foundational work on metrics with G_2 holonomy [8] brought the associative 3-form and its stabilizer to the center of Riemannian geometry and made the linear-algebraic framework of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ indispensable. The treatises [9,10] provide thorough accounts of spinors, calibrations, and the role of G_2 in exceptional holonomy.

The derivation identity

$$D(xy) = D(x)y + xD(y)$$

used in this paper is a concept in the broader setting of non-associative algebras. The octonions are an example of an alternative algebra—a non-associative algebra in which the associator $(x, y, z) = (xy)z - x(yz)$ is an alternating trilinear map, equivalently characterized by the left and right alternative identities $x(xy) = (xx)y$ and $(yx)x = y(xx)$. In this context, Artin’s theorem guarantees that any two elements of an alternative algebra generate an associative subalgebra, which is the main property underlying the octonionic computations of the present paper. The theory of derivations of alternative algebras—including conditions under which multiplicative or additive maps satisfying derivation-type identities on generating subsets are genuine derivations—has been developed in [11,12] for derivations on alternative rings and algebras. For an introduction to the structural theory of alternative algebras and the derivation-type maps they admit, we refer to [13]. From this perspective, the computation of $\mathfrak{g}_2 = \text{Der}(\mathbb{O})$ carried out in the present work is not only a result in the theory of exceptional Lie algebras, but also an explicit illustration of derivation theory within the broader class of alternative algebras. Moreover, exceptional Lie algebras arising from octonionic constructions, such as E_6 , lead to interesting geometric applications, notably in the study of moduli spaces of G -Higgs bundles (see, for example, [14,15]). This highlights the broader geometric significance of the algebraic structures considered here.

The triality of $\text{Spin}(8)$ —the order-three outer automorphism σ of D_4 —was already present in the work of Cartan and was studied in depth by Chevalley [16]. Its precise relationship to the octonionic multiplication, and the fact that $G_2 = \text{Spin}(8)^\sigma$ is the fixed-point subgroup, are treated in [5]. While the abstract structure of triality is well understood, its explicit realization as a flow on $\mathbb{O}$ and its decomposition within the matrix structure of $\mathfrak{so}(8)$ have received less attention. The geometric realization of triality as a flow on $\mathbb{O}$ was studied in [17], where it was shown that the infinitesimal generator A_σ of the triality automorphism does not lie in $\mathfrak{g}_2$, and that there exists a smooth one-parameter family $\tau_t \in G_2$ such that the triality flow $\varphi_t = \exp(tA_\sigma)$ satisfies

$$\varphi_t(x \cdot y) = \tau_t(\varphi_t(x) \cdot \varphi_t(y))$$

for all $x, y \in \mathbb{O}$. This result was existential, as it guaranteed τ_t but did not identify it explicitly in terms of the matrix structure of $\mathfrak{g}_2 \subset \mathfrak{so}(8)$. The present paper addresses this gap by developing a complete matrix-theoretic framework for $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(8)$.

Moduli spaces of principal G -bundles over compact Riemann surfaces have been intensively studied in both algebraic geometry and mathematical physics since the foundational work of Narasimhan and Seshadri [18] for $G = \text{U}(n)$, extended to general semisimple groups by Ramanathan [19–21]. The Atiyah–Bott symplectic structure and the links to conformal field theory further developed their analysis [22]. For the exceptional group G_2 , the moduli space $\mathcal{M}(G_2)$ is an irreducible normal projective variety of dimension $14(g - 1)$ whose geometry has been studied in connection with Higgs bundles and the Hitchin system [23–25]. The action of outer automorphisms on moduli spaces, and the geometry of the resulting fixed-point loci, has been investigated systematically in [26] for

the case of Spin-Higgs bundles. The present paper extends that perspective using the explicit Lie-algebraic decompositions developed here.

The first main contribution of the paper is a reformulation of the derivation condition $\hat{D} \in \mathfrak{g}_2$ as an explicit homogeneous linear system in the 21 entries of $\hat{D} \in \mathfrak{so}(7)$. Using the associative 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ on $\text{Im}(\mathbb{O})$ and the identification $G_2 = \text{Stab}_{\text{SO}(7)}(\varphi)$ [8,9], Theorem 1 shows that the linear map

$$\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*, \quad \hat{D} \mapsto \mathcal{L}_{\hat{D}} \varphi,$$

has rank 7, so that $\mathfrak{g}_2 = \ker \Psi$ has dimension 14. Algorithm 1 provides a procedure for extracting an orthonormal basis $\hat{G}_1, \dots, \hat{G}_{14}$ of $\mathfrak{g}_2$ by Gaussian elimination of the associated sparse 28×21 integer matrix.

Our second contribution is the explicit orthogonal decomposition of $\mathfrak{so}(8)$. The inner product defined by $\langle A, B \rangle = -\frac{1}{2} \text{Tr}(AB)$ induces the decompositions $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}_7$ (Corollary 1) and $\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$ (Proposition 4), where $\mathfrak{m}_7$ is a 7-dimensional irreducible G_2 -module and $\mathfrak{v}_7 = \text{span}\{V_i\}$ with $V_i = e_0 e_i^T - e_i e_0^T$.

The third main contribution of the paper is the explicit computation of

$$\Psi(\hat{A}_\sigma) \in \Lambda^3(\mathbb{R}^7)^*$$

for all 35 triples. Theorem 2 shows that exactly six components are nonzero (Equation (10)) and that

$$\|\Psi(\hat{A}_\sigma)\|_{\Lambda^3}^2 = 12,$$

providing a computational proof that $\hat{A}_\sigma \notin \mathfrak{g}_2$. The $\mathfrak{g}_2$ -component

$$\hat{A}_\sigma^{(\mathfrak{g}_2)} = \pi_{\mathfrak{g}_2}(\hat{A}_\sigma)$$

is identified by Theorem 3 as the infinitesimal generator τ_0 of τ_t , and Corollary 2 establishes $\hat{A}_\sigma^{(\mathfrak{g}_2)} \neq 0$.

Finally, our fourth main contribution applies the above results to the geometry of moduli spaces of principal bundles over a compact Riemann surface X of genus $g \geq 2$. Specifically, the map Ψ globalizes to a morphism of vector bundles, yielding an intrinsic characterization of the G_2 -reducible locus $\mathcal{Z} \subset \mathcal{M}(\text{SO}(7))$ (Theorem 4). The orthogonal decomposition of $\mathfrak{so}(8)$ globalizes to an explicit splitting of the adjoint bundle of any $\text{SO}(8)$ -principal bundle admitting a G_2 -reduction (Theorem 5), from which an explicit Riemann–Roch computation gives the dimensions of all tangent spaces (Corollary 4). Finally, Theorem 6 identifies $\mathcal{M}(G_2)$ as a connected component of the triality fixed-point locus $\mathcal{M}(\text{Spin}(8))^{\sigma^*}$ and provides an explicit decomposition of the tangent and normal spaces at smooth points in terms of the summands $\mathfrak{g}_2, \mathfrak{m}_7, \mathfrak{v}_7$. Corollary 5 computes the normal bundle explicitly.

The paper is organized as follows. Section 2 recalls the octonionic multiplication, the Fano plane, the left-multiplication matrices, the group G_2 , and the triality matrices. In Section 3, we develop the linear system for $\mathfrak{g}_2$ and give the basis algorithm. In Section 4, we construct the orthogonal decomposition of $\mathfrak{so}(8)$, compute $\Psi(\hat{A}_\sigma)$ explicitly, and derive the family τ_t . The main applications of these results to the geometry of moduli spaces of principal bundles are provided in Section 5, where we study the G_2 -reducible locus, the decomposition of adjoint bundles, and the triality fixed-point locus in $\mathcal{M}(\text{Spin}(8))$. Finally, we collect the main conclusions and directions for future research.

2. Preliminaries

This section fixes notation and collects the algebraic background required throughout the paper. We recall the definition of the octonions and their multiplication via the Fano plane, the left-multiplication matrices and their algebraic properties, the group G_2 and its Lie algebra of derivations, and the triality matrices associated with the outer automorphism of $\text{Spin}(8)$.

2.1. The Octonions and the Fano Plane

We begin by recalling the structure of the octonions as a normed division algebra and the combinatorial encoding of their multiplication by the Fano projective plane. The Fano plane provides an explicit description of the structure constants f_{ijk} , which govern octonionic multiplication and underlie all subsequent computations. The seven oriented lines of $\text{PG}(2, 2)$ encode the non-associativity of $\mathbb{O}$.

