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Ricci Semi-Symmetric Bulk Viscous String Fluid Spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -Gravity and Energy Constraints for Penrose Theorem

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Abstract

This study is dedicated to a separable $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity related to the bulk viscous string fluid to extract the equation of state for $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity. In this research, we offer an insight into calculating the density and pressure in terms of string tension in the phantom barrier, stiff fluid, and matter-dominated eras. As demonstrated, if a spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity is full of bulk viscous string fluid matter, it is a generalized quasi-Einstein spacetime. In addition, we determine the equation of state of Ricci semi-symmetric and Ricci pseudo-symmetric spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity filled with bulk viscous string fluid matter. Finally, we try to give the energy constraints in view of Penrose's singularity theorem of black holes for the spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity attached to bulk viscous string fluid.

Keywords: $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity; bulk viscous string fluid; Ricci semi-symmetry; equation of state; energy condition

MSC: 53B30; 53C42; 53C50; 53C80

1. Introduction

The ideal speculative framework for analyzing the vast structure of the universe is the general theory of relativity (*GTR*). It has been found that the *GTR* cannot explain the acceleration of the early and late universes without accounting for dark energy. Gravitation is not properly explained by the *GTR*, hence it is feasible to tweak it to produce theories that accept inflation and mimic *DE* (dark energy). The standard method for interpreting known cosmic dynamics is presented by Einstein's field equations [1,2]. Einstein's field equations yield an approximation to the observed data when a theoretical element of the cosmos called *DM* (dark matter) [3] is included [4]. Likewise, the universe contains an unconventional component called *DE*, which is thought to be the prime cause of the universe's expansion and controls the ratio of matter to energy.

By applying Einstein–Hilbert Lagrangian density to a function $\mathbb{F}(\mathbb{R})$, the *GTR* can be extended to $\mathbb{F}(\mathbb{R})$ -gravity, where $\mathbb{R}$ is the Ricci scalar. For examples, see [5–7]. Nevertheless, $\mathbb{F}(\mathbb{R})$ -gravity has a number of limitations regarding the solar system's stability and cannot accommodate other models (see [8–10] for more information). $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -theory was presented by Harko et al. [11], using the Lagrangian that is an arbitrary function of $\mathbb{T}$ and $\mathbb{R}$. Here, the energy–momentum tensor $\mathbb{T}$ is denoted by its trace. This concept



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was used to effectively explain the universe's late-time fast expansion (for more details, see [12,13]). In contrast to general relativity, $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity theory typically suggests that the energy–momentum tensor is not conserved, adding an extra force to particles. Higher-order curvature resolves the problem of massive neutron stars in the equations of motion of $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity. $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity offers a geometric explanation for dark matter events and has been applied to characterize galaxy rotation curves. It has been used to derive viable cosmological models very similar to Λ CDM.

A spacetime can be conceived as a four-dimensional time-driven Lorentzian manifold (M^4, g) with Lorentzian metric g . Further investigation into the geometric properties of this manifold has been based on the fundamental vector characterization of the Lorentzian manifold. For studying cosmic models, a Lorentzian manifold M is an excellent selection. Much work was performed exploring the energy–momentum distribution of different four-dimensional spacetimes in [14,15].

Definition 1 ([16,17]). A non-flat Riemannian manifold (M^n, g) is designated as a generalized quasi-Einstein (GQE) if its Ricci tensor S obeys the expression

$$S = A_1 g + A_2 \gamma \otimes \gamma + A_3 \sigma \otimes \sigma \quad (1)$$

where A_1, A_2 , and A_3 are non-zero scalars, with $A_2 \neq 0$, $A_3 \neq 0$, and γ, σ are 1-form, defined as

$$g(\zeta, \varsigma) = 0, \quad g(\mu, \zeta) = \gamma(\mu), \quad g(\mu, \varsigma) = \sigma(\mu),$$

for any vector field $\mu \in \chi(M^n, g)$.

The 1-form γ and σ correspond to the unit vector fields ζ (time-like) and ς (space-like), which are orthogonal to one another. (M^n, g) changes to a perfect fluid spacetime if $A_3 = 0$.

However, using basic causal theory, the evolution of viscous pressure in an expanding universe dominated by bulk viscous string matter has been studied. A one-dimensional entity called a string [18,19] can also be used to describe the universe. The extension of cosmic strings, which can reach any point in the universe, seems to be closely linked with the expansion of the cosmos [20]. A string cloud model with bulk viscosity has also been examined in $\mathbb{F}(\mathbb{R})$ -gravity, as explained by Reddy [21].

According to [21], the energy–momentum tensor for a bulk viscous string fluid (BVSF) is

$$\mathbb{T}_{ij} = (\bar{p} + \varrho) \gamma_i \gamma_j - \bar{p} g_{ij} - \lambda \sigma_i \sigma_j, \quad (2)$$

where the whole pressure, including the isotropic pressure, is indicated by $\bar{p}$; p is the string tension; ϱ is the rest density; and

$$\bar{p} = p - 3\bar{U}\mathcal{H} (= \varepsilon\varrho). \quad (3)$$

The bulk viscosity coefficient is $\bar{U}(t)$, the bulk viscous pressure is $3\bar{U}\mathcal{H}$, and the model's Hubble parameter is $\mathcal{H}$.

Additionally, σ_μ is a space-like vector that indicates the anisotropic direction of the string, and γ_μ is the four-velocity vector. Together, they fulfill

$$g^{\mu\nu} \gamma_\mu \gamma_\nu = -\sigma^\mu \sigma_\mu = 1, \quad \gamma^\mu \sigma_\mu = 0. \quad (4)$$

It is believed that scalar fields are essential to cosmology in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity theory. A better understanding of the general characteristics of the gravitational field, however, might result from using scalar fields as a source to create more generic gravitational models. In [22], the authors reconstructed exponential models of $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity. $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity

was studied by Chaubey [23], who presented some of their findings. Some characteristics of the cosmic perfect fluid in $\mathbb{F}(\mathbb{R})$ -gravity were examined by Capozziello et al. in [24,25]. The Danish have also examined $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity in [26–28]. A generalization of the *GTR*, $\mathbb{F}(\mathbb{R})$ -gravity, and $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity is the *EMSG* theory. Siddiqi and his fellow researchers have examined *EMSG* associated with perfect fluid matter in [29,30].

