

## Article

# Block Subgradient Greedy Kaczmarz Methods for Robotic Grasping Problems

Zhiwei Ai <sup>1,2,3,4</sup>  and Chenliang Li <sup>1,5,6,\*</sup> 

<sup>1</sup> School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China; aizhiwei@guet.edu.cn

<sup>2</sup> School of Artificial Intelligence, Guilin University of Aerospace Technology, Guilin 541004, China

<sup>3</sup> Guangxi Zhuang Autonomous Region Engineering Research Center for Artificial Intelligence and Aerospace Information Scenarios, Guilin 541004, China

<sup>4</sup> University Engineering Research Center of Integrated Aerospace-Terrestrial Information and Control, Guilin 541004, China

<sup>5</sup> Center for Applied Mathematics of Guangxi (GUET), Guilin 541004, China

<sup>6</sup> Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin 541004, China

\* Correspondence: chenli@guet.edu.cn

## Abstract

The robotic grasping problem is firstly cast as a linear complementarity problem and reformulated into nonsmooth equations. We propose two block Kaczmarz methods to solve the equations. The K-means clustering algorithm is utilized to partition the nonsmooth equations. Based on the resulting block information, the subgradient greedy randomized Kaczmarz method and the subgradient multi-step greedy Kaczmarz method are constructed. Asymptotic convergence of the proposed block subgradient Kaczmarz methods is given. Numerical results confirm the superior computational accuracy of the new methods.

**Keywords:** robotic grasping problem; complementarity problem; Kaczmarz method; clustering algorithm; convergence

**MSC:** 65H10; 90C33

## 1. Introduction

Robotic grasping lies at the heart of manipulation pipelines that integrate perception, grasp-pose generation, trajectory planning, constraint satisfaction, contact resolution, and feedback control. Recent progress has notably improved perception and grasp planning. Category-level semantically aware methods strengthen generalization through task-related semantics [1], while occlusion-aware grasping explicitly models occlusion and scene context to boost prediction and execution reliability [2]. These works stress the tight coupling between visual perception, high-level grasp planning, and physical hand–object contact. Executing a grasp further involves online numerical computation, manipulator control, and collision avoidance. Robust zeroing neural networks tackle online dynamic matrix solving and trajectory tracking under disturbances and noise [3]. For uncertain manipulators, an Udwadia-based adaptive robust constraint-following framework delivers prescribed tracking performance despite time-varying uncertainties and sensor noise [4]. At the motion-planning level, parallel model predictive path-integral control with a gradient–velocity-modulated signed-distance-field cost enables real-time motion in dynamic obstacle environments [5]. Taken together, these studies demonstrate that grasp



Academic Editor: Behzad Djafari-Rouhani

Received: 3 June 2026

Revised: 25 June 2026

Accepted: 26 June 2026

Published: 30 June 2026

**Copyright:** © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

planning, trajectory tracking, constraint enforcement, and obstacle avoidance form a tightly interwoven core of robotic manipulation.

During robotic grasping, the interplay between grasping displacement and contact force exhibits a complementary relationship. By incorporating the equilibrium equation, friction law, contact and slip condition, maximum dissipation principle, as well as material deformation principle, the grasping process can be formulated as a complementarity problem [6–8]. Through linearization of the friction law, the grasping problem can be further expressed as a linear complementarity problem model [9,10]. With the model, efficient numerical approaches can be used to compute grasping physical quantities, such as grasping stability [11], contact force [8], slippage damage, and so on.

The coefficient matrix of the linear complementarity problem often fails to satisfy the convergence conditions required by conventional solution methods, such as projection [12] or modulus-based matrix splitting [13] iterations. To address this problem, the linear complementarity problem is reformulated as an equivalent system of nonsmooth equations [14]. By integrating the subgradient Newton method with the Kaczmarz algorithm, we establish an effective novel numerical approach to obtain the minimum-norm solution that satisfies the complementarity condition. Here, the most crucial work is designing an appropriate Kaczmarz iterative algorithm.

Ansorge [15] pioneered the greedy residual Kaczmarz algorithm by incorporating a greedy row-selection strategy that prioritizes the equation with the maximal residual component at previous iterations. Strohmer [16] established the exponential convergence rate for a randomized row-selection strategy, which inspired numerous subsequent refinements and extensions. Nutini [17] demonstrated that employing a non-uniform probability distribution for row selection converges at least as rapidly as using a uniform distribution. Bai [18] pioneered the greedy randomized Kaczmarz algorithm by embedding a residual-guided randomized sampling mechanism via a predefined index set. Critically, these algorithms employ exactly one equation per iteration.

Concurrently, block Kaczmarz variants have emerged as a prominent research direction. Needell [19] formulated the theoretical foundation for convergence in block-iterative Kaczmarz methods by deriving orthogonal projection onto a subspace defined by multiple rows of the system matrix. Building on this, Needell [20] proposed the Randomized Block Kaczmarz algorithm. Integrating a residual-guided greedy randomized sampling mechanism into the Randomized Block Kaczmarz framework, Liu [21] presented the Greedy Randomized Block Kaczmarz algorithm for iterative linear system resolution. Jiang [22,23] incorporated the clustering algorithm into the block Kaczmarz framework. More recently, Xiao [24] extended the block greedy Kaczmarz method to systems of smooth nonlinear equations, and further applied it to nonlinear signal recovery problems [25].

The Newton Subgradient Nonsmooth Greedy Randomized Kaczmarz (NSNGRK) method [26] and the Newton Subgradient Nonsmooth Multi-Step Greedy Randomized Kaczmarz (NSNMKGK) method [27] have been employed to compute grasp-related parameters such as grasping forces and relative displacement in robotic grasping problems. We aim to enhance solution accuracy by integrating a block-wise framework based on the K-means clustering algorithm into the aforementioned methods and accordingly propose the Block Newton-Subgradient Nonsmooth Greedy Randomized Kaczmarz (BNSNGRK\_k) method and the Block Newton-Subgradient Nonsmooth Multi-Step Greedy Kaczmarz (BNSNMKGK\_k) method. In these approaches, the residuals are partitioned into several clusters based on Euclidean distance. By integrating a greedy block-selection strategy, this approach prioritizes iteration on blocks containing rows corresponding to larger residual components, thereby promoting more accurate solutions.

This paper proceeds as follows: Section 2 formalizes essential mathematical preliminaries and foundational lemmas. Section 3 details the construction of the proposed block subgradient greedy Kaczmarz methods and derives their convergence guarantees. Section 4 validates computational efficacy through numerical experiments. Section 5 concludes with a comprehensive summary of key contributions.

## 2. Problem Description and Preliminaries

The linear complementarity problem arising from multi-fingered robotic grasping can be expressed as

$$w = Mz + q \geq 0, \quad z \geq 0, \quad w^T z = 0.$$

Here,  $q$  is the right-hand-side vector, and  $z$  and  $w$  form a pair of complementary vectors. The coefficient matrix  $M$  has the following block structure:

$$M = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where  $X$ ,  $Y$ , and  $Z$  are matrix blocks of compatible dimensions, and  $0$  denotes a zero matrix of the appropriate size.

A key tool for reformulating such problems is the Fisher–Burmeister function  $\phi(w, z) = \sqrt{w^2 + z^2} - w - z$ , which possesses the equivalence property  $\phi(w, z) = 0 \iff w \geq 0, z \geq 0, wz = 0$ . Using this function, the complementarity problem is recast as a root-finding problem for the nonsmooth vector equation

$$\Phi(z) = [\phi_1(z), \dots, \phi_n(z)]^T = 0.$$

To solve this nonsmooth equation, a subgradient Newton approach is employed, which requires computing the generalized Jacobian matrix  $J$  of  $\Phi$ . Here,  $J$  is evaluated efficiently via a dual-number automatic differentiation scheme [28]. A dual number  $z$  is defined as  $z = a + b\varepsilon$ , where  $a, b \in \mathbb{R}$  are the primal and tangent components, respectively, and  $\varepsilon$  satisfies  $\varepsilon^2 = 0, \varepsilon \neq 0$ . For a nonsmooth vector function  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , the  $i$ th column of  $J$  is obtained by perturbing the  $i$ th input direction: let  $z_{\text{dual}}^{(i)} = z + e_i\varepsilon$ , where  $e_i$  is the  $i$ th standard basis vector. Substituting into  $\Phi$  gives the dual expansion  $\Phi(z_{\text{dual}}^{(i)}) \approx \Phi(z) + J_{:,i}\varepsilon$ , where  $J_{:,i}$  is precisely the tangent part of the output. Extracting this tangent component for each  $i = 1, \dots, n$  and assembling the results column-wise yields the full generalized Jacobian matrix  $J \in \mathbb{R}^{n \times n}$ .

