

## Article

# Connectivity of Networks Modeled by Weighted Graphs

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## Abstract

A graph  $G = (V, E)$  together with two positive real-valued weight-functions  $w_V : V \rightarrow \mathbb{R}_{>0}$  and  $w_E : E \rightarrow \mathbb{R}_{>0}$  is called a weighted graph and is denoted by  $(G; w_V; w_E)$ . In this paper, we introduce the concepts of connectivity, edge-connectivity, and restricted edge-connectivity for a weighted graph  $(G; w_V; w_E)$ , and we prove general bounds analogous to those in the unweighted case. Furthermore, we study the connectivity and edge-connectivity of the line graph and the  $P_2$ -path graph of a weighted graph, establishing upper and lower bounds for each of these parameters.

**Keywords:** connectivity; edge-connectivity; restricted edge-connectivity; line graphs; path graphs

**MSC:** 05C40

## 1. Introduction

Graph Theory is currently one of the most critical disciplines for advancing research into the behavior of interrelated structures and for developing communication, interconnection, logistics, and social interaction models. All these problems share a common foundation: the design of a network modeled through a graph, where vertices represent nodes and their relationships are represented by the edges of the graph.

Extensive research exists in the field of graphs based on solving optimization problems that involve their invariants (see, for instance, [1–7]). For a comprehensive overview of the progress made, we refer the reader to References [8,9].

In many real-world problems, neither the network components nor their relationships are uniform. Numerous factors influence them, such as the varying size of the areas modeled by each node, the strategic positioning of each node, or the specific importance each one holds within the network. Recent studies highlight that one of the major limitations of traditional theoretical results on unweighted graphs lies in their inability to handle datasets with highly imbalanced density distributions, as they tend to overlook local structural variations (see [10]). By adaptively normalizing edge weight magnitudes in both dense and sparse regions, non-uniform structures can be modeled with greater accuracy. Consequently, this mitigates the distortion of neighborhood relationships and captures the heterogeneity and non-uniform nature of the data, rather than assuming a homogeneous connectivity.

The same applies to the relationships: the actual distance between nodes is generally not uniform, nor is the cost of communication or the actual time required to travel from one node to another. However, most research results stem from the study of parameters in unweighted graphs, where neither vertices nor edges carry weights.



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Advances in Biomolecular Networks require vertex-weighted graphs, as atoms within a molecule have distinct masses and electronegativities that determine molecular stability and reactivity [11]. In [12], weighted graphs are utilized to construct a heterogeneous network by integrating multi-source biological data, where weights represent different similarity values between nodes, thereby enabling the graph neural network to effectively capture local structural details and accurately predict novel drug-disease associations. Graph Theory is also essential for understanding the structure and behavior of proteins, where amino acids and their interactions can be modeled in a graph [13,14]. However, these relationships must be weighted by physical distance or the intensity of interaction between those amino acids. Identifying biomarkers in heterogeneous complex diseases constitutes one of the most pressing challenges in medical research. To address this hurdle, Li et al. [15] developed “DualRank,” a dual ranking framework based on multiplex networks applicable to the diagnosis, prognosis, and classification of such pathologies. The findings of this study convincingly demonstrate that biological systems are characterized by multi-layered architectures and asymmetric interactions. This fact highlights the inadequacy of traditional models based on unweighted connectivity, which oversimplify biological reality, and underscores the critical need to investigate weighted structures that can capture the true disparity and strength of links within complex networks. The need to introduce vertex or edge weights is also present when applying Graph Theory to digital or terrestrial communication networks. Edge weighting provides a better approximation of the real network structure, allowing for the formulation of route optimization problems or the analysis of resilience against connection failures [16,17].

One of the most studied parameters in graphs is connectivity. A graph is connected when we can travel from any vertex to any other through a path formed by intermediate vertices and edges. When nodes or links in a network begin to fail, the network weakens until it breaks into two or more pieces, called sub-networks. In terms of a graph  $G$ , connectivity can be quantified as the minimum number of vertices,  $\kappa(G)$ , or edges,  $\lambda(G)$ , that must be removed from the graph to break it into two or more subgraphs, known as connected components. Two excellent surveys on graph connectivity can be found in [18] and also in Chapter 4, Section 4.1, of [9].

Other variants of the classic connectivity parameter in graphs have also been extensively studied. One such variant is the restricted connectivity, introduced in [19,20], which measures the minimum number of vertices,  $\kappa_1(G)$ , or edges,  $\lambda_1(G)$  (also denoted as  $\lambda'(G)$ ), that must be removed from a graph  $G$  to disconnect it such that each resulting connected component is non-trivial (i.e., it contains two or more vertices). The concepts of connectivity and restricted connectivity will be formally defined in Section 2.

In this article, we extend the study of connectivity, edge-connectivity and restricted edge-connectivity parameters to weighted graphs (graphs with weights on their vertices and/or edges). First, we extend the classic Whitney’s bound [21], proven for unweighted graphs, to the case of weighted graphs. Next, we also extend to weighted graphs the notion of restricted edge-connectivity and give a sufficient condition for a weighted graph to be  $\lambda'$ -connected and a general upper bound in terms of the weighted minimum edge degree. We also show the equivalence between the connectivity of the line graph  $L(G)$  of a weighted graph  $(G; w_V; w_E)$  and the restricted edge-connectivity of  $(G; w_V; w_E)$  and we also provide bounds for the edge-connectivity parameter of the line graph. Finally, we address the study of edge-connectivity in the weighted  $P_2$ -path graph and obtain both lower and upper bounds for this parameter. In all these problems, the proven results extend the unweighted case and improve the known bounds, thereby paving the way for further research into new results when the graph features vertex or edge weights.

## 2. Definitions and Terminology

An unweighted graph  $G$  is a pair  $G = (V, E)$ , where  $V$  is a set of vertices and  $E \subseteq V \times V$  is a set of edges. Two vertices  $u, v \in V$  are said to be adjacent if there is an edge joining them. This edge is denoted as  $(u, v) \in E$ . Note that  $(u, v)$  and  $(v, u)$  represent the same edge, because the graph is undirected. The order and the size of  $G$  are  $|V|$  and  $|E|$ , respectively. A subset  $S \subset V$  of vertices is called independent if there is no edge between any pair of vertices of  $S$ . Given two disjoint subsets  $S, S^* \subset V(G)$ , by  $(S, S^*)$  we denote the subset of  $E(G)$  formed by the edges with one endvertex in  $S$  and the other endvertex in  $S^*$ .

A graph  $G = (V, E)$  together with two positive real-valued weight-functions  $w_V : V \rightarrow \mathbb{R}_{>0}$  and  $w_E : E \rightarrow \mathbb{R}_{>0}$  is called a *weighted graph* and is denoted by  $(G; w_V; w_E)$ . The *(vertex)-weight* of a subset  $S \subseteq V$  of vertices is  $w_V(S) = \sum_{v \in S} w_V(v)$ . Analogously, the *edge-weight* of a subset  $F \subseteq E$  of edges is  $w_E(F) = \sum_{e \in F} w_E(e)$ . The weight of  $(G; w_V; w_E)$  is  $w_V(G) = w_V(V(G))$  whereas the edge-weight of  $(G; w_V; w_E)$  is  $w_E(G) = w_E(E(G))$ .

The set of vertices adjacent to a vertex  $u \in V$  is called the neighborhood of vertex  $u$  and is denoted by  $N_G(u)$  or simply,  $N(u)$ . The neighborhood of a subset  $S \subset V$  is defined analogously as  $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$ . The degree of  $u$  is  $d(u) = |N(u)|$ , and the weighted degree of  $u$  is  $d_w(u) = \sum_{v \in N(u)} w_E((u, v))$ . The minimum degree and the minimum weighted degree of  $(G; w_V; w_E)$  are  $\delta(G) = \min\{d(v) : v \in V(G)\}$  and  $\delta_w(G) = \min\{d_w(v) : v \in V(G)\}$ , respectively. Analogously, the maximum degree and the maximum weighted degree of  $(G; w_V; w_E)$  are  $\Delta(G) = \max\{d(v) : v \in V(G)\}$  and  $\Delta_w(G) = \max\{d_w(v) : v \in V(G)\}$ .

A complete graph with  $n$  vertices is one in which every pair of vertices is joined by an edge. It is denoted by  $K_n$ . A complete bipartite graph, denoted by  $K_{s,t}$ , is a graph whose vertex set is partitioned into two disjoint subsets,  $X$  and  $Y$ , with  $|X| = s$  and  $|Y| = t$ , so that every vertex in  $X$  is adjacent to every vertex in  $Y$ , and no two vertices within the same partition are connected by an edge. A complete bipartite graph  $K_{1,t}$  is also called a star graph. A split graph of order  $n$ , denoted as  $G = (K_r, I_{n-r})$ , is a graph  $G$  whose vertices can be partitioned into a complete graph  $K_r$  and an independent set of  $n - r$  vertices, and its edge set is  $E(G) = E(K_r) \cup (V(K_r), V(I_{n-r}))$ . In this paper we will use the weighted split graph of order  $n$  formed by one copy of  $K_2$  and a set of  $n - 2$  independent vertices, which will be denoted as  $((K_2, I_{n-2}); w_V, w_E)$ .

The subgraph induced by a subset  $S \subset V$ , denoted by  $G[S]$ , is the graph whose vertex set is  $S$ , where two vertices in  $S$  are adjacent in  $G[S]$  if and only if they are adjacent in  $G$ . Analogously, the subgraph of  $(G; w_V; w_E)$  induced by  $S$ , also denoted by  $G[S]$ , is the weighted graph  $(G[S]; w_V; w_E)$ . In this case, the vertex and the edge sets are defined identically to the unweighted case, and the weight function is restricted to  $w : S \rightarrow \mathbb{R}_{>0}$ .