In 2025, Siddiqi and Dayel investigated certain geometric features of the *GTR* filled with an imperfect fluid, such as the bulk viscous string fluid matter, in [31]. Studying $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity combined with the bulk viscous fluid string matter without any geometric flows or solitons is therefore an intriguing challenge. Consequently, we are employing an extended gravity model, and we are utilizing an imperfect fluid matter tensor in place of a perfect fluid.

Therefore, the aforementioned research provides sufficient motivation to explore bulk viscous string fluid spacetime (*BVSFS*) in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity. Contrary to earlier research, we obtain the equation of state of Ricci semi-symmetric and Ricci pseudo-symmetric spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity filled with bulk viscous string fluid matter and describe the universe's development using the pressure, density, and string tension values. Finally, we attempt to provide the energy restrictions for the spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity connected with bulk viscous string fluid in light of Penrose's singularity theorem of black holes. Moreover, under the energy conditions we also find the existence of some Segre class of the Ricci tensor in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity with bulk viscous string fluid matter. In conclusion, $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity gives a solid mathematical framework that offers new solutions for the production of astrophysical structures and cosmic acceleration, frequently bridging the gap between theory and observation in ways that the *GTR* cannot.

2. Bulk Viscous String Fluid in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -Gravity

The *BVSFS* in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity depends on the characteristics of the matter field, hence there are several conjecture models that can be created for distinct values of $\mathbb{R}$ and $\mathbb{T}$ [11]. Some specific $\mathbb{F}(\mathbb{R}, \mathbb{T})$ modified gravity model classes can be produced by specifying the functional form of $\mathbb{F}$. We can obtain the following theoretical models for each choice of $\mathbb{F}$:

$$\mathbb{F}(\mathbb{R}, \mathbb{T}) = \mathbb{R} + 2\mathbb{F}(\mathbb{T}), \quad (5)$$

$$\mathbb{F}(\mathbb{R}, \mathbb{T}) = \mathbb{F}_1(\mathbb{R}) + \mathbb{F}_2(\mathbb{T}), \quad (6)$$

where $\mathbb{F}_1(\mathbb{R})$ and $\mathbb{F}_2(\mathbb{T})$ are arbitrary functions of $\mathbb{R}$ and $\mathbb{T}$, respectively.

$$\mathbb{F}(\mathbb{R}, \mathbb{T}) = \mathbb{F}_1(\mathbb{R}) + \mathbb{F}_2(\mathbb{R})\mathbb{F}_3(\mathbb{T}) \quad (7)$$

is a more general form that generalizes the previous two cases. In the present study we adopt the first model. For example, we use (5) where $\mathbb{F}(\mathbb{T})$ denotes a function $\mathbb{F}$ that depends only on $\mathbb{T}$. It should be noted that the gravitational interaction between space matter and the Ricci scalar $\mathbb{R}$ is altered by the element $2\mathbb{F}(\mathbb{T})$ that emerges in gravitational action.

We use the Einstein–Hilbert action term as our assumption:

$$\mathcal{E}_{\mathcal{H}} = \frac{1}{16\pi} \int [L_m + \mathbb{F}(\mathbb{R}, \mathbb{T})] \sqrt{(-g)} d^4x, \quad (8)$$

where L_m is the scalar space field's matter Lagrangian. The formula for the matter's stress–energy tensor is

$$\mathbb{T}_{\mu\nu} = \frac{-2\delta(\sqrt{-g})L_m}{\sqrt{-g}\delta^{\mu\nu}}. \quad (9)$$

Consider that L_m is exclusively dependent on $g_{\mu\nu}$ and not on its derivatives. The action variation (8) with respect to $g_{\mu\nu}$ implies

$$\begin{aligned} \mathbb{F}_{\mathbb{R}}(\mathbb{R}, \mathbb{T})_{\mu\nu} - \frac{1}{2}\mathbb{F}(\mathbb{R}, \mathbb{T})g_{\mu\nu} + (g_{\mu\nu}\nabla_a\nabla^a - \nabla_\mu\nabla_\nu)\mathbb{F}_{\mathbb{R}}(\mathbb{R}, \mathbb{T}) \\ = 8\pi\mathbb{T}_{\mu\nu} - \mathbb{F}_{\mathbb{T}}(\mathbb{R}, \mathbb{T})\mathbb{T}_{\mu\nu} - \mathbb{F}_{\mathbb{T}}(\mathbb{R}, \mathbb{T})\Psi_{\mu\nu}, \end{aligned} \quad (10)$$

where $\mathbb{F}_{\mathbb{R}} = \frac{\partial\mathbb{F}(\mathbb{R}, \mathbb{T})}{\partial\mathbb{R}}$ and $\mathbb{F}_{\mathbb{T}} = \frac{\partial\mathbb{F}(\mathbb{R}, \mathbb{T})}{\partial\mathbb{T}}$; $\square \equiv \nabla_a\nabla^a$ and ∇_μ symbolize the d'Alembertian operator and the covariant derivative, respectively. In addition, we have

$$\Psi_{\mu\nu} = -2\mathbb{T}_{\mu\nu} + g_{\mu\nu}L_m - 2g^{lk}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{lk}}. \quad (11)$$

Equations (8) and (9), with $\mathbb{F}(\mathbb{R}, \mathbb{T}) = \mathbb{F}(\mathbb{R})$, recover the field equations for $\mathbb{F}(\mathbb{R})$ -gravity.