Let  $|\tau|$  denote the cardinality of a set  $\tau$ , and define  $[n] := \{1, \dots, n\}$ . For a vector  $x$ ,  $\|x\|_2$  denotes the Euclidean norm. For  $J \in \mathbb{R}^{m \times n}$ , define  $\sigma_{\min}(J)$  and  $\sigma_{\max}(J)$  as the minimum and maximum singular values, respectively, and denote  $\|J\|_2$  and  $\|J\|_F$  as the spectral and Frobenius norms and  $J_\tau$  as the submatrix formed by the rows indexed by the set  $\tau$ .

**Definition 1** ([29]). Let  $\phi$  be a proper convex function and point  $z_1 \in \text{dom}\phi$ . A vector  $g \in \mathbb{R}^n$  is called a subgradient of  $\phi$  at  $z_1$  if it satisfies

$$\phi(z_2) \geq \phi(z_1) + g^T(z_2 - z_1), \quad \forall z_2 \in \text{dom}\phi.$$

The set of all such vectors,

$$\partial\phi(z_1) = \left\{ g \in \mathbb{R}^n \mid \phi(z_2) \geq \phi(z_1) + g^T(z_2 - z_1), \forall z_2 \in \text{dom}\phi \right\},$$

is called the subdifferential of  $\phi$  at  $z_1$ .

**Definition 2** ([30]). Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a locally Lipschitz function. The generalized Jacobian of  $\Phi$  at a point  $z$ , denoted  $\partial\Phi(z)$ , is defined by

$$\partial\Phi(z) = \text{conv} \left\{ \lim_{k \rightarrow \infty} \nabla\Phi(z_k) \mid z_k \rightarrow z, z_k \in D_\Phi \right\},$$

where  $D_\Phi$  denotes the set of points at which  $\Phi$  is differentiable, and  $\text{conv}$  represents the convex hull.

**Definition 3** ([31]). A matrix  $A \in \mathbb{R}^{m \times n}$  satisfies the row lower bound property if there exists  $\epsilon > 0$  such that

$$\|A(i, :)\|_2 \geq \epsilon, \quad \forall i \in [m].$$

**Definition 4** ([31]). A function  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfies the local tangential cone condition if there exists  $\eta \in [0, 1/2)$  such that for all  $i \in [n]$  and  $z_1, z_2 \in \mathbb{R}^n$ ,

$$|\phi_i(z_1) - \phi_i(z_2) - (\nabla\phi_i(z_1))^T(z_1 - z_2)| \leq \eta |\phi_i(z_1) - \phi_i(z_2)|.$$

The parameter  $\eta$  in the local tangential cone condition is understood as a uniform upper bound within a neighborhood of a solution rather than as a global constant valid over the entire iteration trajectory. Its value may depend on the nonlinear mapping, the selected generalized Jacobian, and the local neighborhood under consideration. In the numerical experiments, the iteration-dependent quantity  $\hat{\eta}_k$  is used only as an empirical diagnostic of the local linearization error and should not be identified with a rigorously established global value of  $\eta$ .

**Lemma 1** ([32]). The Fisher–Burmeister function is smooth everywhere except at the origin and is locally Lipschitz continuous.

**Lemma 2** ([33]). Let  $A \in \mathbb{R}^{m \times n}$  have full column rank. Then for all  $z \in \mathbb{R}^n$ ,

$$\|Az\|_2^2 \geq \sigma_{\min}^2(A) \|z\|_2^2.$$

**Lemma 3** ([26]). Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a nonsmooth function satisfying the local tangential cone condition at point  $z_2$ . If  $\nabla\phi(z_2) \in \partial\phi(z_1)$ , then for  $\eta \in [0, 1/2)$ , there exists a subgradient  $g(z_1) \in \partial\phi(z_1)$  such that

$$|\phi(z_1) - \phi(z_2) - g^T(z_1)(z_1 - z_2)| \leq \eta |\phi(z_1) - \phi(z_2)|.$$

### 3. Block Subgradient Greedy Kaczmarz Method Based on K-Means Clustering and Convergence Analysis

The condition number  $\kappa(J)$  of the generalized Jacobian matrix  $J$  directly governs both the convergence accuracy and the convergence rate of the Kaczmarz method [34]. In the block variant, the convergence behavior is related to the condition numbers of the block Jacobian matrices  $J_{\tau_i}$ . An appropriate partition may produce block Jacobian matrices with better conditioning than the complete Jacobian matrix, thereby improving the efficiency of the block projection steps.

The K-means clustering algorithm provides an effective approach for partitioning the equations into coherent blocks. In this work, K-means is applied to the rows of the generalized Jacobian matrix. Rows with similar subgradient characteristics are assigned to the same block. The residual vector is not included in the clustering data, thereby avoiding the scale and dimensional inconsistency between derivative information and function values.

The following presents the core derivation and design analysis of the algorithm. First, we utilize the central difference scheme with an optimal step size to numerically replicate the key properties of the dual-number method—specifically, the function value from the

real part and the derivative from the tangent part. For the current iterate  $\mathbf{z}^{(k)}$ , a small perturbation  $h = \sqrt{\text{eps}}$  (the square root of machine precision) is chosen. The  $i$ -th column of the numerical generalized Jacobian is then constructed as

$$J_{:,i}(\mathbf{z}) = \frac{\Phi(\mathbf{z} + h\mathbf{e}_i) - \Phi(\mathbf{z} - h\mathbf{e}_i)}{2h},$$

where  $\mathbf{e}_i$  is the  $i$ -th unit vector. By constructing each column sequentially, an approximate generalized Jacobian matrix  $\mathbf{J} \in \mathbb{R}^{n \times n}$  is obtained. This matrix effectively captures the differential characteristics of  $\Phi$  at  $\mathbf{z}$ , forming the foundation for the subsequent block Newton iterations.

Denote the residual at the current iterate by  $\mathbf{r} = \Phi(\mathbf{z})$ . At each outer iteration, K-means is applied only to the rows of the current generalized Jacobian matrix  $\mathbf{J}$ . Using the sum of squared Euclidean distances as the clustering criterion, the row indices are partitioned into  $p$  disjoint blocks,

$$\{\tau_i\}_{i=1}^p, \quad \bigcup_{i=1}^p \tau_i = \{1, 2, \dots, n\}, \quad \tau_i \cap \tau_j = \emptyset \quad (i \neq j).$$

For the  $i$ -th block, its Jacobian centroid is defined by

$$\mathbf{c}_i = \frac{1}{|\tau_i|} \sum_{j \in \tau_i} J_{j,:}.$$

Since the K-means centroid  $\mathbf{c}_i$  is generally not an actual row of  $\mathbf{J}$ , we select the row in the block that is closest to the centroid:

$$j_i = \arg \min_{j \in \tau_i} \|J_{j,:} - \mathbf{c}_i\|_2.$$

The representative Jacobian row and its corresponding representative residual are then defined as  $\hat{\mathbf{J}}_i = J_{j_i,:}$ ,  $\hat{\mathbf{r}}_i = \mathbf{r}_{j_i}$ . Thus,  $\hat{\mathbf{J}}_i$  and  $\hat{\mathbf{r}}_i$  are associated with the same original nonsmooth equation. The representative quantities of all blocks are assembled as

$$\hat{\mathbf{J}} = [\hat{\mathbf{J}}_1^T, \dots, \hat{\mathbf{J}}_p^T]^T \in \mathbb{R}^{p \times n}, \quad \hat{\mathbf{r}} = [\hat{\mathbf{r}}_1, \dots, \hat{\mathbf{r}}_p]^T \in \mathbb{R}^p.$$

These representative quantities are used for greedy block filtering and randomized block selection. Inspired by the greedy randomized Kaczmarz method, the threshold  $\epsilon_k$  is defined as

$$\epsilon_k = \frac{1}{2} \left( \frac{1}{\|\hat{\mathbf{r}}\|_2^2} \max_{1 \leq i \leq p} \frac{|\hat{\mathbf{r}}_i|^2}{\|\hat{\mathbf{J}}_i\|_2^2} + \frac{1}{\|\hat{\mathbf{J}}\|_F^2} \right).$$

The candidate block set is given by

$$\mathcal{U}_k = \left\{ j \mid |\hat{\mathbf{r}}_j|^2 \geq \epsilon_k \|\hat{\mathbf{r}}\|_2^2 \|\hat{\mathbf{J}}_j\|_2^2 \right\}.$$

A block  $j \in \mathcal{U}_k$  is selected with probability

$$\Pr(i_k = j) = \frac{|\hat{\mathbf{r}}_j|^2}{\sum_{\ell \in \mathcal{U}_k} |\hat{\mathbf{r}}_\ell|^2}.$$

Since the residual vector does not participate in the K-means distance, the convergence of  $\mathbf{r}$  toward zero does not cause scale degeneration in the clustering metric. Nevertheless, residual information is retained in the greedy filtering and randomized block-selection rules.

