

Article

A Formula Similar to a Consequence of the Join Theorem

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Abstract

Let $u : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ and $v : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic function germs that have isolated critical points at the origin. It is known that $\mu(u + v) = \mu(u) \cdot \mu(v)$ holds, where μ denotes the Milnor number. In this paper, we consider a formula similar to the above one. More precisely, for Morse functions $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$, let $C(f + g)$ denote the fiber product of two copies of $f + g$. Two of the main results are as follows: Firstly, under a generic condition on the critical values of f and g , the equation $\chi(C(f + g)) = \chi(C(f)) \cdot \chi(C(g))$ holds. Here χ denotes the Euler characteristic. Secondly, under a certain condition, the converse of the first result is also true.

Keywords: join theorem; fiber product; Morse function; Euler characteristic

MSC: 55R80; 57R70

1. Introduction

1.1. Basics of Morse Theory

Since Morse theory plays a central role in this paper, we first explain the basics of it. (See, for example, [1–3] for more details.) Let X be a connected closed manifold, and

$$\eta : X \rightarrow \mathbb{R} \quad (1)$$

be a smooth function. A point $u \in X$ is a *critical point* of η if the differential $df_u : T_u X \rightarrow \mathbb{R}$ is zero. (Here $T_u X$ denotes the tangent space of X at u .)

The essence of Morse theory lies in elucidating the topology of X by using information about critical points of η . A more accurate statement is that we can study general classes of manifolds using the perspective obtained by considering the critical points of various functions η .

Typically, we do not simply count the number of critical points, but consider more detailed information, such as the second derivative test, to determine which critical points are maxima, which are minima, and so on.

The second derivative test at each critical point $u \in X$ is most properly viewed as the Hessian $\text{Hess}_u(\eta)$, which is a symmetric bilinear form on the tangent space:

$$\text{Hess}_u(\eta) : T_u X \times T_u X \rightarrow \mathbb{R}.$$



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In terms of local coordinates $\{x_1, \dots, x_n\}$ of a neighborhood of $u \in X$, $\text{Hess}_u(\eta)$ is the familiar matrix of second-order partial derivatives

$$\text{Hess}_u(\eta) = \left(\frac{\partial^2 \eta}{\partial x_i \partial x_j}(u) \right).$$

A critical point $u \in X$ is said to be *nondegenerate* if the Hessian $\text{Hess}_u(\eta)$ is nonsingular. The *index* of a nondegenerate critical point of u , $\mu_\eta(u)$, is defined to be the dimension of the negative eigenspace of $\text{Hess}_u(\eta)$. Thus, a local minimum has index 0, and a local maximum has index n , where n denotes the dimension of X . In the case of dimension $n = 2$, local minima have an index of 0, saddles have an index of 1, and local maxima have an index of 2.

A function $\eta : X \rightarrow \mathbb{R}$ whose critical points are all nondegenerate is called a *Morse function*. The underlying idea of Morse theory is to study the topology of X by analyzing the Morse function η . The theory allows one to find CW structures and handle decompositions on X . Below, we show one famous theorem.

Theorem 1 (Morse's Theorem). *Let $\eta : X \rightarrow \mathbb{R}$ be a Morse function. We define the integers $c_0(\eta), c_1(\eta), \dots, c_n(\eta)$ so that $c_i(\eta)$ is the number of critical points of index i . Then we have*

$$\chi(X) = \sum_{i=0}^n (-1)^i c_i(\eta). \quad (2)$$

Throughout this paper, for a finite CW-complex X , let $\chi(X)$ denote the Euler characteristic of X .

Recall that the Euler characteristic of X is defined by

$$\chi(X) = \sum_{i=0}^n (-1)^i b_i(X),$$

where $b_i(X)$ is the i th Betti number of X (the i th Betti number of X is the rank of the i th homology group of X). Thus, Theorem 1 is astonishing in that on the right-hand side of (2) we have an expression that involves η , while the left-hand side of (2) does not depend on η at all, and in fact only depends on the topological information about X .

Morse theory is crucial in the program to classify manifolds. It is also an important tool in a wide range of fields from symplectic geometry to non-abelian gauge theory.

1.2. The Fiber Product

Let $\eta : X \rightarrow \mathbb{R}$ be as in (1). We denote by $C(\eta)$ the fiber product of two copies of η . More precisely, we set

$$C(\eta) := \{(u, v) \in X \times X \mid \eta(u) = \eta(v)\}. \quad (3)$$

The following situation illustrates an example of $C(\eta)$: Let $\eta : S^n \rightarrow \mathbb{R}$ be the height function defined by

$$\eta(x_1, \dots, x_{n+1}) = x_{n+1}. \quad (4)$$

Assume that two particles u and v on S^n move satisfying the condition $\eta(u) = \eta(v)$. Then $C(\eta)$ can be understood as the space which is formed by all possible positions of u and v . In fact, there is a homeomorphism

$$C(\eta) \approx S(S^{n-1} \times S^{n-1}), \quad (5)$$

where S denotes the suspension.

