

Article

Proper Partitions, Graphical Stirling Numbers, and Bell Numbers for Multipartite and Mycielskian Graphs

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Abstract

Explicit formulas for graphical Stirling and Bell numbers are known for relatively few graph families. We derive exact expressions for three classes whose independence structure admits a complete combinatorial description: complete multipartite graphs, the graph obtained from a balanced complete bipartite graph by deleting a perfect matching, and the Mycielskian of a star. For complete multipartite graphs we express the graphical Stirling number as a convolution of classical Stirling numbers across the partite classes, and we recover the known factorization of the graphical Bell number as a product of classical Bell numbers. For the matching-deleted graph we show that its graphical Bell number is a binomial convolution of squared Bell numbers, which we identify as a moment of a product of two independent Poisson random variables with unit mean. This representation yields log-convexity of the sequence, a sharp exponential lower bound, a two-sided estimate, and a Laplace-transform identity. For the Mycielskian of a star, a decomposition according to the block containing the original center vertex, together with Vandermonde's convolution and a Stirling recurrence, gives a single-sum closed form for the graphical Stirling numbers, from which two explicit evaluations follow. Several resulting integer sequences appear in the OEIS, and one Bell-number sequence appears not to be currently recorded there.

Keywords: graphical Bell number; graphical Stirling number; proper partition; partition polynomial; complete multipartite graph; Mycielskian graph; Poisson moments

MSC: 05A18; 05C15; 05C30; 05C69; 60E05

1. Introduction

Let $G = (V, E)$ be a finite graph. A *proper partition* of G is a partition of V into nonempty independent sets. The *graphical Stirling number* $S(G; k)$, introduced by Duncan and Peele [1], counts proper partitions of G into exactly k blocks. The associated *graphical Bell number* is $B(G) = \sum_{k=1}^{|V|} S(G; k)$.

For the edgeless graph E_n , every set partition is proper; hence, $S(E_n; k) = \{n\}_k$ and $B(E_n) = \mathcal{B}_n$. Throughout, $\{n\}_k$ denotes the *Stirling number of the second kind*, that is, the number of partitions of an n -element set into k nonempty blocks. We use the standard boundary conventions $\{0\}_0 = 1$, $\{n\}_0 = 0$ for $n \geq 1$, and $\{n\}_k = 0$ whenever $k < 0$ or $k > n$. In particular, $\{n\}_1 = 1$ for $n \geq 1$. The brace symbol is standard for this quantity and is *not* a binomial coefficient, for which we reserve $\binom{n}{k}$.

The partition polynomial is defined by $F(G; x) = \sum_k S(G; k) x^k$. Its coefficients are related to the chromatic polynomial through the classical falling-factorial expansion



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$\chi(G; x) = \sum_k S(G; k) x^k$, where $x^k = x(x-1) \cdots (x-k+1)$. Consequently, graphical Stirling numbers interpolate between partition enumeration and graph coloring. Applications of graph coloring include register allocation [2], scheduling, and frequency assignment.

Explicit formulas for graphical Stirling and Bell numbers are known for relatively few graph families. Duncan and Peele [1] proved that if T is a tree on n vertices, then $B(T) = \mathcal{B}_{n-1}$. Galvin and Thanh [3] extended this observation to forests by showing that $S(F; k)$ depends only on the order of F and its number of connected components. Kereskenyi-Balogh and Nyul [4] computed graphical Bell numbers for several classical graph families. For complete multipartite graphs, Allagan and Serkan [5] established $B(K(n_1, \dots, n_\ell)) = \prod_{i=1}^\ell \mathcal{B}_{n_i}$. Beyond these families, explicit closed forms remain limited.

The purpose of this paper is to derive exact formulas for graphical Stirling and Bell numbers in three graph families whose independence structure can be described completely.

These three families were selected because their independence structure can be written in closed form, which is what makes exact enumeration possible: for most graphs, no such description is available, and even $S(G; k)$ for $K_{n,n}$ has no elementary product form. Beyond enumeration, the graphical Stirling numbers determine the partition polynomial $F(G; x)$ and, through the falling-factorial expansion above, the chromatic polynomial $\chi(G; x)$; the families studied here therefore supply exactly solvable test cases for both partition counting and coloring, with the latter relevant to applications such as register allocation [2] and frequency assignment.

We distinguish three levels of contribution. The product factorization of $B(K(n_1, \dots, n_\ell))$ is due to Allagan and Serkan [5]; the Stirling-number convolution (1) below is an immediate refinement that follows directly from the partite-class structure of independent sets, and is included for completeness. The formulas for H_n and $M(St_n)$ require additional combinatorial input: the identification of the admissible mixed blocks for H_n , and the center-block decomposition for $M(St_n)$. The probabilistic representation of $B(H_n)$ and its analytic consequences are, to our knowledge, new.

For the complete multipartite graph $K(n_1, \dots, n_\ell)$, we prove

$$S(K(n_1, \dots, n_\ell); k) = \sum_{j_1 + \dots + j_\ell = k} \prod_{i=1}^\ell \left\{ \begin{matrix} n_i \\ j_i \end{matrix} \right\}, \quad (1)$$

which refines the Bell-number identity of Allagan and Serkan [5].

For the graph $H_n = K_{n,n} - M$, obtained by deleting a perfect matching from the complete bipartite graph, we prove

$$B(H_n) = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k^2 = \mathbb{E}[(1 + XY)^n], \quad (2)$$

where X and Y are independent Poisson(1) random variables. This representation yields several consequences, including log-convexity of the sequence $\{B(H_n)\}_{n \geq 0}$ and the lower bound $B(H_n) \geq 2^n$.

For the Mycielskian star graph $M(St_n)$, we obtain

$$S(M(St_n); q) = 2 \sum_{a=0}^{n-1} \binom{n-1}{a} \left\{ \begin{matrix} 2n-2-a \\ q-2 \end{matrix} \right\} + \left\{ \begin{matrix} 2n-1 \\ q-2 \end{matrix} \right\}. \quad (3)$$

Two direct evaluations are $S(M(St_n); 3) = 2^n + 1$ and $S(M(St_n); 2n) = 2n^2 - 3n + 3$.