Finally, after the block index  $i_k$  has been selected using the representative quantities, the actual iteration is performed using the complete Jacobian submatrix  $J_{\tau_{i_k}}$  and the complete residual subvector  $r_{\tau_{i_k}}$ . The block Newton–Kaczmarz update is

$$z^{(k+1)} = z^{(k)} - J_{\tau_{i_k}}^{\dagger} r_{\tau_{i_k}}.$$

The Moore–Penrose pseudoinverse ensures that the update remains well defined when the selected Jacobian block is rank-deficient. Therefore, the representative Jacobian rows and residuals are used only for block selection, whereas all equations in the selected block are used in the solution update.

After each iteration, the residual is updated as  $r = \Phi(z^{(k+1)})$ . The generalized Jacobian is recomputed and its rows are re-clustered at each outer iteration. The procedure terminates when  $\|r\|_2 < \epsilon$ , at which point an approximate solution of the nonsmooth system is obtained. We refer to the resulting method as the BNSNGRK\_k algorithm. Its complete computational procedure is presented in Algorithm 1.

To fully utilize the greedy condition, we extend the BNSNGRK\_k method by modifying its block-selection strategy within each iteration. Instead of choosing only one block per iteration, the proposed variant sequentially utilizes all matrix blocks that satisfy the greedy criterion as working row blocks. This multi-step update scheme gives rise to the BNSNMGK\_k method. The complete computational steps of BNSNMGK\_k are summarized in Algorithm 2.

We now establish the convergence of the proposed methods. First, we present some key lemmas.

**Lemma 4.** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a nonsmooth vector-valued function satisfying the local tangential cone condition, and let  $J(z) \in \partial\Phi(z)$  denote its Clarke generalized Jacobian. For any  $\tau \subseteq [m]$  and  $z_1, z_2 \in \mathbb{R}^n$  with  $\eta \in [0, 1/2)$ ,

$$\|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2) - J_{\tau}(z_1)(z_1 - z_2)\|_2^2 \leq \eta^2 \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2)\|_2^2,$$

and

$$\|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2)\|_2^2 \geq \frac{1}{1 + \eta^2} \|J_{\tau}(z_1)(z_1 - z_2)\|_2^2.$$

**Proof.** Applying Lemma 3 component-wise to each  $i \in \tau$  yields

$$\begin{aligned} & \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2) - J_{\tau}(z_1)(z_1 - z_2)\|_2^2 \\ &= \sum_{i \in \tau} |\phi_i(z_1) - \phi_i(z_2) - g_i^T(z_1 - z_2)|^2 \\ &\leq \sum_{i \in \tau} \eta_i^2 |\phi_i(z_1) - \phi_i(z_2)|^2 \\ &\leq \eta^2 \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2)\|_2^2, \end{aligned}$$

thus proving the first inequality.

For the second inequality, we begin with the derived bound and apply the reverse triangle inequality:

$$\begin{aligned} & \eta^2 \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2)\|_2^2 \\ &\geq \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2) - J_{\tau}(z_1)(z_1 - z_2)\|_2^2 \\ &\geq \|J_{\tau}(z_1)(z_1 - z_2)\|_2^2 - \|\Phi_{\tau}(z_1) - \Phi_{\tau}(z_2)\|_2^2. \end{aligned}$$

Rearranging terms gives

$$\|\Phi_\tau(z_1) - \Phi_\tau(z_2)\|_2^2 \geq \frac{1}{1 + \eta^2} \|J_\tau(z_1)(z_1 - z_2)\|_2^2.$$

□

**Algorithm 1** Block Newton-Subgradient Nonsmooth Greedy Randomized Kaczmarz with K-means (BNSNGRK\_k)

**Require:** Coefficient matrix  $M \in \mathbb{R}^{n \times n}$ , vector  $q \in \mathbb{R}^n$ , tolerance  $\epsilon > 0$ , initial iterate  $z^{(0)}$

**Ensure:** Approximate solution  $z^*$

1: Initialize residual  $r^{(0)} = \Phi(z^{(0)})$  and set iteration counter  $k \leftarrow 0$ .

2: **while**  $\|r^{(k)}\|_2 \geq \epsilon$  **do**

3:   Compute generalized Jacobian

$$J_{:,i}^{(k)} = \frac{\Phi(z^{(k)} + he_i) - \Phi(z^{(k)} - he_i)}{2h}, \quad h = \sqrt{\text{eps}}, \quad i = 1, \dots, n.$$

4:   Partition the rows of  $J^{(k)}$  via K-means into  $p$  blocks:

$$\{\tau_i^{(k)}\}_{i=1}^p.$$

5:   Compute the Jacobian centroid of each block:

$$c_i^{(k)} = \frac{1}{|\tau_i^{(k)}|} \sum_{j \in \tau_i^{(k)}} J_{j,:}^{(k)}, \quad i = 1, \dots, p.$$

6:   Find the actual Jacobian row closest to each centroid and define the corresponding representative quantities:

$$j_i^{(k)} = \arg \min_{j \in \tau_i^{(k)}} \|J_{j,:}^{(k)} - c_i^{(k)}\|_2, \quad \hat{J}_i^{(k)} = J_{j_i^{(k)},:}^{(k)}, \quad \hat{r}_i^{(k)} = r_{j_i^{(k)}}^{(k)}.$$

7:   Assemble the representative Jacobian matrix and residual vector:

$$\hat{J}^{(k)} = [(\hat{J}_1^{(k)})^T, \dots, (\hat{J}_p^{(k)})^T]^T, \quad \hat{r}^{(k)} = [\hat{r}_1^{(k)}, \dots, \hat{r}_p^{(k)}]^T.$$

8:   Determine candidate blocks:

$$\epsilon_k \leftarrow \frac{1}{2} \left( \frac{1}{\|\hat{r}^{(k)}\|_2^2} \max_{1 \leq i \leq p} \frac{|\hat{r}_i^{(k)}|^2}{\|\hat{J}_i^{(k)}\|_2^2} + \frac{1}{\|\hat{J}^{(k)}\|_F^2} \right)$$

$$\mathcal{U}_k \leftarrow \left\{ j \mid |\hat{r}_j^{(k)}|^2 \geq \epsilon_k \|\hat{r}^{(k)}\|_2^2 \|\hat{J}_j^{(k)}\|_2^2 \right\}.$$

9:   Select block  $i_k \in \mathcal{U}_k$  with probability

$$\Pr(i_k = j) = \frac{|\hat{r}_j^{(k)}|^2}{\sum_{\ell \in \mathcal{U}_k} |\hat{r}_\ell^{(k)}|^2}.$$

10:   Update solution:

$$z^{(k+1)} \leftarrow z^{(k)} - \left( J_{\tau_{i_k}^{(k)}}^{(k)} \right)^\dagger r_{\tau_{i_k}^{(k)}}^{(k)}.$$

11:   Update residual  $r^{(k+1)} \leftarrow \Phi(z^{(k+1)})$  and set  $k \leftarrow k + 1$ .