A path  $P$  connecting two vertices  $u, v \in V(G)$ , also called a  $u - v$  path, is a sequence of distinct vertices and edges  $u = x_0, x_1, \dots, x_k = v$ , where  $x_i \in V$  and  $(x_i, x_{i+1}) \in E$  for all  $i$ . When  $x_k = x_0$ , we call it a cycle and denote it by  $C_k$ . The number of edges  $k$  in the path (resp. cycle) is the length of the path  $P_k$  (resp. the length of the cycle  $C_k$ ). The girth of  $G$  is the length of the shortest cycle in  $G$ . The concepts of path, cycle and girth are defined analogously for weighted graphs.

A graph  $G$  or a weighted graph  $(G; w_V; w_E)$  is said to be connected if, for every pair of vertices  $u, v \in V$ , there exists a  $u - v$  path. Otherwise, the graph is said to be disconnected. A subset of vertices  $S \subset V$  of a connected graph  $G$  or weighted graph  $(G; w_V; w_E)$  is called a cut set of  $G$  if  $G - S$  is disconnected or results in an isolated vertex. The latter only occurs when the graph is complete. Each of the maximal connected subgraphs of  $G - S$  is called a connected component of  $G$  or, simply, a component of  $G$ . The connectivity of an unweighted graph  $G$ , denoted by  $\kappa(G)$ , is the minimum cardinality of a cut set of  $G$ . A cut set  $S \subset V(G)$  is minimal if  $G - S'$  remains connected, for all  $S' \subset S$ . Note that, except in the

trivial case where  $G - S$  is an isolated vertex, the graph  $G - S$  resulting from the removal of a cut set  $S$  has at least two connected components, though it may have more.

A subset  $F \subset E$  of edges of a connected graph  $G$  or a weighted connected graph  $(G; w_V; w_E)$  is said to be an edge cut of  $G$  if  $G - F$  is disconnected. Each of the maximal connected subgraphs of  $G - F$  is called a connected component of  $G$  or, simply, a component of  $G$ . The edge-connectivity of an unweighted graph  $G$ , denoted by  $\lambda(G)$ , is the minimum cardinality of an edge cut of  $G$ . An edge cut  $F \subset E(G)$  is minimal if  $G - F'$  remains connected, for all  $F' \subset F$ . The graph  $G - F$  resulting from the removal of an edge cut  $F$  from  $G$  has at least two components. However, it is easy to check that if the set  $F$  is minimal, then  $G - F$  has exactly two components. Besides, every edge cut of minimum cardinality is minimal. Therefore, every minimal edge cut or every edge cut  $F$  of minimum cardinality can be expressed as  $F = (C, \bar{C})$ , where  $C$  and  $\bar{C} = G - V(C)$  are the components of  $G - F$ .

We can generalize the concepts of connectivity and edge-connectivity to weighted graphs, as we will see in the definition that follows.

**Definition 1.** Let  $(G; w_V; w_E)$  be a connected weighted graph.

(i) The weighted connectivity of  $(G; w_V; w_E)$  is defined as

$$\kappa_w(G) = \min\{w_V(S) : S \text{ is a cut set of } (G; w_V; w_E)\}.$$

(ii) The weighted edge-connectivity of  $(G; w_V; w_E)$  is defined as

$$\lambda_w(G) = \min\{w_E(F) : F \text{ is an edge cut of } (G; w_V; w_E)\}.$$

Restricted edge-connectivity, commonly denoted as  $\lambda'(G)$ , is a more refined measure of network resilience. It focuses not only on the disconnection of the graph but also on the specific characteristics of the components resulting from a cut. Specifically, for an unweighted graph  $G$ , the restricted edge-connectivity of  $G$  is defined as the minimum cardinality of a restricted edge cut of  $G$ . A subset  $F \subset E(G)$  is called a restricted edge cut if  $G - F$  is disconnected and every component of  $G - F$  contains at least two vertices. Therefore,

$$\lambda'(G) = \min\{|F| : F \text{ is a restricted edge cut of } G\}.$$

In the following definition, we naturally extend the concept of restricted edge-connectivity to weighted graphs.

**Definition 2.** Let  $(G; w_V; w_E)$  be a connected weighted graph different from a star graph. A subset  $F \subset E(G)$  is called a restricted edge cut of  $(G; w_V; w_E)$  if  $G - F$  is disconnected and every component of  $G - F$  contains at least two vertices. The weighted restricted edge-connectivity of  $(G; w_V; w_E)$  is defined as

$$\lambda'_w(G) = \min\{w_E(F) : F \text{ is a restricted edge cut of } (G; w_V; w_E)\}.$$

The line graph of an unweighted graph  $G$ , denoted by  $L(G)$ , is the graph whose vertex set is  $E(G)$ , in which two vertices  $(uv), (zt) \in V(L(G))$  are adjacent if and only if their corresponding edges,  $(u, v), (z, t) \in E(G)$ , are incident. The extension of the line graph concept for a weighted graph is introduced in the following definition.

**Definition 3.** Let  $(G; w_V; w_E)$  be a weighted graph. The weighted line graph of  $G$ , denoted by  $(L(G); w_V; w_E)$ , is a weighted graph which has the edge set  $E(G)$  as vertex set and two vertices in  $V(L(G))$  are adjacent, whenever the corresponding edges in  $(G; w_V; w_E)$  are incident. In addition,

$w_V((uv)) = w_E((u, v))$  for every  $(uv) \in V(L(G))$ , and  $w_E(((uv), (vz))) = w_V(v)$  for every  $((uv), (vz)) \in E(L(G))$ .

In the line graph  $L(G)$ , the degree of a vertex  $(uv) \in V(L(G))$  is given by  $d((uv)) = (d(u) - 1) + (d(v) - 1) = d(u) + d(v) - 2$ , while its weighted degree is  $d_w((uv)) = (d(u) - 1)w_V(u) + (d(v) - 1)w_V(v)$ . In both an unweighted graph  $G$  and a weighted graph  $(G; w_V; w_E)$ , the minimum edge degree is  $\zeta(G) = \min\{d(u) + d(v) - 2 : (u, v) \in E(G)\}$ , whereas in a weighted graph  $(G; w_V; w_E)$ , the minimum weighted edge degree is  $\zeta_w(G) = \min\{d_w(u) + d_w(v) - 2w_E((u, v)) : (u, v) \in E(G)\}$ .

Given a connected unweighted graph  $G$  and an integer  $k \geq 1$ , the path graph of  $G$ , denoted by  $P_k(G)$ , is defined as the graph where each path  $P : u_0, u_1, \dots, u_k$  of length  $k$  in  $G$  is represented by a vertex  $(u_0u_1 \dots u_k)$ . Two vertices,  $(u_0u_1 \dots u_k)$  and  $(v_0v_1 \dots v_k)$ , are adjacent in  $P_k(G)$  if and only if the intersection of their corresponding paths in  $G$  is a path of length  $k - 1$ , and their union is either a path or a cycle of length  $k + 1$ .

The path graph concept was introduced by Broersma and Hoede [22] as an extension of line graphs. In fact, the line graph of  $G$  is precisely the  $P_1(G)$ -path graph. This concept can be naturally extended to weighted graphs.

**Definition 4.** Given a weighted graph  $(G; w_V; w_E)$ , the weighted path graph of  $(G; w_V; w_E)$ , denoted as  $(P_k(G); w_V; w_E)$ , is defined in the following way:

- (i) Each path  $P : u_0, u_1, \dots, u_k$  of length  $k$  in  $(G; w_V; w_E)$  is represented by a vertex  $(u_0u_1 \dots u_k)$  in  $(P_k(G); w_V; w_E)$ .
- (ii) The vertex weight function  $w_V : V(P_k(G)) \rightarrow \mathbb{R}_{>0}$  is given by

$$w_V((u_0u_1 \dots u_k)) = w_E((u_0u_1 \dots u_{k-1}), (u_1 \dots u_{k-1}u_k)).$$

- (iii) The edge weight function  $w_E : E(P_k(G)) \rightarrow \mathbb{R}_{>0}$  is given by:

$$w_E((xu_1 \dots u_k), (u_1 \dots u_ky)) = w_V(u_1 \dots u_k).$$

In other words, an edge in the weighted path graph  $(P_k(G); w_V; w_E)$  is induced by two paths of length  $k$  in  $(G; w_V; w_E)$  whose intersection is a path of length  $k - 1$  (represented by a vertex in  $P_{k-1}(G)$ ). Therefore, the weight of an edge in  $(P_k(G); w_V; w_E)$  equals the weight of the vertex in  $(P_{k-1}(G); w_V; w_E)$  that represents the common path of length  $k - 1$  in  $(G; w_V; w_E)$ . Conversely, a vertex in  $(P_k(G); w_V; w_E)$  is defined by a path of length  $k$  in  $(G; w_V; w_E)$ ; this path can be viewed as the union of two paths of length  $k - 1$  whose intersection is a path of length  $k - 2$ , which corresponds to an edge in  $(P_{k-1}(G); w_V; w_E)$ . Based on Definition 4, this logic can be iterated to determine the vertex and edge weights of  $(P_i(G); w_V; w_E)$  from  $(P_{i-1}(G); w_V; w_E)$ , eventually yielding the weights of  $(P_k(G); w_V; w_E)$  in terms of  $(G; w_V; w_E)$ . This is presented in the following remark; we omit the proof as it is straightforward.

**Remark 1.** Let  $k \geq 1$  be an integer and  $(G; w_V; w_E)$  be a weighted graph. The vertex and edge weights of the weighted graph  $(P_k(G); w_V; w_E)$  are given by

- (i)  $w_V((u_0u_1 \dots u_k)) = \begin{cases} w_E((u_{\frac{k-1}{2}}, u_{\frac{k+1}{2}})), & \text{if } k \text{ is odd,} \\ w_V(u_{\frac{k}{2}}), & \text{if } k \text{ is even.} \end{cases}$
- (ii)  $w_E((u_0u_1 \dots u_k), (u_1 \dots u_ku_{k+1})) = \begin{cases} w_V(u_{\frac{k+1}{2}}), & \text{if } k \text{ is odd,} \\ w_E((u_{\frac{k}{2}}, u_{\frac{k}{2}+1})), & \text{if } k \text{ is even.} \end{cases}$

### 3. General Bounds on Connectivity and Restricted Edge-Connectivity of Weighted Graphs

The most well-known classical result on unweighted graph connectivity is Whitney's Theorem, which relates the parameters of connectivity, edge-connectivity, and minimum degree of a graph.

