

Article

On 4-Dimensional Lorentzian Manifolds with Pseudo W_2 -Curvature Tensor in General Relativity

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Abstract

The present paper is devoted to the study of Lorentzian spacetimes endowed with the pseudo W_2 -curvature tensor. We investigate the geometric and symmetry properties of pseudo W_2 -flat spacetimes and establish several results concerning their curvature structure and associated energy–momentum tensors. It is shown that a pseudo W_2 -flat spacetime is a spacetime of constant curvature under suitable conditions on the defining parameters of the pseudo W_2 -curvature tensor. Further, for a spacetime satisfying Einstein’s field equation with vanishing pseudo W_2 -curvature tensor, it is proved that a vector field generates a matter collineation if and only if it is a Killing vector field. We also establish that the Lie inheritance property of the energy–momentum tensor is equivalent to the existence of a conformal Killing vector field. Moreover, it is shown that if the energy–momentum tensor is of Codazzi type, then the pseudo W_2 -curvature tensor is divergence-free. To illustrate the obtained results, two explicit Lorentzian spacetime models are examined in detail, namely an isotropic warped spacetime and an FLRW-type exponential (de Sitter) spacetime. For each model, the metric structure, curvature tensors, Ricci operator, scalar curvature, and pseudo W_2 -flatness condition are verified explicitly. These examples provide concrete realizations of the theoretical results and demonstrate the significance of the pseudo W_2 -curvature tensor in the geometric study of Lorentzian manifolds and general relativity.

Keywords: Einstein’s field equation; pseudo W_2 -curvature tensor; Lorentzian manifold; sectional curvature



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1. Introduction

This paper is concerned with the investigation of the pseudo W_2 -curvature tensor and its geometric consequences in the context of general relativity. The study is carried out on a four-dimensional semi-Riemannian manifold (M^4, g) , where g is a Lorentzian metric of signature $(-, +, +, +)$. Such a metric structure distinguishes the temporal dimension from the spatial dimensions and serves as the standard mathematical model of spacetime in Einstein’s theory of gravitation. The Lorentzian character of the manifold introduces a

crucial causal structure, wherein tangent vectors are classified as timelike, spacelike or null. This classification scheme is not merely a mathematical formalism, but rather embodies the physical reality of causal relationships in spacetime, governing both the dynamics of matter and the propagation of physical signals. The causal structure inherent to Lorentzian geometry makes these manifolds particularly well-adapted for modeling relativistic phenomena, as it directly encodes the light-cone structure that is fundamental to Einstein's theory. Within this geometric framework, we examine the pseudo W_2 -curvature tensor, analyzing both its mathematical properties and its physical interpretations. The Lorentzian signature plays a pivotal role in our analysis, as it determines the causal nature of geometric objects and ensures their physical relevance within the theory of general relativity.

The conservation of energy–momentum in Einstein's field equations [1] is characterized by the divergence-free condition $\nabla_p T_j^p = 0$. A stronger requirement is that the energy–momentum tensor be covariantly constant, i.e., $\nabla T = 0$. Chaki and Roy [2] proved that such a condition implies Ricci symmetry ($\nabla S = 0$) in a Lorentzian spacetime, thereby establishing a connection between the invariance of matter fields and the geometric symmetry of spacetime.

Recent years have witnessed substantial progress in the study of curvature tensors and their applications in differential geometry, general relativity, and modified theories of gravity. Among these developments, the investigation of generalized curvature tensors has provided valuable insights into the geometric and physical properties of spacetime manifolds. In particular, the pioneering work of [3] established several fundamental results concerning the role of the W_2 -curvature tensor in both classical relativistic spacetimes and $f(G)$ -gravity models, thereby highlighting its significance in the study of gravitational phenomena.

Motivated by these developments, Mallick and De [4] carried out a detailed investigation of spacetime manifolds admitting the W_2 -curvature tensor and obtained important geometric characterizations. Further contributions were made by Suh et al. [5], who studied pseudo-quasi-conformal curvature tensors and explored their implications in the context of general relativity. Their work revealed interesting connections between generalized curvature restrictions and the geometric structure of Lorentzian manifolds.

The growing interest in curvature-restricted geometric structures has also led to several notable investigations in related directions. For instance, ref. [6] examined the concircular curvature tensor in the framework of $f(R)$ -gravity and derived significant results concerning its geometric and physical implications. Likewise, ref. [7] studied the W^* -curvature tensor in relativistic spacetimes and obtained important characterizations associated with Einstein's field equations. Furthermore, ref. [8] investigated M -projectively flat spacetimes, while [9] analyzed spacetimes admitting Z -symmetric tensor fields, thereby enriching the theory of generalized curvature tensors and their applications. In addition, numerous researchers, including [10–14], have made significant contributions to this rapidly developing area by studying various curvature tensors, symmetry conditions, and geometric structures arising in semi-Riemannian and Lorentzian manifolds. Their collective efforts have considerably enhanced our understanding of the interplay between curvature, spacetime symmetries, and gravitational field equations.

Inspired by these noteworthy contributions and motivated by the continuing development of curvature-restricted geometry, the present work is devoted to the study of Lorentzian spacetimes admitting the pseudo W_2 -curvature tensor within the framework of general relativity.

The concept of pseudo W_2 -curvature tensor $\bar{W}_2$ in an n -dimensional Riemannian manifold (M^n, g) was introduced by Prasad and Maurya [15] and is defined by

$$\begin{aligned}\bar{W}_2(X_u, X_v)X_z = & \psi R(X_u, X_v)X_z + \phi[g(X_v, X_z)QX_u - g(X_u, X_z)QX_v] \\ & - \frac{\theta}{n} \left(\frac{\psi}{n-1} + \phi \right) [g(X_v, X_z)X_u - g(X_u, X_z)X_v],\end{aligned}\quad (1)$$

for all $X_u, X_v, X_z \in (M^n, g)$, where ψ and ϕ are non-zero constants, θ is the scalar curvature and Q is the Ricci operator defined by $g(QX_u, X_v) = S(X_u, X_v)$. If $\psi = 1$ and $\phi = -\frac{1}{n-1}$, then (1) becomes W_2 -curvature tensor defined by [16]. Moreover, the theory of the pseudo W_2 -curvature tensor has been further developed by several researchers. In particular, Prasad [17] and Kumar [18] extended the study of pseudo W_2 -curvature tensors by investigating their geometric properties under various curvature restrictions and symmetry conditions. Their works contributed significantly to a deeper understanding of the relationship between pseudo W_2 -curvature structures, Einstein-type manifolds, and other classes of semi-Riemannian manifolds. These studies not only enriched the theory of generalized curvature tensors but also demonstrated the importance of pseudo W_2 -geometry in the broader context of differential geometry and mathematical physics.

The purpose of the present paper is to investigate Lorentzian spacetimes satisfying geometric conditions involving the pseudo W_2 -curvature tensor. In particular, we study pseudo W_2 -flat spacetimes and examine their relationship with Einstein's field equations, spacetime symmetries, and the energy-momentum tensor. We establish several characterizations of pseudo W_2 -flat spacetimes and derive conditions under which such spacetimes possess constant curvature. Furthermore, we investigate the interplay between pseudo W_2 -flatness and various symmetry properties, including matter collineations, Killing vector fields, conformal Killing vector fields, and Lie inheritance symmetries.