12: **end while**

---

**Algorithm 2** Block Newton-Subgradient Nonsmooth Multi-Step Kaczmarz with K-means (BNSNMKG\_k)

---

**Require:** Coefficient matrix  $M \in \mathbb{R}^{n \times n}$ , vector  $q \in \mathbb{R}^n$ , number of clusters  $p$ , tolerance  $\epsilon > 0$ , initial iterate  $z^{(0)}$

**Ensure:** Approximate solution  $z^*$

1: Initialize residual  $r^{(0)} = \Phi(z^{(0)})$  and set iteration counter  $k \leftarrow 0$ .

2: **while**  $\|r^{(k)}\|_2 \geq \epsilon$  **do**

3:     **Initialization and preprocessing:** Same as Steps 3 to 8 of Algorithm 1; compute  $J^{(k)}$ , construct the blocks  $\{\tau_i^{(k)}\}_{i=1}^p$  and their representative quantities, and determine  $\mathcal{U}_k$ .

4:     Set intermediate solution:

$$z^{(k,0)} \leftarrow z^{(k)}, \quad n_k \leftarrow |\mathcal{U}_k|.$$

5:     **for**  $j = 1$  **to**  $n_k$  **do**

6:         Select the next block:

$$i_j \leftarrow \mathcal{U}_k(j).$$

7:         Multi-step update:

$$z^{(k,j)} \leftarrow z^{(k,j-1)} - \left( J_{\tau_{i_j}^{(k)}}^{(k)} \right)^\dagger r_{\tau_{i_j}^{(k)}}^{(k)}.$$

8:     **end for**

9:     Finalize the outer iteration:

$$z^{(k+1)} \leftarrow z^{(k,n_k)}.$$

10:     Update residual:

$$r^{(k+1)} \leftarrow \Phi(z^{(k+1)}), \quad k \leftarrow k + 1.$$

11: **end while**

---

**Lemma 5.** Let  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a nonsmooth vector-valued function satisfying the local tangential cone condition, with

$$\eta = \max_i \eta_i < \frac{1}{2}.$$

Assume that there exists  $z^*$  such that

$$\Phi(z^*) = 0.$$

At the  $k$ -th iteration, let  $\tau_{i_k}^{(k)}$  denote the selected block. Suppose that there exist a representative subgradient  $\hat{g}_{i_k}^{(k)}$  and a consistency parameter  $\delta_k \in [0, 1)$  such that

$$\|g_i^{(k)} - \hat{g}_{i_k}^{(k)}\|_2 \leq \delta_k \|\hat{g}_{i_k}^{(k)}\|_2, \quad \forall i \in \tau_{i_k}^{(k)},$$

where

$$g_i^{(k)} \in \partial \phi_i(z^{(k)}).$$

Consider the iteration

$$z^{(k+1)} = z^{(k)} - J_{\tau_{i_k}^{(k)}}(z^{(k)})^\dagger \Phi_{\tau_{i_k}^{(k)}}(z^{(k)}).$$

If

$$1 \leq \kappa_2^2 \left( J_{\tau_{i_k}^{(k)}}(z^{(k)}) \right) < \frac{1}{2\eta},$$

then

$$\left\| \mathbf{z}^{(k+1)} - \mathbf{z}^* \right\|_2^2 - \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2 \leq - \frac{(1 - \delta_k)^2}{|\tau_{i_k}^{(k)}| (1 + \delta_k)^2} \left[ 1 - 2\eta \kappa_2^2 \left( J_{\tau_{i_k}^{(k)}}(\mathbf{z}^{(k)}) \right) \right] \frac{|\phi_{\ell_k}(\mathbf{z}^{(k)})|^2}{\left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2^2},$$

where

$$\ell_k = \arg \max_{i \in \tau_{i_k}^{(k)}} |\phi_i(\mathbf{z}^{(k)})|,$$

and  $\kappa_2(\cdot)$  denotes the spectral condition number.

**Proof.** For simplicity, denote

$$J_k := J_{\tau_{i_k}^{(k)}}(\mathbf{z}^{(k)}), \quad \Phi_k := \Phi_{\tau_{i_k}^{(k)}}(\mathbf{z}^{(k)}).$$

Since

$$\Phi_{\tau_{i_k}^{(k)}}(\mathbf{z}^*) = \mathbf{0},$$

expanding the squared Euclidean norm gives

$$\left\| \mathbf{z}^{(k+1)} - \mathbf{z}^* \right\|_2^2 - \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2 = \left\| \mathbf{z}^{(k+1)} - \mathbf{z}^{(k)} \right\|_2^2 + 2 \left\langle \mathbf{z}^{(k+1)} - \mathbf{z}^{(k)}, \mathbf{z}^{(k)} - \mathbf{z}^* \right\rangle.$$

Substituting

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} - J_k^\dagger \Phi_k$$

yields

$$\begin{aligned} & \left\| \mathbf{z}^{(k+1)} - \mathbf{z}^* \right\|_2^2 - \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2 \\ &= \left\| J_k^\dagger \Phi_k \right\|_2^2 - 2 \left\langle J_k^\dagger \Phi_k, \mathbf{z}^{(k)} - \mathbf{z}^* \right\rangle \\ &= \left\| J_k^\dagger \Phi_k \right\|_2^2 - 2 \Phi_k^T \left( J_k^\dagger \right)^T \left( \mathbf{z}^{(k)} - \mathbf{z}^* \right). \end{aligned}$$

Using

$$\left( J_k^\dagger \right)^T = \left( J_k J_k^T \right)^\dagger J_k$$

and

$$\left\| J_k^\dagger \Phi_k \right\|_2^2 = \Phi_k^T \left( J_k J_k^T \right)^\dagger \Phi_k,$$

we obtain

$$\left\| \mathbf{z}^{(k+1)} - \mathbf{z}^* \right\|_2^2 - \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2 = - \left\| J_k^\dagger \Phi_k \right\|_2^2 + 2 \Phi_k^T \left( J_k J_k^T \right)^\dagger \left[ \Phi_k - \Phi_{\tau_{i_k}^{(k)}}(\mathbf{z}^*) - J_k \left( \mathbf{z}^{(k)} - \mathbf{z}^* \right) \right].$$

Applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} & \left\| \mathbf{z}^{(k+1)} - \mathbf{z}^* \right\|_2^2 - \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2 \\ & \leq - \left\| J_k^\dagger \Phi_k \right\|_2^2 + 2 \left\| \left( J_k J_k^T \right)^\dagger \Phi_k \right\|_2 \left\| \Phi_k - \Phi_{\tau_{i_k}^{(k)}}(\mathbf{z}^*) - J_k \left( \mathbf{z}^{(k)} - \mathbf{z}^* \right) \right\|_2. \end{aligned}$$

By the local tangential cone condition,

$$\left\| \Phi_k - \Phi_{\tau_{i_k}^{(k)}}(z^*) - J_k(z^{(k)} - z^*) \right\|_2 \leq \eta \|\Phi_k\|_2.$$

Furthermore,

$$\left\| \left( J_k J_k^T \right)^\dagger \Phi_k \right\|_2 \leq \sigma_{\max}^2(J_k^\dagger) \|\Phi_k\|_2.$$

Therefore,

$$\begin{aligned} & \left\| z^{(k+1)} - z^* \right\|_2^2 - \left\| z^{(k)} - z^* \right\|_2^2 \\ & \leq -\sigma_{\min}^2(J_k^\dagger) \|\Phi_k\|_2^2 + 2\eta \sigma_{\max}^2(J_k^\dagger) \|\Phi_k\|_2^2 \\ & = -\left[ 1 - 2\eta \kappa_2^2(J_k) \right] \sigma_{\min}^2(J_k^\dagger) \|\Phi_k\|_2^2. \end{aligned}$$

Using

$$\sigma_{\min}^2(J_k^\dagger) \geq \frac{1}{\|J_k\|_F^2},$$

we obtain

$$\left\| z^{(k+1)} - z^* \right\|_2^2 - \left\| z^{(k)} - z^* \right\|_2^2 \leq -\left[ 1 - 2\eta \kappa_2^2(J_{\tau_{i_k}^{(k)}}(z^{(k)})) \right] \frac{\left\| \Phi_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_2^2}{\left\| J_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_F^2}. \quad (1)$$