1.3. The Known Results About $C(\eta)$

We recall what is known in [4–6].

(i) In [4], we proved a formula which describes $\chi(C(\eta))$ in terms of the information on the critical points and their indices of η . (See Lemma 2.)

(ii) Using the formula in (i), we determined $\chi(C(\eta))$ when η is the well-known Morse function on the unitary group ([4]), and when η is the well-known Morse function on the Grassmannians ([5]).

(iii) Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions as in (1). We define the map $f * g : M \times N \rightarrow \mathbb{R}$ by

$$(f * g)(p, q) = f(p) \cdot g(q).$$

Putting $\eta = f * g$ in (3), we define $C(f * g)$. It is studied in [6] when

$$\chi(C(f * g)) = \chi(C(f)) \cdot \chi(C(g)) \quad (6)$$

holds. (We will encounter a similar equation to (6) in (9).)

1.4. The Sum of Morse Functions

Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions as in (1). We define the map $f + g : M \times N \rightarrow \mathbb{R}$ by

$$(f + g)(p, q) = f(p) + g(q). \quad (7)$$

The sum $f + g$ is used when we prove the Künneth formula using the Morse homology. (See, for example, Section 4.2 in [7]).

1.5. The Purpose of This Paper

Below we pose Questions 1 and 2. We will explain the motivation for considering these questions in Section 1.7.

Let $f + g$ be as in Section 1.4. Putting $\eta = f + g$ in (3), we define $C(f + g)$. Note that there is a homeomorphism

$$C(f + g) \approx C(g + f). \quad (8)$$

The purpose of this paper is to study the following Questions 1 and 2:

Question 1. (i) Does there exist a sufficient condition on f and g under which

$$\chi(C(f + g)) = \chi(C(f)) \cdot \chi(C(g)) \quad (9)$$

holds?

(ii) Is the converse of (i) also true? More precisely, does (9) imply the sufficient condition mentioned in (i)?

Question 2. Let f and g be the well-known Morse functions on projective spaces or connected closed orientable surfaces, which do not satisfy the sufficient condition mentioned in Question 1 (i). Then what are both sides of (9)?

We give answers to Questions 1 and 2 in Section 3. We also give a summary of the answer to Question 1 in Section 1.6 (i) and (ii).

1.6. The Novelty of This Paper

As we explained in Section 1.3, we already know various results on $C(\eta)$, where η is a given Morse function. The novelty of this paper is as follows: We define the sum of Morse

functions in (7) and study the topological properties of $C(f + g)$. The main questions are Questions 1 and 2 in Section 1.5. The particularly interesting results are as follows:

- (i) We define a condition about the critical values of f and g . Then we show that:
 - The condition is generic for f and g . (See Remark 3 (i).)
 - (9) holds under the condition. (See Theorem 3.)
- (ii) In some case, the converse of (i) is also true. More precisely, if (9) holds, then the condition which was mentioned in (i) holds. (See Theorem 4.)

1.7. Motivations for Considering Questions 1 and 2

In order to state the motivation to consider Questions 1 and 2 in Section 1.5, we recall basic results in [8]: Let $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ which has an isolated critical point at the origin ($d \geq 2$). We fix a sufficiently small positive real number ε . We set $S_\varepsilon = \{z \in \mathbb{C}^d \mid \|z\| = \varepsilon\}$ and $K = S_\varepsilon \cap h^{-1}(0)$. We define the map $\phi : S_\varepsilon - K \rightarrow S^1$ by

$$\phi(z) = \frac{h(z)}{|h(z)|}.$$

A well-known result in [8] states that ϕ is a fiber bundle such that the fiber, which is a $(2d - 2)$ -dimensional manifold, is homotopy equivalent to a space of the form $\bigvee_k S^{d-1}$. The number k is called the *Milnor number* and denoted by $\mu(h)$.

The number $\mu(h)$ admits the following equality: If we write the fiber of ϕ by F_h , then

$$\mu(h) = (-1)^{d-1}(\chi(F_h) - 1).$$

Let $u : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ and $v : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be holomorphic function germs which have isolated critical points at the origin. We define $u + v : (\mathbb{C}^{m+n}, 0) \rightarrow (\mathbb{C}, 0)$ by

$$(u + v)(z, w) = u(z) + v(w).$$

It is proved in ([9] Corollary 4.2) that

$$\mu(u + v) = \mu(u) \cdot \mu(v). \quad (10)$$

The key result to prove (10) is the following join theorem ([10]): The join $F_u * F_v$ and F_{u+v} have the same homotopy type.