Another objective of this work is to analyze the behaviour of the pseudo W_2 -curvature tensor under additional restrictions on the energy-momentum tensor. In particular, we show that when the energy-momentum tensor is of Codazzi type, the pseudo W_2 -curvature tensor becomes divergence-free. These results provide new insights into the geometric structure of Lorentzian manifolds admitting special curvature conditions. To illustrate the theoretical developments, we construct and analyze explicit Lorentzian spacetime models. Two geometrically distinct examples are examined in detail: an isotropic warped spacetime and an FLRW-type exponential (de Sitter) spacetime. For each example, the metric tensor, Christoffel symbols, Riemann curvature tensor, Ricci tensor, Ricci operator, and scalar curvature are computed explicitly, and the pseudo W_2 -flat condition is verified directly. These examples serve as concrete realizations of the theoretical results established in the paper.

The paper is organized as follows. By following the introductory framework that we established, Section 2 investigates pseudo W_2 -flat spacetimes and derives their fundamental geometric properties, Einstein's field equations and the associated symmetry conditions. Section 3 examines the divergence properties of the pseudo W_2 -curvature tensor under Codazzi-type assumptions. The analysis then advances in Section 4 to consider Ricci-recurrent spacetimes satisfying the condition $\text{div } \bar{W}_2 = 0$. Section 5 demonstrates the physical relevance of our theoretical framework, and we present an explicit example of a pseudo W_2 -flat spacetime that substantiates our principal results.

2. Spacetime with Vanishing Pseudo W_2 -Curvature Tensor

Let the spacetime of general relativity be V_4 , then the $(0, 4)$ -type tensor of (1) gives

$$\begin{aligned} {}'\bar{W}_2(X_u, X_v, X_z, X_w) &= \psi \bar{R}(X_u, X_v, X_z, X_w) \\ &+ \phi [g(X_v, X_z)S(X_u, X_w) - g(X_u, X_z)S(X_v, X_w)] \\ &- \frac{\theta}{4} \left(\frac{\psi}{3} + \phi \right) [g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w)], \end{aligned} \quad (2)$$

where

$$\begin{aligned} {}'\bar{W}_2(X_u, X_v, X_z, X_w) &= g(\bar{W}_2(X_u, X_v)X_z, X_w), \\ \bar{R}(X_u, X_v, X_z, X_w) &= g(R(X_u, X_v)X_z, X_w), \end{aligned}$$

and S is the Ricci tensor.

If the spacetime is pseudo W_2 -flat, i.e., ${}'\bar{W}_2(X_u, X_v, X_z, X_w) = 0$, then (2) reduces to

$$\begin{aligned} \psi \bar{R}(X_u, X_v, X_z, X_w) &= \phi [g(X_u, X_z)S(X_v, X_w) - g(X_v, X_z)S(X_u, X_w)] \\ &+ \frac{\theta}{4} \left(\frac{\psi}{3} + \phi \right) [g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w)]. \end{aligned} \quad (3)$$

To provide a more rigorous derivation of Theorem 1, we consider the contraction of the fundamental equation of pseudo W_2 -flat spacetimes. Multiplying (3) by the inverse metric tensor g^{uw} and performing the summation over the indices u and w , the left-hand side of the equation yields the Ricci tensor:

$$g^{uw} \psi \bar{R}(X_u, X_v, X_z, X_w) = \psi S(X_v, X_z).$$

On the right-hand side, the terms involving the Ricci tensor and the metric tensor are contracted as follows:

$$g^{uw} \phi [g(X_u, X_z)S(X_v, X_w) - g(X_v, X_z)S(X_u, X_w)] = \phi [S(X_v, X_z) - g(X_v, X_z)\theta].$$

Now setting $X_u = X_w = e_i$ in (3) and summing over i , $1 \leq i \leq 4$, we have

$$\begin{aligned} \psi \sum_{i=1}^4 \bar{R}(e_i, X_v, X_z, e_i) &= \phi \sum_{i=1}^4 [g(e_i, X_z)S(X_v, e_i) - g(X_v, X_z)S(e_i, e_i)] \\ &+ \frac{\theta}{4} \left(\frac{\psi}{3} + \phi \right) \sum_{i=1}^4 [g(X_v, X_z)g(e_i, e_i) - g(e_i, X_z)g(X_v, e_i)], \end{aligned} \quad (4)$$

where

$$\sum_{i=1}^4 \bar{R}(e_i, X_v, X_z, e_i) = S(X_v, X_z), \quad \sum_{i=1}^4 g(e_i, X_z)S(X_v, e_i) = S(X_v, X_z), \quad (5)$$

$$\sum_{i=1}^4 g(e_i, X_z)S(X_v, e_i) = S(X_v, X_z), \quad \sum_{i=1}^4 S(e_i, e_i) = \theta, \quad (6)$$

$$\sum_{i=1}^4 g(e_i, e_i) = 4, \quad \text{and} \quad \sum_{i=1}^4 g(e_i, X_z)g(X_v, e_i) = g(X_v, X_z). \quad (7)$$

Combining (4)–(7), we obtain

$$(\psi - \phi)S(X_v, X_z) = (\psi - \phi)\frac{\theta}{4}g(X_v, X_z), \quad (8)$$

where $\psi - \phi \neq 0$, where S and θ are the Ricci tensor and the scalar curvature of the manifold, respectively. This mathematical transition highlights that (8) is an Einstein manifold. Thus we have the following theorem:

Theorem 1. A pseudo W_2 -flat spacetime is an Einstein manifold, provided $\psi - \phi \neq 0$.

Next, we prove the following theorem:

Theorem 2. A pseudo W_2 -flat spacetime is a spacetime of constant curvature, provided $\psi - \phi \neq 0$.

Proof. Consider the spacetime (M^4, g) is pseudo W_2 -flat, we have

$$\bar{W}_2(X_u, X_v)X_z = 0, \quad (9)$$

where $\bar{W}_2$ is of $(1,3)$ -type tensor. From (3) and (9), it is obvious that

$$\begin{aligned} \psi R(X_u, X_v)X_z &= \phi \left[g(X_u, X_z)QX_v - g(X_v, X_z)QX_u \right] \\ &+ \frac{\theta}{4} \left(\frac{\psi}{3} + \phi \right) \left[g(X_v, X_z)X_u - g(X_u, X_z)X_v \right]. \end{aligned} \quad (10)$$

Now assuming $\psi \neq 0$, we obtain

$$\begin{aligned} R(X_u, X_v)X_z &= \frac{\phi}{\psi} \left[g(X_u, X_z)QX_v - g(X_v, X_z)QX_u \right] \\ &+ \frac{\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) \left[g(X_v, X_z)X_u - g(X_u, X_z)X_v \right]. \end{aligned} \quad (11)$$

Taking the inner product with X_w in (11), we get

$$\begin{aligned} \bar{R}(X_u, X_v, X_z, X_w) &= \frac{\phi}{\psi} \left[g(X_u, X_z)S(X_v, X_w) - g(X_v, X_z)S(X_u, X_w) \right] \\ &+ \frac{\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) \left[g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w) \right]. \end{aligned} \quad (12)$$

Again contracting (12) with respect to X_u and X_w , we obtain

$$S(X_v, X_z) = \frac{\phi}{\psi} \left[S(X_v, X_z) - \theta g(X_v, X_z) \right] + \frac{3\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) g(X_v, X_z). \quad (13)$$