Several sequences arising from these formulas occur in the OEIS. For $n \geq 2$, the sequence $S(M(St_n); 3) = 2^n + 1$ is A000051 <https://oeis.org/A000051>, accessed on 20 June 2026 [6]. The values $S(M(St_n); 2n)$ coincide with A096376 <https://oeis.org/A096376>,

accessed on 20 June 2026 [7] after the index shift $n \mapsto n - 1$, and therefore also with A116735 <https://oeis.org/A116735>, accessed on 20 June 2026 [8] through the relation recorded there. We do not know a direct combinatorial explanation for this coincidence. The Bell-number sequence $B(M(St_n)) = 11, 106, 1695, 39,325, \dots$ is, to the best of our knowledge, not currently listed in the OEIS; as the database is updated continually, this status may change.

The paper is organized as follows. Section 2 contains the main structural formulas and their proofs. Computational verification and OEIS correspondence tables are collected in Appendix A.

2. Main Results

We begin with complete multipartite graphs, whose independence structure admits an exact decomposition across the partite classes.

2.1. Complete Multipartite Graphs

The structural point is simple: in a complete multipartite graph, no independent set can meet two partite classes, so a proper partition is just an independent choice of an ordinary set partition within each class. The next theorem records the resulting convolution.

Theorem 1. *Let $G = K(n_1, \dots, n_\ell)$ be the complete ℓ -partite graph with partite classes $V_1, \dots, V_\ell$, where $|V_i| = n_i \geq 1$. Then, for every $k \geq 1$,*

$$S(G; k) = \sum_{\substack{j_1 + \dots + j_\ell = k \\ j_i \geq 1}} \prod_{i=1}^{\ell} \left\{ \begin{matrix} n_i \\ j_i \end{matrix} \right\}, \quad (4)$$

and consequently

$$B(G) = \prod_{i=1}^{\ell} \mathcal{B}_{n_i}. \quad (5)$$

Proof. In a complete multipartite graph, every edge joins vertices belonging to distinct partite classes. Hence, an independent set is contained entirely within a single class V_i . It follows that every block of a proper partition of G is contained in exactly one partite class.

Fix a proper partition of G into k blocks and let j_i denote the number of blocks contained in V_i . As $V_i \neq \emptyset$, each j_i satisfies $j_i \geq 1$, and $j_1 + \dots + j_\ell = k$. Conversely, given positive integers $j_1, \dots, j_\ell$ with sum k , a proper partition of G is obtained by independently partitioning each V_i into j_i nonempty blocks. The number of such partitions is $\left\{ \begin{matrix} n_i \\ j_i \end{matrix} \right\}$ for V_i , and the choices are independent across partite classes. Therefore, the number of proper k -partitions is exactly the quantity in (4).

Summing (4) over all k and using the bijection between proper partitions of G and independent choices of set partitions within each partite class gives

$$B(G) = \prod_{i=1}^{\ell} \left(\sum_{j=1}^{n_i} \left\{ \begin{matrix} n_i \\ j \end{matrix} \right\} \right) = \prod_{i=1}^{\ell} \mathcal{B}_{n_i},$$

which proves (5). $\square$

The Bell-number identity (5) was established in [5]. Formula (4) provides the corresponding graphical Stirling numbers and will be used repeatedly in the sequel.

Example 1. For $K_{2,2}$, Theorem 1 gives $S(K_{2,2}; k) = \sum_{\substack{j_1+j_2=k \\ j_1, j_2 \geq 1}} \left\{ \begin{smallmatrix} 2 \\ j_1 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} 2 \\ j_2 \end{smallmatrix} \right\}$. Thus, $S(K_{2,2}; 2) = \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\} = 1$, $S(K_{2,2}; 3) = 2 \left\{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right\} = 2$, and $S(K_{2,2}; 4) = \left\{ \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right\} = 1$. Hence, $B(K_{2,2}) = 1 + 2 + 1 = 4 = \mathcal{B}_2^2$, in agreement with Theorem 1.

We record three special cases that will be referenced later.

Corollary 1. For $n \geq 1$,

$$S(K_{n,n}; 4) = \sum_{j=1}^3 \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ 4-j \end{smallmatrix} \right\} = 2 - 2^{n+1} + 3^{n-1} + 4^{n-1}.$$

Proof. Applying Theorem 1 with $\ell = 2$ and $k = 4$ yields

$$S(K_{n,n}; 4) = \sum_{j=1}^3 \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ 4-j \end{smallmatrix} \right\}.$$

Substituting $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = 1$, $\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\} = 2^{n-1} - 1$, $\left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\} = \frac{1}{6}(3^n - 3 \cdot 2^n + 3)$, and $\left\{ \begin{smallmatrix} n \\ 4 \end{smallmatrix} \right\} = \frac{1}{24}(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4)$ and simplifying gives the stated expression. Direct verification at $n = 1$ gives $S(K_{1,1}; 4) = 0$, and the closed form also evaluates to 0. $\square$

Corollary 2. For $n \geq 1$,

$$S(K_{n,n}; 5) = \sum_{j=1}^4 \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ 5-j \end{smallmatrix} \right\} = 6^{n-1} - \frac{5}{3} 2^{2n-2} - 2 \cdot 3^{n-1} + 2^{n+1} - \frac{4}{3}.$$

Proof. Applying Theorem 1 with $\ell = 2$ and $k = 5$ yields the convolution formula. Substituting the classical expressions for $\left\{ \begin{smallmatrix} n \\ r \end{smallmatrix} \right\}$ with $1 \leq r \leq 5$, including $\left\{ \begin{smallmatrix} n \\ 5 \end{smallmatrix} \right\} = \frac{1}{120}(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5)$, and simplifying proves the identity. For $n = 1$, both sides equal 0. $\square$

Corollary 3. For $n \geq 1$,

$$S(K_{n,n,n}; 5) = 3 \left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\}^2 + 3 \left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\} = \frac{1}{4}(18 - 18 \cdot 2^n + 2 \cdot 3^n + 3 \cdot 4^n).$$