By the definition of  $\ell_k$ ,

$$\left\| \Phi_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_2^2 \geq \left| \phi_{\ell_k}(z^{(k)}) \right|^2. \quad (2)$$

From the consistency condition, for every  $i \in \tau_{i_k}^{(k)}$ ,

$$\left\| \mathbf{g}_i^{(k)} \right\|_2 \leq (1 + \delta_k) \left\| \hat{\mathbf{g}}_{i_k}^{(k)} \right\|_2$$

and

$$\left\| \mathbf{g}_i^{(k)} \right\|_2 \geq (1 - \delta_k) \left\| \hat{\mathbf{g}}_{i_k}^{(k)} \right\|_2.$$

In particular,

$$\left\| \hat{\mathbf{g}}_{i_k}^{(k)} \right\|_2 \leq \frac{1}{1 - \delta_k} \left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2.$$

Hence,

$$\left\| \mathbf{g}_i^{(k)} \right\|_2 \leq \frac{1 + \delta_k}{1 - \delta_k} \left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2, \quad \forall i \in \tau_{i_k}^{(k)}.$$

Therefore,

$$\left\| J_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_F^2 = \sum_{i \in \tau_{i_k}^{(k)}} \left\| \mathbf{g}_i^{(k)} \right\|_2^2 \leq |\tau_{i_k}^{(k)}| \left( \frac{1 + \delta_k}{1 - \delta_k} \right)^2 \left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2^2. \quad (3)$$

Combining (2) and (3) yields

$$\frac{\left\| \Phi_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_2^2}{\left\| J_{\tau_{i_k}^{(k)}}(z^{(k)}) \right\|_F^2} \geq \frac{(1 - \delta_k)^2}{|\tau_{i_k}^{(k)}|(1 + \delta_k)^2} \frac{|\phi_{\ell_k}(z^{(k)})|^2}{\left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2^2}. \quad (4)$$

Substituting (4) into (1) completes the proof.  $\square$

The requirement  $\mathcal{L}_k > 0$  is a sufficient condition for the error bound derived above to be strictly contractive. It is not a necessary condition for the convergence of the numerical method. In particular, when  $\mathcal{L}_k \leq 0$ , the present estimate cannot certify a contraction at that iteration, but this does not imply that the iteration necessarily diverges. It should also be emphasized that the spectral condition number in  $\mathcal{L}_k$  is associated with the selected Jacobian submatrix rather than with the complete saddle-point matrix. Consequently, an ill-conditioned full system may still contain selected local blocks with moderate condition numbers. Nevertheless, the block partition does not guarantee that every selected submatrix is well conditioned, and the condition should therefore be interpreted as a local and conditional assumption.

Using the above lemmas, the following two convergence theorems can be readily proven.

**Theorem 1.** Let  $\Phi : \mathbb{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a nonsmooth function defined on a compact domain  $\mathbb{D}$ . Assume  $\mathbf{z}^* \in \mathbb{D}$  satisfies  $\Phi(\mathbf{z}^*) = \mathbf{0}$ , and for all  $\mathbf{z} \in \mathbb{D}$ ,  $\Phi$  admits a Clarke generalized Jacobian  $J_C(\mathbf{z})$ . The sequence  $\{\mathbf{z}^{(k)}\}$  generated by BNSNGRK\_k converges to  $\mathbf{z}^*$  with the expected squared error satisfying

$$\mathbb{E}[\|\mathbf{z}^{(k+1)} - \mathbf{z}^*\|_2^2] \leq \left( 1 - \frac{\mathcal{L}_k(1 - \delta_k)^2}{|\tau_k| \kappa_F^2(J(\mathbf{z}^{(k)}))(1 + \delta_k)^2(1 + \eta)^2} \right) \mathbb{E}[\|\mathbf{z}^{(k)} - \mathbf{z}^*\|_2^2],$$

for  $k = 0, 1, \dots$ , where  $\mathcal{L}_k$  and  $\kappa_F(\cdot)$  denote spectral and Frobenius condition numbers, respectively.

**Proof.** Let

$$\mathcal{E}_k = \left\| \mathbf{z}^{(k)} - \mathbf{z}^* \right\|_2^2,$$

and let  $\mathcal{F}_k$  denote the information generated before the block selection at iteration  $k$ . By Lemma 5, the update associated with the selected block  $\tau_{i_k}^{(k)}$  satisfies

$$\mathcal{E}_{k+1} - \mathcal{E}_k \leq - \frac{\mathcal{L}_k(1 - \delta_k)^2}{|\tau_{i_k}^{(k)}|(1 + \delta_k)^2} \frac{|\phi_{\ell_k}(z^{(k)})|^2}{\left\| \mathbf{g}_{\ell_k}^{(k)} \right\|_2^2},$$

where

$$\mathcal{L}_k = 1 - 2\eta \kappa_2^2 \left( J_{\tau_{i_k}^{(k)}}^{(k)} \right).$$

Taking the conditional expectation with respect to the block-selection probability and using the definition of the greedy candidate set  $\mathcal{U}_k$  gives the corresponding normalized residual bound. Together with the block-consistency estimate in Lemma 5, this yields

$$\mathbb{E}[\mathcal{E}_{k+1} | \mathcal{F}_k] \leq \mathcal{E}_k - \frac{\mathcal{L}_k(1 - \delta_k)^2}{|\tau_{i_k}^{(k)}|(1 + \delta_k)^2} \frac{\left\| \Phi(z^{(k)}) \right\|_2^2}{\left\| J^{(k)} \right\|_F^2}.$$

By Lemma 4,

$$\left\| \Phi(z^{(k)}) \right\|_2^2 \geq \frac{1}{(1 + \eta)^2} \left\| J^{(k)}(z^{(k)} - \mathbf{z}^*) \right\|_2^2.$$

Furthermore,

$$\left\| J^{(k)} \left( z^{(k)} - z^* \right) \right\|_2^2 \geq \sigma_{\min}^2 \left( J^{(k)} \right) \mathcal{E}_k.$$

Since

$$\kappa_F^2 \left( J^{(k)} \right) = \frac{\left\| J^{(k)} \right\|_F^2}{\sigma_{\min}^2 \left( J^{(k)} \right)},$$

we obtain

$$\mathbb{E}[\mathcal{E}_{k+1} \mid \mathcal{F}_k] \leq \left[ 1 - \frac{\mathcal{L}_k (1 - \delta_k)^2}{|\tau_{i_k}^{(k)}| \kappa_F^2 \left( J^{(k)} \right) (1 + \delta_k)^2 (1 + \eta)^2} \right] \mathcal{E}_k.$$

Taking the total expectation proves the stated result. The conditional expectation argument follows the same principle as that used for the NSNGRK method in [26].  $\square$

**Theorem 2.** Let  $\Phi : \mathbb{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a nonsmooth function defined on a compact domain  $\mathbb{D}$ . Assume  $z^* \in \mathbb{D}$  satisfies  $\Phi(z^*) = 0$ , and for all  $z \in \mathbb{D}$ ,  $\Phi$  admits a Clarke generalized Jacobian  $J_C(z)$ . The sequence  $\{z^{(k)}\}$  generated by BNSNMGK\_k converges to  $z^*$  with error decay satisfying

$$\left\| z^{(k+1)} - z^* \right\|_2^2 \leq \left( 1 - \frac{\frac{1}{2} \mathcal{L}_k \beta_k \gamma_k n_k \left( \frac{\left\| J_C^{(k)} \right\|_F^2}{\nu_k} + 1 \right) (1 - \delta_k)^2}{|\tau_k| \kappa_F^2 \left( J(z^{(k)}) \right) (1 + \delta_k)^2 (1 + \eta)^2} \right) \left\| z^{(k)} - z^* \right\|_2^2,$$

for  $k = 0, 1, \dots$ , where  $\beta_k, \gamma_k, \nu_k$  are problem-specific positive parameters associated with the multi-step update scheme.