Multiplying (13) by ψ and simplifying, we get

$$(\psi - \phi) \left[S(X_v, X_z) - \frac{\theta}{4} g(X_v, X_z) \right] = 0, \quad (14)$$

since $\psi - \phi \neq 0$, it follows that

$$S(X_v, X_z) = \frac{\theta}{4} g(X_v, X_z).$$

Therefore, the spacetime is an Einstein, and

$$QX_u = \frac{\theta}{4} X_u. \quad (15)$$

Substituting (15) into (11), we obtain

$$R(X_u, X_v)X_z = \frac{\phi\theta}{4\psi} [g(X_u, X_z)X_v - g(X_v, X_z)X_u] + \frac{\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) [g(X_v, X_z)X_u - g(X_u, X_z)X_v]. \quad (16)$$

After cancellation of the ϕ -terms, (16) reduces to

$$R(X_u, X_v)X_z = \frac{\theta}{12} [g(X_v, X_z)X_u - g(X_u, X_z)X_v].$$

Consequently,

$$\bar{R}(X_u, X_v, X_z, X_w) = \frac{\theta}{12} [g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w)]. \quad (17)$$

Thus the curvature tensor has the form

$$\bar{R}(X_u, X_v, X_z, X_w) = K[g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w)],$$

where

$$K = \frac{\theta}{12}.$$

This is precisely the curvature tensor of a manifold of constant sectional curvature K . Hence (17) shows that the spacetime is of constant curvature. Therefore, a pseudo W_2 -flat spacetime is a spacetime of constant curvature whenever $\psi - \phi \neq 0$. $\square$

Remark 1. Depending on the sign of the curvature, these spaces can be spherical (positive curvature), flat (zero curvature), or hyperbolic (negative curvature), each corresponding to different possible geometries of the universe. When applying Einstein's field equations to such symmetric conditions, the resulting spacetime geometry is described by the Robertson–Walker (RW) metric, which models a dynamic universe whose spatial curvature may evolve over time. This metric is central to the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological models, which describe expanding or contracting universes based on different energy contents. Moreover, when the condition of constant curvature is extended to the entire 4-dimensional spacetime, the solution leads to the de Sitter universe, an important model in cosmology that represents an exponentially expanding universe, often associated with the presence of a cosmological constant or dark energy [19,20].

Einstein's field equation with cosmological constant can be written as

$$S(X_u, X_v) - \frac{\theta}{2}g(X_u, X_v) + \lambda g(X_u, X_v) = kT(X_u, X_v), \quad (18)$$

where λ , k and T denote the cosmological constant, the Einstein gravitational constant and the energy–momentum tensor, respectively. With the help of (8) and (18), we obtain

$$T(X_u, X_v) = \frac{1}{k} \left[\lambda - \frac{\theta}{4} \right] g(X_u, X_v). \quad (19)$$

Taking the covariant derivative of Equation (19) with respect to X_z , we obtain

$$(\nabla_{X_z} T)(X_u, X_v) = \frac{1}{4k} d\theta(X_z)g(X_u, X_v), \quad (20)$$

where $d\theta(X_z) = \nabla_{X_z} \theta$. Since a pseudo W_2 -flat spacetime is Einstein. Therefore, the scalar curvature θ is constant. Hence

$$d\theta(X_z) = 0, \quad (21)$$

for all X_z .

From (20) and (21), we get

$$(\nabla_{X_z} T)(X_u, X_v) = 0. \quad (22)$$

Thus we can state the following:

Theorem 3. *In pseudo W_2 -flat spacetime the energy–momentum tensor T , which fulfills Einstein’s field equation with cosmological constant λ , is observed to be covariantly constant.*

In a pioneering work, Katzin et al. [21] carried out a detailed analysis of curvature collineations (CCs) in relation to the particle and field conservation laws arising in standard general relativity. If a spacetime admits a Killing vector field that also generates a curvature collineation, then the following result follows.

Theorem 4. *If a spacetime admits a pseudo W_2 -curvature tensor with respect to a Killing vector field ξ admitting curvature collineation, then the Lie derivative of pseudo W_2 -curvature tensor vanishes along ξ .*

Proof. The geometrical symmetries of a spacetime can be written as

$$\mathcal{L}_\xi A - 2\Omega A = 0, \quad (23)$$

where A is the physical or geometrical quantity, Ω is a smooth scalar and $\mathcal{L}_\xi$ represents the Lie derivative with respect to ξ . For the metric inheritance symmetry, we put $A = g_{ij}$ in Equation (23). Therefore, we get

$$(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega g(X_u, X_v). \quad (24)$$

Clearly, in this case if $\Omega = 0$, then ξ becomes a Killing vector field. If Ω is a smooth scalar, then ξ becomes a conformal Killing vector field.

A spacetime is said to admit a symmetry called a curvature collineation, provided there exists a vector field ξ such that [22,23]

$$(\mathcal{L}_\xi R)(X_u, X_v)X_z = 0, \quad (25)$$

where R is the Riemannian curvature tensor.

Motivated by the preceding discussion, we proceed to examine the inheritance of such symmetries in spacetimes that admit a pseudo W_2 -curvature tensor.

Let a spacetime with a pseudo W_2 -curvature tensor with respect to a Killing vector field ξ be a curvature collineation. Then we have

$$(\mathcal{L}_\xi g)(X_u, X_v) = 0. \quad (26)$$

Again, since spacetime admitting a curvature collineation. Then from (25), we have

$$(\mathcal{L}_\xi S)(X_u, X_v) = 0, \quad (27)$$

where S is the Ricci tensor of the manifold. By taking the Lie derivative of (1) and using (25)–(27), we get

$$(\mathcal{L}_\xi \bar{W}_2)(X_u, X_v) = 0. \quad (28)$$

Thus the proof is complete. $\square$

Theorem 5. *Let a spacetime obeying Einstein's field equation have vanishing pseudo W_2 -curvature tensor. Then the spacetime admits a matter collineation with respect to a vector field ξ if and only if ξ is a Killing vector field.*

Proof. Suppose that the pseudo W_2 -curvature tensor vanishes, i.e., $\bar{W}_2 = 0$. From (10), we obtain

$$\begin{aligned} \psi R(X_u, X_v)X_z &= \phi \left[g(X_u, X_z)QX_v - g(X_v, X_z)QX_u \right] \\ &+ \frac{\theta}{4} \left(\frac{\psi}{3} + \phi \right) \left[g(X_v, X_z)X_u - g(X_u, X_z)X_v \right]. \end{aligned} \quad (29)$$

Taking the inner product of (29) with X_w , we have

$$\begin{aligned} \bar{R}(X_u, X_v, X_z, X_w) &= \frac{\phi}{\psi} \left[g(X_u, X_z)S(X_v, X_w) - g(X_v, X_z)S(X_u, X_w) \right] \\ &+ \frac{\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) \left[g(X_v, X_z)g(X_u, X_w) - g(X_u, X_z)g(X_v, X_w) \right]. \end{aligned} \quad (30)$$

Contracting (30) with respect to X_u and X_w , we obtain

$$S(X_v, X_z) = \frac{\phi}{\psi} \left[S(X_v, X_z) - \theta g(X_v, X_z) \right] + \frac{3\theta}{4\psi} \left(\frac{\psi}{3} + \phi \right) g(X_v, X_z). \quad (31)$$

After simplification, (31) directly follows that

$$S(X_v, X_z) = \frac{\theta}{4} g(X_v, X_z). \quad (32)$$

Thus the spacetime is an Einstein spacetime.