Proof. Applying Theorem 1 with $\ell = 3$ and $k = 5$ gives

$$S(K_{n,n,n}; 5) = \sum_{\substack{j_1+j_2+j_3=5 \\ j_i \geq 1}} \left\{ \begin{smallmatrix} n \\ j_1 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ j_2 \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} n \\ j_3 \end{smallmatrix} \right\}.$$

The only compositions of 5 into three positive parts are $(1, 1, 3)$ and $(1, 2, 2)$, each occurring in three permutations. Using $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = 1$ yields

$$S(K_{n,n,n}; 5) = 3 \left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\} + 3 \left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\}^2.$$

Substituting the classical formulas for $\left\{ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\}$ and simplifying gives the closed form. For $n = 1$, both sides equal 0. $\square$

2.2. Complete Bipartite Graphs with a Removed Perfect Matching

We next consider the graph obtained from $K_{n,n}$ by deleting a fixed perfect matching. The deleted matching is the only source of possible mixed independent blocks, and this yields an exact convolution formula. Specifically, the only edges removed are $u_i v_i$, so the

sole way an independent block can span both sides is to consist of a single matched pair $\{u_i, v_i\}$; everything else is partitioned within one side. Counting how many matched pairs are used as blocks produces the binomial convolution below.

Theorem 2. Let $H_n = K_{n,n} - M$, where $M = \{u_i v_i : 1 \leq i \leq n\}$ is a perfect matching between the bipartition classes $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. For every $k \geq 1$,

$$S(H_n; k) = \sum_{s=0}^n \binom{n}{s} \sum_{j=0}^{k-s} \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\} \left\{ \begin{matrix} n-s \\ k-s-j \end{matrix} \right\}, \quad (6)$$

where $\left\{ \begin{matrix} m \\ r \end{matrix} \right\} = 0$ if $r < 0$ or $r > m$. Consequently,

$$B(H_n) = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k^2. \quad (7)$$

Proof. In H_n , the edge $u_i v_j$ is present exactly when $i \neq j$. Hence, a proper block meeting both U and V can contain only vertices of the form u_i and v_i for a common index i . Such a mixed block cannot contain any further vertex, as $u_j v_i$ and $u_i v_j$ are edges for $j \neq i$. Thus, the mixed blocks in any proper partition are precisely some collection of isolated pairs $\{u_i, v_i\}$.

Choose s of these matched pairs. This gives $\binom{n}{s}$ choices. After the chosen pairs are fixed as blocks, the remaining vertices in U and V cannot form additional mixed blocks, as s was chosen to be the total number of mixed blocks. The remaining $n-s$ vertices of U may therefore be partitioned into j nonempty blocks, and the remaining $n-s$ vertices of V into $k-s-j$ nonempty blocks. This gives $\left\{ \begin{matrix} n-s \\ j \end{matrix} \right\} \left\{ \begin{matrix} n-s \\ k-s-j \end{matrix} \right\}$ choices. Summing over j and s proves (6).

Summing (6) over k gives $B(H_n) = \sum_{s=0}^n \binom{n}{s} \sum_{r \geq 0} \sum_{j=0}^r \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\} \left\{ \begin{matrix} n-s \\ r-j \end{matrix} \right\}$, where $r = k-s$. The inner double sum is the Cauchy product $(\sum_{j \geq 0} \left\{ \begin{matrix} n-s \\ j \end{matrix} \right\})^2 = \mathcal{B}_{n-s}^2$. Hence, $B(H_n) = \sum_{s=0}^n \binom{n}{s} \mathcal{B}_{n-s}^2$, and the change of index $k = n-s$ gives (7). $\square$

The Bell-number convolution also has a compact probabilistic form. For the statements involving $n = 0$, we regard H_0 as the empty graph and use the standard convention $B(H_0) = 1$.

Theorem 3 (Poisson moment representation). Let X and Y be independent Poisson(1) random variables. For every $n \geq 0$,

$$B(H_n) = \mathbb{E}[(1 + XY)^n]. \quad (8)$$

Proof. By the binomial theorem and independence, $\mathbb{E}[(1 + XY)^n] = \sum_{k=0}^n \binom{n}{k} \mathbb{E}[X^k] \mathbb{E}[Y^k]$. Dobinski's identity gives $\mathbb{E}[X^k] = \mathcal{B}_k$ for $X \sim \text{Poisson}(1)$; see, for example, [9]. Thus, $\mathbb{E}[(1 + XY)^n] = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k^2$, which is $B(H_n)$ by (7). $\square$

Remark 1. Representation (8) is not simply a restatement of the convolution (7). By turning a combinatorial sum into a moment, it makes standard moment inequalities available: Jensen and Cauchy–Schwarz applied to $Z = 1 + XY$ yield, with no further computation, the sharp bound $B(H_n) \geq 2^n$ and the log-convexity of $\{B(H_n)\}$ (Corollaries 4 and 5), neither of which is apparent from (7). Whether other families admit analogous moment representations is open; the construction here uses the bipartite product structure of H_n , which is what makes the two sides contribute independent Poisson factors. Graphs built from two interacting independent families of vertices are the natural place to look for similar identities.

Corollary 4 (Log-convexity). For all $n \geq 1$, $B(H_{n-1})B(H_{n+1}) \geq B(H_n)^2$.