Note that the Milnor number μ is determined by the information of h near the origin. On the other hand, $C(\eta)$ contains global information on η . The motivation of this paper is to study whether $C(f + g)$ behaves like $\mu(u + v)$. That is, we study whether (9) holds true, similar to (10).

1.8. Highly Relevant Works

This paper is highly relevant to [11,12]. Below, we explain it.

(i) *The relevance to [11]*. This paper develops analytical techniques involving structured dynamical systems and may help position the present work within a broader framework of modern mathematical analysis and system-theoretic methodologies.

(ii) *The relevance to [12]*. This recent contribution on approximation theory and operator-based mathematical analysis would broaden the mathematical context of the manuscript and strengthen its connection with contemporary developments in applied and theoretical mathematics.

1.9. Organization of This Paper

This paper is organized as follows. In Section 2, we recall auxiliary lemmas. In Section 3 we state our main results. In Section 4 we prove the main results. In Section 5 we state conclusions.

2. Preliminaries

2.1. Table of Notations

The notations used in this paper are summarized in the following Table 1.

Table 1. Table of notations.

Notation	Definition	Defined Place
$C(\eta)$	The fiber product of η	(3)
$f + g$	The sum of Morse functions	(7)
$\mathcal{P}(\eta)$	The set of critical points of η	Definition 1
$\mathcal{V}(\eta)$	The set of critical values of η	Definition 1
$\mu_\eta(u)$	The index of η at u	Definition 1
$\Omega(\eta, \xi)$	The integer associated with a critical value ξ	(11)
$\Lambda(\eta)$	An analogue of Laurent polynomials	(14)

2.2. Auxiliary Lemmas

Definition 1. If η is a Morse function as in (1), then $\mathcal{P}(\eta)$ means the set of critical points of η and $\mathcal{V}(\eta)$ stands for the set of critical values of η . Furthermore, if $u \in \mathcal{P}(\eta)$, then $\mu_\eta(u)$ denotes the index of η at u .

Lemma 1. Let f and g be as in Section 1.4. Then the following results hold:

- (i) Let $(p, q) \in M \times N$. Then $(p, q) \in \mathcal{P}(f + g)$ if and only if $p \in \mathcal{P}(f)$ and $q \in \mathcal{P}(g)$.
- (ii) For $(p, q) \in \mathcal{P}(f + g)$, we have

$$\mu_{f+g}(p, q) = \mu_f(p) + \mu_g(q).$$

Proof. The lemma is well-known. (See, for example, [7], p. 18, Exercise 4.) $\square$

Lemma 2 ([4], Theorem A). Let η be a Morse function as in (1). For $\xi \in \mathcal{V}(\eta)$, let k be the number of the critical points contained in $\eta^{-1}(\xi)$. We write the critical points contained in $\eta^{-1}(\xi)$ by u_i for $1 \leq i \leq k$.

We write $\mu_\eta(u_i)$ by λ_i and define the integer $\Omega(\eta, \xi)$ by

$$\Omega(\eta, \xi) := \sum_{i=1}^k (-1)^{\lambda_i}. \quad (11)$$

Then we have

$$\chi(C(\eta)) = (-1)^{\dim X} \sum_{\xi \in \mathcal{V}(\eta)} (\Omega(\eta, \xi))^2.$$

Proof. The lemma is obtained by combining Morse surgery and Mayer–Vietoris argument. (See [4] for more details.) $\square$

Example 1. (i) Let $\eta : S^n \rightarrow \mathbb{R}$ be the Morse function in (4). Then

$$\chi(C(\eta)) = (-1)^n \cdot 2. \quad (12)$$

Note that (12) is in agreement with (5).

(ii) For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let $\eta : \mathbb{F}P^n \rightarrow \mathbb{R}$ be the well-known Morse function (see (21) and (24)). Then

$$\chi(C(\eta)) = (-1)^{dn} \cdot (n+1), \quad (13)$$

where $d := \dim_{\mathbb{R}} \mathbb{F}$.

Proof. About η in (i) and (ii), the critical values of the critical points are mutually distinct. Hence $\Omega(\eta, \xi) \in \{-1, 1\}$ for all $\xi \in \mathcal{V}(\eta)$. Then Example 1 follows from Lemma 2. $\square$

The following Lemma 3 is a paraphrase of Lemma 2, but sometimes useful in calculations.