Now from (18) and (32), we obtain

$$kT(X_u, X_v) = \left(\lambda - \frac{\theta}{4} \right) g(X_u, X_v)$$

and

$$T(X_u, X_v) = \mu g(X_u, X_v), \quad (33)$$

where $\mu = \frac{1}{k} \left(\lambda - \frac{\theta}{4} \right)$. Now taking the Lie derivative of (33) along ξ , we get

$$(\mathcal{L}_\xi T)(X_u, X_v) = (\mathcal{L}_\xi \mu) g(X_u, X_v) + \mu (\mathcal{L}_\xi g)(X_u, X_v). \quad (34)$$

Suppose that ξ generates a matter collineation. Then (34) becomes

$$(\mathcal{L}_\xi T)(X_u, X_v) = 0.$$

Consequently,

$$\mu (\mathcal{L}_\xi g)(X_u, X_v) = 0.$$

Since $\mu \neq 0$, it follows that $(\mathcal{L}_\xi g)(X_u, X_v) = 0$. Thus ξ is a Killing vector field. Conversely, if ξ is a Killing vector field, then

$$(\mathcal{L}_\xi g)(X_u, X_v) = 0.$$

Using $T(X_u, X_v) = \mu g(X_u, X_v)$, we obtain

$$(\mathcal{L}_\xi T)(X_u, X_v) = \mu (\mathcal{L}_\xi g)(X_u, X_v) = 0.$$

Hence ξ generates a matter collineation. Therefore

$$(\mathcal{L}_\xi T)(X_u, X_v) = 0 \iff (\mathcal{L}_\xi g)(X_u, X_v) = 0.$$

Thus the spacetime admits a matter collineation with respect to ξ if and only if ξ is a Killing vector field. Hence the theorem is proved. $\square$

Theorem 6. *Let a spacetime satisfying Einstein's field equation be of vanishing pseudo W_2 -curvature tensor. Then the vector field ξ is a conformal Killing vector field if and only if the energy-momentum tensor has the Lie inheritance property with respect to ξ .*

Proof. Suppose that ξ is a conformal Killing vector field. Then

$$(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega g(X_u, X_v), \quad (35)$$

where Ω is a smooth scalar function.

Taking the Lie derivative of (33) along ξ , we have

$$(\mathcal{L}_\xi T)(X_u, X_v) = (\mathcal{L}_\xi \mu)g(X_u, X_v) + \mu(\mathcal{L}_\xi g)(X_u, X_v), \quad (36)$$

since μ is constant, i.e., $\mathcal{L}_\xi \mu = 0$, then (36) becomes

$$(\mathcal{L}_\xi T)(X_u, X_v) = \mu(\mathcal{L}_\xi g)(X_u, X_v). \quad (37)$$

From (35) and (37), it follows that

$$(\mathcal{L}_\xi T)(X_u, X_v) = 2\Omega\mu g(X_u, X_v). \quad (38)$$

Now, combining (33) and (38), we get

$$(\mathcal{L}_\xi T)(X_u, X_v) = 2\Omega T(X_u, X_v).$$

Hence the energy-momentum tensor satisfies the Lie inheritance property with respect to ξ .

Conversely, suppose that the energy-momentum tensor admits the Lie inheritance property with respect to ξ . Then

$$(\mathcal{L}_\xi T)(X_u, X_v) = 2\Omega T(X_u, X_v). \quad (39)$$

With the help of (33), Equation (39) becomes

$$(\mathcal{L}_\xi \mu)g(X_u, X_v) + \mu(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega\mu g(X_u, X_v), \quad (40)$$

since $\mathcal{L}_\xi \mu = 0$. Therefore, (40) yields

$$\mu(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega\mu g(X_u, X_v).$$

Assuming $\mu \neq 0$ in the above equation, we obtain

$$(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega g(X_u, X_v).$$

Thus ξ is a conformal Killing vector field. Therefore, we conclude that

$$(\mathcal{L}_\xi g)(X_u, X_v) = 2\Omega g(X_u, X_v) \iff (\mathcal{L}_\xi T)(X_u, X_v) = 2\Omega T(X_u, X_v).$$

Hence the vector field ξ is a conformal Killing vector field if and only if the energy–momentum tensor has the Lie inheritance property with respect to ξ . $\square$

Next, we analyze a spacetime filled with a perfect fluid in which the pseudo W_2 -curvature tensor vanishes. The energy–momentum tensor $T(X_u, X_v)$ of a perfect fluid is specified by [1]

$$T(X_u, X_v) = (\beta + \gamma)\eta(X_u)\eta(X_v) + \gamma g(X_u, X_v), \quad (41)$$

where β is the energy density, γ is the isotropic pressure, η is a non-zero 1-form such that $g(X_u, \xi) = \eta(X_u)$ for all X_u , and the characteristic vector field ξ is unit time-like with respect to the Lorentzian metric, that is, $g(\xi, \xi) = \eta(\xi) = -1$.

From (19) and (41), we obtain

$$\left[\lambda - \frac{\theta}{4} - \gamma k\right]g(X_u, X_v) = k(\beta + \gamma)\eta(X_u)\eta(X_v). \quad (42)$$

By applying a frame field and performing contraction over the vector fields X_u and X_v , we obtain

$$\theta = 4\lambda + k\beta - 3\gamma k. \quad (43)$$

Setting $X_u = X_v = \xi$ in (42), we get

$$\theta = 4(\lambda + k\beta). \quad (44)$$

From (43) and (44), we deduce that

$$\beta + \gamma = 0,$$

which implies that the isotropic pressure equals the negative of the energy density ($\gamma = -\beta$). In the context of cosmological models, this specific equation of state characterizes a vacuum energy or a cosmological constant, which acts as dark energy driving the accelerated expansion of the universe. However, for an ordinary perfect fluid, $\beta + \gamma \neq 0$, leading to a contradiction. Thus the statement can be written as:

Theorem 7. *If a perfect fluid spacetime obeying Einstein’s field equation with cosmological constant λ has a vanishing pseudo W_2 -curvature tensor, then its matter content cannot be an ordinary perfect fluid with $\beta + \gamma \neq 0$, where β is the energy density, and γ is the isotropic pressure. Instead, it corresponds to a dark energy-like vacuum fluid state.*

Now, Einstein’s field equation without cosmological constant λ is given by

$$S(X_u, X_v) - \frac{\theta}{2}g(X_u, X_v) = kT(X_u, X_v), \quad (45)$$

where $k \neq 0$. Taking a frame field over X_u and X_v in (45), we have

$$\theta = -k\delta, \quad (46)$$

where $\delta = \text{trace } T$. Using (46) in (45), we obtain

$$S(X_u, X_v) = k\left[T(X_u, X_v) - \frac{\delta}{2}g(X_u, X_v)\right]. \quad (47)$$

Einstein’s field equation without cosmological constant λ for a purely electromagnetic distribution takes the form [24]

$$S(X_u, X_v) = kT(X_u, X_v). \quad (48)$$

Therefore (47) and (48) give $\delta = 0$, Equation (28) gives $\theta = 0$ and Equation (17) yields $\bar{R}(X_u, X_v, X_z, X_w) = 0$, which shows that the spacetime is a locally flat Lorentzian spacetime. Thus we conclude the following:

Theorem 8. *A pseudo W_2 -flat spacetime satisfying Einstein's field equation is a locally flat Lorentzian spacetime.*

In pseudo W_2 -flat perfect fluid spacetime, the curvature tensor R of type (1, 3) is given by

$$R(X_u, X_v)X_z = \frac{\theta}{12} [g(X_v, X_z)X_u - g(X_u, X_z)X_v], \quad (49)$$

where θ denotes the scalar curvature of the spacetime. Since pseudo W_2 -flat spacetime is an Einstein space, it follows that the scalar curvature θ is constant.