Proof. Set $Z = 1 + XY$. As $Z \geq 1$, Cauchy–Schwarz gives $\mathbb{E}[Z^{n-1}]\mathbb{E}[Z^{n+1}] \geq \mathbb{E}[Z^n]^2$. The claim now follows from Theorem 3. $\square$

Corollary 5 (Sharp lower bound). *For all $n \geq 0$, $B(H_n) \geq 2^n$, with equality if and only if $n \leq 1$.*

Proof. Let $Z = 1 + XY$. Then $\mathbb{E}[Z] = 1 + \mathbb{E}[X]\mathbb{E}[Y] = 2$. For $n \geq 1$, Jensen’s inequality gives $B(H_n) = \mathbb{E}[Z^n] \geq \mathbb{E}[Z]^n = 2^n$. The case $n = 0$ is immediate. Equality holds for $n = 0$ and $n = 1$. For $n \geq 2$, the function $t \mapsto t^n$ is strictly convex on $[0, \infty)$, while Z is not almost surely constant because $\mathbb{P}(Z = 1) > 0$ and $\mathbb{P}(Z = 2) > 0$. Hence, the inequality is strict for $n \geq 2$. $\square$

Corollary 6 (Two-sided bound). *For all $n \geq 0$,*

$$2^n \leq B(H_n) \leq \mathcal{B}_{n+1}^2.$$

Proof. The lower bound is Corollary 5. For the upper bound, as $X, Y \geq 0$ we have $1 + XY \leq (1 + X)(1 + Y)$; hence, $(1 + XY)^n \leq (1 + X)^n(1 + Y)^n$. Taking expectations and using independence,

$$B(H_n) = \mathbb{E}[(1 + XY)^n] \leq \mathbb{E}[(1 + X)^n] \mathbb{E}[(1 + Y)^n] = \mathbb{E}[(1 + X)^n]^2.$$

By the binomial theorem, Dobinski’s identity $\mathbb{E}[X^k] = \mathcal{B}_k$, and the Bell recurrence $\sum_{k=0}^n \binom{n}{k} \mathcal{B}_k = \mathcal{B}_{n+1}$, we obtain $\mathbb{E}[(1 + X)^n] = \mathcal{B}_{n+1}$, which gives the stated upper bound. $\square$

Theorem 4 (Moment–Laplace identity). *Let X and Y be independent Poisson(1) random variables, and define $F(t) = \mathbb{E}[e^{t(1+XY)}]$ for $t \leq 0$. Then, for every real $t \leq 0$,*

$$F(t) = e^{t-1} \sum_{k=0}^{\infty} \frac{e^{ekt}-1}{k!}, \quad (9)$$

and the series converges absolutely. Moreover, F is C^∞ on $(-\infty, 0)$, all left derivatives at 0 exist, and

$$F_-^{(n)}(0) = B(H_n) \quad (n \geq 0). \quad (10)$$

For every $t > 0$, the series in (9) diverges.

Proof. Fix $t \leq 0$. As $X, Y \geq 0$ almost surely, $e^{tXY} \leq 1$, so $F(t) = e^t \mathbb{E}[e^{tXY}]$ is finite. Conditioning on Y and using the moment generating function $\mathbb{E}[e^{sX}] = e^{e^s-1}$ for a Poisson(1) random variable (see, e.g., [10]) gives $\mathbb{E}[e^{tXY}] = \mathbb{E}_Y[e^{te^Y}-1] = e^{-1} \sum_{k=0}^{\infty} \frac{e^{ekt}-1}{k!}$, which proves (9). As $t \leq 0$ implies $e^{kt} - 1 \leq 0$, the summand is bounded by $1/k!$, so the convergence is absolute.

For $t < 0$, differentiation under the expectation is justified by dominated convergence on compact subintervals of $(-\infty, 0)$; for each fixed n , the derivative is $F^{(n)}(t) = \mathbb{E}[(1 + XY)^n e^{t(1+XY)}]$. Taking $t \uparrow 0$ and using monotone convergence gives $F_-^{(n)}(0) = \mathbb{E}[(1 + XY)^n]$. Theorem 3 then gives (10).

If $t > 0$, the ratio of successive terms in the series in (9) is $e^{ekt(e^t-1)}/(k+1)$, which tends to infinity as $k \rightarrow \infty$. Thus, the series diverges. $\square$

Remark 2 (Laplace transform and moment recovery). *Theorem 4 provides a Laplace-transform representation of the Bell numbers $B(H_n)$. For $t \leq 0$, $\mathbb{E}[e^{t(1+XY)}] = e^{t-1} \sum_{k=0}^{\infty} \frac{e^{ekt}-1}{k!}$. Moreover,*

the left derivatives at the origin recover the moments: $\left. \frac{d^n}{dt^n} \right|_{t=0^-} \mathbb{E}[e^{t(1+XY)}] = B(H_n)$ for every $n \geq 0$. Thus, $B(H_n)$ is the moment sequence of the random variable $1 + XY$. Furthermore, Theorem 4 shows that the series representation above converges for all $t \leq 0$ and diverges for every $t > 0$.

Example 2. Consider $H_3 = K_{3,3} - M$ with vertex sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ and matching $M = \{a_i b_i : 1 \leq i \leq 3\}$. The graph is depicted in Figure 1. Applying (6) with $n = 3$ and $k = 3$ gives $S(H_3; 3) = 10$. The ten valid proper 3-partitions are the following:

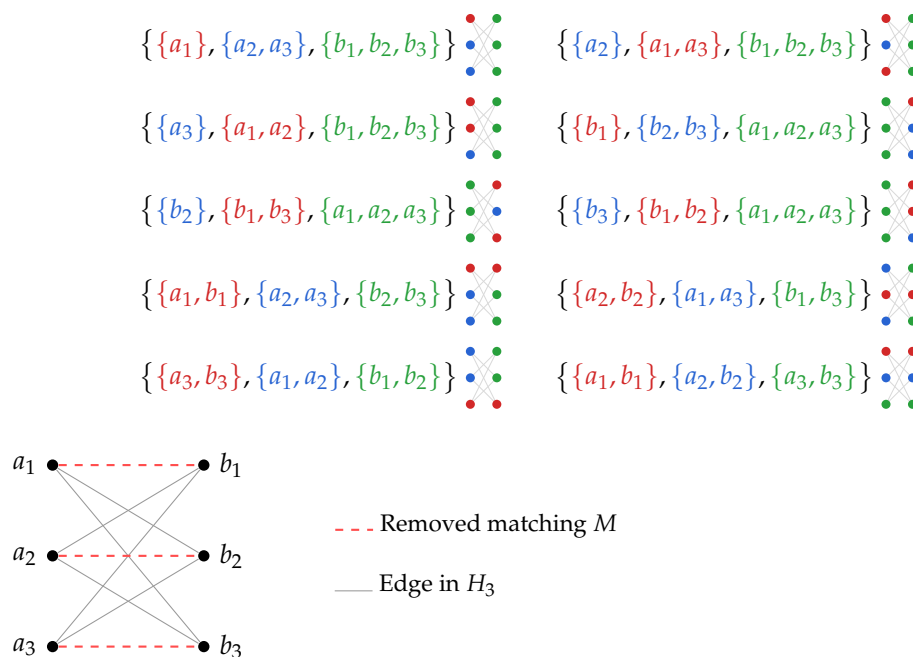


Figure 1. The graph $H_3 = K_{3,3} - M$. Dashed lines are the removed perfect matching; solid lines form H_3 .