Definition 1. The octonions $\mathbb{O}$ are the unique (up to isomorphism) 8-dimensional normed division algebra over $\mathbb{R}$ [1]. As a vector space, $\mathbb{O} = \mathbb{R}e_0 \oplus \text{Im}(\mathbb{O})$ where $\text{Im}(\mathbb{O}) = \text{span}_{\mathbb{R}}\{e_1, \dots, e_7\}$, with $e_0 = 1$ the multiplicative identity. The norm is

$$N(x) = \sum_{i=0}^7 x_i^2$$

and the conjugate is

$$\bar{x} = x_0 e_0 - \sum_{i=1}^7 x_i e_i.$$

Definition 2. The Fano plane $\text{PG}(2, 2)$ has point set $\{1, 2, 3, 4, 5, 6, 7\}$ and seven oriented lines:

$$(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 5, 6).$$

The structure constants $f_{ijk} \in \{-1, 0, +1\}$ satisfy $f_{ijk} = +1$ if (i, j, k) is a cyclic permutation of an oriented line, $f_{ijk} = -1$ if anticyclic, and $f_{ijk} = 0$ otherwise. The octonionic product of imaginary units is

$$e_i \cdot e_j = -\delta_{ij} e_0 + \sum_{k=1}^7 f_{ijk} e_k, \quad i, j \in \{1, \dots, 7\}. \quad (1)$$

Remark 1. The structure constants satisfy

$$f_{ijk} = f_{jki} = f_{kij}$$

and

$$f_{ijk} = -f_{jik}.$$

The nonzero values with $i < j < k$ are

$$f_{123} = f_{145} = f_{167} = f_{246} = f_{257} = f_{347} = f_{356} = 1.$$

Table 1 gives the multiplication table of $\text{Im}(\mathbb{O})$.

Table 1. Multiplication table of $\text{Im}(\mathbb{O})$. Entry (i, j) gives $e_i \cdot e_j$. Notation: $-k$ means $-e_k$ and -1 means $-e_0$.

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	$-e_6$	$-e_1$	-1	e_3	e_2
e_6	$-e_7$	e_4	e_5	$-e_2$	$-e_3$	-1	e_1
e_7	e_6	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	-1

The left-multiplication matrices L_i provide an 8×8 real matrix representation of the octonionic algebra. Their skew-symmetry and the Clifford-type relation

$$L_i L_j + L_j L_i = -2\delta_{ij} I_8$$

are the algebraic properties that connect octonionic multiplication to the spin representation theory of $\text{SO}(8)$ and make the left-multiplication matrices useful for computing with $\mathfrak{g}_2 \subset \mathfrak{so}(8)$.

Definition 3. For $i = 1, \dots, 7$, the left multiplication matrix $L_i \in \mathbb{R}^{8 \times 8}$ represents $x \mapsto e_i \cdot x$ in the basis $\{e_0, e_1, \dots, e_7\}$:

$$(L_i)_{ab} = \begin{cases} -\delta_{ib} & a = 0, \\ \delta_{ib} & a \neq 0, b = 0, \\ f_{iab} & a \neq 0, b \neq 0. \end{cases}$$

Remark 2. Each L_i is skew-symmetric ($L_i^T = -L_i$) and satisfies

$$L_i^2 = -I_8$$

and

$$L_i L_j + L_j L_i = -2\delta_{ij} I_8.$$

2.2. The Group G_2 and Triality

Now we recall the definition of G_2 as the automorphism group of $\mathbb{O}$ and its basic properties as a compact Lie group of dimension 14. The structural fact exploited throughout the paper is that every derivation $D \in \mathfrak{g}_2$ annihilates the unit e_0 , so D is entirely determined by its restriction $\hat{D} = D|_{\text{Im}(\mathbb{O})}$, which lies in $\mathfrak{so}(7)$ and satisfies an explicit system of linear equations derived from the Leibniz rule.

Definition 4. The compact exceptional Lie group $G_2 = \text{Aut}(\mathbb{O})$ consists of all $\phi \in \text{GL}(\mathbb{O})$ satisfying $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for all $x, y \in \mathbb{O}$. Its Lie algebra is

$$\mathfrak{g}_2 = \text{Der}(\mathbb{O}) = \{D \in \mathfrak{gl}(\mathbb{O}) : D(xy) = D(x)y + xD(y)\}.$$

Proposition 1. The group G_2 satisfies:

- (i) $G_2 \subset \text{SO}(7)$, preserving e_0 and the norm N ;
- (ii) $\dim G_2 = 14$;
- (iii) Every $D \in \mathfrak{g}_2$ satisfies $D(e_0) = 0$, so

$$\mathfrak{g}_2 \subset \mathfrak{so}(7) \subset \mathfrak{so}(8).$$

Proof. Parts (i) and (ii) are classical and can be found in [5]. For part (iii), notice that

$$D(1) = D(1 \cdot 1) = 2D(1),$$

so $D(e_0) = 0$. $\square$

Remark 3. Since $D(e_0) = 0$, every $D \in \mathfrak{g}_2$ is determined by its restriction

$$\hat{D} = D|_{\text{Im}(\mathbb{O})} \in \mathfrak{so}(7),$$

which satisfies the derivative condition explicitly established in Equation (3) below.

The outer automorphism σ of order three of $\text{Spin}(8)$ —triality—has an explicit infinitesimal generator $A_\sigma \in \mathfrak{so}(8)$ whose restriction to $\text{Im}(\mathbb{O})$ consists of three simultaneous rotation planes [17]. We record the precise matrix form of $\hat{A}_\sigma$ and note the fact that, since σ is an outer automorphism, A_σ cannot belong to $\mathfrak{g}_2$. This will be confirmed computationally in Section 4, thus providing a computational approach to this fact.

Definition 5. The infinitesimal generator of the triality automorphism of $\text{Spin}(8)$ is $A_\sigma \in \mathfrak{so}(8)$, with zero 0-th row and column and 7×7 imaginary block

$$\hat{A}_\sigma = F_{12}^{(7)} + F_{34}^{(7)} + F_{56}^{(7)} \in \mathfrak{so}(7), \quad (2)$$

where

$$F_{ab}^{(7)} = e_a e_b^T - e_b e_a^T.$$

Thus $\hat{A}_\sigma$ consists of three rotation blocks

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in planes (e_1, e_2) , (e_3, e_4) , (e_5, e_6) , with e_7 fixed. The square is

$$\hat{A}_{\sigma^2} = \hat{A}_\sigma^2 = \text{diag}(-1, -1, -1, -1, -1, -1, 0).$$

Remark 4. Since triality is an outer automorphism of $\text{Spin}(8)$, its infinitesimal generator cannot belong to $\mathfrak{g}_2$. This is confirmed computationally in Theorem 2 of the paper.

3. A Matrix Basis for $\mathfrak{g}_2$

This section constructs an explicit basis for $\mathfrak{g}_2$ as a subspace of $\mathfrak{so}(7)$, establishes $\dim \mathfrak{g}_2 = 14$ via the associative 3-form, and provides an algorithm for computing the basis.

We translate the Leibniz rule

$$\hat{D}(e_i \cdot e_j) = \hat{D}(e_i) \cdot e_j + e_i \cdot \hat{D}(e_j)$$

into an explicit sparse linear system in the 21 independent entries of the skew-symmetric matrix $\hat{D} \in \mathfrak{so}(7)$. The sparsity of the system—at most two nonzero coefficients per equation—reflects the binary incidence structure of the Fano plane and is the key feature that makes Gaussian elimination of the 28×21 coefficient matrix efficient.

A matrix $\hat{D} \in \mathfrak{so}(7)$ belongs to $\mathfrak{g}_2$ if and only if

$$\hat{D}(e_i \cdot e_j) = \hat{D}(e_i) \cdot e_j + e_i \cdot \hat{D}(e_j), \quad 1 \leq i < j \leq 7, \quad (3)$$

with products via Table 1. Parametrizing $\hat{D}$ by $d_{ab} := \hat{D}_{ab}$ ($a < b$) and projecting onto e_n :

$$f_{ijk} \hat{D}_{nk} = \sum_{m=1}^7 \hat{D}_{mi} f_{mjn} + \sum_{m=1}^7 \hat{D}_{mj} f_{imn}, \quad n = 1, \dots, 7. \quad (4)$$

Since each pair $\{j, n\}$ belongs to exactly one Fano line, each right-hand side has at most two nonzero summands.

Definition 6. For distinct $p, q \in \{1, \dots, 7\}$, $m(p, q)$ denotes the unique third index of the Fano line through $\{p, q\}$, and

$$\varepsilon(p, q) := f_{m(p,q),p,q} \in \{\pm 1\}.$$

Table 2 records all 21 values.

Table 2. Third Fano index $m(p, q)$ and sign $\varepsilon(p, q) = f_{m(p,q),p,q}$ for all 21 pairs $\{p, q\} \subset \{1, \dots, 7\}$.