Let $L^\perp$ be the 3-dimensional distribution in pseudo W_2 -flat perfect fluid spacetime orthogonal to L . Then

$$R(X_u, X_v)X_z = \frac{\theta}{12} [g(X_v, X_z)X_u - g(X_u, X_z)X_v] \quad (50)$$

for all $X_u, X_v, X_z \in L^\perp$ and

$$R(X_u, L)L = -\frac{\theta}{12}X_u, \quad (51)$$

for every $X_u \in L^\perp$.

In [25], a Lorentzian manifold is called infinitesimally spatially isotropic relative to time-like unit vector field L if its curvature tensor R satisfies

$$R(X_u, X_v)X_z = q[g(X_v, X_z)X_u - g(X_u, X_z)X_v], \quad (52)$$

for all $X_u, X_v, X_z \in L^\perp$ and $R(X_u, L)L = tX_u$ for all $X_u \in L^\perp$, where q and t are real valued functions on the manifold. Thus from (50) and (51), the following theorem arises:

Theorem 9. *A pseudo W_2 -flat perfect fluid spacetime satisfying the Einstein's field equation without cosmological constant λ and having the vector field L as the velocity vector field is infinitesimally spatially isotropic relative to the unit time-like vector field L .*

Let $X_u, X_v \in L^\perp$, A_1 denotes the sectional curvature of the spacetime determined by X_u and X_v , whereas A_2 denotes the sectional curvature of the spacetime determined by X_u and L . Then

$$A_1 = \frac{g(R(X_u, X_v)X_v, X_u)}{g(X_u, X_u)g(X_v, X_v) - \{g(X_u, X_v)\}^2} = \frac{\theta}{12}$$

and

$$A_2 = \frac{g(R(X_u, L)L, X_u)}{g(X_u, X_u)g(L, L) - \{g(X_u, L)\}^2} = \frac{\theta}{12}.$$

Thus we are able to state the following:

Theorem 10. *In pseudo W_2 -flat perfect fluid spacetime, the sectional curvature determined by two vectors $X_u, X_v \in L^\perp$ and the sectional curvature determined by two vectors X_u and L are equal.*

3. Divergence-Free Pseudo W_2 -Curvature Tensor and Fluid Spacetimes

Definition 1. A symmetric $(0, 2)$ type tensor field T on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_{X_u} T)(X_v, X_z) = (\nabla_{X_v} T)(X_u, X_z) \quad (53)$$

for all X_u, X_v and X_z .

The significance of the Codazzi tensor on a Riemannian manifold was studied by [26]. Now, taking the covariant derivative of (45) with X_u , we have

$$(\nabla_{X_u} S)(X_v, X_z) - \frac{1}{2}g(X_v, X_z)d\theta(X_u) = k(\nabla_{X_u} T)(X_v, X_z), \quad (54)$$

where $d\theta(X_u) = \nabla_{X_u} \theta$. Interchanging X_u and X_v in (54), we get

$$(\nabla_{X_v} S)(X_u, X_z) - \frac{1}{2}g(X_u, X_z)d\theta(X_v) = k(\nabla_{X_v} T)(X_u, X_z). \quad (55)$$

Now, subtracting (55) from (54), we obtain

$$\begin{aligned} (\nabla_{X_u} S)(X_v, X_z) - (\nabla_{X_v} S)(X_u, X_z) &= k[(\nabla_{X_u} T)(X_v, X_z) - (\nabla_{X_v} T)(X_u, X_z)] \\ &\quad + \frac{1}{2}[g(X_v, X_z)d\theta(X_u) - g(X_u, X_z)d\theta(X_v)]. \end{aligned} \quad (56)$$

If T is a Codazzi tensor, then (56) yields

$$(\nabla_{X_u} S)(X_v, X_z) - (\nabla_{X_v} S)(X_u, X_z) = \frac{1}{2}[g(X_v, X_z)d\theta(X_u) - g(X_u, X_z)d\theta(X_v)]. \quad (57)$$

Now, contracting the Codazzi condition on T with respect to X_v and X_z , we obtain

$$\nabla^j T_{ij} = \nabla_i \delta,$$

where $\delta = \text{trace } T$. Since the energy-momentum tensor is divergence-free, i.e., $\nabla^j T_{ij} = 0$, it follows that δ is constant. Taking the trace of (45), we get

$$\theta = -k\delta,$$

which shows that θ is constant. Therefore, we obtain

$$d\theta(X_u) = d\theta(X_v) = 0. \quad (58)$$

From (57) and (58), we arrive at

$$(\nabla_{X_u} S)(X_v, X_z) = (\nabla_{X_v} S)(X_u, X_z).$$

Thus the Ricci tensor S is of Codazzi type.

Conversely, suppose that S is of Codazzi type. Then from (56) and (58), we have

$$k[(\nabla_{X_u} T)(X_v, X_z) - (\nabla_{X_v} T)(X_u, X_z)] = 0,$$

since $k \neq 0$. Therefore, the above equation becomes

$$(\nabla_{X_u} T)(X_v, X_z) = (\nabla_{X_v} T)(X_u, X_z),$$

which shows that T is of Codazzi type. Hence, the energy–momentum tensor T is of Codazzi type if and only if the Ricci tensor S is of Codazzi type. Thus the following theorem can be stated as:

Theorem 11. *If a spacetime satisfying Einstein’s field equation without cosmological constant λ has an energy–momentum tensor T of Codazzi type, then the Ricci tensor S is of Codazzi type and conversely.*

Next, we take the covariant derivative of (1) with respect to X_u , we get

$$\begin{aligned} (\nabla_{X_u}' \bar{W}_2)(X_v, X_z, X_w, X_p) &= \psi(\nabla_{X_u}' R)(X_v, X_z, X_w, X_p) \\ &+ \phi \left[g(X_z, X_w)(\nabla_{X_u} S)(X_v, X_p) \right. \\ &- g(X_v, X_w)(\nabla_{X_u} S)(X_z, X_p) \left. \right] \\ &- \frac{(\nabla_{X_u} \theta)}{4} \left(\frac{\psi}{3} + \phi \right) \left[g(X_z, X_w)g(X_v, X_p) \right. \\ &- g(X_v, X_w)g(X_z, X_p) \left. \right], \end{aligned} \quad (59)$$

for all the vector fields X_u, X_v, X_z, X_w, X_p on (M^4, g) . Contracting (59) over $X_u = e_i$ and $X_p = e_i$, we obtain

$$\begin{aligned} (div \bar{W}_2)(X_v, X_z, X_w) &= \psi(div R)(X_v, X_z, X_w) \\ &+ \phi \left[g(X_z, X_w)(div S)(X_v) - g(X_v, X_w)(div S)(X_z) \right] \\ &- \frac{1}{4} \left(\frac{\psi}{3} + \phi \right) \left[g(X_z, X_w)(\nabla_{X_v} \theta) - g(X_v, X_w)(\nabla_{X_z} \theta) \right], \end{aligned} \quad (60)$$

where div represents divergence [27]. The divergence of $\bar{W}_2$ is given by

$$(div \bar{W}_2)(X_v, X_z, X_w) = \sum_{i=1}^4 (\nabla_{e_i}' \bar{W}_2)(X_v, X_z, X_w, e_i),$$

where $\{e_i\}_{i=1}^4$ is a local orthonormal frame. By the contracted second Bianchi identity, it is known that

$$(div R)(X_v, X_z, X_w) = (\nabla_{X_v} S)(X_z, X_w) - (\nabla_{X_z} S)(X_v, X_w). \quad (61)$$