2.3. Mycielskian Star Graphs

Let St_n be the star on $n \geq 2$ vertices, with center c and leaves $v_1, \dots, v_m$, where $m = n - 1$. Its Mycielskian $M(St_n)$ (in the sense of Mycielski [11]) has vertex set $\{c\} \cup L \cup \{c'\} \cup L' \cup \{u\}$, where $L = \{v_1, \dots, v_m\}$ and $L' = \{v'_1, \dots, v'_m\}$. Its edge set is $\{cv_i, cv'_i : 1 \leq i \leq m\} \cup \{v_i c' : 1 \leq i \leq m\} \cup \{v'_i u : 1 \leq i \leq m\} \cup \{c' u\}$. Thus, $c \not\sim c'$, $c \not\sim u$, and $c' \sim u$. Figure 2 shows St_3 and $M(St_3)$.

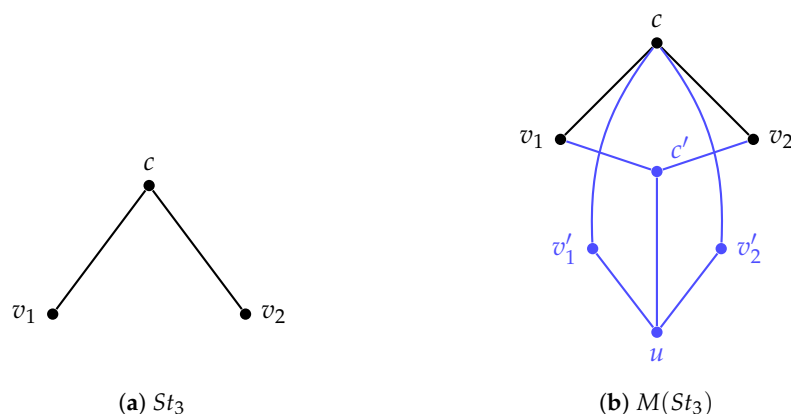


Figure 2. The star St_3 and its Mycielskian $M(St_3)$.

Remark 3. Unlike the trees of Duncan–Peele [1] and the forests of Galvin–Thanh [3], where $B(G)$ depends only on the order and the number of components, $M(St_n)$ is not a tree, and, in

fact, not chordal: for $n \geq 3$ the vertices c, v_1, c', v_2 induce a 4-cycle. It is, however, triangle-free, being the Mycielskian of a triangle-free graph. The closed form below therefore reflects the specific center-block structure of $M(St_n)$ rather than membership in a known class for which graphical Stirling numbers factor.

The next result counts proper partitions by isolating the block containing the original center c . The idea is that c is adjacent to every leaf and to every leaf copy, so its block is forced to be one of only three possibilities; conditioning on which one occurs reduces the count, in each case, to an ordinary set partition of an edgeless remainder.

Theorem 5 (Structural formula). *Let $n \geq 2$ and $m = n - 1$. For every integer q ,*

$$S(M(St_n); q) = 2 \sum_{a=0}^m \binom{m}{a} \left\{ \begin{matrix} 2m-a \\ q-2 \end{matrix} \right\} + \sum_{a=0}^m \sum_{b=0}^m \binom{m}{a} \binom{m}{b} \left\{ \begin{matrix} 2m-a-b \\ q-3 \end{matrix} \right\}, \quad (11)$$

where $\left\{ \begin{matrix} r \\ s \end{matrix} \right\} = 0$ if $s < 0$ or $s > r$.

Proof. The only edges of $M(St_n)$ are $cv_i, cv'_i, v_i c', v'_i u$, and $c' u$, for $1 \leq i \leq m$. In particular, every subset of $L \cup L'$ is independent: L and L' are each independent, and there are no edges between L and L' , because every edge incident to a vertex of $L \cup L'$ goes to one of c, c' , or u . Let P_c be the block containing c in a proper partition. As c is adjacent to every vertex in $L \cup L'$, no such vertex lies in P_c . As $c' \sim u$, the block P_c is exactly one of $\{c, c'\}$, $\{c, u\}$, and $\{c\}$.

Suppose first that $P_c = \{c, c'\}$. The block containing u has the form $\{u\} \cup A$ for a unique subset $A \subseteq L$, because u is nonadjacent to all vertices of L and adjacent to all vertices of L' . If $|A| = a$, the remaining vertices form the edgeless set $(L \setminus A) \cup L'$ of size $2m - a$, and they must be partitioned into $q - 2$ nonempty blocks. This gives $\left\{ \begin{matrix} 2m-a \\ q-2 \end{matrix} \right\}$ partitions for each fixed A ; hence, $\sum_{a=0}^m \binom{m}{a} \left\{ \begin{matrix} 2m-a \\ q-2 \end{matrix} \right\}$ in this case.

The case $P_c = \{c, u\}$ gives the same contribution. Indeed, the map fixing c and interchanging c' with u and v_i with v'_i is an automorphism of $M(St_n)$, as it sends the edge types $cv_i, cv'_i, v_i c', v'_i u$, and $c' u$ to edge types of the same graph.