Let the *BVSF* matter have total isotropic pressure $\bar{p}$, rest energy density ϱ , four-velocity vector γ^μ , and space-like vector σ^μ . We have the advantage of choosing L_m . Consequently, we set $L_m = \bar{p}$. Since the term “matter Lagrangian” has no universally accepted definition, we consider

$$\mathbb{T}_{\mu\nu} = (\bar{p} + \varrho)\gamma_\mu\gamma_\nu - \bar{p}g_{\mu\nu} - \lambda\sigma_\mu\sigma_\nu \quad (12)$$

where

$$\gamma^\mu\nabla_\nu\gamma_\mu = 0, \quad \gamma_\mu \cdot \gamma^\mu = -\sigma^\mu \cdot \sigma_\mu = 1, \quad \gamma^\mu \cdot \sigma_\mu = 0. \quad (13)$$

From (11) and (12), we obtain

$$\Psi_{\mu\nu} = \bar{p}g_{\mu\nu} - 2\mathbb{T}_{\mu\nu}. \quad (14)$$

After adopting (5) and (10), we get

$$S_{\mu\nu} = \frac{\mathbb{R}}{2}g_{\mu\nu} - 2\mathbb{F}'(\mathbb{T})\mathbb{T}_{\mu\nu} - 2\mathbb{F}'(\mathbb{T})\Psi_{\mu\nu} + \mathbb{f}(\mathbb{T})g_{\mu\nu} + 8\pi\mathbb{T}_{\mu\nu}. \quad (15)$$

In light of (12), (13), and (14), Equation (15) is transformed into

$$\begin{aligned} S_{\mu\nu} = \left\{ \frac{1}{2}\mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})] \right\} g_{\mu\nu} + \left\{ (\bar{p} + \varrho)(8\pi + 2\mathbb{F}'(\mathbb{T})) \right\} \gamma_\mu\gamma_\nu \\ - \left\{ 2\lambda(4\pi + \mathbb{F}'(\mathbb{T})) \right\} \sigma_\mu\sigma_\nu, \end{aligned} \quad (16)$$

$$\mathbb{R} = (\bar{p} + \varrho)[2\mathbb{F}'(\mathbb{T}) + 8\pi] + 16\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi] + 2\lambda[\mathbb{F}'(\mathbb{T}) + 4\pi] - 4\mathbb{F}(\mathbb{T}). \quad (17)$$

Thus, for *BVSFS* in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity, the Ricci tensor is

$$S_{\mu\nu} = A_1g_{\mu\nu} + A_2\gamma_\mu\gamma_\nu + A_3\sigma_\mu\sigma_\nu, \quad (18)$$

where

$$A_1 = \frac{1}{2}\mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})] \quad A_2 = (\bar{p} + \varrho)(8\pi + 2\mathbb{F}'(\mathbb{T})), \quad (19)$$

$$A_3 = 2\lambda(4\pi + \mathbb{F}'(\mathbb{T})).$$

We assume throughout that A_1 , A_2 , and A_3 are not simultaneously zero, and throughout the remainder of the paper $\Xi^{\mathbb{F}}$ stands for “the $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity model with *BVSF* matter”.

As a result, we can draw the conclusions below.

Theorem 1. The Ricci curvature tensor for $\Xi^{\mathbb{F}}$ is of the form

$$S_{\mu\nu} = \left\{ \frac{1}{2}\mathbb{R} - 4\bar{\mathfrak{p}}(\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})) \right\} g_{\mu\nu} + \left\{ (\bar{\mathfrak{p}} + \varrho)(8\pi + 2\mathbb{F}'(\mathbb{T})) \right\} \gamma_{\mu}\gamma_{\nu} - \left\{ 2\lambda(4\pi + \mathbb{F}'(\mathbb{T})) \right\} \sigma_{\mu}\sigma_{\nu}.$$

Corollary 1. In $\Xi^{\mathbb{F}}$, the scalar curvature is

$$\mathbb{R} = 2(\varrho + 9\bar{\mathfrak{p}})\mathbb{F}'(\mathbb{T}) + 8\pi(\varrho + 5\bar{\mathfrak{p}}) + 2\lambda(\mathbb{F}'(\mathbb{T}) + 4\pi) - 4\mathbb{F}(\mathbb{T}). \quad (20)$$

Next, Theorem 1 and Definition 1 entail the following result.

Theorem 2. A spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ is a GQE-spacetime.

The following outcome can also be stated.

Theorem 3. A BVFS (M^4, g) in $\Xi^{\mathbb{F}}$ is a GQE-spacetime.

Now, considering Equation (20), we get

$$\bar{\mathfrak{p}} + \varrho = \frac{\frac{\mathbb{R}}{2} + 2\mathbb{F}(\mathbb{T}) - \mathbb{F}'(\mathbb{T})(\lambda + 8\bar{\mathfrak{p}}) - 4\pi(\lambda + 4\bar{\mathfrak{p}})}{\mathbb{F}'(\mathbb{T}) + 4\pi}, \quad (21)$$

provided $\mathbb{F}'(\mathbb{T}) + 4\pi \neq 0$. The $\mathcal{EoS}$ (equation of state) for the dark matter era is defined by $\bar{\mathfrak{p}} = -\varrho + \mathbb{F}(\beta)$, where $\mathbb{F}(\beta)$ is a function of the scale factor β with the cosmic time t according to [32].

Remark 1. In the case of the imposed condition not being satisfied, i.e., $\mathbb{F}'(\mathbb{T}) + 4\pi = 0$, then from (21), we get $\bar{\mathfrak{p}} + \varrho \rightarrow \infty$, which explains a physical singularity when an unlimited density gradient or endless compression results from the breakdown of normal physics. This situation frequently aligns with the gravitational collapse endpoint. The emergence of a spacetime singularity in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity, like the core of a black hole, is suggested when pressure and density approach infinity.