$\{p, q\}$	m	ε	$\{p, q\}$	m	ε	$\{p, q\}$	m	ε
$\{1, 2\}$	3	−1	$\{2, 4\}$	6	+1	$\{4, 5\}$	1	+1
$\{1, 3\}$	2	−1	$\{2, 5\}$	7	+1	$\{4, 6\}$	2	+1
$\{1, 4\}$	5	+1	$\{2, 6\}$	4	−1	$\{4, 7\}$	3	+1
$\{1, 5\}$	4	−1	$\{2, 7\}$	5	−1	$\{5, 6\}$	3	+1
$\{1, 6\}$	7	+1	$\{3, 4\}$	7	+1	$\{5, 7\}$	2	+1
$\{1, 7\}$	6	−1	$\{3, 5\}$	6	+1	$\{6, 7\}$	1	+1
$\{2, 3\}$	1	+1	$\{3, 6\}$	5	−1	$\{3, 7\}$	4	−1

Remark 5. For the line $(1, 4, 5)$:

- $m(1, 4) = 5, \varepsilon(1, 4) = f_{5,1,4} = +1$ (cyclic);
- $m(4, 5) = 1, \varepsilon(4, 5) = +1$;
- $m(1, 5) = 4, \varepsilon(1, 5) = f_{4,1,5} = -1$ (anticyclic).

In this notation, system (4) becomes

$$f_{ijk} \hat{D}_{nk} = \varepsilon(j, n) \hat{D}_{m(j,n),i} + \varepsilon(i, n) \hat{D}_{m(i,n),j}, \quad j \neq n, i \neq n. \quad (5)$$

Rather than computing the rank of the derivation system directly by elimination, we use the associative 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ to give a conceptual proof of $\dim \mathfrak{g}_2 = 14$ via the rank–nullity theorem. The key observation is that the derivation condition is equivalent to the vanishing of the Lie derivative $\mathcal{L}_{\hat{D}} \varphi$, so that $\mathfrak{g}_2$ is the kernel of the linear map $\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*$. The classical result $G_2 = \text{Stab}_{\text{SO}(7)}(\varphi)$, together with the known dimension of G_2 , then immediately gives $\text{rank } \Psi = 7$.

Definition 7. The associative 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ has components $\varphi_{ijk} = f_{ijk}$. With $e^{ijk} = e^i \wedge e^j \wedge e^k$:

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} + e^{257} + e^{347} + e^{356}. \quad (6)$$

Proposition 2. For $\hat{D} \in \mathfrak{so}(7)$, condition (3) is equivalent to $\mathcal{L}_{\hat{D}} \varphi = 0$, where

$$(\mathcal{L}_{\hat{D}} \varphi)_{abc} = \sum_{d=1}^7 (\hat{D}_{da} \varphi_{dbc} + \hat{D}_{db} \varphi_{adc} + \hat{D}_{dc} \varphi_{abd}). \quad (7)$$

Proof. Each sum reduces to a single term:

$$\sum_d \hat{D}_{da} \varphi_{dbc} = \hat{D}_{m(b,c),a} \varepsilon(b, c).$$

For Fano triples (a, b, c) : $m(b, c) = a$, $m(a, c) = b$, $m(a, b) = c$, so all summands involve $\hat{D}_{pp} = 0$, giving $(\mathcal{L}_{\hat{D}}\varphi)_{abc} = 0$ for every $\hat{D} \in \mathfrak{so}(7)$.

For a non-Fano triple (a, b, c) , since $\varphi_{dbc} \neq 0$ only when $d = m(b, c)$, and similarly for the other sums, Equation (7) reduces to

$$(\mathcal{L}_{\hat{D}}\varphi)_{abc} = \varepsilon(b, c) \hat{D}_{m(b,c),a} + \varepsilon(a, c) \hat{D}_{m(a,c),b} + \varepsilon(a, b) \hat{D}_{m(a,b),c} = 0.$$

Using the skew-symmetry $\hat{D}_{m(a,b),c} = -\hat{D}_{c,m(a,b)}$ and noting that $\varepsilon(a, b) = f_{m(a,b),a,b}$, this rearranges to

$$\varepsilon(b, c) \hat{D}_{m(b,c),a} + \varepsilon(a, c) \hat{D}_{m(a,c),b} = f_{ab m(a,b)} \hat{D}_{c,m(a,b)},$$

which is precisely Equation (5) with the substitution $i = a$, $j = b$, $n = c$, $k = m(a, b)$. Hence the two conditions are equivalent for each non-Fano triple (see also [8,9]). $\square$

Theorem 1. *The map*

$$\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*, \quad \hat{D} \mapsto \mathcal{L}_{\hat{D}}\varphi,$$

satisfies $\mathfrak{g}_2 = \ker \Psi$, $\text{rank } \Psi = 7$, and $\dim \mathfrak{g}_2 = 14$.

Proof. By Proposition 2, $\ker \Psi = \mathfrak{g}_2$. The classical result $G_2 = \text{Stab}_{\text{SO}(7)}(\varphi)$ [8,9] gives $\mathfrak{g}_2 = \ker \Psi$. Since $\dim G_2 = 14$ [5], $\text{rank } \Psi = 21 - 14 = 7$. $\square$

Corollary 1. *There is an orthogonal direct sum decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}_7$, with respect to the inner product defined by*

$$\langle A, B \rangle = -\frac{1}{2} \text{Tr}(AB),$$

where

$$\mathfrak{m}_7 = \mathfrak{g}_2^\perp \cap \mathfrak{so}(7)$$

has dimension 7.

Proof. For $A \neq 0$ in $\mathfrak{so}(7)$,

$$\langle A, A \rangle = -\frac{1}{2} \text{Tr}(A^2) = \frac{1}{2} \sum_{i,j} A_{ij}^2 > 0,$$

so $\langle \cdot, \cdot \rangle$ is positive-definite. The orthogonal direct sum follows, with $\dim \mathfrak{m}_7 = 21 - 14 = 7$. $\square$

Remark 6. *The subspace $\mathfrak{m}_7$ is irreducible as a $\mathfrak{g}_2$ -module, isomorphic to the 7-dimensional standard representation of G_2 [5,9]. The map Ψ is G_2 -equivariant and $\Psi|_{\mathfrak{m}_7}$ is injective, so $\Psi(\mathfrak{m}_7)$ is a 7-dimensional G_2 -invariant subspace of $\Lambda^3(\mathbb{R}^7)^*$.*

We now give an explicit constructive procedure for extracting a basis of $\mathfrak{g}_2$ from the linear system of Section 3. The algorithm reduces to Gaussian elimination of a sparse integer matrix, followed by a Gram–Schmidt orthogonalization, and can be executed in full by any computer algebra system.

Proposition 3. *Algorithm 1 produces a well-defined orthonormal basis $\hat{G}_1, \dots, \hat{G}_{14}$ of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$.*

Proof. Each $\hat{D}_k \in \ker \Psi = \mathfrak{g}_2$ by construction. Independence follows from the identity structure of free-parameter coordinates. Since $\dim \mathfrak{g}_2 = 14$, they span $\mathfrak{g}_2$. Gram–Schmidt preserves the span within $\mathfrak{g}_2$. $\square$

Algorithm 1 Basis of $\mathfrak{g}_2$

Step 1. Write the 28 non-Fano equations $(\mathcal{L}_{\hat{D}}\varphi)_{abc} = 0$ using (7) and Table 2; convert to variables d_{pq} via

$$\hat{D}_{pq} = \text{sgn}(q - p) d_{\min(p,q), \max(p,q)}.$$

Step 2. Form the 28×21 sparse integer matrix M (three nonzero entries ± 1 per row).

Step 3. Gaussian-eliminate M ; by Theorem 1, $\text{rank}(M) = 7$, leaving 14 free columns.

Step 4. For each free parameter, set it to 1 (others to 0), solve for constrained parameters, and read off $\hat{D}_k \in \mathfrak{so}(7)$.

Step 5. Gram–Schmidt with

$$\langle A, B \rangle = -\frac{1}{2} \text{Tr}(AB)$$

yields the orthonormal basis $\hat{G}_1, \dots, \hat{G}_{14}$ of $\mathfrak{g}_2$.

Remark 7. The execution of Algorithm 1 through a computer algebra system (e.g., SAGE, GAP, or MATHEMATICA) would carry out the Gaussian elimination and Gram–Schmidt steps on the sparse 28×21 integer matrix M .

4. Orthogonal Decomposition of $\mathfrak{so}(8)$ and Decomposition of the Triality Generator

Having established the splitting

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}_7,$$

we extend the orthogonal decomposition to the full algebra $\mathfrak{so}(8)$ by identifying the seven-dimensional complement $\mathfrak{v}_7$ spanned by the matrices

$$V_i = e_0 e_i^T - e_i e_0^T$$

that encode the coupling between the real and imaginary parts of $\mathbb{O}$. The resulting three-way splitting

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$$

is the Lie-algebraic foundation for the geometric applications in Section 5.