**Theorem 1** (See [21]). *For any graph  $G$ ,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

When the weight of each vertex is less than or equal to the weight of every edge incident to that vertex, Theorem 1 can be extended to weighted graphs, as seen in the following result.

**Theorem 2.** *Let  $(G; w_V; w_E)$  be a weighted graph. Then  $\lambda_w(G) \leq \delta_w(G)$  and, furthermore, if  $\max\{w_V(u), w_V(v)\} \leq w_E((u, v))$  for every edge  $(u, v) \in E(G)$ , then  $\kappa_w(G) \leq \lambda_w(G)$ .*

**Proof.** Both inequalities clearly hold when the weighted graph  $(G; w_V; w_E)$  is disconnected, since in such a case  $\kappa_w(G) = \lambda_w(G) = 0$  and  $\delta_w(G) \geq 0$ . Therefore, let us assume that  $(G; w_V; w_E)$  is a connected weighted graph.

If we remove the set  $F$  of edges incident to any vertex  $v \in V(G)$ , the said vertex becomes disconnected from  $G - v$ . Thus,  $\lambda_w(G) \leq w_E(F) = d_w(v)$  for each  $v \in V(G)$ , which implies that  $\lambda_w(G) \leq \delta_w(G)$ .

Now let us prove the first inequality. Let  $F \subseteq E(G)$  be an edge cut of  $(G; w_V; w_E)$  with weight  $w_E(F) = \lambda_w(G)$ , and let  $C$  and  $\bar{C}$  be the components of  $G - F$ . First, assume that  $G$  is a complete graph  $K_n$ , with  $V(K_n) = \{u_1, \dots, u_n\}$ , such that  $d_w(u_1) = \delta_w(K_n)$ . If  $K_n - F$  isolates a vertex from the rest of the graph, we can assume that  $F$  is the set of edges incident to the vertex  $u_1$ , since  $F$  has minimum weight. In this case,  $S = \{u_2, \dots, u_n\}$  is a cut set of  $K_n$  whose weight satisfies

$$\kappa_w(K_n) \leq w_V(S) = \sum_{j=2}^n w_V(u_j) \leq \sum_{j=2}^n w_E((u_1, u_j)) = w_E(F) = \lambda_w(K_n).$$

If  $C$  and  $\bar{C}$  are both non-trivial components, then  $n \geq 4$  and we can assume without loss of generality that  $V(C) = \{u_1, \dots, u_k\}$  and  $V(\bar{C}) = \{u_{k+1}, \dots, u_n\}$ , for some  $2 \leq k \leq n - 2$ . In such a case, the same set  $S = \{u_2, \dots, u_n\}$  is a cut set of  $K_n$ , following that

$$\begin{aligned} \kappa_w(K_n) \leq w_V(S) &= \sum_{j=2}^n w_V(u_j) \\ &\leq \sum_{j=2}^k \max\{w_V(u_j), w_V(u_n)\} + \sum_{j=k+1}^n \max\{w_V(u_1), w_V(u_j)\} \\ &\leq \sum_{j=2}^k w_E((u_j, u_n)) + \sum_{j=k+1}^n w_E((u_1, u_j)) \\ &\leq w_E(V(C), V(\bar{C})) = w_E(F) = \lambda_w(K_n). \end{aligned}$$

Therefore,  $\kappa_w(K_n) \leq \lambda_w(K_n)$  holds.

We shall now prove it for any weighted graph  $(G; w_V; w_E)$  other than a complete graph. Let  $S$  and  $\bar{S}$  denote the sets of endpoints of the edges in  $F$  that lie in  $C$  and  $\bar{C}$ , respectively. Suppose first that there exists a vertex  $u \in V(C) \setminus S$ . Then,  $S$  is a cut set of  $G$  that separates vertex  $u$  from the component  $\bar{C}$ . Since no edge in  $F$  has both end-vertices in  $S$ , we deduce

that for each vertex  $v \in S$  there exists at least one edge  $(v, \bar{v}) \in F$ . Since by hypothesis  $w_V(v) \leq w_E((v, \bar{v}))$ , we have

$$\kappa_w(G) \leq w_V(S) = \sum_{v \in S} w_V(v) \leq \sum_{v \in S} w_E((v, \bar{v})) \leq w_E(F) = \lambda_w(G).$$

Similarly, if there exists a vertex  $\bar{u} \in V(\bar{C}) \setminus \bar{S}$ , then  $\bar{S}$  is a cut set of  $G$  satisfying  $\kappa_w(G) \leq w_V(\bar{S}) \leq w_E(F) = \lambda_w(G)$ . Finally, suppose that  $V(C) = S$  and  $V(\bar{C}) = \bar{S}$ . Then, for each vertex  $u \in V(C)$ ,  $(\{u\}, V(\bar{C})) \cup (N(u) \cap V(C), V(\bar{C})) \subseteq F$ .  $N(u)$  is a cut set of  $G$ ; consequently,  $\kappa_w(G) \leq w_V(N(u))$ , and we have

$$\begin{aligned} \kappa_w(G) \leq w_V(N(u)) &= \sum_{v \in N(u) \cap V(\bar{C})} w_V(v) + \sum_{v \in N(u) \cap V(C)} w_V(v) \\ &\leq w_E((\{u\}, V(\bar{C}))) + w_E((N(u) \cap V(C), V(\bar{C}))) \\ &\leq w_E(F) = \lambda_w(G). \end{aligned}$$

This completes the proof of the result.  $\square$

The restricted edge-connectivity parameter was introduced by Esfahanian and Hakimi [19] for unweighted graphs. As discussed in Section 2, restricted edge-connectivity is a more refined measure of network resilience, as it focuses on edge cuts that partition the network into non-trivial components. In [19], the authors proved that  $\lambda'(G)$  exists for any connected graph  $G$  with at least four vertices, provided that  $G$  is not a star graph, and established an upper bound for this parameter in terms of the minimum edge degree.

**Lemma 1** (See [19]). *Every connected graph  $G$  of order  $n \geq 4$  except a star, is  $\lambda'$ -connected and satisfies  $\lambda'(G) \leq \xi(G)$ .*

Lemma 1 can be extended to weighted graphs, where the minimum weighted edge degree is an upper bound for the restricted edge-connectivity.

**Lemma 2.** *Let  $(G; w_V; w_E)$  be a connected weighted graph of order  $n \geq 4$  other than a star graph  $K_{1,n-1}$ . Then  $\lambda'_w(G)$  exists. Furthermore,  $\lambda'_w(G) \leq \xi_w(G)$  provided that one of the following conditions holds:*

- $(G; w_V; w_E) \neq ((K_2, I_{n-2}); w_V; w_E)$ , i.e.,  $(G; w_V; w_E)$  is not a weighted split graph where the clique is  $K_2$ .
- $(G; w_V; w_E) = ((K_2, I_{n-2}); w_V; w_E)$  and  $\Delta_w(G) \leq (n-2)\delta_w(G)$ .

**Proof.** For each  $(u, v) \in E(G)$ , let  $E_{uv} = (\{u, v\}, V(G) \setminus \{u, v\}) \subset E(G)$ . It is clear that  $E_{uv}$  is an edge cut of  $(G; w_V; w_E)$  which separates the edge  $(u, v)$  from  $G - \{u, v\}$ .

First, assume there exists an edge  $(u_0, v_0) \in E(G)$  such that  $G - E_{u_0v_0}$  has at least one component  $C$ , other than the edge  $(u_0, v_0)$ , that is not an isolated vertex. Then  $E'_{u_0v_0} = (V(C), V(\bar{C})) \subseteq E_{u_0v_0}$  is a restricted edge cut, since  $\bar{C}$  is also a non-trivial component as it contains the edge  $(u_0, v_0)$ . Therefore,  $\lambda'_w(G)$  exists.

Now, suppose that for every edge  $(u, v) \in E(G)$ , all components of  $G - E_{uv}$  other than the edge  $(u, v)$  are isolated vertices. Consider an arbitrary edge  $(u, v) \in E(G)$ . If  $d(u) = 1$ , then  $G$  must be a star centered at vertex  $v$ . The same holds if  $d(v) = 1$ . Since  $G$  is not a star, it follows that  $d(u) \geq 2$  and  $d(v) \geq 2$ . In fact, since  $|V(G)| = n \geq 4$ , it must hold that  $d(u) \geq 3$  or  $d(v) \geq 3$ . Without loss of generality, assume  $d(v) \geq 3$ . Let  $z \in N(u) \setminus \{v\}$  and  $t \in N(v) \setminus \{u, z\}$ . Then, one of the components of  $G - E_{uz}$  other than the edge  $(u, z)$  contains the edge  $(v, t)$  and is therefore non-trivial. This contradicts the assumption that

for every edge  $(u, v) \in E(G)$ , all components of  $G - E_{uv}$  except  $(u, v)$  are isolated vertices. Thus,  $\lambda'_w(G)$  exists.

We now show that  $\lambda'_w(G) \leq \xi_w(G)$  if either condition (i) or (ii) of the lemma is satisfied.

First, suppose  $(G; w_V; w_E) \neq ((K_2, I_{n-2}); w_V; w_E)$ . Let  $(u, v) \in E(G)$  such that  $w(E_{uv}) = \xi_w(G)$ . Since  $(G; w_V; w_E) \neq ((K_2, I_{n-2}); w_V; w_E)$ , the graph  $G - E_{uv}$  must contain at least one non-trivial component  $C$ . Otherwise,  $G$  would consist of the complete graph  $K_2$ , formed by the edge  $(u, v)$ , and  $n - 2$  independent vertices. Since  $C$  is non-trivial and  $\bar{C}$  contains the edge  $(u, v)$ , it follows that  $\bar{C}$  is also non-trivial. Furthermore,  $E'_{uv} = (V(C), V(\bar{C}))$  is a subset of  $E_{uv}$  that separates  $(G; w_V; w_E)$  into two non-trivial components. Thus,  $E'_{uv} \subseteq E_{uv}$  is a restricted edge cut of  $G$ , and consequently,

$$\lambda'_w(G) \leq w_E(E'_{uv}) \leq w_E(E_{uv}) = \xi_w(G).$$

Second, suppose  $(G; w_V; w_E) = ((K_2, I_{n-2}); w_V; w_E)$  and  $\Delta_w(G) \leq (n - 2)\delta_w(G)$ . Let  $V(K_2) = \{x, y\}$  and  $V(I_{n-2}) = \{z_1, \dots, z_{n-2}\}$ . We prove the following claim.