Since the spacetime satisfies Einstein’s field equation. Therefore, taking covariant differentiation of (18) with respect to X_u , we obtain

$$(\nabla_{X_u} S)(X_v, X_z) = k(\nabla_{X_u} T)(X_v, X_z) + \frac{1}{2}(\nabla_{X_u} \theta)g(X_v, X_z). \quad (62)$$

As T is of Codazzi type, then we have

$$(\nabla_{X_u} T)(X_v, X_z) = (\nabla_{X_v} T)(X_u, X_z). \quad (63)$$

Using (62) and (63), we obtain

$$(\nabla_{X_u} S)(X_v, X_z) = (\nabla_{X_v} S)(X_u, X_z), \quad (64)$$

which shows that the Ricci tensor S is also of Codazzi type. Hence

$$(\nabla_{X_v} S)(X_z, X_w) = (\nabla_{X_z} S)(X_v, X_w). \quad (65)$$

Therefore, from (61), (64) and (65), it is obvious that

$$(\operatorname{div} R)(X_v, X_z, X_w) = 0. \quad (66)$$

Substituting (66) into (60), we get

$$\begin{aligned} (\operatorname{div} \bar{W}_2)(X_v, X_z, X_w) = & \phi \left[g(X_z, X_w)(\operatorname{div} S)(X_v) - g(X_v, X_w)(\operatorname{div} S)(X_z) \right] \\ & - \frac{1}{4} \left(\frac{\psi}{3} + \phi \right) \left[g(X_z, X_w)(\nabla_{X_v} \theta) - g(X_v, X_w)(\nabla_{X_z} \theta) \right]. \end{aligned} \quad (67)$$

Again, by the contracted Bianchi identity, we have

$$(\operatorname{div} S)(X_u) = \frac{1}{2}(\nabla_{X_u} \theta) = \frac{1}{2}d\theta(X_u). \quad (68)$$

Now, from (67) and (68), we get

$$(\operatorname{div} \bar{W}_2)(X_v, X_z, X_w) = \left[\frac{\phi}{2} - \frac{1}{4} \left(\frac{\psi}{3} + \phi \right) \right] \left[g(X_z, X_w)(\nabla_{X_v} \theta) - g(X_v, X_w)(\nabla_{X_z} \theta) \right]. \quad (69)$$

Since T is a Codazzi tensor, it follows from (18) that the scalar curvature θ is constant. Therefore, we obtain

$$\nabla_{X_v} \theta = d\theta(X_v) = 0. \quad (70)$$

Consequently, (69) and (70) follow that

$$(\operatorname{div} \bar{W}_2)(X_v, X_z, X_w) = 0,$$

which implies

$$\operatorname{div} \bar{W}_2 = 0.$$

Hence the pseudo W_2 -curvature tensor is divergence-free whenever the energy-momentum tensor is of Codazzi type and $\psi \neq 0$. Thus the theorem states:

Theorem 12. *If in a spacetime the energy-momentum tensor T is of Codazzi type, then the pseudo W_2 -curvature tensor is divergence-free, provided $\psi \neq 0$.*

Also, taking the frame field after contraction and from (41), we have

$$\delta = 3\gamma - \beta. \quad (71)$$

If $(\operatorname{div} \bar{W}_2)(X_v, X_z, X_w) = 0$, and $T(X_v, X_z)$ is of Codazzi type, then from (47), (60) and (71), we get

$$g(X_z, X_w)d\delta(X_v) - g(X_v, X_w)d\delta(X_z) = 0, \quad (72)$$

which means that $\delta = \text{constant}$, i.e., $3\gamma - \beta = \text{constant}$. Hence we can state the following:

Theorem 13. *If in a perfect fluid spacetime the divergence of pseudo W_2 -curvature tensor vanishes and the energy-momentum tensor T is of Codazzi type, then $3\gamma - \beta$ is constant.*

If $3\gamma - \beta = 0$, then the isotropic pressure relates to the energy density by $\gamma = \frac{\beta}{3}$. In relativistic cosmology, this specific equation of state indicates that the spacetime is filled with pure radiation. This represents the radiation-dominated era of the early universe, where relativistic particles and photons dictated the expansion dynamics [24].

4. Divergence-Free Pseudo W_2 -Curvature Tensor in Ricci-Recurrent Spacetimes

Definition 2. A semi-Riemannian manifold (M^n, g) ($n > 2$) is called a Ricci-recurrent manifold [28] if the Ricci tensor S satisfies

$$(\nabla_{X_u} S)(X_v, X_z) = \eta(X_u)S(X_v, X_z), \quad (73)$$

where η is a non-vanishing 1-form associated with the vector field ξ , as $\eta(X_u) = g(X_u, \xi)$. If the Ricci tensor fulfills the condition (73), a Lorentzian manifold is referred to as Ricci-recurrent spacetime. In this case, suppose the vector field ξ related to the 1-form η is treated as a unit time-like vector field, i.e., $g(\xi, \xi) = \eta(\xi) = -1$.

If $\text{div } \bar{W}_2 = 0$, then from Theorem 11, we find

$$(\nabla_{X_u} S)(X_v, X_z) = (\nabla_{X_v} S)(X_u, X_z). \quad (74)$$

Applying (73) in (74), we have

$$\eta(X_u)S(X_v, X_z) = \eta(X_v)S(X_u, X_z). \quad (75)$$

Setting $X_u = \xi$ in (75), we get

$$S(X_v, X_z) = -\eta(X_v)S(X_z, \xi). \quad (76)$$

Performing $X_v = X_z = e_i$ in (75) and summing over i , $1 \leq i \leq 4$, we have

$$\sum_{i=1}^4 S(e_i, e_i) = \theta,$$

and

$$\sum_{i=1}^4 g(e_i, \xi)S(X_u, e_i) = S(X_u, \xi).$$

Therefore, with the help of the above equations and (75), we obtain

$$\eta(X_u)\theta = S(X_u, \xi), \quad (77)$$

where θ is the scalar curvature of the manifold. Inserting (77) in (76), we deduce that

$$S(X_v, X_z) = -\theta\eta(X_v)\eta(X_z),$$

which implies that the spacetime is Ricci simple [29].

As a consequence of the above discussion, we obtain the following proposition:

Proposition 1. Let (M^4, g) be a Ricci-recurrent spacetime admitting a divergence-free pseudo W_2 -curvature tensor, i.e., $\text{div } \bar{W}_2 = 0$. Then the Ricci tensor S is of the Ricci simple form

$$S(X_u, X_v) = -\theta\eta(X_u)\eta(X_v),$$

for some non-zero 1-form η and scalar curvature θ . If $\theta \neq 0$ and η is non-null, then (M^4, g) is a quasi-Einstein spacetime of rank-one type, i.e.,

$$S(X_u, X_v) = ag(X_u, X_v) + b\omega(X_u)\omega(X_v),$$

with $a = 0$, $b = -\theta$, and $\omega = \eta$.