Definition 8. For $i = 1, \dots, 7$ define

$$V_i = e_0 e_i^T - e_i e_0^T \in \mathfrak{so}(8),$$

with $(V_i)_{0i} = 1$, $(V_i)_{i0} = -1$, and all other entries zero. Set $\mathfrak{v}_7 = \text{span}\{V_1, \dots, V_7\}$.

Proposition 4. With respect to $\langle A, B \rangle = -\frac{1}{2} \text{Tr}(AB)$ on $\mathfrak{so}(8)$,

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$$

orthogonally, with $14 + 7 + 7 = 28$. The matrices $V_1, \dots, V_7$ are orthonormal and $\mathfrak{v}_7 \perp \mathfrak{so}(7)$.

Proof. Using $e_p^T e_q = \delta_{pq}$,

$$V_i V_j = -\delta_{ij} e_0 e_0^T - e_i e_j^T,$$

so

$$\text{Tr}(V_i V_j) = -2\delta_{ij}$$

and

$$\langle V_i, V_j \rangle = \delta_{ij}.$$

For $\hat{D} \in \mathfrak{so}(7)$,

$$(V_i)_{ac} = \delta_{a0}\delta_{ci} - \delta_{ai}\delta_{c0},$$

so

$$\text{Tr}(V_i \hat{D}) = \hat{D}_{i0} - \hat{D}_{0i} = 0.$$

The decomposition follows from Corollary 1. $\square$

Remark 8. Since A_σ fixes e_0 , it has zero 0-th row and column, so $A_\sigma \in \mathfrak{so}(7)$ and

$$\hat{A}_\sigma = \hat{A}_\sigma^{(\mathfrak{g}_2)} + \hat{A}_\sigma^{(\mathfrak{m}_7)},$$

where

$$\hat{A}_\sigma^{(\mathfrak{g}_2)} = \pi_{\mathfrak{g}_2}(\hat{A}_\sigma)$$

and

$$\hat{A}_\sigma^{(\mathfrak{m}_7)} = \pi_{\mathfrak{m}_7}(\hat{A}_\sigma).$$

We now apply the map Ψ to the triality generator $\hat{A}_\sigma$ and compute explicitly, for all 35 ordered triples (a, b, c) with $a < b < c$, the component $(\mathcal{L}_{\hat{A}_\sigma} \varphi)_{abc}$. The computation uses the Fano data of Table 2 and the nonzero entries of $\hat{A}_\sigma$, and the outcome is precise: six nonzero values, arranged symmetrically, with squared norm 12. This provides a self-contained computational proof that $\hat{A}_\sigma \notin \mathfrak{g}_2$, independent of any group-theoretic argument.

The nonzero entries of $\hat{A}_\sigma$ are

$$\begin{aligned} (\hat{A}_\sigma)_{21} = 1, (\hat{A}_\sigma)_{12} = -1, (\hat{A}_\sigma)_{43} = 1 \\ (\hat{A}_\sigma)_{34} = -1, (\hat{A}_\sigma)_{65} = 1, (\hat{A}_\sigma)_{56} = -1. \end{aligned} \quad (8)$$

Since $\hat{A}_\sigma^2 = \text{diag}(-1, -1, -1, -1, -1, -1, 0)$,

$$\|\hat{A}_\sigma\|^2 = -\frac{1}{2} \text{Tr}(\hat{A}_\sigma^2) = 3. \quad (9)$$

Theorem 2. Among the 35 components $(\mathcal{L}_{\hat{A}_\sigma} \varphi)_{abc}$ with $a < b < c$, exactly six are nonzero:

$$\begin{aligned} (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{124} = -1, (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{135} = +1, \\ (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{157} = +2, (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{236} = +1, \\ (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{267} = -2, (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{456} = -1. \end{aligned} \quad (10)$$

All seven Fano-triple and all remaining 22 non-Fano components vanish. In particular, $\Psi(\hat{A}_\sigma) \neq 0$, so

$$\hat{A}_\sigma \notin \mathfrak{g}_2 = \ker \Psi.$$

Proof. By Proposition 2 and Table 2:

$$\begin{aligned} (\mathcal{L}_{\hat{A}_\sigma} \varphi)_{abc} &= (\hat{A}_\sigma)_{m(b,c),a} \varepsilon(b, c) + (\hat{A}_\sigma)_{m(a,c),b} f_{a,m(a,c),c} \\ &\quad + (\hat{A}_\sigma)_{m(a,b),c} f_{a,b,m(a,b)}. \end{aligned}$$

For Fano triples, $m(b, c) = a$, $m(a, c) = b$, $m(a, b) = c$, so all summands involve $(\hat{A}_\sigma)_{pp} = 0$.

For non-Fano triples, we have that $(\hat{A}_\sigma)_{pq} \neq 0$ only for

$$(p, q) \in \{(2, 1), (1, 2), (4, 3), (3, 4), (6, 5), (5, 6)\}.$$

We compute the six nonzero cases explicitly:

- $(1, 2, 4)$: $m(2, 4) = 6$, $m(1, 4) = 5$, $m(1, 2) = 3$; $f_{1,5,4} = -1$ (anticyclic), $f_{1,2,3} = +1$. Thus,

$$0 \cdot (+1) + 0 \cdot (-1) + (-1)(+1) = -1.$$

- $(1, 3, 5)$: $m(3, 5) = 6$, $m(1, 5) = 4$, $m(1, 3) = 2$; $f_{1,4,5} = +1$, $f_{1,3,2} = -1$. Thus,

$$0 + (1)(+1) + 0 = +1.$$

- $(1, 5, 7)$: $m(5, 7) = 2$, $m(1, 7) = 6$, $m(1, 5) = 4$; $\varepsilon(5, 7) = f_{2,5,7} = +1$, $f_{1,6,7} = +1$, $f_{1,5,4} = -1$. Thus,

$$(1)(+1) + (1)(+1) + 0 = +2.$$

- $(2, 3, 6)$: $m(3, 6) = 5$, $m(2, 6) = 4$, $m(2, 3) = 1$; $\varepsilon(3, 6) = f_{5,3,6} = -1$, $f_{2,4,6} = +1$, $f_{2,3,1} = +1$. Thus,

$$0 + (1)(+1) + 0 = +1.$$

- $(2, 6, 7)$: $m(6, 7) = 1$, $m(2, 7) = 5$, $m(2, 6) = 4$; $\varepsilon(6, 7) = f_{1,6,7} = +1$, $f_{2,5,7} = +1$, $f_{2,6,4} = -1$. Thus,

$$(-1)(+1) + (-1)(+1) + 0 = -2.$$

- $(4, 5, 6)$: $m(5, 6) = 3$, $m(4, 6) = 2$, $m(4, 5) = 1$; $\varepsilon(5, 6) = f_{3,5,6} = +1$, $f_{4,2,6} = -1$, $f_{4,5,1} = +1$. Thus,

$$(-1)(+1) + 0 + 0 = -1.$$

For the 22 remaining non-Fano triples, all three summands involve $(\hat{A}_\sigma)_{pq} = 0$. For brevity, we verify four representative cases:

- $(1, 2, 5)$: all entries zero;
- $(2, 4, 5)$: $(-1)(+1) + 0 + (1)(+1) = 0$;
- $(1, 4, 6)$: $(1)(+1) + 0 + (-1)(+1) = 0$;
- $(3, 4, 5)$: all entries zero.

Since $\Psi(\hat{A}_\sigma) \neq 0$, it follows that

$$\hat{A}_\sigma \notin \ker \Psi = \mathfrak{g}_2.$$

□

Remark 9. The squared norm of $\Psi(\hat{A}_\sigma)$ with respect to the inner product defined by

$$\langle \omega_1, \omega_2 \rangle_{\Lambda^3} = \sum_{a < b < c} (\omega_1)_{abc} (\omega_2)_{abc}$$

is

$$\|\Psi(\hat{A}_\sigma)\|_{\Lambda^3}^2 = (-1)^2 + 1^2 + 2^2 + 1^2 + (-2)^2 + (-1)^2 = 12. \quad (11)$$

The nonvanishing of $\Psi(\hat{A}_\sigma)$ determines the $\mathfrak{m}_7$ -component of $\hat{A}_\sigma$. We now compute its $\mathfrak{g}_2$ -component and use it to identify explicitly the unique smooth one-parameter family $\tau_t \in G_2$ whose existence was established in the literature [17]. The main result of this subsection is that

$$\hat{A}_\sigma^{(\mathfrak{g}_2)} = \dot{\tau}_0$$

is nonzero, so that the triality flow involves a G_2 -rotation and is not merely the obstruction to A_σ being a derivation.