**Proof.** Let

$$z^{(k,0)} = z^{(k)}, \quad z^{(k+1)} = z^{(k,n_k)},$$

and define

$$\mathcal{E}_{k,j} = \left\| z^{(k,j)} - z^* \right\|_2^2.$$

For the  $j$ -th inner block update, Lemma 5 gives

$$\mathcal{E}_{k,j} - \mathcal{E}_{k,j-1} \leq -\mathcal{D}_{k,j}, \quad j = 1, \dots, n_k,$$

where  $\mathcal{D}_{k,j}$  denotes the nonnegative error-decrease term obtained from Lemma 5 for the block used in the  $j$ -th inner update. Summing these inequalities over all inner updates gives the telescoping relation

$$\mathcal{E}_{k,n_k} - \mathcal{E}_{k,0} \leq -\sum_{j=1}^{n_k} \mathcal{D}_{k,j}.$$

Since

$$\mathcal{E}_{k,n_k} = \left\| z^{(k+1)} - z^* \right\|_2^2, \quad \mathcal{E}_{k,0} = \left\| z^{(k)} - z^* \right\|_2^2,$$

it remains to estimate the sum of the inner error decreases. Using Lemma 4, the consistency estimate in Lemma 5, and the definitions of  $\beta_k, \gamma_k$ , and  $\nu_k$ , we obtain

$$\sum_{j=1}^{n_k} \mathcal{D}_{k,j} \geq \frac{\frac{1}{2} \mathcal{L}_k \beta_k \gamma_k n_k \left( \frac{\left\| J_C^{(k)} \right\|_F^2}{\nu_k} + 1 \right) (1 - \delta_k)^2}{|\tau_{i_k}^{(k)}| \kappa_F^2 \left( J^{(k)} \right) (1 + \delta_k)^2 (1 + \eta)^2} \mathcal{E}_{k,0}.$$

Substituting this estimate into the telescoping relation yields

$$\|z^{(k+1)} - z^*\|_2^2 \leq \left[ 1 - \frac{\frac{1}{2} \mathcal{L}_k \beta_k \gamma_k n_k \left( \frac{\|J_c^{(k)}\|_F^2}{v_k} + 1 \right) (1 - \delta_k)^2}{|\tau_{i_k}^{(k)}| \kappa_F^2(J^{(k)}) (1 + \delta_k)^2 (1 + \eta)^2} \right] \|z^{(k)} - z^*\|_2^2.$$

This is the claimed estimate. The telescoping multi-step argument is analogous to the analysis of the NSNMGK method in [27].  $\square$

A comparison of the convergence results for the NSNGRK method and the BN-SNGRK\_k method (Theorem 1) reveals a critical refinement: the convergence bound for the block-based method incorporates the squared spectral condition number  $\kappa_2^2(J_{\tau_k}(z^{(k)}))$  of the current block Jacobian. Given that  $\frac{1-2\eta}{(1+\eta)^2 \kappa_F^2(J(z^{(k)}))}$  is positive and typically small, the following inequality holds

$$0 < 1 - \frac{1 - 2\eta \kappa_2^2(J_{\tau_k}(z^{(k)}))}{(1 + \eta)^2 \kappa_F^2(J(z^{(k)}))} < 1 - \frac{1 - 2\eta}{(1 + \eta)^2 \kappa_F^2(J(z^{(k)}))} < 1.$$

This demonstrates a strictly smaller contraction factor for the block greedy strategy. Therefore, when solving the nonsmooth nonlinear system  $\Phi(z) = 0$  ( $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ) with a Kaczmarz-type iteration, employing a greedy rule to select row blocks after partitioning yields faster convergence than applying a greedy rule to individual rows alone.

## 4. Experimental Results

In this section, we focus on comparing our new methods, i.e., the BNSNGRK\_k and BNSNMGK\_k methods, with the current methods for solving grasping problems in terms of the grasping force and displacement. To verify the effectiveness of the proposed methods in practical grasping scenarios, we use the robot dexterous hand to grasp three typical objects [27]: an L-shaped planar block, a prism, and a cube, as shown in Figure 1. The numerical experiments are based on the grasping of these objects. In Figures 2–6, the horizontal axis labeled “IT” denotes the outer iteration index  $k$ . For the BNSNMGK\_k method, all block updates performed within one outer iteration are counted together as one iteration, rather than as separate iterations. The vertical axis reports the base-10 logarithm of the nonlinear residual norm,

$$\log_{10} \|\Phi(z^{(k)})\|_2.$$

Consequently, a lower value on the vertical axis indicates a smaller residual and hence a more accurate approximation to the solution. In particular, vertical-axis values of  $-6$  and  $-8$  correspond to residual norms of  $10^{-6}$  and  $10^{-8}$ , respectively.

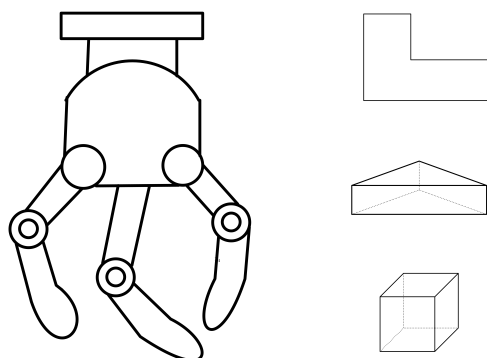
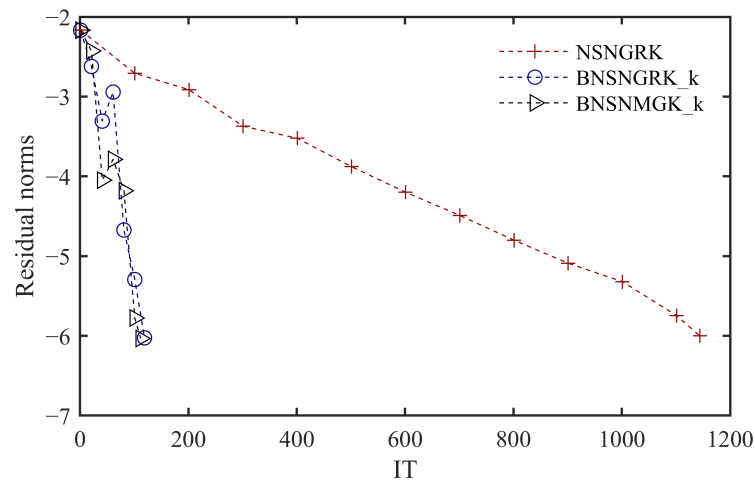
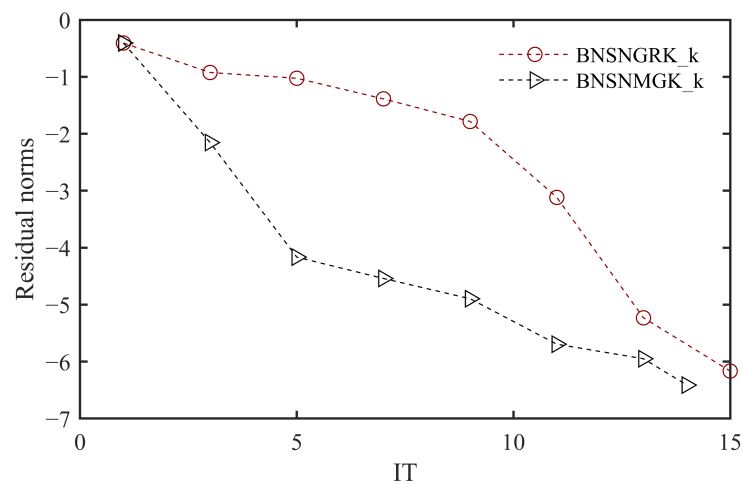


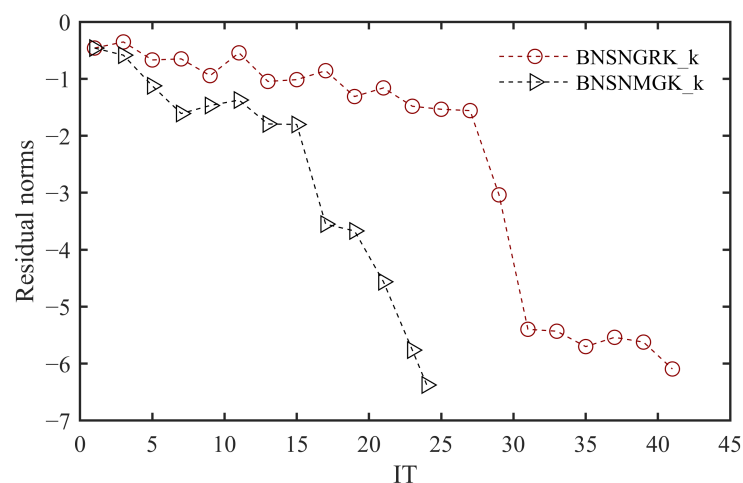
Figure 1. Grasping scenarios.