Given (3), the $\mathcal{EoS}$ gives an expression for the relationship between isotropic pressure and energy density:

$$\bar{\mathfrak{p}} = \varepsilon\varrho, \quad \mathfrak{p} = \varepsilon_0\varrho, \quad \text{where } \varepsilon = \varepsilon_0 - \Omega, \quad (0 \leq \varepsilon \leq 1), \quad (22)$$

where Ω and ε_0 are constants. The stiff fluid era, matter-dominated era, radiation-dominated era, and vacuum-dominated era are represented by the values $\varepsilon = -1, 0, \frac{1}{3}, 1$ [33]. This yields the following.

Theorem 4. Consider the spacetime (M^4, g) in $\Xi^{\mathbb{F}}$, then the $\mathcal{EoS}$ is given by (21).

$$\mathfrak{p} = \frac{(\mathbb{R} + 4\mathbb{F}(\mathbb{T}))(\varepsilon + \Omega)}{2\mathbb{F}'(\mathbb{T})(1 + 9\varepsilon + \lambda) + 8\pi(1 + 5\varepsilon + \lambda)} \quad (23)$$

and

$$\varrho = \frac{\mathbb{R} + 4\mathbb{F}(\mathbb{T})}{2\mathbb{F}'(\mathbb{T})(1 + 9\varepsilon + \lambda) + 8\pi(1 + 5\varepsilon + \lambda)},$$

where $2\mathbb{F}'(\mathbb{T})(1 + 9\varepsilon + \lambda) + 8\pi(1 + 5\varepsilon + \lambda) \neq 0$.

Corollary 2. For $\Xi^{\mathbb{F}}$, the pressure p and density ρ are governed by (23).

Corollary 3. If the spacetime (M^4, g) in $\Xi^{\mathbb{F}}$, then we have the following.

BVSF in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity	Equation of state ($\mathcal{EoS}$)	Pressure p and density ρ
Vacuum-dominated era	$p = -\rho$	$p = \frac{(\mathbb{R}+4\mathbb{F}(\mathbb{T}))(\Omega-1)}{2\mathbb{F}'(\mathbb{T})(\lambda-8)+8\pi(\lambda-4)}$ and $\rho = \frac{\mathbb{R}+4\mathbb{F}(\mathbb{T})}{2\mathbb{F}'(\mathbb{T})(\lambda-8)+8\pi(\lambda-4)}$
Stiff fluid era	$p = \rho$	$p = \frac{(\mathbb{R}+4\mathbb{F}(\mathbb{T}))(\Omega+1)}{2\mathbb{F}'(\mathbb{T})(\lambda+10)+8\pi(\lambda+5)}$ and $\rho = \frac{\mathbb{R}+4\mathbb{F}(\mathbb{T})}{2\mathbb{F}'(\mathbb{T})(\lambda+10)+8\pi(\lambda+5)}$
Radiation era	$p = \frac{\rho}{3}$	$p = \frac{(3\mathbb{R}+12\mathbb{F}(\mathbb{T}))(3\Omega+1)}{6\mathbb{F}'(\mathbb{T})(\lambda+4)+8\pi(\lambda+\frac{8}{3})}$ and $\rho = \frac{\mathbb{R}+4\mathbb{F}(\mathbb{T})}{2\mathbb{F}'(\mathbb{T})(\lambda+4)+8\pi(\lambda+\frac{8}{3})}$
Matter-dominated era	$p = 0$	$p = \frac{(\mathbb{R}+4\mathbb{F}(\mathbb{T}))\Omega}{[2\mathbb{F}'(\mathbb{T})+8\pi](\lambda+1)}$ and $\rho = \frac{\mathbb{R}+4\mathbb{F}(\mathbb{T})}{[2\mathbb{F}'(\mathbb{T})+8\pi](\lambda+1)}$

Next, consider that if $\mathbb{F}(\mathbb{T}) = 0$, then we retrieve $\mathbb{F}(\mathbb{R})$ -gravity. Then, we get the following.

Corollary 4. If the spacetime (M^4, g) matter in $\mathbb{F}(\mathbb{R})$ -gravity is the BVSF, then the pressure and density are

$$p = \frac{\mathbb{R}(\varepsilon + \Omega)}{8\pi(1 + 5\varepsilon + \lambda)} \quad \text{and} \quad \rho = \frac{\mathbb{R}}{8\pi(1 + 5\varepsilon + \lambda)}, \quad (24)$$

Corollary 5. If the spacetime (M^4, g) in $\mathbb{F}(\mathbb{R})$ -gravity is filled with BVSF matter, then we have the following.

BVSF in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity	Equation of state ($\mathcal{EoS}$)	Pressure p and density ρ
Vacuum-dominated era	$p = -\rho$	$p = \frac{\mathbb{R}(\Omega-1)}{8\pi(\lambda-4)}$ and $\rho = \frac{\mathbb{R}}{8\pi(\lambda-4)}$
Stiff fluid era	$p = \rho$	$p = \frac{\mathbb{R}(\Omega+1)}{8\pi(\lambda+5)}$ and $\rho = \frac{\mathbb{R}}{8\pi(\lambda+5)}$
Radiation era	$p = \frac{\rho}{3}$	$p = \frac{3\mathbb{R}(3\Omega+1)}{8\pi(\lambda+\frac{8}{3})}$ and $\rho = \frac{\mathbb{R}}{8\pi(\lambda+\frac{8}{3})}$
Matter-dominated era	$p = 0$	$p = \frac{\mathbb{R}\Omega}{8\pi(\lambda+1)}$ and $\rho = \frac{\mathbb{R}}{8\pi(\lambda+1)}$

Once again, Equation (23) entails the next outcome.

Theorem 5. Let a spacetime (M^4, g) in $\Xi^{\mathbb{F}}$, then the following table describes the universe's evolution through the $\mathcal{EoS}$ of the BVSF-spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity.