Lemma 3. (i) Let η be a Morse function as in (1). For a variable t , we set

$$\Lambda(\eta) := \sum_{u \in \mathcal{P}(\eta)} (-1)^{\mu_{\eta}(u)} t^{\eta(u)}. \quad (14)$$

It is in general not true that $\Lambda(\eta)$ is a Laurent polynomial. Indeed, it is not in general true that $\eta(u) \in \mathbb{Z}$. But we introduce a similar identification as in the Laurent polynomial. More precisely, we identify $at^{\xi} + bt^{\xi}$ with $(a+b)t^{\xi}$. Then we have

$$\Lambda(\eta) = \sum_{\xi \in \mathcal{V}(\eta)} \Omega(\eta, \xi) t^{\xi},$$

where $\Omega(\eta, \xi)$ is defined in (11).

(ii) We have

$$\chi(C(\eta)) = (-1)^{\dim X} \sum_{\xi \in \mathcal{V}(\eta)} (\Omega(\eta, \xi))^2.$$

Proof. We shall prove that Lemmas 2 and 3 are equivalent.

(i) First, to each critical point u of η , we correspond the integer

$$(-1)^{\mu_{\eta}(u)}. \quad (15)$$

Let $\xi \in \mathcal{V}(\eta)$. Then the sum of (15) over critical points contained in $\eta^{-1}(\xi)$ is $\Omega(\eta, \xi)$. (See (11).)

Second, generalizing (15), we consider the pair

$$((-1)^{\mu_{\eta}(u)}, \eta(u)) \in \mathbb{Z} \times \mathbb{R}. \quad (16)$$

We identify (16) with

$$(-1)^{\mu_{\eta}(u)} t^{\eta(u)}. \quad (17)$$

Summing (17) over all critical points of η , we obtain $\Lambda(\eta)$. We remark the following items:

- From a monomial of $\Lambda(t)$, we can read its critical value as the exponent of t .
- In $\Lambda(t)$, the sum of the coefficients of t^{ξ} is equal to $\Omega(\eta, \xi)$. (See the first paragraph of the present proof.)

Hence the item (i) of the lemma holds.

(ii) The item is a repetition of Lemma 2. $\square$

Remark 1. Let $P(\eta)$ be the Morse polynomial of η . More precisely, we set

$$P(\eta) := \sum_{u \in \mathcal{P}(\eta)} t^{\mu_{\eta}(u)}.$$

In Morse theory, $P(\eta)$ is often mentioned. Note that $P(\eta)$ does not have information on the critical value $\eta(u)$.

On the other hand, for $\Lambda(\eta)$ in (14), $\eta(u)$ is crucial. The reason for this is that in order to compute $\chi(C(\eta))$ from the Mayer–Vietoris sequence, we consider the following stratification of $C(\eta)$:

$$C(\eta) = \bigcup_{\xi \in \mathbb{R}} \eta^{-1}(\xi) \times \eta^{-1}(\xi).$$

3. Statements of the Main Results

Theorem 2. Let f and g be as in Section 1.4.

(i) Recall that $\Lambda(\eta)$ is defined in (14). We define the product $\Lambda(f) \cdot \Lambda(g)$ in the same way as in the product of Laurent polynomials. Then we have

$$\Lambda(f + g) = \Lambda(f) \cdot \Lambda(g).$$

(ii) If we define $\Omega(f + g, \xi)$ so as to satisfy

$$\Lambda(f) \cdot \Lambda(g) = \sum_{\xi \in \mathcal{V}(f+g)} \Omega(f + g, \xi) t^{\xi}. \quad (18)$$

Then we have

$$\chi(C(f + g)) = (-1)^{\dim(M \times N)} \sum_{\xi \in \mathcal{V}(f+g)} (\Omega(f + g, \xi))^2.$$

Remark 2. (i) Theorem 2 (i) seems to be interesting itself. Moreover, the item is the key result to prove Theorem 2 (ii). On the other hand, Theorem 2 (ii) is useful when we compute $\chi(C(f + g))$ for various f and g . We will state Theorems 5, 6 and 7 later in this section, and in Section 4, we will see how Theorem 2 (ii) is used for the proofs of these theorems. In particular, we have given a detailed proof of Theorem 5.

(ii) At present, we give an easy application of Theorem 2 in the following Example 2.