Remark 2. The physical interpretation of a Ricci simple spacetime is rigorously explored in [29], where the authors proved that a Ricci simple spacetime dynamically behaves as a stiff matter fluid. A stiff fluid, characterized by the equation of state where pressure equals energy density ($\beta = \gamma$), corresponds to the Zel'dovich cosmological model. This model is critical for describing the ultra-dense, extreme conditions of the very early universe. Thus, we conclude that a Ricci-recurrent spacetime admitting a divergence-free pseudo W_2 -curvature tensor represents a Zel'dovich stiff matter universe. A stiff matter fluid is characterized by an equation of state where the pressure γ is exactly equal to the energy density β ($\gamma = \beta$). This implies that the speed of sound in such a medium is equal to the speed of light ($v_s = c$). Such extreme conditions are not found in the contemporary universe but are theorized to have existed during the very early stages of the Big Bang, specifically during the Planck epoch or within the core of ultra-dense relativistic stars. It is established that a Ricci-recurrent spacetime with divergence-free pseudo W_2 -curvature tensor corresponds to a Zel'dovich stiff matter universe.

5. Examples of Pseudo W_2 -Flat Spacetimes and Comparative Discussion

To illustrate the theoretical results, we present two explicit Lorentzian spacetimes whose pseudo W_2 -curvature tensor vanishes. The first example is an isotropic warped spacetime, while the second is the de Sitter spacetime. In each case, the metric, Christoffel symbols, curvature tensors, Ricci operator, scalar curvature, and pseudo W_2 -flatness condition are verified explicitly.

Example 1. Isotropic Warped Spacetime:

Consider the 4-dimensional Lorentzian manifold $(\mathbb{R}^4, g)$ with metric expressed in the coordinates (x^1, x^2, x^3, x^4) as follows

$$ds^2 = g_{ij}dx^i dx^j = -(dx^1)^2 + e^{x^1} [(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \quad (78)$$

for $i, j = 1, 2, 3, 4$. The metric components are

$$g_{11} = -1, \quad g_{22} = g_{33} = g_{44} = e^{x^1}, \quad g_{ij} = 0 \text{ for } i \neq j,$$

with inverse

$$g^{11} = -1, \quad g^{22} = g^{33} = g^{44} = e^{-x^1}.$$

The non-zero Christoffel symbols of the second kind are

$$\Gamma_{22}^1 = \Gamma_{33}^1 = \Gamma_{44}^1 = -\frac{e^{x^1}}{2}, \quad \Gamma_{12}^2 = \Gamma_{13}^3 = \Gamma_{14}^4 = \frac{1}{2}.$$

The non-zero Riemann curvature components are

$$R_{1212} = R_{1313} = R_{1414} = -\frac{e^{x^1}}{4}, \quad R_{2323} = R_{2424} = R_{3434} = \frac{e^{2x^1}}{4}.$$

Also, the non-zero Ricci components are

$$R_{11} = \frac{3}{4}, \quad R_{22} = R_{33} = R_{44} = -\frac{3e^{x^1}}{4}.$$

For the metric given in (78) and with the help of the above results, the scalar curvature θ is

$$\theta = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} = -3,$$

which can be easily seen from (17). Now, substituting $QX_u = -\frac{3}{4}X_u$, $\theta = -3$ and $n = 4$ in (2), we obtain

$$\bar{W}_2(X_u, X_v)X_z = 0.$$

Hence the isotropic warped spacetime is pseudo $\bar{W}_2$ -flat.

Example 2. *FLRW-type exponential spacetime (de Sitter):*

Consider the de Sitter metric with the standard Cartesian spatial coordinates (x, y, z) given by

$$ds^2 = -dt^2 + e^{2\alpha t}(dx^2 + dy^2 + dz^2), \quad \alpha \neq 0,$$

where α is the Hubble constant, and t is the cosmological time coordinate. The metric components are

$$g_{00} = -1, \quad g_{11} = g_{22} = g_{33} = e^{2\alpha t}, \quad g_{ij} = 0 \quad (i \neq j),$$

$$g^{00} = -1, \quad g^{11} = g^{22} = g^{33} = e^{-2\alpha t}.$$

The non-zero Christoffel symbols are obtained as follows

$$\Gamma_{11}^0 = \Gamma_{22}^0 = \Gamma_{33}^0 = \alpha e^{2\alpha t}, \quad \Gamma_{10}^1 = \Gamma_{20}^2 = \Gamma_{30}^3 = \alpha.$$

The non-zero Riemann curvature tensor are

$$R_{0i0i} = -\alpha^2 e^{2\alpha t}, \quad R_{ijkl} = \alpha^2 e^{4\alpha t}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad i, j, k, l = 1, 2, 3.$$

Moreover, the Ricci tensor are obtained by

$$R_{00} = -3\alpha^2, \quad R_{11} = R_{22} = R_{33} = 3\alpha^2 e^{2\alpha t},$$

and the scalar curvature θ is

$$\theta = 12\alpha^2.$$

Also, the Ricci operator Q is

$$Q(\partial_0) = -3\alpha^2 \partial_0, \quad Q(\partial_i) = 3\alpha^2 \partial_i, \quad i = 1, 2, 3.$$

Finally, from (1) and the above results, we obtain

$$\bar{W}_2(X_u, X_v)X_z = 0.$$

Hence the FLRW-type exponential spacetime (de Sitter) is pseudo $\bar{W}_2$ -flat.

Proposition 2. *The spacetimes described in Examples 1 and 2 are pseudo W_2 -flat spacetimes of constant sectional curvature. Consequently, they satisfy the conclusion of Theorem 1.*

Proof. For Example 1, direct computation shows that

$$K_{12} = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{-e^{x^1}/4}{-e^{x^1}} = \frac{1}{4}.$$

Similarly,

$$K_{13} = K_{14} = K_{23} = K_{24} = K_{34} = \frac{1}{4}.$$

Hence the sectional curvature K is constant. Moreover,

$$S = -\frac{3}{4}g,$$

so the spacetime is Einstein. Since $\bar{W}_2 = 0$, the conclusion of Theorem 1 follows.

For Example 2, the de Sitter spacetime satisfies

$$\bar{R}(X_u, X_v, X_z, X_w) = \alpha^2 [g(X_u, X_z)g(X_v, X_w) - g(X_u, X_w)g(X_v, X_z)],$$

which implies constant sectional curvature $K = \alpha^2$. Thus de Sitter spacetime is a constant-curvature spacetime. Furthermore,

$$S = 3\alpha^2 g.$$

Since $\bar{W}_2 = 0$, the theorem is again verified. $\square$

From the examples provided above, we got the results in the Table 1:

Table 1. Comparison of pseudo W_2 -flat spacetimes.

Property	Example 1	Example 2
Metric type	Isotropic warped	FLRW exponential (de Sitter)
Non-zero Christoffel symbols	2 sets	6 sets
Riemann curvature	Non-zero, coordinate-dependent	Constant sectional curvature
Ricci tensor	$S = -\frac{3}{4}g$	$S = 3\alpha^2 g$
Scalar curvature	-3	$12\alpha^2$
Ricci operator	$Q = -\frac{3}{4}I$	$Q = 3\alpha^2 I$
Physical interpretation	Isotropic warped spacetime	de Sitter universe
Pseudo W_2 -flat	Yes	Yes

All the above spacetimes satisfy $\bar{W}_2 = 0$, thereby confirming their pseudo W_2 -flatness. Example 1 represents an isotropic warped spacetime with non-constant Riemann curvature components, whereas Example 2 corresponds to the de Sitter spacetime, a Lorentzian manifold of constant positive curvature satisfying $S = \lambda g$. Despite their distinct geometric structures, both examples satisfy the pseudo W_2 -flat condition. These examples demonstrate that pseudo W_2 -flat spacetimes arise in different geometric settings, ranging from isotropic warped geometries to FLRW-type cosmological models. Consequently, the pseudo W_2 -flat condition provides a useful geometric framework for the study of physically relevant Lorentzian spacetimes.