It remains to consider $P_c = \{c\}$. As $c' \sim u$, the vertices c' and u lie in distinct blocks. The block containing c' has the form $\{c'\} \cup A$ for a unique subset $A \subseteq L'$, and the block containing u has the form $\{u\} \cup B$ for a unique subset $B \subseteq L$. If $|A| = a$ and $|B| = b$, the remaining vertices form the edgeless set $(L \setminus B) \cup (L' \setminus A)$ of size $2m - a - b$, and they must be partitioned into $q - 3$ nonempty blocks. This gives the double sum in (11). The three cases are mutually exclusive and exhaustive, so the formula follows. $\square$

Lemma 1 (Stirling–binomial recurrence). *For integers $N \geq 0$ and $k \geq 0$, $\sum_{j=0}^N \binom{N}{j} \left\{ \begin{matrix} j \\ k+1 \end{matrix} \right\} = \left\{ \begin{matrix} N+1 \\ k+1 \end{matrix} \right\}$.*

Proof. Both sides count partitions of $[N + 1]$ into $k + 1$ nonempty blocks. Choose the j elements of $[N]$ that do not lie in the block containing $N + 1$; these j elements must form the remaining k blocks. Summing over j gives the identity. $\square$

Proposition 1 (Simplified closed form). *For $n \geq 2$, with $m = n - 1$,*

$$S(M(St_n); q) = 2 \sum_{a=0}^m \binom{m}{a} \left\{ \begin{matrix} 2m-a \\ q-2 \end{matrix} \right\} + \left\{ \begin{matrix} 2n-1 \\ q-2 \end{matrix} \right\}, \quad (12)$$

and, consequently,

$$B(M(St_n)) = 2 \sum_{a=0}^m \binom{m}{a} \mathcal{B}_{2m-a} + \mathcal{B}_{2n-1}. \quad (13)$$

Proof. The double sum in (11) satisfies $\sum_{a=0}^m \sum_{b=0}^m \binom{m}{a} \binom{m}{b} \{ \frac{2m-a-b}{q-3} \} = \sum_{s=0}^{2m} \binom{2m}{s} \{ \frac{2m-s}{q-3} \}$ by Vandermonde's identity. Reindexing with $r = 2m - s$ gives $\sum_{r=0}^{2m} \binom{2m}{r} \{ \frac{r}{q-3} \}$, which equals $\{ \frac{2m+1}{q-2} \} = \{ \frac{2n-1}{q-2} \}$ by Lemma 1 with $N = 2m$ and $k = q - 3$. This proves (12). Summing over q gives (13). $\square$

Remark 4. The double sum in Theorem 5 collapses to the single Stirling number appearing in Proposition 1. The remaining single sum is left in convolution form. This remaining sum $\sum_{a=0}^m \binom{m}{a} \{ \frac{2m-a}{q-2} \}$ is a Stirling analogue of a binomial transform; the two indices a and $2m - a$ both vary with a , so Lemma 1 and Vandermonde's identity do not apply, and it does not reduce to a single Stirling number for general q . We therefore regard (12) as the simplest closed form. It does collapse for specific q , as the two corollaries below show.

Corollary 7 (Bell numbers of $M(St_n)$). For $n \geq 2$, $B(M(St_n)) = 2 \sum_{a=0}^{n-1} \binom{n-1}{a} \mathcal{B}_{2n-2-a} + \mathcal{B}_{2n-1}$.

Proof. This is (13) with $m = n - 1$. $\square$

Corollary 8 (Growth bracket for $B(M(St_n))$). For $n \geq 2$, with $m = n - 1$,

$$\mathcal{B}_{2n-1} \leq B(M(St_n)) \leq \mathcal{B}_{2n-1} + 2^n \mathcal{B}_{2n-2}.$$

Proof. Both bounds use Corollary 7. The lower bound drops the nonnegative sum. For the upper bound, $\mathcal{B}_{2m-a} \leq \mathcal{B}_{2m} = \mathcal{B}_{2n-2}$ for $0 \leq a \leq m$, so $2 \sum_{a=0}^m \binom{m}{a} \mathcal{B}_{2m-a} \leq 2 \cdot 2^m \mathcal{B}_{2n-2} = 2^n \mathcal{B}_{2n-2}$. $\square$

Corollary 9 (Partition polynomial of $M(St_n)$). Let $B_r(x) = \sum_{j \geq 0} \{ \frac{r}{j} \} x^j$. For $m = n - 1$, $F(M(St_n); x) = x^2 (2 \sum_{a=0}^m \binom{m}{a} \mathcal{B}_{2m-a}(x) + \mathcal{B}_{2n-1}(x))$.

Proof. Multiply (12) by x^q and sum over q . As $\sum_q \{ \frac{r}{q-2} \} x^q = x^2 B_r(x)$, the asserted formula follows. $\square$

Corollary 10. For $n \geq 2$, $S(M(St_n); 3) = 2^n + 1$.

Proof. Set $q = 3$ in (12). As $2m - a \geq 1$ for $0 \leq a \leq m$ and $\{ \frac{r}{1} \} = 1$ for $r \geq 1$, we obtain $S(M(St_n); 3) = 2 \sum_{a=0}^m \binom{m}{a} + \{ \frac{2n-1}{1} \} = 2^{m+1} + 1 = 2^n + 1$. $\square$

Corollary 11. For $n \geq 2$, $S(M(St_n); 2n) = 2n^2 - 3n + 3$.

Proof. Set $q = 2n$ in (12). As $\{ \frac{2n-2-a}{2n-2} \}$ is nonzero only when $a = 0$, the single sum contributes 2. Also, $\{ \frac{2n-1}{2n-2} \} = \binom{2n-1}{2} = (2n-1)(n-1)$. Hence, $S(M(St_n); 2n) = 2 + (2n-1)(n-1) = 2n^2 - 3n + 3$. $\square$

Example 3. For $n = 2$ the star St_2 is a single edge (K_2), and its Mycielskian $M(St_2)$ is the 5-cycle C_5 . Proposition 1 with $m = 1$ gives $S(M(St_2); 3) = 2 \sum_{a=0}^1 \binom{1}{a} \{ \frac{2-a}{1} \} + \{ \frac{3}{1} \} = 2(1+1) + 1 = 5$, matching $2^2 + 1$, and $B(M(St_2)) = 2(\mathcal{B}_2 + \mathcal{B}_1) + \mathcal{B}_3 = 2(2+1) + 5 = 11$, in agreement with Table A2.