Example 2. Similarly to (4), we define the Morse function $F_k : S^k \rightarrow \mathbb{R}$ by

$$F_k(x_1, \dots, x_{k+1}) = x_{k+1}.$$

Then we have

$$\chi(C(F_m + F_n)) = \begin{cases} 6, & \text{if } m + n \text{ is even,} \\ -2, & \text{if } m + n \text{ is odd.} \end{cases} \quad (19)$$

Proof. Since $\Lambda(F_k) = t^{-1} + (-1)^k t$, we have

$$\begin{aligned} \Lambda(F_m) \cdot \Lambda(F_n) &= (t^{-1} + (-1)^m t) \cdot (t^{-1} + (-1)^n t) \\ &= t^{-2} + ((-1)^m + (-1)^n) t^0 + (-1)^{m+n} t^2. \end{aligned} \quad (20)$$

Comparing (20) with (18), we have

$$\Omega(F_m + F_n, \xi) = \begin{cases} 1, & \text{if } \xi = -2, \\ (-1)^m + (-1)^n, & \text{if } \xi = 0, \\ (-1)^{m+n}, & \text{if } \xi = 2. \end{cases}$$

Now using Theorem 2 (ii), (19) follows. $\square$

The following theorem is an answer to Question 1 (i) in Section 1.5.

Theorem 3. Let f and g be as in Section 1.4. We define the condition (C) as follows:

Condition (C). We define the map

$$P : \mathcal{V}(f) \times \mathcal{V}(g) \rightarrow \mathbb{R}$$

by $P(a, b) = a + b$. Then P is injective.

If f and g satisfy the condition (C), then (9) holds. Namely, we have

$$\chi(C(f + g)) = \chi(C(f)) \cdot \chi(C(g)).$$

Remark 3. (i) The condition (C) in Theorem 3 is generic in an appropriate sense. Indeed, for given Morse functions f , g and a generic real value $\varepsilon \in \mathbb{R}$, f and $g + \varepsilon$ satisfy the condition (C).

(ii) More concretely, we give in the following Example 3 two examples of (f, g) which satisfy the condition (C).

Example 3. Firstly, regarding $\mathbb{C}P^n = S^{2n+1}/S^1$, we define $f_n : \mathbb{C}P^n \rightarrow \mathbb{R}$ by

$$f_n([z_0, z_1, \dots, z_n]) := \sum_{i=0}^n i |z_i|^2. \quad (21)$$

(i) We fix a positive irrational number α and define $\phi_n : \mathbb{C}P^n \rightarrow \mathbb{R}$ by

$$\phi_n([z_0, z_1, \dots, z_n]) := \alpha \sum_{i=0}^n i |z_i|^2.$$

Then $(f, g) = (f_n, \phi_n)$ also satisfies the condition (C) because α is an irrational number.

(ii) We fix a positive transcendental number β . (For example, we can set $\beta = e$ or π .) We define $\psi_n : \mathbb{C}P^n \rightarrow \mathbb{R}$ by

$$\psi_n([z_0, z_1, \dots, z_n]) := \sum_{i=0}^n \beta^i |z_i|^2.$$

Then $(f, g) = (f_n, \psi_n)$ also satisfies the condition (C) because β is a transcendental number.

The following theorem asserts that under a certain condition, the converse of Theorem 3 is also true. The theorem is an answer to Question 1 (ii) in Section 1.5.

Theorem 4. Let f and g be as in Section 1.4. We assume that f and g satisfy the following conditions (i) and (ii):

- (i) The indices of the critical points of f and g are all even.
- (ii) (9) holds.

Then f and g satisfy the condition (C) in Theorem 3.

Remark 4. Theorem 4 does not hold without the assumption (i). We give a counterexample as follows: Let η be as in (1) and let ξ be a critical value of η . Assume that $\eta^{-1}(\xi)$ contains α critical points of even indices and β critical points of odd indices. In this case, we write the information by (ξ, α, β) .

Let M and N be three-dimensional manifolds such that the f and g have the following information:

- (i) About f , $(-5, 1, 0)$, $(-1, 3, 3)$, $(1, 1, 1)$ and $(20, 0, 1)$.
- (ii) About g , $(-10, 1, 0)$, $(-1, 1, 1)$, $(1, 1, 1)$ and $(30, 0, 1)$.

We show that the above f and g are in fact a counterexample. Firstly, note that

$$\Lambda(f) = t^{-5} - t^{20} \text{ and } \Lambda(g) = t^{-10} - t^{30}. \quad (22)$$

By Lemma 3, we have $\chi(C(f)) = \chi(C(g)) = -2$. Hence $\chi(C(f)) \cdot \chi(C(g)) = 4$.

Secondly, substituting (22) into (18), we have

$$\Omega(f + g, \xi) \in \{-1, 1\} \text{ if } \xi \in \{-15, 10, 25, 50\}.$$

Hence we have from Theorem 2 (ii) that $\chi(C(f + g)) = 4$.

Thirdly, since $P(-1, 1) = P(1, -1) = 0$, f and g do not satisfy the condition (C) in Theorem 3.

Summarizing the above results, f and g satisfy (9) but do not satisfy the condition (C). Hence f and g are a counterexample.

Next, we give an answer to Question 2 in Section 1.5. About the case of spheres, see (12) and (19).