**Claim 1.** If  $d(y) = 2$  and there exists  $j \in \{1, \dots, n - 2\}$  such that  $N(z_j) = \{x, y\}$ , then the unweighted split graph  $(K_2, I_{n-2})$  is isomorphic to  $(K_2^*, I_{n-2}^*)$ , where  $V(K_2^*) = \{x, z_j\}$  and  $V(I_{n-2}^*) = (V(I_{n-2}) \setminus \{z_j\}) \cup \{y\}$ .

**Proof of Claim 1.** By the definition of a split graph,  $N(z_j) \subseteq \{x, y\}$  for all  $j \in \{1, \dots, n - 2\}$ . Therefore, if  $d(y) = 2$ , there exists a unique  $j \in \{1, \dots, n - 2\}$  such that  $z_j \in N(y)$ . This implies that  $y$  is non-adjacent to every vertex in  $V(I_{n-2}) \setminus \{z_j\}$ . Consequently,  $V(I_{n-2}^*) = (V(I_{n-2}) \setminus \{z_j\}) \cup \{y\}$  is a set of independent vertices, which proves the claim.  $\square$

Let  $(u, v) \in E(G)$  be an edge such that  $w_E(E_{uv}) = \xi_w(G)$ . By hypothesis, we have

$$\Delta_w(G) \leq (n - 2)\delta_w(G). \quad (1)$$

We distinguish between two cases:

Case 1. Suppose  $(u, v) = (x, y)$ . In this case, note that

$$w_E(E_{uv}) = \xi_w(G) = w_E(G) - w_E((u, v)). \quad (2)$$

Since  $n \geq 4$  and  $(G; w_V; w_E)$  is not a star graph, there exist  $j, k \in \{1, \dots, n - 2\}$ , with  $j \neq k$ , such that  $z_j \in N(u)$  and  $z_k \in N(v)$ . Then  $E_{uz_j}$  is an edge cut of  $(G; w_V; w_E)$ , and the edge  $(v, z_k)$  is contained in some non-trivial component  $C$  of  $G - E_{uz_j}$ . Consequently,  $(u, z_j)$  is contained in  $\bar{C}$ , leading  $E'_{uz_j} = (V(C), V(\bar{C}))$  a restricted edge cut of  $(G; w_V; w_E)$ . From (1) and (2), it follows that

$$\begin{aligned} \lambda'_w(G) \leq w_E(E'_{uz_j}) &\leq w_E(G) - w_E((u, z_j)) - \sum_{\substack{i=1 \\ i \neq j}}^{n-2} w_E((v, z_i)) \\ &= \xi_w(G) + w_E((u, v)) - w_E((u, z_j)) - \sum_{\substack{i=1 \\ i \neq j}}^{n-2} w_E((v, z_i)) \\ &\leq \xi_w(G) + \Delta_w(G) - (n - 2)\delta_w(G) \\ &\leq \xi_w(G). \end{aligned}$$

This completes the proof of Case 1.

Case 2. Suppose  $(u, v) \neq (x, y)$ . Since  $(G; w_V; w_E)$  is a split graph with a clique  $K_2$  and  $V(K_2) = \{x, y\}$ , the edges  $(u, v)$  and  $(x, y)$  must be incident. We may assume  $u = x$  and  $v = z_j$  for some  $j \in \{1, \dots, n - 2\}$ , so that

$$w_E(E_{xz_j}) = \xi_w(G). \quad (3)$$

If  $N(y) \cap V(I_{n-2}) = \{z_j\}$ , then, by Claim 1, we can represent  $G$  as  $(K_2^*, I_{n-2}^*)$  with  $V(K_2^*) = \{x, z_j\}$ . In this representation,  $(x, z_j)$  is the edge of the clique  $K_2^*$  and  $w_E(E_{xz_j}) = \xi_w(G)$ . Thus, the edge  $(x, z_j)$  satisfies the hypothesis of Case 1, which implies  $\lambda'_w(G) \leq \xi_w(G)$ .

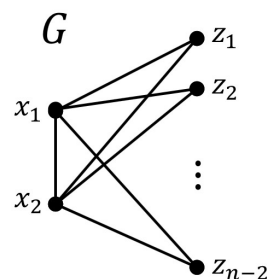
If there exists  $k \in \{1, \dots, n-2\}$ , with  $k \neq j$ , such that  $z_k \in N(y)$ , then some component  $C$  of  $G - E_{xz_j}$  contains the edge  $(y, z_k)$ . Thus,  $E'_{xz_j} = (V(C), V(\bar{C})) \subset E_{xz_j}$  is a restricted edge cut of  $(G; w_V; w_E)$ , as  $\bar{C}$  contains the edge  $(x, z_j)$ . Therefore, from (3), it follows that

$$\lambda'_w(G) \leq w_E(E'_{xz_j}) \leq w_E(E_{xz_j}) = \xi_w(G).$$

This concludes the proof of the lemma.  $\square$

Lemma 2 guarantees that, as in the unweighted case, the restricted edge-connectivity parameter exists for any weighted graph  $(G; w_V; w_E)$  other than a weighted star graph. The minimum weighted edge-degree  $\xi_w(G)$  is almost always an upper bound for  $\lambda'_w(G)$ ; however, this inequality fails for certain weighted split graphs with clique  $K_2$ . In item (ii) of this lemma, we provide a sufficient condition on the minimum and maximum weighted degrees that guarantees that the restricted edge-connectivity of a weighted split graph with clique  $K_2$  is upper-bounded by its minimum weighted edge-degree. Below, we show that this sufficient condition is tight (the best possible), but not necessary.

Consider the split graph  $G = (K_2, I_{n-2})$  with  $n$  vertices, as shown in Figure 1. Note that every restricted edge-cut  $F$  of  $G$  must contain the edge  $(x_1, x_2)$ . That is,  $x_1$  and  $x_2$  must belong to different components of  $G - F$ ; otherwise, the graph  $G - F$  would contain isolated vertices. In fact, for any edge  $(x_i, z_j)$  with  $i = 1, 2$  and  $j = 1, \dots, n-2$ , the set  $E_{x_i z_j}$  is a restricted edge-cut of  $G$ , regardless of whether it is considered as an unweighted or a weighted graph.



**Figure 1.** An example of a split graph  $((K_2, I_{n-2}); w_V; w_E)$  with two different weight functions,  $w_E$  and  $w_E^*$ , which shows that the hypothesis of Lemma 2 is a sufficient but not necessary condition to assure  $\lambda'_w \leq \xi'_w$ .

To show that the hypothesis of item (ii) in Lemma 2 cannot be relaxed, let us assign to  $G$  the edge-weight function  $w_E : E(G) \rightarrow \mathbb{R}_{>0}$  defined as  $w_E((x_1, x_2)) = (n-2)\alpha + \varepsilon$ , where  $\alpha, \varepsilon > 0$ , and  $w_E((x_i, z_j)) = \alpha$  for  $i = 1, 2$  and  $j = 1, \dots, n-2$ . The graph  $G$  together with the weight function  $w_E$  has maximum weighted degree  $\Delta_w(G) = 2(n-2)\alpha + \varepsilon$  and minimum weighted degree  $\delta_w(G) = 2\alpha$ , because  $n \geq 4$ . Thus,  $G$  does not satisfy the hypothesis of item (ii) in Lemma 2 by a value  $\varepsilon > 0$  that can be arbitrarily chosen small. Since all edges with one endvertex in  $\{x_1, x_2\}$  and the other in  $\{z_1, \dots, z_{n-2}\}$  have the same weight, we can assume without loss of generality that  $F = (V(C), V(\bar{C}))$ , where  $V(C) = \{x_1\} \cup \{z_1, \dots, z_k\}$  and  $V(\bar{C}) = \{x_2\} \cup \{z_{k+1}, \dots, z_{n-2}\}$  for some integer  $k \in \{1, \dots, n-3\}$ . Consequently,

$$\lambda'_w(G) = w_E(F) = w_E((x_1, x_2)) + \sum_{j=1}^k w_E((x_2, z_j)) + \sum_{j=k+1}^{n-2} w_E((x_1, z_j)) = 2(n-2)\alpha + \varepsilon$$

and yet  $\xi_w(G) = w_E(E_{x_1, x_2}) = 2(n-2)\alpha$ , leading to  $\lambda'_w(G) > \xi_w(G)$ .

To show that this sufficient condition is not necessary, let us now assign to  $G$  the edge-weight function  $w_E^* : E(G) \rightarrow \mathbb{R}_{>0}$  defined as  $w_E^*((x_1, x_2)) = (n-2)\alpha$ ,  $w_E^*((x_1, z_1)) = \alpha + \varepsilon$ , and  $w_E^*((x_i, z_j)) = \alpha$  for  $i = 1, 2$  and  $j = 1, \dots, n-2$  with  $(i, j) \neq (1, 1)$ . Under this function, the graph  $G$  has maximum weighted degree  $\Delta_w(G) = d_w(x_1) = 2(n-2)\alpha + \varepsilon$  and minimum weighted degree  $\delta_w(G) = 2\alpha$ . Therefore,  $G$  does not satisfy the hypothesis  $\Delta_w \leq (n-2)\delta_w$  of item (ii) by a value  $\varepsilon > 0$  that can be as large as desired. Furthermore, we find that  $\xi_w(G) = w_E^*(E_{x_1 z_1}) = 2(n-2)\alpha$ . Since  $E_{x_1 z_1}$  is indeed a restricted edge-cut of  $G$  together with  $w_E^*$ , we deduce that  $\lambda'_w(G) \leq w_E^*(E_{x_1 z_1}) = \xi_w(G)$ , even though  $\Delta_w(G) > (n-2)\delta_w(G)$ .