2.4. Sequence Identifications

The sequences from Sections 2.1–2.3 are identified with OEIS entries below; numerical evidence is given in Appendix A.

By Corollary 1, $S(K_{n,n}; 4) = 2 - 2^{n+1} + 3^{n-1} + 4^{n-1}$ is OEIS A384980 <https://oeis.org/A384980>, accessed on 20 June 2026 [12] ($n \geq 1: 0, 1, 11, 61, 275, \dots$). By Corollary 2, $S(K_{n,n}; 5)$ is OEIS A384981 <https://oeis.org/A384981>, accessed on 20 June 2026 [13]. By Corollary 3, $\frac{1}{3}S(K_{n,n,n}; 5)$ is OEIS A384988 <https://oeis.org/A384988>, accessed on 20 June 2026 [14]. The triangle $T(n, k) = S(K_{n,n,n}; k)$ is OEIS A385432 <https://oeis.org/A385432>, accessed on 20 June 2026 [15], and the triangle $T(n, k) = S(H_n; k)$ is OEIS A385437 <https://oeis.org/A385437>, accessed on 20 June 2026 [16]; its row sums give $B(H_n)$.

Proposition 2. For $n \geq 2$, the values $S(M(St_n); 3) = 2^n + 1$ agree with the corresponding terms of OEIS A000051 <https://oeis.org/A000051>, accessed on 20 June 2026 [6].

In the next statement, $a_{A096376}(n)$ denotes the n -th term of the OEIS sequence A096376, and $a_{A116735}(n)$ the n -th term of A116735.

Proposition 3. For $n \geq 2$, the graphical Stirling number satisfies $S(M(St_n); 2n) = 2n^2 - 3n + 3$, which coincides with the shifted OEIS sequence $a_{A096376}(n - 1)$. Moreover, as OEIS records the relation $a_{A116735}(m) = a_{A096376}(m - 2)$ for $m > 1$, we obtain $S(M(St_n); 2n) = a_{A116735}(n + 1)$.

Proof. As $a_{A096376}(n) = 2n^2 + n + 2$, substitution gives $a_{A096376}(n - 1) = 2(n - 1)^2 + (n - 1) + 2 = 2n^2 - 3n + 3 = S(M(St_n); 2n)$. The OEIS relation then yields $a_{A116735}(n + 1) = a_{A096376}(n - 1) = 2n^2 - 3n + 3$. $\square$

Remark 5. The sequence $B(M(St_n))$: 11, 106, 1695, 39,325, 1,224,464, $\dots$ for $n = 2, 3, 4, 5, 6, \dots$ is, to the best of our knowledge, not currently listed in the OEIS and is a natural candidate for submission; as the database is updated continually, this status may change. The closed form is given by Corollary 7.

3. Conclusions

We established explicit formulas for graphical Stirling and Bell numbers for three graph families whose independence structure admits a complete combinatorial description; see Table 1 for a summary.

Table 1. Summary of the graphical Stirling and Bell number formulas established in this paper.

Graph	$S(G; k)$	$B(G)$	Reference
$K(n_1, \dots, n_\ell)$	$\sum_{j_1 + \dots + j_\ell = k} \prod_i \binom{n_i}{j_i}$	$\prod_i \mathcal{B}_{n_i}$	Theorem 1
$H_n = K_{n,n} - M$	$\sum_s \binom{n}{s} \sum_j \binom{n-s}{j} \binom{n-s}{k-s-j}$	$\sum_k \binom{n}{k} \mathcal{B}_k^2 = \mathbb{E}[(1 + XY)^n]$	Theorems 2 and 3
$M(St_n)$	$2 \sum_a \binom{m}{a} \left\{ \begin{matrix} 2m-a \\ q-2 \end{matrix} \right\} + \left\{ \begin{matrix} 2n-1 \\ q-2 \end{matrix} \right\}$	$2 \sum_a \binom{m}{a} \mathcal{B}_{2m-a} + \mathcal{B}_{2n-1}$	Proposition 1, Corollary 7

For the complete multipartite graph $K(n_1, \dots, n_\ell)$, every independent set is contained in a single partite class. This yields the convolution Formula (4) for the graphical Stirling numbers and the product identity $B(K(n_1, \dots, n_\ell)) = \prod_{i=1}^\ell \mathcal{B}_{n_i}$.

For the graph $H_n = K_{n,n} - M$, where M is a perfect matching, the mixed blocks of a proper partition are exactly the matched pairs $\{u_i, v_i\}$. This structure leads to the convolution Formula (7),

$$B(H_n) = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k^2.$$

Theorem 3 identifies this quantity as the moment $\mathbb{E}[(1 + XY)^n]$ of two independent Poisson(1) random variables X and Y . The moment representation yields the log-convexity inequality $B(H_{n-1})B(H_{n+1}) \geq B(H_n)^2$ and the sharp lower bound $B(H_n) \geq 2^n$. It also gives the Laplace-transform identity (9) for $t \leq 0$.

For the Mycielskian star graph $M(St_n)$, the partition structure is governed by the block containing the original center c . The three-case decomposition of Theorem 5, combined with Vandermonde's convolution and Lemma 1, yields the closed form (12),

$$S(M(St_n); q) = 2 \sum_{a=0}^{n-1} \binom{n-1}{a} \left\{ \begin{matrix} 2n-2-a \\ q-2 \end{matrix} \right\} + \left\{ \begin{matrix} 2n-1 \\ q-2 \end{matrix} \right\}.$$

Summing over q gives the Bell-number Formula (13). Two special evaluations follow directly: $S(M(St_n); 3) = 2^n + 1$ (A000051 <https://oeis.org/A000051>, accessed on 20 June 2026) and $S(M(St_n); 2n) = 2n^2 - 3n + 3$ (A116735 <https://oeis.org/A116735>, accessed on 20 June 2026).