In the following Theorems 5 and 6, we consider the case of projective spaces. About $\chi(C(\eta))$, see (13).

Theorem 5. Let f_n be as in (21). Then the following results hold:

(i) By (8), we may assume that $m \leq n$. Then

$$\chi(C(f_m + f_n)) = \frac{(m+1)(-m^2 + 3mn + m + 3n + 3)}{3}.$$

(ii) In particular, we have

$$\chi(C(f_n + f_n)) = \frac{(n+1)(2(n+1)^2 + 1)}{3}. \quad (23)$$

Remark 5. (i) Setting $n + 1 = k$ in (23), we define the number

$$a_k := \frac{k(2k^2 + 1)}{3}.$$

The numbers a_k ($k = 1, 2, 3, \dots$) are called octahedral numbers and their properties are listed in [13].

(ii) We have from (13) that $\chi(C(f_n)) = n + 1$. Hence

$$\chi(C(f_m + f_n)) \neq \chi(C(f_m)) \cdot \chi(C(f_n)) \text{ for all } m, n \geq 1.$$

Theorem 6. Similarly to (21), we define $g_n : \mathbb{R}P^n \rightarrow \mathbb{R}$ by

$$g_n([x_0, x_1, \dots, x_n]) := \sum_{i=0}^n i x_i^2. \quad (24)$$

Then the following results hold:

(i) We have

$$\chi(C(g_m + g_n)) = (-1)^{m+n} \cdot \chi(C(f_m + f_n)).$$

(ii) In particular, we have

$$\chi(C(g_n + g_n)) = \chi(C(f_n + f_n)).$$

As a final answer to Question 2 in Section 1.5, we consider the case of connected closed orientable surfaces.

Theorem 7. For $n \geq 1$, let $T(n)$ be the connected sum of n copies of $S^1 \times S^1$. Let $h_n : T(n) \rightarrow \mathbb{R}$ be the usual height function. More precisely, h_n has $2n + 2$ critical points such that their critical values are $\{i \in \mathbb{Z} \mid 0 \leq i \leq 2n + 1\}$. Then the following results hold:

(i) We have $\chi(C(h_n)) = 2n + 2$.

(ii) We have

$$\chi(C(h_n + h_n)) = \frac{2}{3}(8n^3 - 24n^2 + 49n + 9).$$

4. Proofs of the Main Results

Proof of Theorem 2. (i) Using Lemma 1, we have

$$\begin{aligned} \Lambda(f + g) &= \sum_{(p,q) \in \mathcal{P}(f+g)} (-1)^{\mu_{f+g}(p,q)} t^{(f+g)(p,q)} \\ &= \sum_{\substack{p \in \mathcal{P}(f) \\ q \in \mathcal{P}(g)}} (-1)^{\mu_f(p)} (-1)^{\mu_g(q)} t^{f(p)+g(q)} \\ &= \left(\sum_{p \in \mathcal{P}(f)} (-1)^{\mu_f(p)} t^{f(p)} \right) \cdot \left(\sum_{q \in \mathcal{P}(g)} (-1)^{\mu_g(q)} t^{g(q)} \right) \\ &= \Lambda(f) \cdot \Lambda(g). \end{aligned}$$

(ii) If $\eta = f + g$, Lemma 3 is stated as follows: If we define $\Omega(f + g, \xi)$ so as to satisfy

$$\Lambda(f + g) = \sum_{\xi \in \mathcal{V}(f+g)} \Omega(f + g, \xi) t^{\xi}. \quad (25)$$

Then we have

$$\chi(C(f + g)) = (-1)^{\dim(M \times N)} \sum_{\xi \in \mathcal{V}(f+g)} (\Omega(f + g, \xi))^2.$$

By Theorem 2 (i), we can replace the left-hand side of (25) by $\Lambda(f) \cdot \Lambda(g)$. Hence, Theorem 2 (ii) follows. $\square$

Proof of Theorem 3. Let $\xi \in \mathcal{V}(f + g)$. By the condition (C), there exists $(a, b) \in \mathcal{V}(f) \times \mathcal{V}(g)$ uniquely such that $a + b = \xi$. We shall prove that

$$\Omega(f + g, \xi) = \Omega(f, a) \cdot \Omega(g, b). \quad (26)$$

In order to prove (26), we return to the definition of $\Omega(\eta, \xi)$ in (11). We write the critical points in $f^{-1}(a)$ by p_i for $1 \leq i \leq \ell$. Similarly, we write the critical points in $g^{-1}(b)$ by q_j for $1 \leq j \leq m$. Then Lemma 1 tells us that the critical points in $(f + g)^{-1}(\xi)$ are given by (p_i, q_j) for $1 \leq i \leq \ell$ and $1 \leq j \leq m$.