$\mathcal{EoS}$ is $\frac{p}{\rho} = \omega$	Restrictions on ε and Ω	Evolution of the Universe or "Era"
$\omega = 1$	$\Omega = 1 - \varepsilon$	Ultra-relativistic
$\omega > -1$	$\Omega > -(1 + \varepsilon)$	Quintessence
$\omega < -1$	$\Omega < -(1 + \varepsilon)$	Phantom
$\omega = 0$	$\Omega = -\varepsilon$	Dust

3. BVSTS in $\mathbb{F}((\mathbb{R}, \mathbb{T}))$ -Gravity with Codazzi-Type Ricci Tensor

In a bulk viscous string fluid spacetime in $\Xi^{\mathbb{F}}$, suppose the Ricci tensor S is of the Codazzi type [34], meaning that

$$(\nabla_{\mu} S)(\nu, \varsigma) = (\nabla_{\nu} S)(\mu, \varsigma), \quad (25)$$

for all vector fields $\mu, \nu, \varsigma \in \chi(M^4)$.

Equation (25) defines a Codazzi-type Ricci tensor, which indicates a high degree of symmetry in a manifold's curvature. Geometrically stated, this requirement suggests that the Ricci tensor can coexist with unique structural arrangements like harmonic curvature or particular immersions.

A Codazzi-type Ricci tensor, which frequently results from highly symmetric spaces, particularly embedded surfaces or specific physical situations like irrotational and shear-free fluid flows, indicates that the Ricci tensor is “well-behaved” under the covariant derivative.

By using Equations (18) and (25), we have

$$\begin{aligned} & dA_1(\mu)g(v, \varsigma) + dA_2(\mu)\gamma(v)\gamma(\varsigma) + dA_3(\mu)\sigma(v)\sigma(\varsigma) \\ & + A_2[(\nabla_\mu\gamma)v\gamma(\varsigma) + \gamma(v)(\nabla_\mu\gamma)\varsigma] + A - 3[(\nabla_\mu\sigma)v\sigma(\varsigma) + \sigma(v)(\nabla_\mu\sigma)\varsigma] \\ & = dA_1(v)g(\mu, \varsigma) + dA_2(v)\gamma(\mu)\gamma(\varsigma) + dA_3(v)\sigma(\mu)\sigma(\varsigma) \\ & + A_2[(\nabla_v\gamma)\mu\gamma(\varsigma) + \gamma(\mu)(\nabla_v\gamma)\varsigma] + A_3[(\nabla_v\sigma)\mu\sigma(\varsigma) + \sigma(\mu)(\nabla_v\sigma)\varsigma]. \end{aligned} \quad (26)$$

By contracting over v and ς in a frame field, we obtain

$$\begin{aligned} & 4dA_1(\mu) - dA_2(\mu) + A_2[(\nabla_\mu\gamma)\zeta + \gamma(v)(\nabla_\mu\gamma)\zeta] \\ & = dA_1(\mu) + dA_2(\zeta)\gamma(\mu) + A_2[(\nabla_\zeta\gamma)\mu + \text{div}\zeta\gamma(\mu)], \end{aligned} \quad (27)$$

here, the divergence of ζ is indicated by $\text{div}(\zeta)$.

When ζ is substituted for μ , the following equation is produced:

$$3dA_1(\zeta) + A_2[(\nabla_\zeta\gamma)\zeta + (\nabla_\zeta\gamma)\zeta] = A_2[(\nabla_\zeta\gamma)\zeta - \text{div}\zeta]. \quad (28)$$

Given that $(\nabla_\zeta\gamma)\zeta = 0$, the preceding equation gives

$$3dA_1(\zeta) + \text{div}\zeta = 0. \quad (29)$$

Consider the velocity vector field. ζ does not affect the scalar A_1 . Following the previous equation, we may conclude that either $A_2 = 0$ or $\text{div}\zeta = 0$.

Using (19), we get

$$\bar{\mathbf{p}} + \varrho = 0. \quad (30)$$

It is also possible to state the following result.

Theorem 6. *If a BVFS in $\mathbb{E}^4$ is endowed with a Codazzi-type Ricci tensor, then the BVFS represents the dark matter era, provided A_1 is invariant under the four-velocity vector field ζ .*

Now, from (3) and (30), we get

$$\mathbf{p} + \varrho = 3\mathcal{U}\mathcal{H}. \quad (31)$$

According to [35], “Yang pure space” is a Lorentzian manifold (M^4, g) in which Yang’s equations are solved by the metric tensor of the manifold:

$$(\nabla_\mu S)(v, \varsigma) - (\nabla_v S)(\mu, \varsigma) = 0.$$

Additionally, if and only if the spacetime is an RW-spacetime, the authors of the same work demonstrated that a perfect fluid spacetime with $\mathbf{p} + \varrho \neq 0$ is a Yang pure space.

Therefore, we deduce that an RW -spacetime is a bulk viscous string fluid spacetime with a Codazzi-type Ricci tensor.

A four-dimensional Lorentzian manifold that satisfies particular geometric requirements (25) established from Yang's gravitational theory and where the gravitational field acts as a pure gauge is referred to as a Yang pure space in the context of modified gravity. This condition frequently places stringent restrictions on the geometry within gravity, such as demanding a constant scalar curvature or particular conformal characteristics. Q -flat spacetimes are defined in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity like Yang pure spacetime, which frequently indicates a certain conformal collineation, where the gravitational field equations take a simplified form, allowing for analytical solutions in modified $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity theory.

As a result, we may conclude the following.