Proposition 5. With $\hat{A}_\sigma^{(\mathfrak{g}_2)} = \pi_{\mathfrak{g}_2}(\hat{A}_\sigma)$ and $\hat{A}_\sigma^{(\mathfrak{m}_7)} = \pi_{\mathfrak{m}_7}(\hat{A}_\sigma)$:

- (i) $\hat{A}_\sigma^{(\mathfrak{m}_7)} \neq 0$.
- (ii) $\|\hat{A}_\sigma^{(\mathfrak{g}_2)}\|^2 + \|\hat{A}_\sigma^{(\mathfrak{m}_7)}\|^2 = \|\hat{A}_\sigma\|^2 = 3$.
- (iii) $\Psi(\hat{A}_\sigma^{(\mathfrak{m}_7)}) = \Psi(\hat{A}_\sigma)$, with nonzero components (10).

Proof. Part (i) follows from

$$\Psi(\hat{A}_\sigma^{(\mathfrak{m}_7)}) = \Psi(\hat{A}_\sigma) \neq 0.$$

Part (ii) follows from the Pythagorean identity. Part (iii) follows from

$$\Psi(\hat{A}_\sigma^{(\mathfrak{g}_2)}) = 0$$

and the linearity of Ψ . $\square$

Remark 10. The explicit matrix $\hat{A}_\sigma^{(\mathfrak{g}_2)}$ is computed from the basis of Algorithm 1 via

$$\hat{A}_\sigma^{(\mathfrak{g}_2)} = \sum_{k=1}^{14} \langle \hat{A}_\sigma, \hat{G}_k \rangle \hat{G}_k, \quad \hat{A}_\sigma^{(\mathfrak{m}_7)} = \hat{A}_\sigma - \hat{A}_\sigma^{(\mathfrak{g}_2)}. \quad (12)$$

Theorem 3. There exists a unique smooth one-parameter family $\{\tau_t\}_{t \in \mathbb{R}}$ in G_2 such that:

- (i) $\tau_0 = \text{id}_{\mathbb{O}}$;
- (ii) $\varphi_t(x \cdot y) = \tau_t(\varphi_t(x) \cdot \varphi_t(y))$ for all $x, y \in \mathbb{O}$, $t \in \mathbb{R}$, where φ_t is defined by $\varphi_t = \exp(tA_\sigma)$;
- (iii) $\dot{\tau}_0 = \hat{A}_\sigma^{(\mathfrak{g}_2)} \in \mathfrak{g}_2$.

Proof. Existence and part (ii) are proved in [17].

Part (iii): Differentiating (ii) at $t = 0$ with $\varphi_0 = \tau_0 = \text{id}$:

$$A_\sigma(xy) = \dot{\tau}_0(xy) + A_\sigma(x)y + xA_\sigma(y).$$

Setting $B = A_\sigma - \dot{\tau}_0$ and using the Leibniz rule for $\dot{\tau}_0 \in \mathfrak{g}_2$, one finds

$$B(xy) - B(x)y - xB(y) = \dot{\tau}_0(xy).$$

The identification $\dot{\tau}_0 = \hat{A}_\sigma^{(\mathfrak{g}_2)}$ then follows directly from the global construction of [17].

For uniqueness, notice that any $\sigma_t \in G_2$ satisfying (i) and (ii) has

$$\dot{\sigma}_0 = \hat{A}_\sigma^{(\mathfrak{g}_2)}$$

by the same differentiation, hence satisfies the same first-order ODE on G_2 with the same initial condition. Then, standard ODE uniqueness gives $\sigma_t = \tau_t$. $\square$

Remark 11. The family

$$\tau_t = \exp\left(t \hat{A}_\sigma^{(\mathfrak{g}_2)}\right)$$

is the constructive realization of the existing family of [17].

Corollary 2. The family τ_t of Theorem 3 satisfies:

- (i) $\|\hat{A}_\sigma^{(\mathfrak{g}_2)}\|^2 + \|\hat{A}_\sigma^{(\mathfrak{m}_7)}\|^2 = 3.$
- (ii) $\Psi(\hat{A}_\sigma^{(\mathfrak{m}_7)}) = \Psi(\hat{A}_\sigma)$ measures the failure of A_σ to be a derivation.
- (iii) $\hat{A}_\sigma^{(\mathfrak{g}_2)} \neq 0.$

Proof. Parts (i) and (ii) follow directly from Proposition 5.

For part (iii), suppose for contradiction that $\hat{A}_\sigma^{(\mathfrak{g}_2)} = 0$. Then $\tau_t \equiv \text{id}$, and Theorem 3(ii) gives $\varphi_t \in G_2$ for all t , hence

$$A_\sigma = \dot{\varphi}_t|_{t=0} \in \mathfrak{g}_2,$$

contradicting Theorem 2. $\square$

5. Applications to Moduli Spaces of Principal Bundles

Let X be a compact Riemann surface of genus $g \geq 2$. This section applies the algebraic results of the above section to derive consequences for the geometry of moduli spaces of principal bundles over X . Specifically, we use the orthogonal decomposition

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7,$$

the map

$$\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*,$$

and the identification $G_2 = \text{Spin}(8)^\sigma$ to study the G_2 -reducible locus in $\mathcal{M}(\text{SO}(7))$, to decompose adjoint bundles of $\text{SO}(8)$ -principal bundles, and to identify $\mathcal{M}(G_2)$ as a component of the triality fixed-point locus in $\mathcal{M}(\text{Spin}(8))$.

We denote by $\mathcal{M}(G)$ the moduli space of semistable principal G -bundles of topological type zero over X . For G semisimple and compact, $\mathcal{M}(G)$ is a normal projective variety of dimension $\dim(G) \cdot (g - 1)$ [19].

The algebraic map

$$\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*$$

of Theorem 1 globalizes naturally to a morphism of associated vector bundles over a compact Riemann surface X . The locus of principal $\text{SO}(7)$ -bundles for which the pointwise kernel of this morphism forms a globally well-defined rank-14 subbundle of the adjoint bundle is precisely the set of bundles admitting a reduction of structure group to G_2 , providing an intrinsic criterion for G_2 -reducibility in the moduli space $\mathcal{M}(\text{SO}(7))$.

Definition 9. For a principal $\text{SO}(7)$ -bundle P over X , the fibred Lie derivative map is

$$\Psi_P: \text{ad}(P) \longrightarrow \Lambda_P^3, \quad (13)$$

defined fibrewise by $\Psi_P|_{P_x} = \Psi$ for each $x \in X$, where

$$\text{ad}(P) = P \times_{\text{SO}(7)} \mathfrak{so}(7)$$

has rank 21 and

$$\Lambda_P^3 = P \times_{\text{SO}(7)} \Lambda^3(\mathbb{R}^7)^*$$

has rank 35.

Remark 12. The map Ψ_P is well-defined because $\Psi = \mathcal{L}_{(\cdot)}\varphi$ is $\text{SO}(7)$ -equivariant: for $g \in \text{SO}(7)$ and $\hat{D} \in \mathfrak{so}(7)$,

$$\Psi(g\hat{D}g^{-1}) = \mathcal{L}_{g\hat{D}g^{-1}}\varphi = g_*(\mathcal{L}_{\hat{D}}\varphi) = g \cdot \Psi(\hat{D}),$$

where the last equality uses the natural $\mathrm{SO}(7)$ -action on $\Lambda^3(\mathbb{R}^7)^*$.

Theorem 4. Let P be a principal $\mathrm{SO}(7)$ -bundle over X . The following are equivalent:

- (i) P is G_2 -reducible, i.e., it admits a reduction of structure group to $G_2 \subset \mathrm{SO}(7)$.
- (ii) The pointwise kernel $\ker(\Psi_P) \subset \mathrm{ad}(P)$ is a globally well-defined rank-14 subbundle of $\mathrm{ad}(P)$, i.e., the assignment $x \mapsto \ker(\Psi_P|_{P_x})$ defines a locally free subsheaf of rank 14 of the sheaf of sections of $\mathrm{ad}(P)$.
- (iii) The bundle Λ_P^3 admits a global section $\varphi_P \in H^0(X, \Lambda_P^3)$ whose value at every $x \in X$ is $\mathrm{SO}(7)$ -conjugate to the associative 3-form $\varphi \in \Lambda^3(\mathbb{R}^7)^*$ of Definition 7.

Proof. We note first that the linear map $\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*$ has rank 7 and kernel $\mathfrak{g}_2$ by Theorem 1. The fibrewise map $\Psi_P|_{P_x} = \Psi$ therefore has rank 7 at every point $x \in X$ for every principal $\mathrm{SO}(7)$ -bundle P ; rank alone does not distinguish G_2 -reducible bundles. The meaningful condition is whether the family of kernels $\{\ker(\Psi_P|_{P_x})\}_{x \in X}$ patches together into a globally trivializable subbundle of $\mathrm{ad}(P)$, which requires the transition functions of P to preserve $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, i.e., to take values in $N_{\mathrm{SO}(7)}(\mathfrak{g}_2) = G_2$.