The resulting formulas provide exact enumerative descriptions for each family and reduce their graphical Bell and Stirling numbers to classical Stirling and Bell numbers. Several associated integer sequences appear in the OEIS, while the Bell-number sequence $B(M(St_n))$ appears not to be currently recorded.

Two limitations should be noted. First, our exact formulas rest on a closed description of the independent sets, and the three families above are chosen precisely because such a description is known. The same method does not directly apply to graph families whose independent sets do not admit a tractable closed-form description. Second, while we obtain the bounds $2^n \leq B(H_n) \leq \mathcal{B}_{n+1}^2$ from Corollary 6 and $\mathcal{B}_{2n-1} \leq B(M(St_n)) \leq \mathcal{B}_{2n-1} + 2^n \mathcal{B}_{2n-2}$ from Corollary 8, we do not determine precise first-order asymptotics. We expect that a saddle-point analysis of the Poisson moment $\mathbb{E}[(1 + XY)^n]$ should yield the leading growth of $B(H_n)$. We also leave open the asymptotic question of whether $B(M(St_n)) = \mathcal{B}_{2n-1}(1 + o(1))$. Natural further directions include extending the center-block method to Mycielskians of arbitrary trees, where the leaf structure is no longer uniform, and to iterated Mycielskians $M^k(G)$, where even the independence structure of $M^2(St_n)$ is already substantially more involved.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Computational Tables

Tables A1–A4 verify the formulas of Sections 2.1–2.3.

Table A1. Bell numbers and Stirling numbers for $H_n = K_{n,n} - M$ (Theorem 2, Corollaries 4 and 5). The inequality $B(H_n) \geq 2^n$ is verified in the final column.

n	$B(H_n) = \sum_{k=0}^n \binom{n}{k} \mathcal{B}_k^2$	$S(H_n; 3)$	$S(H_n; 4)$	2^n
1	2	0	0	2
2	7	4	1	4
3	41	10	20	8
4	354	18	92	16
5	4125	35	355	32
6	61,375	68	1336	64
7	1,123,218	133	5026	128
8	24,615,179	262	19,097	256

Table A2. Graphical Bell and Stirling numbers for $M(St_n)$, $n \geq 2$ (Proposition 1, Corollaries 10 and 11). Column 4 reproduces A000051 <https://oeis.org/A000051>, accessed on 20 June 2026 for comparison.

n	$B(M(St_n))$	$S(M(St_n); 3) = 2^n + 1$	A000051	$S(M(St_n); 2n) = 2n^2 - 3n + 3$
2	11	5	5	5
3	106	9	9	12
4	1695	17	17	23
5	39,325	33	33	38
6	1,224,464	65	65	57
7	48,674,731	129	129	80
8	2,385,316,419	257	257	107

Table A3. Values of A384980 <https://oeis.org/A384980>, A384981 <https://oeis.org/A384981>, accessed on 20 June 2026, and A384988 <https://oeis.org/A384988>, accessed on 20 June 2026 for $1 \leq n \leq 8$ (Corollaries 1–3).

n	A384980: $S(K_{n,n}; 4)$	A384981: $S(K_{n,n}; 5)$	A384988
1	0	0	0
2	1	0	1
3	11	6	10
4	61	86	55
5	275	770	250
6	1141	5710	1051
7	4571	38,626	4270
8	18,061	248,766	17,095

Table A4. Triangle $T(n, k) = S(K_{n,n}; k)$ for $3 \leq k \leq 3n$ (OEIS A385432 <https://oeis.org/A385432>, accessed on 20 June 2026; Corollary 3).

n	Row Entries
1	1
2	1, 3, 3, 1
3	1, 9, 30, 45, 30, 9, 1
4	1, 21, 165, 598, 1032, 939, 471, 129, 18, 1

Table A5. OEIS sequences submitted or referenced in this paper. An asterisk (*) denotes a sequence not currently in the OEIS.

Sequence	Description and Connection to This Paper
A000051	$a(n) = 2^n + 1$. Equals $S(M(St_n); 3)$ (Corollary 10; Table A2).
A096376	$a(n) = 2n^2 + n + 2$. Satisfies $a(n - 1) = 2n^2 - 3n + 3 = S(M(St_n); 2n)$ (Proposition 3; Table A2).
A116735	Permutations of $[n]$ with π and π^2 avoiding 132 and 3421. Satisfies $a(n + 1) = S(M(St_n); 2n)$ (Proposition 3; Table A2).
A384980	$S(K_{n,n}; 4) = 2 - 2^{n+1} + 3^{n-1} + 4^{n-1}$: proper 4-partitions of $K_{n,n}$ (Corollary 1; Table A3).
A384981	$S(K_{n,n}; 5)$: proper 5-partitions of $K_{n,n}$ (Corollary 2; Table A3).
A384988	$\frac{1}{3}S(K_{n,n,n}; 5) = \frac{1}{12}(18 - 18 \cdot 2^n + 2 \cdot 3^n + 3 \cdot 4^n)$ (Corollary 3; Table A3).
A385432	Triangle $T(n, k) = S(K_{n,n,n}; k)$, $n \geq 1, 3 \leq k \leq 3n$: graphical Stirling numbers of $K_{n,n,n}$ (Corollary 3; Table A4).
A385437	Triangle $T(n, k) = S(H_n; k)$, $n \geq 1, 2 \leq k \leq 2n$; row sums give $B(H_n)$: graphical Stirling numbers of H_n (Theorem 2; Table A1).
*	$B(M(St_n))$: 11, 106, 1695, 39,325, 1,224,464, $\dots$, $n = 2, 3, 4, 5, 6, \dots$ (Corollary 7; Table A2). Not in OEIS; candidate for submission.

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16. OEIS Foundation Inc. Sequence A385437: Triangle of Graphical Stirling Numbers $S(H_n; k)$, $n \geq 1$, $2 \leq k \leq 2n$; Row Sums Give $B(H_n)$. In *The On-Line Encyclopedia of Integer Sequences*; OEIS Foundation Inc.: Highland Park, NJ, USA, 2024. Available online: <https://oeis.org/A385437> (accessed on 15 December 2025).
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