Theorem 7. *If a BVSFS (M^4, g) in $\Xi^{\mathbb{F}}$ has isotropic pressure p and rest energy density ρ , and is endowed with a Codazzi-type Ricci tensor, then the BVSFS is a Yang pure spacetime.*

Theorem 8. *If a BVSFS (M^4, g) in $\Xi^{\mathbb{F}}$ has isotropic pressure p and rest energy density ρ , and is endowed with a Codazzi-type Ricci tensor, then the BVSFS is an RW -spacetime.*

Theorem 9. *If a spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ has isotropic pressure p , rest energy density ρ , and is endowed with a Codazzi-type Ricci tensor, then the $\mathcal{EoS}$ of BVSFS is given by (31).*

The source is then assumed to be of the radiation kind. When used with (31), this information provides

$$p = \frac{3}{4}\mathcal{U}\mathcal{H}, \quad \rho = \frac{9}{4}\mathcal{U}\mathcal{H}. \quad (32)$$

Corollary 6. *If the source of BVSFS (M^4, g) in $\Xi^{\mathbb{F}}$ is a radiation type with a Codazzi-type Ricci tensor, then the isotropic pressure p is $\frac{3}{4}\mathcal{U}\mathcal{H}$ and the rest energy density ρ is $\frac{9}{4}\mathcal{U}\mathcal{H}$.*

4. Symmetries in $\Xi^{\mathbb{F}}$

The bulk viscous string fluid spacetime (M^4, g) is assumed to be Ricci semi-symmetric in this section.

Definition 2 ([36]). *The Riemannian manifold is designated as Ricci semi-symmetric if its curvature tensor satisfies*

$$\mathcal{R}im \cdot S = 0, \quad (33)$$

where $\mathcal{R}im$ indicates the Riemannian curvature and S is the Ricci curvature tensor.

The difference between the volume of an extremely small area in curved space and flat space is described by the Ricci tensor. Understanding gravitational theories, especially $\mathbb{F}(\mathbb{R})$ -gravity and $\Xi^{\mathbb{F}}$, which can describe complex astrophysical objects like massive neutron stars, is aided by the study of Ricci semi-symmetric manifolds. In particular, at higher dimensions or under modified gravity theories such as $\Xi^{\mathbb{F}}$, the definition of Ricci semi-symmetric manifolds helps mathematicians and physicists in their research of spacetime geometry by defining a class of manifolds that are not entirely symmetric but have a significant geometric structure.

Now,

$$(\mathcal{R}im(\mu, \nu) \cdot S)(\alpha, \beta) = S(\mathcal{R}im(\mu, \nu)\alpha, \beta) - S(\alpha, \mathcal{R}im(\mu, \nu)\beta) \quad (34)$$

for all $\mu, \nu, \alpha, \beta \in \chi(M^4)$.

Next, Equations (18) and (4) can be written in index-free notation as

$$S(\mu, \nu) = A_1 g(\mu, \nu) + A_2 \gamma(\mu) \gamma(\nu) + A_3 \sigma(\mu) \sigma(\nu). \quad (35)$$

In addition, we have $g(\mu, \zeta) = \gamma(\mu)$, $g(\mu, \xi) = \sigma(\mu)$, and $\gamma(\zeta) = -1$, $\sigma(\xi) = 1$, where ζ and ξ are orthogonal vector fields such that $g(\zeta, \xi) = 0$.

Now, from (34) we get

$$\begin{aligned} & A_1 g(\mathcal{R}im(\mu, \nu) \alpha, \beta) + A_2 \gamma(\mathcal{R}im(\mu, \nu) \alpha) \gamma(\beta) + A_3 \sigma(\mathcal{R}im(\mu, \nu) \alpha) \sigma(\beta) \\ & + A_1 g(\alpha, \mathcal{R}im(\mu, \nu) \beta) + A_2 \gamma(\alpha) \gamma(\mathcal{R}im(\mu, \nu) \beta) + A_3 \sigma(\alpha) \sigma(\mathcal{R}im(\mu, \nu) \beta) = 0. \end{aligned} \quad (36)$$

Inserting $\beta = \zeta$ and $\alpha = \xi$ in (36), we get

$$A_2 \gamma(\mathcal{R}im(\mu, \nu) \xi) + A_3 \sigma(\mathcal{R}im(\mu, \nu) \zeta) = 0$$

or

$$(A_3 - A_2) \bar{\mathcal{R}}im(\mu, \nu, \zeta, \xi) = 0, \quad (37)$$

where $\bar{\mathcal{R}}im(\mu, \nu, \zeta, \xi) = g(\mathcal{R}im(\mu, \nu) \zeta, \xi)$.

Consequently, Equation (37) argues either $A_3 - A_2 = 0$ or $\bar{\mathcal{R}}im(\mu, \nu, \zeta, \xi) = 0$. Therefore, we have the following situations.

Case (i): If $A_3 = A_2$, then $\bar{p} + \varrho = \lambda$.

Case (ii): Let $\bar{\mathcal{R}}im(\mu, \nu, \zeta, \xi) = 0$ for all $\mu, \nu, \zeta, \xi \in \chi(M^4)$.

Again, setting $\mu = \nu = \zeta$, after contraction we get $S(\zeta, \zeta) = 0$ for all $\zeta \in \chi(M^4)$. Thus, (18) entails that $(A_1 - A_2) \gamma(\zeta) = 0$, so $A_1 = A_2$. Hence, $A_1 = A_2$ signifies

$$\bar{p} + \varrho = \frac{\frac{1}{2} \mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]}{2[\mathbb{F}'(\mathbb{T}) + 4\pi]}. \quad (38)$$

where $2[\mathbb{F}'(\mathbb{T}) + 4\pi] \neq 0$. Thus, we can articulate the following.

Theorem 10. Let the spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ be a Ricci semi-symmetric spacetime, then the $\mathcal{EoS}$ is $\bar{p} + \varrho = \frac{\frac{1}{2} \mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]}{2[\mathbb{F}'(\mathbb{T}) + 4\pi]}$, provided $2[\mathbb{F}'(\mathbb{T}) + 4\pi] \neq 0$.