(i) $\Rightarrow$ (ii): If $P = Q \times_{G_2} \mathrm{SO}(7)$ for a G_2 -bundle Q , the transition functions of P take values in $G_2 \subset \mathrm{SO}(7)$. Since G_2 preserves $\mathfrak{g}_2 = \ker \Psi$ under the adjoint action, the family of subspaces $\ker(\Psi_P|_{P_x}) \cong \mathfrak{g}_2$ is preserved by the transition functions and defines a globally well-defined rank-14 subbundle of $\mathrm{ad}(P)$, isomorphic to $\mathrm{ad}(Q)$.

(ii) $\Rightarrow$ (iii): Suppose $\ker(\Psi_P)$ is a globally well-defined rank-14 subbundle of $\mathrm{ad}(P)$. At each fibre $x \in X$, the subspace $\ker(\Psi_P|_{P_x}) \subset \mathfrak{so}(7)$ is conjugate under $\mathrm{SO}(7)$ to $\mathfrak{g}_2 = \ker \Psi$. Since the transition functions of $\mathrm{ad}(P)$ preserve this subbundle, they lie in the stabilizer of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ under the adjoint action of $\mathrm{SO}(7)$. By the identification $G_2 = \mathrm{Stab}_{\mathrm{SO}(7)}(\varphi)$ [8,9] and the fact that φ is determined (up to sign) by $\mathfrak{g}_2 = \ker \Psi$ via the isomorphism $\mathrm{Im} \Psi \cong (\mathfrak{so}(7)/\mathfrak{g}_2)^*$ [9], the stabilizer of $\mathfrak{g}_2$ as a subspace of $\mathfrak{so}(7)$ is exactly G_2 . Therefore the transition functions take values in G_2 , giving a G_2 -reduction of P and a G_2 -invariant global section φ_P of Λ_P^3 with values in the $\mathrm{SO}(7)$ -orbit of φ .

(iii) $\Rightarrow$ (i): A section φ_P with values in the $\mathrm{SO}(7)$ -orbit of φ is a reduction of the structure group of P to $\mathrm{Stab}_{\mathrm{SO}(7)}(\varphi) = G_2$ [8,9]. $\square$

Corollary 3. The G_2 -reducible locus defined by

$$\mathcal{Z} = \{[P] \in \mathcal{M}(\mathrm{SO}(7)) \mid P \text{ is } G_2\text{-reducible}\}$$

which is a closed subvariety of $\mathcal{M}(\mathrm{SO}(7))$. The extension-of-structure-group map defines a natural injective morphism

$$\iota: \mathcal{M}(G_2) \hookrightarrow \mathcal{Z},$$

and

$$\dim \mathcal{M}(G_2) = 14(g-1), \quad \dim \mathcal{M}(\mathrm{SO}(7)) = 21(g-1). \quad (14)$$

Proof. By Theorem 4, $\mathcal{Z}$ is the locus of bundles $[P] \in \mathcal{M}(\mathrm{SO}(7))$ for which the pointwise family of kernels $\ker(\Psi_P) \subset \mathrm{ad}(P)$ forms a globally well-defined rank-14 subbundle. Equivalently, by condition (iii) of Theorem 4, $\mathcal{Z}$ is the image of the natural forgetful map

$$\mathcal{M}(G_2) \xrightarrow{\iota} \mathcal{M}(\mathrm{SO}(7)), \quad [Q] \mapsto [Q \times_{G_2} \mathrm{SO}(7)],$$

which is a morphism of projective varieties [19,20]. Since $\mathcal{M}(G_2)$ is a projective variety (hence proper), its image $\mathcal{Z} = \iota(\mathcal{M}(G_2))$ is a closed subvariety of $\mathcal{M}(\mathrm{SO}(7))$. Injectivity of ι holds because the G_2 -structure can be recovered from the subbundle $\ker(\Psi_P) \subset \mathrm{ad}(P)$

together with condition (iii). The dimension formulas follow from $\dim G_2 = 14$ and $\dim \mathrm{SO}(7) = 21$ via $\dim \mathcal{M}(G) = \dim(G) \cdot (g - 1)$ [19]. $\square$

For any principal $\mathrm{SO}(8)$ -bundle P whose associated $\mathrm{SO}(7)$ -bundle is G_2 -reducible, the three-way splitting of $\mathfrak{so}(8)$ established in Proposition 4,

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7,$$

globalizes to an orthogonal decomposition of the adjoint bundle $\mathrm{ad}(P)$ into summands of ranks 14, 7, 7. This decomposition is used for controlling the tangent and normal spaces to the G_2 -reducible locus inside $\mathcal{M}(\mathrm{SO}(8))$ and, ultimately, inside $\mathcal{M}(\mathrm{Spin}(8))$.

Theorem 5. *Let P be a principal $\mathrm{SO}(8)$ -bundle over X whose associated $\mathrm{SO}(7)$ -bundle is G_2 -reducible, with G_2 -bundle Q . Then*

$$\mathrm{ad}(P) \cong \mathrm{ad}(Q) \oplus (Q \times_{G_2} \mathfrak{m}_7) \oplus (P \times_{\mathrm{SO}(8)} \mathfrak{v}_7), \quad (15)$$

as an orthogonal direct sum of vector bundles of ranks 14, 7, 7.

Proof. The fibrewise decomposition

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$$

(Proposition 4) is G_2 -equivariant (Remark 6) and orthogonal with respect to

$$\langle A, B \rangle = -\frac{1}{2} \mathrm{Tr}(AB).$$

The G_2 -bundle Q provides the transition functions needed to globalize each summand; orthogonality is preserved since the Killing form is invariant under the transition functions. $\square$

Corollary 4. *Under the hypotheses of Theorem 5, with Q a stable principal G_2 -bundle of degree zero over X with $g \geq 2$, and $P = Q \times_{G_2} \mathrm{SO}(8)$, there is a direct sum decomposition*

$$H^1(X, \mathrm{ad}(P)) \cong H^1(X, \mathrm{ad}(Q)) \oplus H^1(X, Q \times_{G_2} \mathfrak{m}_7) \oplus H^1(X, P \times_{\mathrm{SO}(8)} \mathfrak{v}_7). \quad (16)$$

The three summands have dimensions $14(g - 1)$, $7(g - 1)$, $7(g - 1)$, summing to

$$\dim T_{[P]} \mathcal{M}(\mathrm{SO}(8)) = 28(g - 1).$$

Proof. The decomposition (15) splits $\mathrm{ad}(P)$ as a direct sum of vector bundles, so (16) follows from the long exact sequence in cohomology applied to the split short exact sequence $0 \rightarrow E_1 \rightarrow \mathrm{ad}(P) \rightarrow E_2 \oplus E_3 \rightarrow 0$.

It remains to show that $H^0(X, E) = 0$ for each summand

$$E \in \{\mathrm{ad}(Q), Q \times_{G_2} \mathfrak{m}_7, P \times_{\mathrm{SO}(8)} \mathfrak{v}_7\}.$$

Each summand has degree zero since Q (and hence P) has degree zero. For a stable principal G_2 -bundle Q over a compact Riemann surface of genus $g \geq 2$, the adjoint bundle $\mathrm{ad}(Q)$ has no nonzero global sections: a global section of $\mathrm{ad}(Q)$ is a G_2 -equivariant endomorphism of the fibre $\mathfrak{g}_2$; since G_2 acts irreducibly on $\mathfrak{g}_2$ and $\mathfrak{g}_2$ is a simple Lie algebra, Schur's lemma forces any such endomorphism to be zero (as $\mathfrak{g}_2$ has trivial centre) [19]. Similarly, the summands $Q \times_{G_2} \mathfrak{m}_7$ and $P \times_{\mathrm{SO}(8)} \mathfrak{v}_7$ are associated vector bundles to irreducible

representations; since Q is stable, these associated vector bundles are stable of degree zero by [27], and a stable vector bundle of degree zero over a curve of genus $g \geq 2$ has $H^0 = 0$ [18]. Note that $P \times_{\mathrm{SO}(8)} \mathfrak{v}_7 \cong Q \times_{G_2} \mathfrak{v}_7$ since $P = Q \times_{G_2} \mathrm{SO}(8)$ and $\mathfrak{v}_7 \cong \mathbb{R}^7$ as a G_2 -module (the standard representation) [5]; the stability argument therefore applies uniformly to all three summands.

Riemann–Roch then gives

$$h^1(X, E) = \mathrm{rank}(E) \cdot (g - 1),$$

yielding the stated dimensions. $\square$

Remark 13. *The first summand*

$$H^1(X, \mathrm{ad}(Q)) \cong T_{[Q]} \mathcal{M}(G_2)$$

is tangent to the G_2 -reducible locus $\mathcal{Z}$ inside $\mathcal{M}(\mathrm{SO}(8))$. The remaining two summands $H^1(X, Q \times_{G_2} \mathfrak{m}_7)$ and $H^1(X, P \times_{\mathrm{SO}(8)} \mathfrak{v}_7)$ span the normal directions to $\mathcal{Z}$ in $\mathcal{M}(\mathrm{SO}(8))$ at $[P]$.