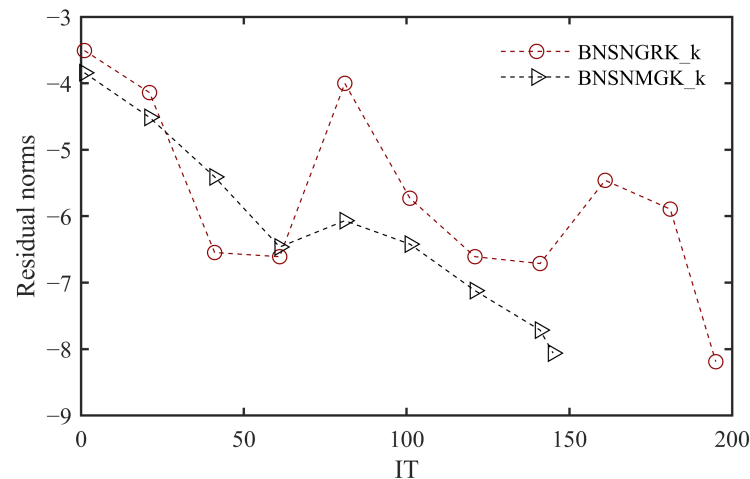
**Figure 2.** Residual error vs. iteration count for the planar grasping problem.



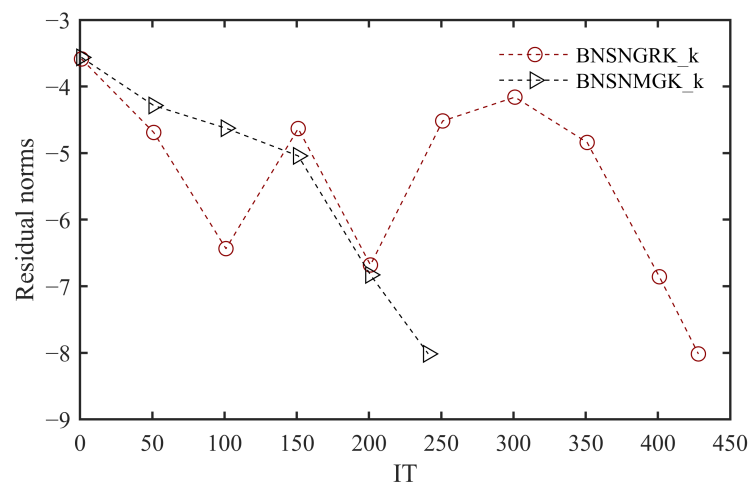
**Figure 3.** Iteration process for three-finger grasping of a prism with tetrahedral (4-sided) cone approximation.



**Figure 4.** Iteration process for three-finger grasping of a prism with octahedral (8-sided) cone approximation.



**Figure 5.** Iteration process for three-finger grasping of a cube with tetrahedral (4-sided) cone approximation.



**Figure 6.** Iteration process for three-finger grasping of a cube with octahedral (8-sided) cone approximation.

#### 4.1. Planar Grasping Problem

We begin with the planar grasping problem. The planar grasping problem can be modeled as a linear complementarity problem

$$\begin{bmatrix} \varepsilon \\ v^- \\ v^+ \end{bmatrix} = \begin{bmatrix} E & G^T & -G^T \\ -G & 0 & 0 \\ G & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \dot{u}^+ \\ \dot{u}^- \end{bmatrix} + \begin{bmatrix} \delta^* \\ P \\ -P \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} r \\ \dot{u}^+ \\ \dot{u}^- \end{bmatrix} \geq 0, \begin{bmatrix} \varepsilon \\ v^- \\ v^+ \end{bmatrix} \geq 0, \begin{bmatrix} r & \dot{u}^+ & \dot{u}^- \end{bmatrix} \begin{bmatrix} \varepsilon \\ v^- \\ v^+ \end{bmatrix} = 0,$$

where  $v^-$  and  $v^+$  are parametric variables,  $E$  is the compliance matrix,  $G$  is the grasping matrix,  $P$  is the external wrench vector,  $\delta^*$  is the initial gap,  $\varepsilon$  is the relative displacement vector,  $r$  is the grasping force vector, and  $\dot{u}$  is the object displacement vector, with  $\dot{u}^+ = \frac{|\dot{u}| + \dot{u}}{2}$ ,  $\dot{u}^- = \frac{|\dot{u}| - \dot{u}}{2}$ . The coefficient matrix takes the form of a skew-symmetric saddle-point matrix  $\begin{bmatrix} X & Y \\ -Y^T & O \end{bmatrix}$ .

Through regularization, the regularized model can be solved directly by the Lemke method, which provides an exact reference solution. With the convergence tolerance set to  $10^{-6}$ , the comparisons between numerical and exact results are summarized in Table 1, and the decay of the residual versus iteration count is plotted in Figure 2. Table 1 and Figure 2 demonstrate that the block Kaczmarz approach is effective for systems with a skew-symmetric saddle-point coefficient matrix. Furthermore, the proposed algorithms exhibit a clear advantage over the NSNGRK method in terms of iteration numbers.

**Table 1.** Planar grasping results (tol.  $10^{-6}$ ).

| Physical Quantity      | NSNGRK                | BNSNGRK_k              | BNSNMKG_k              |
|------------------------|-----------------------|------------------------|------------------------|
| Max Force Error        | $1.00 \times 10^{-6}$ | $4.38 \times 10^{-10}$ | $7.29 \times 10^{-16}$ |
| Max Displacement Error | $2.00 \times 10^{-6}$ | $4.38 \times 10^{-11}$ | $7.83 \times 10^{-13}$ |

To examine the practical relevance of the sufficient contraction condition, the exact solution calculated by Lemke is used as the reference solution and empirical values for the Jacobian submatrix selected at each iteration of the first numerical example are recorded. Specifically,

$$\hat{\eta}_k = \max_{i \in \tau_{i_k}^{(k)}} \frac{|\phi_i(\mathbf{z}^{(k)}) - \phi_i(\mathbf{z}^*) - J_{i,:}^{(k)}(\mathbf{z}^{(k)} - \mathbf{z}^*)|}{|\phi_i(\mathbf{z}^{(k)}) - \phi_i(\mathbf{z}^*)|},$$

where components with denominators close to machine precision were excluded.

For the first numerical example, empirical values of  $\hat{\eta}_k$  and  $\hat{\kappa}_{2,k}$  are evaluated along the computed iteration trajectory. The first quartile, median, and third quartile of  $\hat{\eta}_k$  were  $8.22 \times 10^{-6}$ ,  $5.55 \times 10^{-2}$ ,  $1.67 \times 10^{-1}$ , respectively, while the corresponding values of  $\hat{\kappa}_{2,k}$  were 1.00, 1.00, 2.18. For the empirical contraction quantity  $\hat{\mathcal{L}}_k$ , the first quartile, median, and third quartile were 0.23, 0.79, 0.99, respectively. These representative values show that the derived contraction condition is practically attainable along the computed trajectory.

#### 4.2. Three-Finger Grasping of an Isosceles Prism

The grasping problem in three-dimensional space can be modeled as the following linear complementarity problem:

$$\begin{bmatrix} \varepsilon_N \\ \gamma \\ \bar{w} \end{bmatrix} = \begin{bmatrix} E_N - G_N^T U G_N & G_N^T A T^T & e \\ T_N^T - T_T^T A^T G_N & \mathbf{0} & \mathbf{0} \\ -e^T & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} r_N \\ \lambda \\ \bar{z} \end{bmatrix} + \begin{bmatrix} d_N + G_N^T U P \\ T_T^T A^T P \\ \bar{r}_N \end{bmatrix}.$$

The associated nonnegativity and complementarity conditions are

$$\begin{bmatrix} r_N \\ \lambda \\ \bar{z} \end{bmatrix} \geq \mathbf{0}, \quad \begin{bmatrix} \varepsilon_N \\ \gamma \\ \bar{w} \end{bmatrix} \geq \mathbf{0}, \quad \begin{bmatrix} r_N \\ \lambda \\ \bar{z} \end{bmatrix}^T \begin{bmatrix} \varepsilon_N \\ \gamma \\ \bar{w} \end{bmatrix} = 0.$$

Here,  $r_N$  denotes the normal force,  $G_N$  and  $G_T$  are the grasp matrices associated with the normal and tangential forces, respectively, and  $E_N$  and  $E_T$  are the corresponding normal and tangential compliance matrices. The vector  $P$  denotes the external wrench,  $\varepsilon_N$  denotes the normal relative displacement, and  $d_N$  is the vector of initial normal distances between the fingertips and the object. Moreover,  $\lambda$  is the slip parameter associated with the approximated friction cone,  $\gamma$  is the corresponding friction-cone force parameter, and  $\bar{r}_N$  is

the prescribed maximum normal force. The variables  $\bar{w}$  and  $\bar{z}$  form a complementary pair, while  $T_T$  and  $T_N$  are matrices arising from the linearization of the friction law.