We set $\alpha_i := \mu_f(p_i)$ and $\beta_j := \mu_g(q_j)$. By (11), we have

$$\begin{aligned} \Omega(f + g, \xi) &= \sum_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (-1)^{\alpha_i + \beta_j} \\ &= \left(\sum_{i=1}^{\ell} (-1)^{\alpha_i} \right) \cdot \left(\sum_{j=1}^m (-1)^{\beta_j} \right) \\ &= \Omega(f, a) \cdot \Omega(g, b). \end{aligned}$$

Hence (26) holds.

Now we complete the proof of Theorem 3. Combining Lemma 2 and (26), we have

$$\begin{aligned}\chi(C(f+g)) &= (-1)^{\dim(M \times N)} \sum_{\xi \in \mathcal{V}(f+g)} (\Omega(f+g, \xi))^2 \\ &= \left((-1)^{\dim M} \sum_{a \in \mathcal{V}(f)} (\Omega(f, a))^2 \right) \cdot \left((-1)^{\dim N} \sum_{b \in \mathcal{V}(g)} (\Omega(g, b))^2 \right) \\ &= \chi(C(f)) \cdot \chi(C(g)).\end{aligned}$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. Assume that f and g satisfy the assumption (i) of Theorem 4. Since the method of proof remains the same in all cases, we assume that $\mathcal{V}(f) = \{\xi_1, \dots, \xi_4\}$, and let n_i be the number of critical points contained in $f^{-1}(\xi_i)$. Similarly, we assume that $\mathcal{V}(g) = \{\zeta_1, \dots, \zeta_4\}$, and let N_j be the number of critical points contained in $g^{-1}(\zeta_j)$. Then

$$\chi(C(f)) \cdot \chi(C(g)) = (n_1^2 + \dots + n_4^2) \cdot (N_1^2 + \dots + N_4^2). \quad (27)$$

To prove by contradiction, assume that f and g do not satisfy the condition (C). By changing the order, we may assume that $P(\xi_2 + \zeta_3) = P(\xi_3 + \zeta_2)$. We assume that this is the only relation for P not to be injective. Then

$$\chi(C(f+g)) = \left(\sum_{(i,j) \neq (2,3) \text{ nor } (3,2)} (n_i N_j)^2 \right) + (n_2 N_3 + n_3 N_2)^2. \quad (28)$$

Since

$$n_2^2 N_3^2 + n_3^2 N_2^2 < (n_2 N_3 + n_3 N_2)^2,$$

we have (27) < (28). This contradicts the assumption (ii) of Theorem 4. Hence, under the assumptions (i) and (ii) of Theorem 4, the condition (C) holds. $\square$

In order to prove Theorem 5, we first prove the following:

Lemma 4. Assume $m \leq n$. Then the following equation holds in $\mathbb{Z}[t]$:

$$\begin{aligned}& \left(\sum_{i=0}^m t^i \right) \cdot \left(\sum_{j=0}^n t^j \right) \\ &= \left(\sum_{q=0}^{m-1} (q+1) t^q \right) + \left(\sum_{q=m}^n (m+1) t^q \right) + \left(\sum_{q=n+1}^{m+n} (m+n-q+1) t^q \right).\end{aligned} \quad (29)$$

Proof. We write

$$\left(\sum_{i=0}^m t^i \right) \cdot \left(\sum_{j=0}^n t^j \right) = \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} t^{i+j}$$

as $\sum_{q=0}^{m+n} a_q t^q$. Below we determine a_q .

(i) The case that $0 \leq q \leq m-1$. Since $i+j=q$, we have $0 \leq i \leq q$. Hence $a_q = q+1$.

(ii) The case that $m \leq q \leq n$. We have from $j=q-i$ that $0 \leq i \leq m$ and $0 \leq q-i \leq n$.

They are equivalent to $0 \leq i \leq m$. Hence $a_q = m+1$.

(iii) The case that $n + 1 \leq q \leq m + n$. We have from $j = q - i$ that $0 \leq i \leq m$ and $0 \leq q - i \leq n$. They are equivalent to $q - n \leq i \leq m$. Hence $a_q = m + n - q + 1$. $\square$

Proof of Theorem 5. (i) Firstly, since $\Lambda(f_k) = \sum_{i=0}^k t^i$, we have

$$\Lambda(f_m) \cdot \Lambda(f_n) = \left(\sum_{i=0}^m t^i \right) \cdot \left(\sum_{j=0}^n t^j \right).$$

Note that the right-hand side of (30) is computed in (29).

Secondly, comparing the result in the first paragraph of the present proof with (18), we can determine $\Omega(f_m + f_n, \xi)$. (This is where Theorem 2 (ii) comes into play. Indeed, we can determine $\Omega(f_m + f_n, \xi)$ just by computing the product of polynomials as done in Lemma 4.)