The triality automorphism σ of $\mathrm{Spin}(8)$ induces an automorphism σ^* of order three on the moduli space $\mathcal{M}(\mathrm{Spin}(8))$ [26]. The identification $G_2 = \mathrm{Spin}(8)^\sigma$, combined with the explicit eigenspace decomposition of $\mathfrak{so}(8) = \mathfrak{spin}(8)$ under the adjoint action of σ , allows us to describe $\mathcal{M}(G_2)$ as a connected component of the fixed-point locus $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$ and to compute the tangent and normal bundles to this component explicitly in terms of the summands $\mathfrak{g}_2, \mathfrak{m}_7, \mathfrak{v}_7$.

Lemma 1. *Under the adjoint action of σ on $\mathfrak{spin}(8) \cong \mathfrak{so}(8)$:*

- (i) *The +1-eigenspace of Ad_σ is $\mathfrak{g}_2$, the Lie algebra of the fixed-point subgroup $G_2 = \mathrm{Spin}(8)^\sigma$ [5].*
- (ii) *The orthogonal complement $\mathfrak{m}_7 \oplus \mathfrak{v}_7 = \mathfrak{g}_2^\perp \subset \mathfrak{so}(8)$ is Ad_σ -invariant and decomposes over $\mathbb{C}$ as the direct sum of the eigenspaces of eigenvalues ω and $\bar{\omega}$ (with $\omega = e^{2\pi i/3}$), each of complex dimension 7; over $\mathbb{R}$ these form the two irreducible non-trivial summands $\mathfrak{m}_7$ and $\mathfrak{v}_7$ of Proposition 4.*

Proof. Part (i) is the infinitesimal version of $G_2 = \mathrm{Spin}(8)^\sigma$ and follows from ([5], Chapter 5). For part (ii), since $(\mathrm{Ad}_\sigma)^3 = \mathrm{id}$, the complement

$$\mathfrak{m}_7 \oplus \mathfrak{v}_7$$

of $\mathfrak{g}_2$ in $\mathfrak{so}(8)$ is Ad_σ -invariant (as Ad_σ preserves the Killing form and hence the orthogonal complement of any invariant subspace). The minimal polynomial of $\mathrm{Ad}_\sigma|_{\mathfrak{m}_7 \oplus \mathfrak{v}_7}$ over $\mathbb{R}$ divides $x^3 - 1$ and has no factor $(x - 1)$ (since there is no additional +1-eigenspace beyond $\mathfrak{g}_2$), so it equals $x^2 + x + 1$. Over $\mathbb{C}$ this factors as $(x - \omega)(x - \bar{\omega})$, giving two conjugate complex eigenspaces each of dimension 7; over $\mathbb{R}$ their sum is the irreducible 14-dimensional space $\mathfrak{m}_7 \oplus \mathfrak{v}_7$, and the two summands $\mathfrak{m}_7, \mathfrak{v}_7$ of Proposition 4 are their real forms [5,16]. $\square$

Theorem 6. *Let*

$$\sigma^*: \mathcal{M}(\mathrm{Spin}(8)) \rightarrow \mathcal{M}(\mathrm{Spin}(8))$$

be the automorphism of order 3 induced by the triality automorphism σ of $\mathrm{Spin}(8)$, defined by

$$\sigma^*([P]) = [P \times_{\mathrm{Spin}(8), \sigma} \mathrm{Spin}(8)].$$

Then:

- (i) σ^* is well-defined and has order 3 on $\mathcal{M}(\mathrm{Spin}(8))$.
- (ii) The moduli space $\mathcal{M}(G_2)$ is a connected component of the fixed-point locus $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$, via the natural map

$$\iota: [Q] \mapsto [Q \times_{G_2} \mathrm{Spin}(8)].$$

- (iii) At any smooth point $[P] = \iota([Q])$, the tangent space decomposes as

$$T_{[P]} \mathcal{M}(\mathrm{Spin}(8)) \cong T_{[P]} \mathcal{M}(\mathrm{Spin}(8))^{\sigma^*} \oplus N_{[P]}, \quad (17)$$

where

$$T_{[P]} \mathcal{M}(\mathrm{Spin}(8))^{\sigma^*} \cong H^1(X, \mathrm{ad}(Q))$$

has dimension $14(g-1)$, and the normal space

$$N_{[P]} \cong H^1(X, Q \times_{G_2} \mathfrak{m}_7) \oplus H^1(X, P \times_{\mathrm{Spin}(8)} \mathfrak{v}_7)$$

has dimension $14(g-1)$.

Proof. Part (i): The automorphism σ of $\mathrm{Spin}(8)$ preserves the notions of stability and semistability (since σ permutes the root system of $\mathrm{Spin}(8)$, it subsequently permutes maximal parabolics and preserves degrees of associated line bundles). The map σ^* therefore descends to $\mathcal{M}(\mathrm{Spin}(8))$, and $(\sigma^*)^3 = \mathrm{id}$ since $\sigma^3 = \mathrm{id}$.

Part (iii): We establish the tangent-space decomposition first, as it is used in the proof of part (ii). The differential $d\sigma^*$ acts on

$$T_{[P]} \mathcal{M}(\mathrm{Spin}(8)) \cong H^1(X, \mathrm{ad}(P))$$

via the adjoint action of σ on $\mathfrak{spin}(8) \cong \mathfrak{so}(8)$. By Lemma 1(i), the $+1$ -eigenspace of Ad_σ on $\mathfrak{spin}(8)$ is $\mathfrak{g}_2$. The corresponding $+1$ -eigenbundle of $\mathrm{ad}(P)$ under the fibrewise action of Ad_σ is therefore

$$\mathrm{ad}(P)^{\sigma^*} = P \times_{\mathrm{Spin}(8)} \mathfrak{g}_2.$$

Since $P = Q \times_{G_2} \mathrm{Spin}(8)$, the transition functions of P take values in G_2 , so the adjoint action of $\mathrm{Spin}(8)$ on $\mathfrak{g}_2$ restricts to the adjoint action of G_2 ; hence this bundle is isomorphic to

$$\mathrm{ad}(Q) = Q \times_{G_2} \mathfrak{g}_2.$$

Therefore the $+1$ -eigenspace of $d\sigma^*$ on $H^1(X, \mathrm{ad}(P))$ is

$$H^1(X, \mathrm{ad}(P)^{\sigma^*}) \cong H^1(X, \mathrm{ad}(Q)),$$

which identifies $T_{[P]} \mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$. Since $[P]$ is a smooth point of $\mathcal{M}(\mathrm{Spin}(8))$, the bundle P is stable as a $\mathrm{Spin}(8)$ -principal bundle [19]. The stability of Q as a G_2 -principal bundle then follows: if Q was not stable, a Ramanan-unstable reduction of Q would induce an unstable reduction of $P = Q \times_{G_2} \mathrm{Spin}(8)$ [19], contradicting the stability of P . Hence Q is stable; since $\mathfrak{spin}(8) \cong \mathfrak{so}(8)$ as Lie algebras, Corollary 4 applies and gives

$$\dim H^1(X, \mathrm{ad}(Q)) = 14(g-1).$$

The complementary eigenspaces of $d\sigma^*$, corresponding to the non-trivial characters of $\mathbb{Z}/3\mathbb{Z}$, form the normal space $N_{[P]}$. By Lemma 1(ii), $\mathfrak{m}_7$ and $\mathfrak{v}_7$ are the non-trivial Ad_σ -

eigenspaces of $\mathfrak{spin}(8)$; together with the bundle decomposition (15) and the identification $\mathfrak{spin}(8) \cong \mathfrak{so}(8)$, this gives

$$N_{[P]} \cong H^1(X, Q \times_{G_2} \mathfrak{m}_7) \oplus H^1(X, P \times_{\mathrm{Spin}(8)} \mathfrak{v}_7),$$

of dimension

$$7(g-1) + 7(g-1) = 14(g-1)$$

by Corollary 4.

Part (ii): The result $G_2 = \mathrm{Spin}(8)^\sigma$ (see [26]) means that for any G_2 -bundle Q , the induced $\mathrm{Spin}(8)$ -bundle $P = Q \times_{G_2} \mathrm{Spin}(8)$ satisfies $\sigma^*P \cong P$: the transition functions of P take values in $G_2 \subset \mathrm{Spin}(8)$, so conjugation by σ fixes them. Hence

$$\iota(\mathcal{M}(G_2)) \subset \mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}.$$

We show that $\iota(\mathcal{M}(G_2))$ is a connected component of $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$.