#### 4. Results on Connectivity and Edge-Connectivity of Weighted Line Graphs

In [23], Chartrand and Stewart established lower bounds for the connectivity and edge-connectivity of the line graph of a connected unweighted graph  $G$  in terms of its edge-connectivity. This result regarding connectivity was later improved by Helwig et al. in [24], who showed that the exact value of the connectivity of the line graph of a connected unweighted graph  $G$  equals the restricted edge-connectivity of  $G$ . In the following theorem, we address this problem for connected weighted graphs and prove a result analogous to that of [24].

**Theorem 3.** *Let  $(G; w_V; w_E)$  be a connected weighted graph of order  $n \geq 4$  different from a star  $(K_{1, n-1}; w_V; w_E)$ . Then*

$$\kappa_w(L(G)) = \lambda'_w(G).$$

**Proof.** Let  $S \subset V(L(G))$  be a minimum cut set of  $(L(G); w_V; w_E)$ . Since  $(G; w_V; w_E)$  is not a star and  $n \geq 4$ , we can ensure that  $L(G)$  is not a complete weighted graph; thus,  $L(G) - S$  is disconnected into two or more components. Let  $C_1$  and  $C_2$  be two components of  $L(G) - S$ , and denote by  $C_1^*$  and  $C_2^*$  the corresponding subgraphs of  $G$  such that  $L(C_i^*) = C_i$  for  $i = 1, 2$ . Note that each component of  $L(G) - S$  corresponds to a non-trivial subgraph in  $G$ , since each vertex of  $L(G)$  represents an edge in  $G$ . Therefore, both  $C_1^*$  and  $C_2^*$  contain at least one edge. If an edge  $(u, v) \in E(G)$  has one endvertex  $u \in V(C_1^*)$  and the other endvertex  $v \in V(C_2^*)$ , then the vertex  $(uv) \in V(L(G))$  must be adjacent to at least one vertex in  $V(C_1)$  and one vertex in  $V(C_2)$ . This implies that  $(uv) \in S$ , since  $S$  disconnects  $C_1$  from  $C_2$ . Consequently,  $F = \{(u, v) \in E(G) : (uv) \in S\}$  is an edge cut of  $G$ . Since the components of  $G - F$  have at least one edge,  $F$  is indeed a restricted edge cut of  $G$  and this yields  $\lambda'_w(G) \leq w_E(F) = w_V(S) = \kappa_w(L(G))$ .

Conversely, let  $F \subset E(G)$  be a minimum restricted edge cut of  $(G; w_V; w_E)$ , and denote by  $C_1^*$  and  $C_2^*$  the components of  $G - F$ . Let us consider the set  $S = \{(uv) \in V(L(G)) : (u, v) \in F\}$ . Since both  $C_1^*$  and  $C_2^*$  contain at least one edge,  $C_i = L(C_i^*)$  for  $i = 1, 2$ , and  $S$  are well-defined. Indeed,  $V(L(G)) = V(C_1) \cup S \cup V(C_2)$  due to the minimality of  $F$ . Since there is no edge joining  $C_1^*$  to  $C_2^*$  in  $G - F$ , there cannot be any edge joining  $C_1$  to  $C_2$  in  $L(G) - S$ . This implies that  $S$  is a cut set of  $L(G)$ . Thus,  $\kappa_w(L(G)) \leq w_V(S) = w_E(F) = \lambda'_w(G)$  and this concludes the proof.  $\square$

We will now focus on establishing lower and upper bounds for the weighted edge-connectivity parameter of a weighted connected graph. In the proofs of the theorems where these bounds are derived, we will utilize the concepts and terminology introduced in the following definition:

**Definition 5.** Let  $(G; w_V; w_E)$  be a weighted connected graph and let  $F = (C, \bar{C})$  be a minimum edge cut of  $(L(G); w_V; w_E)$ . A vertex  $v \in V(G)$  is called a mediator vertex of  $(G; w_V; w_E)$  with respect to  $F$  if there exist two neighbors  $u, w \in N_G(v)$  such that  $((uv), (vw)) \in F$ . Each edge  $((uv), (vw)) \in F$ , with  $u, w \in N_G(v)$ , is referred to as a  $v$ -edge. The subset of  $F$  consisting of  $v$ -edges is denoted by  $F_v$ , and the set of mediator vertices of  $(G; w_V; w_E)$  with respect to  $F$  is denoted by  $S'_F$ .

Definition 5 provides a way to express the weight of an edge cut. Indeed,

$$w_E(F) = \sum_{v \in S'_F} |F_v| \cdot w_V(v). \quad (4)$$

Regarding identity (4), our aim is to estimate the factor  $|F_v|$  for each mediator vertex  $v \in S'_F$ . To this end, we shall make use of the following remark:

**Remark 2.** Let  $F = (C, \bar{C})$  be a minimal edge cut of the line graph  $L(G)$  of a connected graph  $G$  (weighted or unweighted). If  $v \in V(G)$  is a mediator vertex of  $G$  with respect to  $F$ , then the number of edges in  $F$  for which  $v$  is the mediator satisfies the inequality:

$$|F_v| \geq d(v) - 1.$$

**Proof.** Let  $\mathcal{V}_v = \{(uv) : u \in N_G(v)\}$  denote the subset of vertices of  $L(G)$  corresponding to the edges incident to  $v$  in  $G$ ; clearly,  $|\mathcal{V}_v| = d(v)$ . Furthermore, it is evident that  $\mathcal{V}_v \cap V(C) \neq \emptyset$  and  $\mathcal{V}_v \cap V(\bar{C}) \neq \emptyset$ , since there exists at least one edge in  $F$  whose mediator vertex is  $v$ . Therefore, using the property  $a \cdot b \geq a + b - 1$  for all  $a, b \geq 1$ , we deduce:

$$|F_v| = |\mathcal{V}_v \cap V(C)| \cdot |\mathcal{V}_v \cap V(\bar{C})| \geq |\mathcal{V}_v \cap V(C)| + |\mathcal{V}_v \cap V(\bar{C})| - 1 = |\mathcal{V}_v| - 1 = d(v) - 1.$$

□

In the following two results, we establish both a lower bound and an upper bound for the weighted edge-connectivity of the line graph of a connected weighted graph  $(G; w_V; w_E)$ . These bounds depend on the weighted vertex connectivity of  $(G; w_V; w_E)$  and the minimum degree and weighted degree and the maximum degree of the original weighted graph.

**Theorem 4.** Let  $(G; w_V; w_E)$  be a connected weighted graph with minimum degree  $\delta(G) \geq 2$ . Then

$$\lambda_w(L(G)) \geq \min\{(\delta(G) - 1)\kappa_w(G), \delta_w(L(G))\}.$$

**Proof.** It is well known that the line graph of a connected graph is always connected. Let  $F \subset E(L(G))$  be a minimum weighted edge cut of  $(L(G); w_V; w_E)$ ; that is,  $w_E(F) = \lambda_w(L(G))$ . By the minimality of  $F$ ,  $L(G) - F$  has exactly two components,  $C$  and  $\bar{C}$ , so that  $F = (C, \bar{C})$ .

Let  $S'_F \subseteq V(G)$  be the set of mediator vertices of  $(G; w_V; w_E)$  with respect to  $F$ ; i.e.,  $v \in S'_F$  if and only if there exist  $u, w \in N_G(v)$  such that  $((uv), (vw)) \in F$ .

We distinguish two cases to obtain a lower bound for  $\lambda_w(L(G))$ :

Case 1. No two vertices in  $S'_F$  are adjacent in  $G$ . For any  $v_0 \in S'_F$ , there exist  $u, w \in N_G(v_0)$  such that  $(u, v_0) \in E(C)$  and  $(v_0, w) \in E(\bar{C})$ . Note that  $\{u, w\} \cap S'_F = \emptyset$ , because  $S'_F$  is a set of independent vertices. Let us see that  $S'_F$  is a cut set of  $(G; w_V; w_E)$ . We reason by contradiction supposing that  $G - S'_F$  is connected. Since  $u, w \notin S'_F$ , there exists a path  $P' : u = x_0, x_1, \dots, x_k = w$  of length  $k$  connecting  $u$  and  $w$  in  $G - S'_F$ . Note that

$x_i \neq v_0$  for all  $i = 0, \dots, k$ , because  $v_0 \in S'_F$ . From this path  $P'$ , we can construct a path  $P : (v_0u), (ux_1), \dots, (x_{k-1}w), (wv_0)$  in  $L(G) - F$  that connects  $(uv_0) \in V(C)$  with  $(v_0w) \in V(\overline{C})$ , which contradicts the fact that  $V(C)$  and  $V(\overline{C})$  are disconnected in  $L(G) - F$ . Thus,  $S'_F$  is a cut set of  $(G; w_V; w_E)$  and from (4) and Remark 2, it follows that

$$\begin{aligned} \lambda_w(L(G)) = w_E(F) &= \sum_{v \in S'_F} |F_v| w_V(v) \\ &\geq \sum_{v \in S'_F} (d(v) - 1) w_V(v) \\ &\geq (\delta(G) - 1) \sum_{v \in S'_F} w_V(v) \\ &\geq (\delta(G) - 1) \kappa_w(G). \end{aligned}$$

Case 2. There exist two adjacent vertices  $u_0, v_0 \in S'_F$ . This implies that  $(u_0v_0) \in V(L(G))$ . By Remark 2, we know that  $|F_{u_0}| \geq d(u_0) - 1$  and  $|F_{v_0}| \geq d(v_0) - 1$ . Therefore, from (4) it follows that

$$\begin{aligned} \lambda(L(G)) = w_E(F) &= \sum_{v \in S'_F} |F_v| w_V(v) \\ &\geq |F_{u_0}| w_V(u_0) + |F_{v_0}| w_V(v_0) \\ &\geq (d(u_0) - 1) w_V(u_0) + (d(v_0) - 1) w_V(v_0) \\ &= d_w((u_0v_0)) \\ &\geq \delta_w(L(G)). \end{aligned}$$

This completes the proof.  $\square$

It is easy to verify that the lower bound of Theorem 4 is tight. Let us consider the graph  $G$  from Figure 2. On the vertex set of  $G$ , we define two weight functions  $w_V, w_V^* : V(G) \rightarrow \mathbb{R}_{>0}$  given by the following:

- $w_V(v_1) = w_V(v_4) = 1, w_V(v_2) = w_V(v_5) = 2$ , and  $w_V(v_3) = w_V(v_6) = 3$ .
- $w_V^*(v_1) = w_V^*(v_2) = w_V^*(v_3) = 1, w_V^*(v_4) = 2$ , and  $w_V^*(v_5) = w_V^*(v_6) = 3$ .