Thirdly, using Theorem 2 (ii), we have

$$\begin{aligned} \chi(C(f_m + f_n)) &= \left(\sum_{q=0}^{m-1} (q+1)^2 \right) + \left(\sum_{q=m}^n (m+1)^2 \right) + \left(\sum_{q=n+1}^{m+n} (m+n-q+1)^2 \right) \\ &= 2 \cdot \left(\sum_{r=1}^m r^2 \right) + (m+1)^2(n-m+1) \\ &= \frac{m(m+1)(2m+1)}{3} + (m+1)^2(n-m+1) \\ &= \frac{(m+1)(-m^2 + 3mn + m + 3n + 3)}{3}. \end{aligned}$$

(ii) The item follows from (i). $\square$

Proof of Theorem 6. (i) Note that $\Lambda(g_k) = \sum_{i=0}^k (-1)^i \cdot t^i$. Hence, if we replace t by $-t$ in (29), then we obtain $\Lambda(g_m) \cdot \Lambda(g_n)$. Using this result and Theorem 2 (ii), we see that the difference between $\chi(C(f_n + f_n))$ and $\chi(C(g_m + g_n))$ is only the fact that $\dim(\mathbb{R}P^m \times \mathbb{R}P^n) = m + n$. Hence (i) holds.

(ii) The item follows from (i). $\square$

Proof of Theorem 7. Note that

$$\Lambda(h_n) = 1 - \left(\sum_{i=1}^{2n} t^i \right) + t^{2n+1}. \quad (30)$$

Combining Lemma 3 and (30), (i) follows.

We prove the item (ii) in the same way as in the proof of Theorem 5 (i). We define $\Omega(h_n + h_n, \xi)$ so as to satisfy

$$(\Lambda(h_n))^2 = \sum_{\xi=0}^{4n+2} \Omega(h_n + h_n, \xi) t^\xi.$$

Direct computations show

$$\begin{aligned} &(\Omega(h_n + h_n, 0), \Omega(h_n + h_n, 1), \dots, \Omega(h_n + h_n, 4n + 2)) \\ &= (1, -2, -1, 0, 1, 2, \dots, 2n - 4, 2n - 3, 2n + 2, 2n - 3, 2n - 4, \dots, 2, 1, 0, -1, -2, 1). \end{aligned} \quad (31)$$

Here, $2n + 2$ is the center of (31). Using Theorem 2 (ii), we have

$$\begin{aligned}\chi(C(h_n + h_n)) &= 2 \cdot (1^2 + (-2)^2 + (-1)^2) + 2 \cdot \left(\sum_{i=1}^{2n-3} i^2 \right) + (2n + 2)^2 \\ &= \frac{2}{3} (8n^3 - 24n^2 + 49n + 9). \quad \square\end{aligned}$$

5. Conclusions

The main results of this paper are summarized in the following (i) and (ii):

(i) The condition (C) in Theorem 3 is generic in an appropriate sense. (See Remark 3 (i).) In connection with the condition (C), we have obtained the following results:

- If f and g satisfy the condition (C), then (9) holds (see Theorem 3).
- Conversely, if the indices of the critical points of f and g are all even, then (9) implies the condition (C) (see Theorem 4).

(ii) We have computed both sides of (9) for the cases where f and g are the well-known Morse functions on projective spaces or connected closed orientable surfaces. (See Theorems 5, 6 and 7.)

We mention further problems. So far, we have considered Morse functions as defined in Section 1.1. A smooth function $\eta : X \rightarrow \mathbb{R}$ is called a *Bott–Morse function* if its critical point set is a finitely disjoint union of connected submanifolds called *critical manifolds*, and the Hessian of η is nondegenerate (fiberwise) on the normal bundle of critical manifolds.

In [14], results on Morse functions were generalized to the assertions on Bott–Morse functions (see also [15,16], which are excellent references on Bott–Morse theory). We introduce two famous papers that used Bott–Morse theory effectively.

- (i) In [17], Bott–Morse functions on classical Lie groups were constructed. The results were applied to prove the Bott periodicity theorem, which is a periodicity in the homotopy groups of classical Lie groups.
- (ii) In [18], Bott–Morse functions on spatial polygon spaces were constructed. The results were applied to determine the Poincaré polynomials of the spaces.

Now we pose the following:

Question 3. (i) Can we generalize Lemma 2 to the case of Bott–Morse functions?

(ii) What is $\chi(C(\eta))$ when η is a famous Bott–Morse function such as the above (i) and (ii)?

(iii) Can we generalize Theorem 3 to the case of Bott–Morse functions?

Studying Question 3 is our further task.

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