Next, we presume that the source is of the radiation type in the stiff matter era or vacuum-dominated era. When combined with (38), this information yields

$$\bar{p} = \frac{\mathbb{R}}{8[\mathbb{F}'(\mathbb{T}) + 10\pi + \mathbb{F}(\mathbb{T})]}, \quad \varrho = \frac{3\mathbb{R}}{8[\mathbb{F}'(\mathbb{T}) + 10\pi + \mathbb{F}(\mathbb{T})]} \quad (39)$$

$$\bar{p} = \varrho = \frac{\mathbb{R}}{4[3\mathbb{F}'(\mathbb{T}) + 8\pi + 2\mathbb{F}(\mathbb{T})]}, \quad (40)$$

$$\bar{p} = -\varrho = \frac{\mathbb{R}}{8[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]}, \quad (41)$$

Consequently, with the help of (39), (40), and (41), we get the outcomes below.

Theorem 11. If the source of Ricci semi-symmetric BVSF-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ is stiff matter fluid, then the density ϱ and total pressure $\bar{p}$ is given by (39).

Corollary 7. If the source of Ricci semi-symmetric BVSF-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ is of the radiation type, then the density ϱ and total pressure $\bar{p}$ is given by (40).

Corollary 8. *If the source of Ricci semi-symmetric BVSF-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ is of the vacuum type, then the density ϱ and total pressure $\bar{p}$ is given by (41).*

Remark 2. *The converse, that any Ricci semi-symmetric manifold is Ricci pseudo-symmetric (see [36]), is not true in this case.*

Therefore, in light of the above remark and Theorem 10, we get the following result.

Theorem 12. *Let the spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ with the $\mathcal{EoS}$ be $\bar{p} + \varrho = \frac{\frac{1}{2}\mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]}{2[\mathbb{F}'(\mathbb{T}) + 4\pi]}$, then the spacetime (M^4, g) is Ricci pseudo-symmetric bulk viscous string fluid spacetime in $\Xi^{\mathbb{F}}$ provided $2[\mathbb{F}'(\mathbb{T}) + 4\pi] \neq 0$.*

5. Energy Constraint and Penrose Theorem in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -Gravity

Any black hole will always have some kind of geodesic incompleteness if the matter meets adequate energy requirements according to the Penrose theorem [37]. When the energy of matter is non-negative, the weak energy condition required by the black hole singularity theorem is valid because light rays are always focused together by gravity and never blow away. To be more precise, this theorem only holds true when matter meets the strong energy requirement, which is accomplished when the energy is greater than the pressure.

If we can ascertain whether the Ricci tensor S in the spacetime satisfies the criterion, then Equation (42) is referred to as the *TCC (time-like convergence condition)*:

$$S(\mu, \mu) > 0, \quad (42)$$

for all time-like vector fields $\mu \in \chi(M^4)$.

In light of Theorem 2, we consider that the spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ is a GQE-spacetime with unit time-like velocity vector field ζ , thus we get $g(\zeta, \zeta) = -1$. Now, adopting $\mu = \nu = \zeta$ in (16), we get

$$S(\zeta, \zeta) = (\rho + \bar{p})(8\pi + 2\mathbb{F}'(\mathbb{T})) - \frac{1}{2}\mathbb{R} - 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]. \quad (43)$$

We can now assert the subsequent theorem.

Theorem 13. *A GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the TCC if $(\rho + \bar{p})(8\pi + 2\mathbb{F}'(\mathbb{T})) > [\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]]$.*

TCC is the geometric representation of the strong energy condition (SEC) [38,39]. Furthermore, the null convergence condition (NCC) is implied by the TCC, implying the null energy condition (NEC) [38]. Considering (43) and Theorem 13, we now arrive at

Corollary 9. *A GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the NCC if $(\bar{p} + \varrho)(8\pi + 2\mathbb{F}'(\mathbb{T})) > [\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]]$, provided the four-velocity vector field ζ .*

Again, (43) and Corollary 9 entail the following

Theorem 14. *If GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the NEC $(\bar{p} + \varrho) > 0$, this gives $\frac{[\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]]}{(8\pi + 2\mathbb{F}'(\mathbb{T}))} > 0$.*

The spacetime M fulfills the $SCC \Rightarrow NCC$ according to Penrose's singularity theorem [37], as shown by Vilenkin and Wall ([40]). This means the following:

- (i) The Cauchy surface that is connected to spacetime M is not compact;
- (ii) There are some black holes in spacetime M , and there is a trapped surface outside of them.

Thus, we obtain the following two findings by applying the aforementioned facts (1) and (2) in conjunction with Corollary 9.

Theorem 15. A GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the NCC if

$$(\bar{p} + q)(8\pi + 2\mathbb{F}'(\mathbb{T})) > \left[\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]\right], \quad (44)$$

thus GQE-spacetime (M^4, g) admits a connected and non-compact Cauchy surface.

Theorem 16. If a GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the NCC (44), then GQE-spacetime (M^4, g) admits black holes and there is a trapped surface outside of them.

Additionally, the Ricci tensor S and the energy–momentum tensor $\mathbb{T}$ are examples of symmetric second-order tensors defined on the four-dimensional Lorentzian spacetime manifold (M^4, g) that are classified algebraically as *Segre classes* (for more information, see [41,42]). Restrictions are imposed by the energy condition on the energy–momentum tensors or equivalently on the Ricci tensor of the GTR for matter fields like those represented by Segre classes. The most common energy condition is *SEC* [38]. The *SEC* is the assertion that for any time-like vector field $S(\mu, \nu) > 0$, then S belongs to the Segre class [(11),(11)] [41,42].

Considering the above facts and using Theorem 13, we are able to state the following results.

Theorem 17. If a GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the SEC with time-like vector field ζ , with condition

$$(\bar{p} + q)(8\pi + 2\mathbb{F}'(\mathbb{T})) > \left[\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]\right],$$

then Ricci tensor S belongs to the Segre class [(11), (11)].