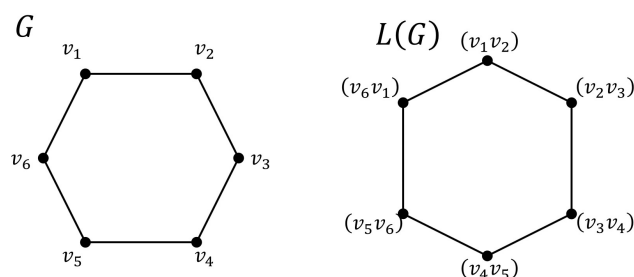


Figure 2. An example for which the upper bound of Theorem 4 can be reached for some weight function.

The graph  $G$  together with the weight function  $w_V$  has connectivity  $\kappa_w(G) = 2$ , since  $S = \{v_1, v_4\}$  is a cut set of minimum weight. Its corresponding weighted line graph  $(L(G); w_E)$  has edge-connectivity  $\lambda_w(L(G)) = 2 = (\delta(G) - 1) \kappa_w(G)$ , since  $F = \{((v_6v_1), (v_1v_2)), ((v_3v_4), (v_4v_5))\}$  is an edge cut of minimum weight.

On the other hand, the graph  $G$  together with the weight function  $w_V^*$  has connectivity  $\kappa_w(G) = 2$ , since  $S = \{v_1, v_3\}$  is a cut set of minimum weight. Its corresponding weighted line graph  $(L(G); w_E^*)$  has minimum weighted degree  $\delta_w(L(G)) = d_w((v_1v_2)) = 2$  and edge-connectivity  $\lambda_w(L(G)) = 2 = \delta_w(L(G))$ , as  $F = \{((v_6v_1), (v_1v_2)), ((v_1v_2), (v_2v_3))\}$  is an edge cut of minimum weight.

**Theorem 5.** Let  $(G; w_V; w_E)$  be a connected weighted graph with maximum degree  $\Delta(G) \geq 2$ . Then:

$$\lambda_w(L(G)) \leq \min \left\{ \frac{\Delta(G)^2}{4} \kappa_w(G), \delta_w(L(G)) \right\}.$$

**Proof.** The line graph of any connected graph is also connected. By applying Theorem 2, the inequality  $\lambda_w(L(G)) \leq \delta_w(L(G))$  holds. Let  $S \subset V(G)$  be a minimum cut set of  $(G; w_V; w_E)$ , that is, such that  $w_V(S) = \kappa_w(G)$ . Let  $C$  be a component of  $G - S$  and denote  $\bar{C} = (G - S) - V(C)$ .

Consider the subgraphs  $H$  and  $\bar{H}$  of  $L(G)$  with vertex sets  $V(H) = E(C) \cup (V(C), S)$  and  $V(\bar{H}) = E(\bar{C}) \cup (S, V(\bar{C})) \cup E(G[S])$ . Thus, the vertex set of  $L(G)$  is  $V(L(G)) = V(H) \cup V(\bar{H})$  and the edge set is  $E(L(G)) = E(H) \cup F \cup E(\bar{H})$ , where  $F = (V(H), V(\bar{H}))$  is an edge cut of  $L(G)$ .

Let us consider the set  $S'_F$  of mediator vertices of  $G$  with respect to  $F$  and take any vertex  $v \in S'_F$ . Then there exists an edge  $((uv), (vw)) \in F$ , with  $(uv) \in V(H)$  and  $(vw) \in V(\bar{H})$ . If  $(u, v) \in E(C)$  in  $G$ , then the edge  $(u, v)$  could only be incident with edges in  $E(C) \cup (V(C), S)$ ; consequently,  $N_{L(G)}((uv)) \subset V(H)$ . This contradicts the fact that the edge  $((uv), (vw))$  belongs to  $F$ . Therefore,  $(u, v) \notin E(C)$  in  $G$ , which implies that  $(u, v) \in (V(C), S)$ . Analogously, it follows that  $(v, w) \in (S, V(\bar{C})) \cup E(G[S])$ , because  $(v, w) \notin E(\bar{C})$ .

Since  $v$  is the common vertex of edges  $(u, v)$  and  $(v, w)$  in  $G$ , and given that  $V(C) \cap V(\bar{C}) = \emptyset$ , it must be that  $v \in S$ . Consequently,  $S'_F \subseteq S$ , and therefore,

$$\begin{aligned} \lambda_w(L(G)) \leq w_E(F) &= \sum_{v \in S'_F} |F_v| w_V(v) \\ &\leq \sum_{v \in S} |F_v| w_V(v) \\ &\leq \sum_{v \in S} |N_G(v) \cap V(C)| \cdot (d(v) - |N_G(v) \cap V(C)|) \cdot w_V(v) \\ &\leq \sum_{v \in S} \frac{d(v)}{2} \left( d(v) - \frac{d(v)}{2} \right) w_V(v) \\ &= \sum_{v \in S} \left( \frac{d(v)}{2} \right)^2 w_V(v) \\ &\leq \frac{\Delta(G)^2}{4} w_V(S) \\ &= \frac{\Delta(G)^2}{4} \kappa_w(G). \end{aligned}$$

This completes the proof.  $\square$

The upper bound of Theorem 5 is tight, as shown below. Figure 3 represents a regular graph  $G$  of degree 4 and its corresponding line graph  $L(G)$ . Let us consider any weight function  $w_V : V(G) \rightarrow \mathbb{R}_{>0}$  on the vertex set of  $G$  such that vertex 1 has the minimum weight under  $w_V$ . In this case,  $S = \{1\}$  is the unique cut set of minimum weight of  $(G; w_V)$ , implying that  $\kappa_w(G) = w_V(S) = w_V(1)$ .

The corresponding line graph,  $(L(G); w_E)$ , depicted in Figure 3, is a regular graph of degree 6 whose edge-connectivity is  $\lambda_w(L(G)) = 4w_V(1) = \frac{\Delta(G)^2}{4} \kappa_w(G)$ , since  $F = \{((12), (12')), ((12), (6'1)), ((61), (12')), ((61), (6'1))\}$  is an edge cut of minimum weight (the weight of each edge of  $F$  is  $w_V(1) = 1$ ).

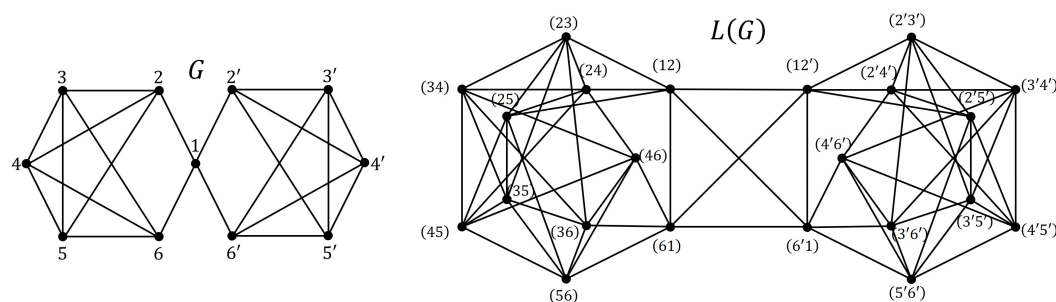


Figure 3. An example for which the upper bound of Theorem 5 is reached.

## 5. Results on Edge-Connectivity of Weighted $P_2$ -Path Graphs

In this section, we focus on the edge-connectivity of the weighted path graph  $(P_2(G); w_V; w_E)$  of a weighted graph  $(G; w_V; w_E)$ . For unweighted graphs, it was proven in [25] that a necessary and sufficient condition for the path graph  $P_2(G)$  of a connected graph  $G$  to be disconnected is that  $G$  contains at least two distinct paths of length 2, where the end-vertices of one of these paths have degree 1. From this condition, it follows that if  $G$  is a connected graph with minimum degree  $\delta(G) \geq 2$ , then  $P_2(G)$  is connected. The result established in [25] remains applicable to weighted graphs, as connectivity depends on the underlying structure rather than the specific weights assigned to vertices and edges.

Our goal in what follows is to provide bounds for the edge-connectivity of the weighted path graph  $(P_2(G); w_V; w_E)$ , assuming  $(G; w_V; w_E)$  is connected and  $\delta(G) \geq 2$ . To achieve this, we utilize the notion of a mediator edge, which is defined analogously to Definition 5.

**Definition 6.** Let  $(G; w_V; w_E)$  (resp.  $G$ ) be a connected weighted (resp. unweighted) graph such that  $(P_2(G); w_V; w_E)$  (resp.  $P_2(G)$ ) is also connected, and let  $F = (C, \bar{C})$  be a minimal edge cut of  $(P_2(G); w_V; w_E)$  (resp.  $P_2(G)$ ). An edge  $(u, v) \in E(G)$  is called a mediator edge of  $(G; w_V; w_E)$  (resp.  $G$ ) with respect to  $F$  if there exist  $x \in N_G(u) \setminus \{v\}$  and  $y \in N_G(v) \setminus \{u\}$  such that  $((xuv), (uvy)) \in F$ . Each such edge  $((xuv), (uvy)) \in F$  is referred to as a  $(u, v)$ -edge. We denote the subset of  $F$  consisting of  $(u, v)$ -edges by  $F_{(u,v)}$ , and the set of all mediator edges of  $(G; w_V; w_E)$  (resp.  $G$ ) with respect to  $F$  by  $F'$ .