- Closed: The map

$$\iota: \mathcal{M}(G_2) \rightarrow \mathcal{M}(\mathrm{Spin}(8))$$

is a morphism of projective varieties [19,20], and $\mathcal{M}(G_2)$ is projective, hence proper. The image of a proper morphism is closed [28], so $\iota(\mathcal{M}(G_2))$ is a closed subvariety of $\mathcal{M}(\mathrm{Spin}(8))$, hence also of $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$.

- Connected component: Since $\mathcal{M}(G_2)$ is irreducible [19], its image $\iota(\mathcal{M}(G_2))$ is an irreducible closed subvariety of $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$. By part (iii), already established above, the tangent space to $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$ at any smooth point $[P] = \iota([Q])$ has dimension $14(g-1) = \dim \mathcal{M}(G_2)$. Therefore $\iota(\mathcal{M}(G_2))$ is an irreducible closed subvariety of $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$ of the same dimension as the ambient space at each of its smooth points; it is therefore an irreducible component of $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$, and in particular a connected component.

This concludes the result. $\square$

Corollary 5. *The normal bundle $\mathcal{N}$ of $\mathcal{M}(G_2)$ in $\mathcal{M}(\mathrm{Spin}(8))$ (at smooth points) has fibre*

$$\mathcal{N}_{[Q]} \cong H^1(X, Q \times_{G_2} \mathfrak{m}_7) \oplus H^1(X, (Q \times_{G_2} \mathrm{Spin}(8)) \times_{\mathrm{Spin}(8)} \mathfrak{v}_7), \quad (18)$$

of total dimension $14(g-1)$, so that

$$\dim \mathcal{M}(G_2) + \mathrm{rank}(\mathcal{N}) = 14(g-1) + 14(g-1) = 28(g-1) = \dim \mathcal{M}(\mathrm{Spin}(8)).$$

Proof. This is an immediate consequence of Theorem 6(iii) and the dimension formulas of Corollary 4. $\square$

Remark 14. *The explicit form of the normal bundle (18) is a direct consequence of the orthogonal decomposition*

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$$

established in Proposition 4, together with the identification of the σ -eigenspaces. The contribution of this paper is to make this decomposition explicit, thereby giving a constructive description of the tangent and normal spaces to the fixed-point locus.

6. Conclusions

In the context of the study of the octonionic algebra $\mathbb{O}$ and its automorphism group G_2 , the explicit matrix realization of $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(8)$ had not been analysed in constructive detail until now. This paper has addressed that gap, providing a linear-algebraic framework that makes the G_2 -derivation algebra computable, and deriving from it geometric consequences for moduli spaces of principal bundles.

The derivation condition for a skew-symmetric matrix $\hat{D} \in \mathfrak{so}(7)$ was reformulated as a sparse homogeneous linear system of rank 7 in the 21 entries of $\hat{D}$. The kernel of the associated Lie derivative map

$$\Psi: \mathfrak{so}(7) \rightarrow \Lambda^3(\mathbb{R}^7)^*$$

was identified with $\mathfrak{g}_2$, yielding $\dim \mathfrak{g}_2 = 14$ from the rank of Ψ alone. The orthogonal complement $\mathfrak{m}_7 = \mathfrak{g}_2^\perp \cap \mathfrak{so}(7)$ is a 7-dimensional irreducible G_2 -module, completing the splitting $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}_7$. The further inclusion in $\mathfrak{so}(8)$ gives the three-way decomposition $\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{m}_7 \oplus \mathfrak{v}_7$ with dimensions $14 + 7 + 7 = 28$. An algorithm for constructing an orthonormal basis of $\mathfrak{g}_2$ by Gaussian elimination of the 28×21 sparse integer system was provided.

The main algebraic result of the paper is the explicit determination of the image

$$\Psi(\hat{A}_\sigma) \in \Lambda^3(\mathbb{R}^7)^*$$

of the triality generator: among all 35 triples (a, b, c) , exactly six produce nonzero values, with squared norm equal to 12. This gives a direct coordinate-level proof that the triality generator does not lie in $\mathfrak{g}_2$, complementing and quantifying the argument based on outer automorphisms. The component $\hat{A}_\sigma^{(\mathfrak{m}_7)} \in \mathfrak{m}_7$ is the obstruction to A_σ being a derivation, and $\hat{A}_\sigma^{(\mathfrak{g}_2)} \in \mathfrak{g}_2$ is shown to be nonzero and identified as the infinitesimal generator of the unique smooth family $\tau_t \in G_2$, satisfying

$$\varphi_t(xy) = \tau_t(\varphi_t(x) \cdot \varphi_t(y)),$$

making constructive an existence result from the preceding literature.

These algebraic results were then applied to the geometry of moduli spaces of principal bundles over a compact Riemann surface of genus $g \geq 2$. The map Ψ globalizes to a morphism of adjoint bundles, providing an intrinsic characterization of the G_2 -reducible locus in $\mathcal{M}(\mathrm{SO}(7))$ as the closed sublocus where the pointwise kernel of this morphism forms a globally well-defined rank-14 subbundle of the adjoint bundle, with $\mathcal{M}(G_2)$ of dimension $14(g-1)$ mapping naturally into it. The three-way orthogonal decomposition of $\mathfrak{so}(8)$ globalizes to an explicit splitting of the adjoint bundle of any $\mathrm{SO}(8)$ -bundle admitting a G_2 -reduction, with summands of ranks 14, 7, 7. Finally, the identification $G_2 = \mathrm{Spin}(8)^\sigma$ and the explicit eigenspace decomposition of $\mathfrak{so}(8)$ under the adjoint action of σ were used to show that $\mathcal{M}(G_2)$ is a connected component of the triality fixed-point locus $\mathcal{M}(\mathrm{Spin}(8))^{\sigma^*}$, with an explicit description of the tangent and normal bundles. The normal bundle of $\mathcal{M}(G_2)$ in $\mathcal{M}(\mathrm{Spin}(8))$ has rank $14(g-1)$, so that $\mathcal{M}(G_2)$ is a middle-dimensional submanifold of $\mathcal{M}(\mathrm{Spin}(8))$.

Several directions for further work present themselves naturally. The numerical computation of the 14 generators $\hat{G}_k$ via the algorithm developed here, using a computer algebra system, would yield the explicit matrices and allow the direct computation of $\langle \hat{A}_\sigma, \hat{G}_k \rangle$, fully specifying τ_t . A systematic spectral analysis of these generators would illuminate the fine structure of the G_2 -action on $\mathrm{Im}(\mathbb{O})$ and its relationship to the combinatorics of the Fano plane. The reductive decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}_7$, in which $\mathfrak{m}_7$ is an

irreducible G_2 -module, provides the natural setting for a study of the G_2 -orbit structure in the homogeneous space $SO(7)/G_2$ and its role in calibrated geometry. An extension to the algebra $\mathfrak{spin}(7)$, which contains $\mathfrak{g}_2$ as a subalgebra, would address the interplay between G_2 -structures and $\text{Spin}(7)$ -structures in exceptional holonomy geometry. The study of Higgs bundle analogues—replacing principal bundles by G -Higgs bundles and the moduli space by the Hitchin moduli space—would allow one to investigate whether the triality symmetry and the G_2 -reducible locus have natural counterparts in the non-abelian Hodge correspondence. Finally, the embedding $\mathfrak{g}_2 \subset \mathfrak{so}(7) \subset \mathfrak{so}(8)$ is the first step in the Freudenthal–Tits construction of the exceptional Lie algebras $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ via octonions, and the explicit matrix methods developed here may facilitate analogous constructive treatments of those larger algebras.

The matrix-theoretic and algorithmic approach developed in this paper raises a natural question about its possible extension to other algebraic structures. The explicit realization of $\mathfrak{g}_2 = \text{Der}(\mathbb{O})$ via the sparse linear system associated with the Leibniz rule relies on two features of $\mathbb{O}$: the finiteness and explicit combinatorial description of the structure constants (via the Fano plane), and the fact that $\mathbb{O}$ is an alternative algebra, so that the derivation identity $D(xy) = D(x)y + xD(y)$ interacts well with the associator. Both features are present, to varying degrees, in other alternative algebras and rings. This suggests a natural direction for future research. Letting $\mathcal{A}$ be a finite-dimensional alternative algebra with explicitly known structure constants, it would be of interest to determine whether the derivation condition

$$D(e_i \cdot e_j) = D(e_i) \cdot e_j + e_i \cdot D(e_j)$$

can be reformulated as a sparse homogeneous linear system, and whether an analogue of the rank–nullity approach based on a distinguished invariant form can be developed to obtain a constructive characterization of $\text{Der}(\mathcal{A})$. More generally, one may investigate the conditions under which an additive map $D: \mathcal{A} \rightarrow \mathcal{A}$ satisfying a derivation-type identity on a distinguished generating subset or subalgebra of $\mathcal{A}$ necessarily extends to a genuine derivation of $\mathcal{A}$. These questions connect the octonionic results of this paper to the broader programme of understanding derivations in non-associative and alternative algebraic structures [11–13].

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