6. Conclusions

The pseudo W_2 -curvature tensor serves as a fundamental object in geometric gravity theories, extending beyond classical Riemannian geometry. By encoding richer geometric information than standard curvature tensors, it provides crucial insights into non-Einsteinian spacetimes—particularly in modified gravity, anisotropic cosmological models and scenarios involving unconventional matter. In general relativity, the curvature of spacetime is directly linked to the energy and momentum of matter through Einstein’s field equations. Additionally, investigating the role of this tensor in anisotropic Bianchi-type cosmological models could offer deeper insights into the pre-inflationary phase of the universe, where spatial isotropy was not yet established. To highlight the novelty and significance of the present work, a comparison with related studies is presented in Table 2.

Table 2. Comparison of the present work with related studies.

Aspect	W_2 -Curvature Tensor	Pseudo Quasi-Conformal Curvature Tensor	Pseudo W_2 -Curvature Tensor (Present Work)
Geometric setting	Relativistic spacetimes and modified gravity models [3,4]	Lorentzian manifolds with pseudo quasi-conformal structures [5]	Lorentzian spacetimes satisfying Einstein's field equations
Curvature structure	Based on the Riemann, Ricci, and scalar curvatures	Generalized conformal-type curvature tensor	Parameterized extension of the W_2 -curvature tensor involving ψ and ϕ
Flatness condition	W_2 -flatness and related geometric properties	Pseudo quasi-conformal flatness	Pseudo W_2 -flatness and its geometric consequences
Einstein field equations	Limited investigation	Not a principal focus	Direct characterization via the energy–momentum tensor
Symmetry properties	Curvature restrictions and symmetry conditions	Conformal-type geometric symmetries	Matter collineations, Killing vector fields, conformal Killing vector fields, and Lie inheritance symmetries
Divergence properties	Not emphasized	Not emphasized	Divergence-free pseudo W_2 -curvature tensor under Codazzi-type conditions
Main characterization	Curvature properties of W_2 -spacetimes	Pseudo quasi-conformal geometric structures	Constant-curvature characterization of pseudo W_2 -flat spacetimes
Examples	Selected spacetime models	Mainly theoretical studies	Explicit isotropic warped and de Sitter spacetime models

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References

1. O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic: New York, NY, USA, 1983.
2. Chaki, M.C.; Roy, S. Spacetimes with covariant constant energy momentum tensor. *Int. J. Theor. Phys.* **1996**, *35*, 1027–1032.
3. De, U.C.; Hazra, D. Effect of W_2 -curvature tensor in spacetimes and $f(G)$ -gravity. *Chin. J. Phys.* **2023**, *85*, 421–430.
4. Mallick, S.; De, U.C. Spacetime admitting W_2 -curvature tensor. *Int. J. Geom. Methods Mod. Phys.* **2014**, *11*, 1450030.
5. Suh, Y.; Chavan, V.; Pundeer, N.A. Pseudo-quasi-conformal curvature tensor and spacetimes of general relativity. *Filomat* **2021**, *35*, 657–666.
6. De, U.C.; Shenawy, S.; Abu-Donia, H.M.; Turki, N.B.; Alsaeed, S.; Syied, A.A. Spacetimes admitting concircular curvature tensor in $f(R)$ gravity. *Front. Phys.* **2022**, *9*, 800060.

7. Abu-Donia, H.M.; Shenawy, S.; Syied, A.A. The W^* -curvature tensor on relativistic space-times. *Kyungpook Math. J.* **2020**, *60*, 185–195.
8. Zengin, F.Ö. M -projectively flat spacetimes. *Math. Rep.* **2012**, *14*, 363–370.
9. Zengin, F.Ö.; Taşci, A.Y. Spacetimes admitting the Z -symmetric tensor. *Quaest. Math.* **2020**, *44*, 1613–1623.
10. De, U.C.; Hazra, D. Spacetimes admitting divergence free m -projective curvature tensor. *Commun. Korean Math. Soc.* **2024**, *39*, 201–210.
11. Mofarreh, F.; De, K.; De, U.C. Characterizations of a spacetime admitting Ψ -conformal curvature tensor. *Filomat* **2023**, *37*, 10265–10274.
12. Pundeer, N.A.; Ali, M. The semiconformal curvature tensor on Relativistic spacetimes. *Palest. J. Math.* **2023**, *12*, 301–308.
13. De, K.; Woo, C.; De, U.C. Geometric and physical characterizations of a spacetime concerning a novel curvature tensor. *Filomat* **2024**, *38*, 3535–3546.
14. Dey, C. On generalized Ricci recurrent spacetimes. *Kyungpook Math. J.* **2020**, *60*, 571–584.
15. Prasad, B.; Maurya, A.M. A pseudo W_2 -curvature tensor on a Riemannian manifold. *J. Pure Math.* **2004**, *21*, 81–85.
16. Pokhariyal, G.P.; Mishra, R.S. The curvature tensor and their relativistic significance. *Yokohama Math. J.* **1970**, *18*, 105–108.
17. Prasad, B. W_2 -curvature tensor on Kenmotsu manifold. *Indian J. Math.* **1997**, *39*, 287–291.
18. Kumar, R. On pseudo W_2 -flat LP-Sasakian manifold with coefficient- α . *Sci. Vis.* **2012**, *12*, 17–21.
19. Narlikar, J.V. *General Relativity and Gravitation*; Macmillan Co. of India: Noida, India, 1978.
20. Stephani, H. *General Relativity—An Introduction to the Theory of Gravitational Field*; Cambridge University Press: Cambridge, UK, 1978.
21. Katzin, G.H.; Levine, J.; Davis, W.R. Curvature collineations: A fundamental symmetry property of the spacetime of general relativity defined by the vanishing Lie derivative of the Riemannian curvature tensor. *J. Math. Phys.* **1969**, *10*, 617–629.
22. Duggal, K.L. Curvature Collineations and Conservation Laws of General Relativity. In Proceedings of the Canadian Conference on General Relativity and Relativistic Astro-Physics, Halifax, NS, Canada, 31 May–2 June 1985.
23. Duggal, K.L. Curvature inheritance symmetry in Riemannian spaces with applications to fluid space times. *J. Math. Phys.* **1992**, *33*, 2989–2997.
24. Ahsan, Z.; Siddiqui, S.A. Conircular curvature tensor and fluid spacetimes. *Int. J. Theor. Phys.* **2009**, *35*, 3202–3212.
25. Karchar, H. Infinitesimal characterization of Friedman universes. *Arch. Math.* **1992**, *38*, 58–64.
26. Derdzinski, A.; Shen, C.L. Codazzi tensor fields, curvature and Pontryagin forms. *Proc. Lond. Math. Soc.* **1983**, *47*, 15–26.
27. Hicks, N.J. *Notes on Differential Geometry*; D. Van Nostrand Company, Inc.: Princeton, NJ, USA, 1965.
28. Patterson, E.M. Some theorems on Ricci-recurrent spaces. *J. London Math. Soc.* **1952**, *27*, 287–295.
29. Mantica, C.A.; Suh, Y.J.; De, U.C. A note on generalized Robertson-Walker space-times. *Int. J. Geom. Methods Mod. Phys.* **2016**, *13*, 1650079.

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