Following the notation of Definition 6, let  $(G; w_V; w_E)$  be a connected weighted graph such that  $(P_2(G); w_V; w_E)$  is also connected. By applying Remark 1, the weight of an edge cut  $F$  of  $(P_2(G); w_V; w_E)$  is given by

$$w_E(F) = \sum_{(u,v) \in F'} w_E(F_{(u,v)}) = \sum_{(u,v) \in F'} |F_{(u,v)}| \cdot w_E((u, v)). \quad (5)$$

From (5), it follows that to determine the weight of an edge cut  $F = (C, \bar{C})$  of the graph  $(P_2(G); w_V; w_E)$  for a connected weighted graph  $(G; w_V; w_E)$ , it is essential to find the cardinality of  $F_{(u,v)}$  for each mediator edge  $(u, v) \in F'$ . Let

$$\mathcal{V}_{(u,v)}^u = \{(zuv) \in V(P_2(G)) : z \in N_G(u) \setminus \{v\}\}$$

and

$$\mathcal{V}_{(u,v)}^v = \{(uvt) \in V(P_2(G)) : t \in N_G(v) \setminus \{u\}\}.$$

Then,

$$F_{(u,v)} = (\mathcal{V}_{(u,v)}^u \cap V(C), \mathcal{V}_{(u,v)}^v \cap V(\bar{C})) \cup (\mathcal{V}_{(u,v)}^v \cap V(C), \mathcal{V}_{(u,v)}^u \cap V(\bar{C})) \quad (6)$$

and therefore,

$$|F_{(u,v)}| = |\mathcal{V}_{(u,v)}^u \cap V(C)| \cdot |\mathcal{V}_{(u,v)}^v \cap V(\bar{C})| + |\mathcal{V}_{(u,v)}^v \cap V(C)| \cdot |\mathcal{V}_{(u,v)}^u \cap V(\bar{C})|. \quad (7)$$

From (6) and (7), we derive a lower bound for the weight of the subset  $F_{(u,v)} \subseteq F$  for each mediator edge  $(u, v) \in F'$ .

**Lemma 3.** *Let  $(G; w_V; w_E)$  be a connected weighted graph with  $\delta(G) \geq 2$ , and let  $F = (C, \bar{C})$  be a minimal edge cut of  $P_2(G)$ . Then, for each mediator edge  $(u, v) \in E(G)$  with respect to  $F$ , the following assertions hold:*

- (i) *If either  $\mathcal{V}_{(u,v)}^u \cap V(C) = \emptyset$  or  $\mathcal{V}_{(u,v)}^u \cap V(\bar{C}) = \emptyset$ , then  $|F_{(u,v)}| \geq d(u) - 1$ .*
- (ii) *If either  $\mathcal{V}_{(u,v)}^v \cap V(C) = \emptyset$  or  $\mathcal{V}_{(u,v)}^v \cap V(\bar{C}) = \emptyset$ , then  $|F_{(u,v)}| \geq d(v) - 1$ .*
- (iii) *Otherwise,  $|F_{(u,v)}| \geq d(u) + d(v) - 4$ .*

**Proof.** Since  $F$  is a minimal edge cut of  $(P_2(G); w_V; w_E)$ ,  $P_2(G) - F$  has exactly two components,  $C$  and  $\bar{C}$ . The proof follows by applying the hypotheses to Equation (7).

If  $\mathcal{V}_{(u,v)}^u \cap V(C) = \emptyset$ , then  $\mathcal{V}_{(u,v)}^u \subseteq V(\bar{C})$  and  $\mathcal{V}_{(u,v)}^v \cap V(C) \neq \emptyset$ , this last because  $(u, v)$  is a mediator edge. Therefore, from (7) it follows that

$$|F_{(u,v)}| = |\mathcal{V}_{(u,v)}^v \cap V(C)| \cdot |\mathcal{V}_{(u,v)}^u \cap V(\bar{C})| = |\mathcal{V}_{(u,v)}^v \cap V(C)| \cdot (d(u) - 1) \geq d(u) - 1.$$

The same result holds if  $\mathcal{V}_{(u,v)}^u \cap V(\bar{C}) = \emptyset$ , which proves item (i). Item (ii) is proven analogously by swapping the roles of  $u$  and  $v$ .

Finally, if all factors in (7) are positive, using the inequality  $a \cdot b \geq a + b - 1$  for  $a, b \geq 1$ , we deduce the following:

$$\begin{aligned} |F_{(u,v)}| &= |\mathcal{V}_{(u,v)}^u \cap V(C)| \cdot |\mathcal{V}_{(u,v)}^v \cap V(\bar{C})| + |\mathcal{V}_{(u,v)}^v \cap V(C)| \cdot |\mathcal{V}_{(u,v)}^u \cap V(\bar{C})| \\ &\geq |\mathcal{V}_{(u,v)}^u \cap V(C)| + |\mathcal{V}_{(u,v)}^v \cap V(\bar{C})| - 1 + |\mathcal{V}_{(u,v)}^v \cap V(C)| + |\mathcal{V}_{(u,v)}^u \cap V(\bar{C})| - 1 \\ &= (d(u) - 1) + (d(v) - 1) - 2 = d(u) + d(v) - 4, \end{aligned}$$

which proves item (iii) and completes the proof.  $\square$

It follows immediately from Lemma 3 that if  $F$  is a minimal edge cut of the graph  $(P_2(G); w_V; w_E)$  for a connected weighted graph  $(G; w_V; w_E)$  with minimum degree  $\delta(G) \geq 2$ , then for each edge  $(u, v) \in F'$ , the cardinality of  $F_{(u,v)}$  can be lower bounded in terms of the degrees of its end-vertices. The proof is straightforward.

**Theorem 6.** *Let  $(G; w_V; w_E)$  be a connected weighted graph with  $\delta(G) \geq 2$ , and let  $F = (C, \bar{C})$  be a minimal edge cut of  $P_2(G)$ . Then,  $|F_{(u,v)}| \geq \min\{d(u), d(v)\} - 1$  for every  $(u, v) \in F'$ .*

By combining (5) and Theorem 6, we obtain the following corollary, the proof of which is straightforward:

**Corollary 1.** *Let  $(G; w_V; w_E)$  be a connected weighted graph with  $\delta(G) \geq 2$ , and let  $F = (C, \bar{C})$  be a minimum weight edge cut of  $P_2(G)$ . Then,*

$$\lambda_w(P_2(G)) \geq \sum_{(u,v) \in F'} (\min\{d(u), d(v)\} - 1) w_E((u, v)).$$

Given a connected graph  $G$  with minimum degree of at least two and a minimal edge cut  $F$  of  $P_2(G)$ , the following lemma, proven in [26], provides a sufficient condition for the

set  $F'$  of mediator edges of  $G$  with respect to  $F$  to be an edge cut of  $G$ . The result is also valid for weighted graphs.

**Lemma 4** (See [26]). *Let  $G$  be a connected graph with  $\delta(G) \geq 2$ . Let  $F = (C, \overline{C})$  be a minimal edge cut of  $P_2(G)$ , and consider the set  $F' \subset E(G)$  of mediator edges of  $G$  with respect to  $F$ .*

- (a) *If  $((yab), (abx)) \in F$  with  $(y, a) \notin F'$  and  $(b, x) \notin F'$ , then  $y$  is not connected with  $x$  in  $G - F'$ .*
- (b) *If there exist both  $(uvw) \in V(C)$  and  $(u'v'w') \in V(\overline{C})$  with  $(u, v) \notin F'$  or  $(v, w) \notin F'$ , and  $(u', v') \notin F'$  or  $(v', w') \notin F'$ , then  $F'$  is an edge cut of  $G$ .*

In the following lemma, we prove that if a minimum weight edge cut of  $(P_2(G); w_V; w_E)$  does not satisfy hypothesis (b) of Lemma 4, then the weighted edge-connectivity of the path graph  $(P_2(G); w_V; w_E)$  attains its maximum possible value, namely, the minimum weighted degree.

**Lemma 5.** *Let  $(G; w_V; w_E)$  be a connected weighted graph, with  $\delta(G) \geq 2$ . Let  $F$  be a minimum weight edge cut of  $(P_2(G); w_V; w_E)$ , and let  $F' \subset E(G)$  be the set of mediator edges of  $G$  with respect to  $F$ . If there exists a component  $C$  of  $P_2(G) - F$  such that for every vertex  $(uvw) \in V(C)$  it holds that both  $(u, v) \in F'$  and  $(v, w) \in F'$ , then*

$$\lambda_w(P_2(G)) = \delta_w(P_2(G)).$$

**Proof.** We know, by Theorem 2, that  $\lambda(P_2(G)) \leq \delta_w(P_2(G))$ . Therefore, it suffices to prove the reverse inequality.