Corollary 10. A GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$ obeys the SEC if condition

$$(\bar{p} + q)(8\pi + 2\mathbb{F}'(\mathbb{T})) > \left[\frac{1}{2}\mathbb{R} + 4\bar{p}[\mathbb{F}'(\mathbb{T}) + 2\pi + \mathbb{F}(\mathbb{T})]\right],$$

thus GQE-spacetime (M^4, g) is of Ludwig–Scanlan types A_1 and A_2 [43].

For a time-like vector field ζ , through Equation (18) we have

$$S(\mu, \zeta) = (A_1 - A_2)g(\mu, \zeta) \quad \text{for all } \mu. \quad (45)$$

$$\bar{p} = -\frac{q(8\pi + 2\mathbb{F}'(\mathbb{T}))}{[16\pi + 6\mathbb{F}'(\mathbb{T}) + 4\mathbb{F}(\mathbb{T})]} + \frac{\mathbb{R}}{2[16\pi + 6\mathbb{F}'(\mathbb{T}) + 4\mathbb{F}(\mathbb{T})]}, \quad (46)$$

provided $16\pi + 6\mathbb{F}'(\mathbb{T}) + 4\mathbb{F}(\mathbb{T}) \neq 0$. Consequently, we arrive at the following conclusion.

Theorem 18. In a GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$, the generator ζ is an eigenvector of the Ricci tensor corresponding to the eigenvalue $A_1 - A_2$, thus $\mathcal{EoS}$ is furnished by (46).

In addition, if we set $\mathbb{R} = 0$ in (46), then we obtain $\mathcal{EoS}, \frac{\bar{p}}{\bar{\rho}} = -\frac{(8\pi+2\mathbb{F}'(\mathbb{T}))}{[16\pi+6\mathbb{F}'(\mathbb{T})+4\mathbb{F}(\mathbb{T})]}$. This leads us to the next corollary.

Corollary 11. Consider a GQE-spacetime (M^4, g) in $\Xi^{\mathbb{F}}$. If the generator ζ is an eigenvector of the Ricci tensor and $\mathbb{R} = 0$, then the evolution of the universe is given in the following table through the $\mathcal{EoS}$ of the bulk viscous string fluid spacetime in $\Xi^{\mathbb{F}}$ as

Equation of state $\frac{\bar{p}}{\bar{\rho}} = \omega$	Restrictions on $\mathbb{F}(\mathbb{T})$ and $\mathbb{F}'(\mathbb{T})$	Evolution of the Universe
$\omega = 1$	$\mathbb{F}(\mathbb{T}) = -2(3\pi + \mathbb{F}'(\mathbb{T}))$	Ultra-relativistic era
$\omega > -1$	$\mathbb{F}'(\mathbb{T}) < -(3\pi + \frac{\mathbb{F}(\mathbb{T})}{2})$	Quintessence era
$\omega < -1$	$\mathbb{F}'(\mathbb{T}) > 2\pi + \mathbb{F}(\mathbb{T})$	Phantom era
$\omega = 0$	$\mathbb{F}'(\mathbb{T}) = -4\pi$	Dust era

6. Conclusions

$\mathbb{F}(\mathbb{R})$ -gravity is a family of modified gravity theories that are a generalization of the GTR obtained by replacing the Ricci scalar in the Einstein–Hilbert action with a generic function of $\mathbb{R}$. Another interesting extension is $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity, where the gravitational action is an arbitrary function of the trace of the energy–momentum tensor $\mathbb{T}$ and the Ricci scalar $\mathbb{R}$. Modified gravity theory has several models, such as $\mathbb{F}(\mathbb{R})$ and $\mathbb{F}(\mathbb{R}, \mathbb{T})$. Higher-order derivative problems are introduced by the modified field equations of $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity, which are often fourth-order and provide richer geometric structures than the GTR, indicating that $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity allows for matter particle motion that is non-geodesic (influenced by an extra force). The additional elements $\mathbb{T}$, which give dark components a geometric origin, can be understood as either an effective cosmological constant or as an effective description of dark matter. Because the updated field equations provide distinct behaviors for gravitational radiation, the theory of $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity offers a new framework for studying gravitational waves.

The modified $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity hypothesis provides an effective representation of the universe's abrupt expansion during the late cosmic eras. Numerous phenomena have been revealed by our study of bulk viscous string fluid matter in modified gravity, most notably its fascinating relationship to certain geometric symmetries like Ricci semi-symmetry and Ricci pseudo-symmetry. In the particular $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity model, we obtain the formula for the Ricci scalar for the spacetime of the bulk viscous fluid string matter in conjunction with the field equation of the $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity model. Additionally, using the $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity framework, we calculate the density and pressure across the radiation and phantom barrier eras of the universe and obtain the equation of state for the bulk viscous fluid string matter fluid. We demonstrate that bulk viscous fluid string spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity with a Codazzi-type Ricci tensor is both a Robertson–Walker spacetime and a Yang pure spacetime. In addition, a bulk viscous fluid string spacetime in $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity holds for the dark matter era with a Codazzi-type Ricci tensor.

Lastly, we propose certain conditions for the bulk viscous fluid string matter using the given $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity, taking into consideration different energy sources, the existence of black holes, and some Segre class of Ricci tensor with the bulk viscous fluid string matter.

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Acronyms

$\boxtimes^{\mathbb{F}}$	The $\mathbb{F}(\mathbb{R}, \mathbb{T})$ -gravity model with <i>BVSF</i> matter
<i>GTR</i>	General theory of relativity
<i>BVSF</i>	Bulk viscous string fluid
<i>SEC</i>	Strong energy condition
<i>TCC</i>	Time-like convergence condition
<i>NCC</i>	Null convergence condition
<i>NEC</i>	Null energy condition

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