Since  $F$  is a minimum weight edge cut,  $P_2(G) - F$  consists of two components,  $C$  and  $\overline{C}$ , where  $F = (C, \overline{C})$ . Suppose that the hypothesis of the lemma holds for component  $C$ ; that is, for every vertex in  $C$ , the edges of its corresponding path of length 2 in  $G$  are mediators. We shall prove the following claim:

**Claim 2.** There exists a vertex  $(u_0v_0w_0) \in V(C)$  such that  $|F_{(u_0v_0)}| \geq d(u_0) - 1$ .

**Proof of Claim 2.** Let us consider any vertex  $(uvw) \in V(C)$ . If  $|F_{(u,v)}| < d(u) - 1$ , as  $(u, v) \in F'$ , applying Lemma 3, we have  $d(v) < d(u)$  and either  $\mathcal{V}_{(u,v)}^v \cap V(C) = \emptyset$  or  $\mathcal{V}_{(u,v)}^v \cap V(\overline{C}) = \emptyset$ , and, moreover,  $|F_{(u,v)}| \geq d(v) - 1$ . In fact,  $\mathcal{V}_{(u,v)}^v \cap V(\overline{C}) = \emptyset$ , because  $(uvw) \in \mathcal{V}_{(u,v)}^v \cap V(C)$ , which implies that  $\mathcal{V}_{(u,v)}^v \subseteq V(C)$ . If  $\mathcal{V}_{(u,v)}^u \subseteq V(\overline{C})$ , then  $|F_{(u,v)}| = |(\mathcal{V}_{(u,v)}^v, \mathcal{V}_{(u,v)}^u)| = (d(u) - 1)(d(v) - 1) \geq d(u) - 1$ , since  $\delta(G) \geq 2$ . This contradicts hypothesis  $|F_{(u,v)}| < d(u) - 1$ . Therefore, there exists  $x \in N_G(u) \setminus \{v\}$  such that  $(xuv) \in \mathcal{V}_{(u,v)}^u \cap V(C)$ . Setting  $v = u_0$ ,  $u = v_0$ , and  $x = w_0$ , we have  $(u_0v_0w_0) \in V(C)$  and  $|F_{(u_0v_0)}| = |F_{(v,u)}| \geq d(v) - 1 = d(u_0) - 1$ . This completes the proof of the claim.  $\square$

Applying Claim 2, there exists a vertex  $(xv_0v_1) \in V(C)$  such that  $|F_{(x,v_0)}| \geq d(x) - 1$ . If, additionally,  $|F_{(v_0,v_1)}| \geq d(v_1) - 1$ , then

$$\begin{aligned} \lambda(P_2(G)) = w_E(F) &\geq |F_{(x,v_0)}|w_E((x, v_0)) + |F_{(v_0,v_1)}|w_E((v_0, v_1)) \\ &\geq (d(x) - 1)w_E((x, v_0)) + (d(v_1) - 1)w_E((v_0, v_1)) \\ &= d_w((xv_0v_1)) \\ &\geq \delta_w(P_2(G)). \end{aligned} \quad (8)$$

If  $|F_{(v_0,v_1)}| < d(v_1) - 1$ , as  $(v_0, v_1) \in F'$ , by Lemma 3, we have  $\mathcal{V}_{(v_0,v_1)}^{v_1} \cap V(C) \neq \emptyset$  and  $d(v_0) < d(v_1)$ , because  $|F_{(v_0,v_1)}| \geq \min\{d(v_0), d(v_1)\} - 1$ . Thus, there exists  $v_2 \in N_G(v_1) \setminus$

$\{v_0\}$  such that  $(v_0v_1v_2) \in V(C)$ . We already know  $|F_{(v_0,v_1)}| \geq \min\{d(v_0), d(v_1)\} - 1 = d(v_0) - 1$ . If, furthermore,  $|F_{(v_1,v_2)}| \geq d(v_2) - 1$ , then, reasoning as in (8), we obtain

$$\lambda(P_2(G)) = w_E(F) \geq d_w((v_0v_1v_2)) \geq \delta_w(P_2(G)).$$

Otherwise, if  $|F_{(v_1,v_2)}| < d(v_2) - 1$ , then applying Lemma 3 again, we have  $\mathcal{V}_{(v_1,v_2)}^{v_2} \cap V(C) \neq \emptyset$  and  $d(v_0) < d(v_1) < d(v_2)$ , since  $|F_{(v_1,v_2)}| \geq \min\{d(v_1), d(v_2)\} - 1$ . Therefore, there exists  $v_3 \in N_G(v_2) \setminus \{v_1\}$  such that  $(v_1v_2v_3) \in V(C)$  and, moreover,  $|F_{(v_1,v_2)}| \geq d(v_1) - 1$ . If  $|F_{(v_2,v_3)}| \geq d(v_3) - 1$ , then by the same reasoning as in (8), we conclude that  $\lambda(P_2(G)) \geq \delta_w(P_2(G))$ . If not, we repeat the argument.

Since  $(G; w_V; w_E)$  is finite and thus its maximum degree is also finite, there must exist an integer  $k \geq 2$  such that  $d(v_k) \leq d(v_{k-1})$  and  $(v_{k-2}v_{k-1}v_k) \in V(C)$ , where  $|F_{(v_{k-2},v_{k-1})}| \geq d(v_{k-2}) - 1$ . Furthermore, we have  $|F_{(v_{k-1},v_k)}| \geq \min\{d(v_{k-1}), d(v_k)\} - 1 = d(v_k) - 1$ , which leads to

$$\begin{aligned} \lambda(P_2(G)) = w_E(F) &\geq |F_{(v_{k-2},v_{k-1})}|w_E((v_{k-2}, v_{k-1})) + |F_{(v_{k-1},v_k)}|w_E((v_{k-1}, v_k)) \\ &\geq (d(v_{k-2}) - 1)w_E((v_{k-2}, v_{k-1})) + (d(v_k) - 1)w_E((v_{k-1}, v_k)) \\ &= d_w((v_{k-2}v_{k-1}v_k)) \\ &\geq \delta_w(P_2(G)). \end{aligned}$$

This completes the proof of the result.  $\square$

The following theorem provides upper and lower bounds for the edge-connectivity of the weighted path graph  $(P_2(G); w_V; w_E)$  of a connected weighted graph  $(G; w_V; w_E)$  with minimum degree at least 2.

**Theorem 7.** Let  $(G; w_V; w_E)$  be a connected weighted graph with minimum degree  $\delta(G) \geq 2$ . Then

$$\min\{\lambda_w(G)(\delta(G) - 1), \delta_w(P_2(G))\} \leq \lambda_w(P_2(G)) \leq \delta_w(P_2(G)).$$

**Proof.** We know that  $P_2(G)$  is connected because  $\delta(G) > 1$ . Furthermore, inequality  $\lambda_w(P_2(G)) \leq \delta_w(P_2(G))$  clearly holds by applying Theorem 2. Let therefore prove the other inequality.

Let  $F = (C, \bar{C})$  be a minimum weight edge cut of  $(P_2(G); w_V; w_E)$ . If  $C$  or  $\bar{C}$  is an isolated vertex, then  $F$  is the set of edges incident to a vertex of minimum weighted degree, hence,  $\lambda_w(P_2(G)) = w_E(F) = \delta_w(P_2(G))$ . From now on, let us assume that both  $C$  and  $\bar{C}$  have at least two vertices.

First, suppose there exist  $(uvw) \in V(C)$  and  $(u'v'w') \in V(\bar{C})$  such that  $(u, v) \notin F'$  or  $(v, w) \notin F'$ , and  $(u', v') \notin F'$  or  $(v', w') \notin F'$ . Then, by Lemma 4,  $F'$  is an edge cut of  $G$ . Thus, applying Theorem 6 in (5), we have

$$\begin{aligned} \lambda_w(P_2(G)) = w_E(F) &= \sum_{(u,v) \in F'} |F_{(u,v)}| \cdot w_E((u, v)) \\ &\geq \sum_{(u,v) \in F'} (\min\{d(u), d(v)\} - 1) \cdot w_E((u, v)) \\ &\geq (\delta(G) - 1) \sum_{(u,v) \in F'} w_E((u, v)) \\ &\geq (\delta(G) - 1) \lambda_w(G). \end{aligned}$$

Second, suppose that in one of the components—for instance, in  $C$ —it holds that  $(u, v) \in F'$  and  $(v, w) \in F'$  for every vertex  $(uvw) \in V(C)$ . Then, by applying Lemma 5,  $\lambda_w(P_2(G)) = \delta_w(P_2(G))$ , which completes the proof.  $\square$

Note that, for any weighted cycle  $(C_n; w_E)$ , the weighted  $P_2$ -path graph  $(P_2(C_n); w_E)$  is isomorphic to  $(C_n; w_E)$ . If  $\lambda_w(C_n) < \delta_w(C_n)$ , then  $\lambda_w(P_2(C_n)) = \lambda_w(C_n) = (\delta(C_n) - 1)\lambda_w(C_n)$ . Conversely, if  $\lambda_w(C_n) = \delta_w(C_n)$ , then  $\lambda_w(P_2(C_n)) = \delta_w(P_2(C_n))$ . Therefore, the bounds of Theorem 7 are tight.

## 6. Discussion and Future Works

Vertex- and edge-weighted graphs provide a better model for network behavior, as nodes and their interconnections are often non-uniform. In this paper, we have extended the concepts of connectivity and restricted edge-connectivity to weighted graphs.

Regarding the classical parameters of connectivity and edge-connectivity, we have proven Whitney's general bounds [21]. We have also provided a sufficient condition for the existence of restricted edge-connectivity in a weighted graph and bounded this parameter by the minimum weighted edge degree of the graph. In the last two sections of the article, we studied the connectivity and edge-connectivity of the weighted line graph and the edge-connectivity of the weighted  $P_2$ -path graph, establishing upper and lower bounds for these parameters.

An interesting future work is to generalize the study of edge-connectivity for the  $P_k$ -path graph of a weighted graph  $(G; w_V; w_E)$  with girth at least  $k + 1$ . In [27], it was shown that for every integer  $k \geq 3$ ,  $\lambda(P_k(G)) \geq 2\delta(G) - 2$  holds for every connected graph  $G$  with minimum degree  $\delta(G) \geq 3$  and girth  $g(G) \geq k + 1$ . This result can likely be extended to graphs using a result similar to Theorem 7.

Regarding the results proven in Section 3, there are two promising lines of future research. One would consist of finding sufficient conditions for a weighted graph  $(G; w_V; w_E)$  to be maximally connected (resp. maximally edge-connected), i.e.,  $\kappa_w(G) = \delta_w(G)$  (resp.  $\lambda_w(G) = \delta_w(G)$ ). The other is to find sufficient conditions for a weighted graph  $(G; w_V; w_E)$  to be  $\lambda'_w$ -optimal—that is,  $\lambda'_w(G) = \xi_w(G)$ .

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