The matrices  $A$  and  $U$  and the vector  $e$  are defined by

$$\begin{aligned} A &= \left[ E_T^{-1} G_T^T \left( G_T E_T^{-1} G_T^T \right)^{-1} \right]^T, \\ U &= - \left( G_T E_T^{-1} G_T^T \right)^{-1}, \\ e &= \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T. \end{aligned}$$

The coefficient matrix takes the form of a general saddle-point block structure

$$M = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}.$$

For the three-finger grasping of an isosceles prism in 3D space, we approximate the friction cone with a 4-sided pyramid and an 8-sided pyramid, respectively, when formulating it as a linear complementarity problem. When using the 4-sided pyramid to approximate the friction cone, the Lemke method can obtain a set of exact solutions. We set the convergence tolerance for the numerical algorithms to  $10^{-6}$ . The results of numerical algorithms are compared with those of the Lemke algorithm in Table 2. The iteration process of the block Kaczmarz methods is shown in Figure 3. When using the 8-sided pyramid to approximate the friction cone, the Lemke method fails to find a solution. The similarity of the results computed by the two block-iterative numerical methods, as denoted by the mean absolute error, is shown in Table 3, and the iteration process is depicted in Figure 4. From Tables 2 and 3, we observe that the same convergence tolerance does not guarantee the same solution quality. The grasping forces computed by the NSNGRK algorithm show noticeable errors compared to the exact solution. In contrast, both block Kaczmarz methods not only achieve high accuracy in computing grasping forces but also maintain high accuracy in computing grasping displacements. This advantage is particularly evident in ill-conditioned problems such as the grasping complementarity model, making the block Kaczmarz methods not only faster algorithms but also capable of providing higher-quality solutions.

**Table 2.** Three-finger prism grasping results (4-sided cone, tol.  $10^{-6}$ ).

| Physical Quantity               | NSNGRK               | BNSNGRK_k             | BNSNMGK_k             |
|---------------------------------|----------------------|-----------------------|-----------------------|
| Max Normal Force Error          | $3.0 \times 10^{-2}$ | $8.2 \times 10^{-5}$  | $8.7 \times 10^{-5}$  |
| Max Tangential Force Error      | $1.2 \times 10^{-2}$ | $6.0 \times 10^{-5}$  | $4.6 \times 10^{-5}$  |
| Max Relative Displacement Error | $1.6 \times 10^{-6}$ | $4.3 \times 10^{-9}$  | $5.7 \times 10^{-9}$  |
| Max Object Displacement Error   | $8.1 \times 10^{-7}$ | $5.3 \times 10^{-10}$ | $1.7 \times 10^{-10}$ |

**Table 3.** Three-finger prism grasping results (8-sided cone, tol.  $10^{-6}$ ).

| Physical Quantity     | Mean Absolute Error   |
|-----------------------|-----------------------|
| Normal Force          | $8.69 \times 10^{-5}$ |
| Tangential Force      | $4.34 \times 10^{-5}$ |
| Relative Displacement | $3.54 \times 10^{-7}$ |
| Object Displacement   | $2.92 \times 10^{-7}$ |

#### 4.3. Three-Finger Grasping of a Cube

For the problem of three-finger grasping a cube in three-dimensional space, we again approximate the friction cone using 4- and 8-sided pyramids. For this problem, we further tighten the convergence tolerance for the algorithms to  $10^{-8}$ . The Lemke method provides exact solutions when approximating the friction cone with a 4-sided pyramid, but not with an 8-sided pyramid. The NSNGRK method does not converge to the required tolerance in either case. We present the computational results and convergence iteration process for the three algorithms that successfully yield solutions under the tetrahedral pyramid approximation in Table 4 and Figure 5, and under the octahedral pyramid approximation in Table 5 and Figure 6. We observe that the grasping forces computed by the proposed methods are essentially consistent. The multi-step block Kaczmarz method requires fewer iterations than the block Kaczmarz method and demonstrates that the new algorithms effectively improve convergence accuracy.

**Table 4.** Three-finger cube-grasping results (4-sided cone, tol.  $10^{-8}$ ).

| Physical Quantity               | BNSNGRK_k             | BNSNMKG_k             |
|---------------------------------|-----------------------|-----------------------|
| Max Normal Force Error          | $1.8 \times 10^{-13}$ | $4.4 \times 10^{-11}$ |
| Max Tangential Force Error      | $3.0 \times 10^{-13}$ | $8.8 \times 10^{-11}$ |
| Max Relative Displacement Error | $7.8 \times 10^{-7}$  | $8.9 \times 10^{-8}$  |
| Max Object Displacement Error   | $1.2 \times 10^{-7}$  | $1.2 \times 10^{-7}$  |

**Table 5.** Three-finger cube-grasping results (8-sided cone, tol.  $10^{-8}$ ).

| Physical Quantity     | Mean Absolute Error    |
|-----------------------|------------------------|
| Normal Force          | $3.50 \times 10^{-12}$ |
| Tangential Force      | $4.12 \times 10^{-12}$ |
| Relative Displacement | $6.64 \times 10^{-9}$  |
| Object Displacement   | $7.95 \times 10^{-9}$  |

The results in Tables 1–5 show that the proposed block methods attain higher numerical accuracy and require fewer outer iterations than the comparison method for the tested robotic grasping problems. These results demonstrate advantages in solution accuracy and iteration count. However, because K-means clustering is performed during the outer iterations, the reduction in the number of iterations does not by itself establish a reduction in total wall-clock execution time.

## 5. Conclusions

This work addresses accuracy in multi-fingered robotic grasping complementarity problems by introducing two novel algorithms: BNSNGRK\_k and BNSNMKG\_k. Leveraging K-means clustering for adaptive block partitioning of the generalized Jacobian matrix, these methods significantly enhance computational performance. Convergence is established and numerical experiments validate their efficacy in grasp planning for multi-fingered robotic hands. Although the proposed methods achieve higher numerical accuracy and require fewer outer iterations in the reported experiments, the repeated K-means clustering step introduces an additional computational overhead. Therefore, fewer outer iterations do not necessarily imply a shorter total wall-clock time for every implementation or computational platform. Reducing the clustering overhead is an important direction for future work. Possible strategies include warm-starting K-means with the centroids obtained at the preceding iteration, using incremental clustering, updating the

block partition only when the generalized Jacobian changes significantly, and performing re-clustering periodically rather than at every outer iteration.

**Author Contributions:** Conceptualization, Z.A. and C.L.; methodology, Z.A. and C.L.; software, Z.A.; validation, Z.A.; formal analysis, Z.A.; investigation, Z.A. and C.L.; resources, Z.A. and C.L.; data curation, Z.A. and C.L.; writing—original draft preparation, Z.A.; writing—review and editing, C.L.; visualization, Z.A.; supervision, C.L.; project administration, C.L.; funding acquisition, Z.A. and C.L. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported by the National Natural Science Foundation of China (12161027) and partially supported by the Science and Technology Project of Guangxi, China (GuikeAD23023002).

**Data Availability Statement:** The data are contained within the article.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References

1. Ye, C.; Yao, Y.; Lin, W.; Yang, X.; Qiu, J. CSGrasp: Category-Level Semantically-Aware Grasping Method. *IEEE Trans. Ind. Electron.* **2026**, *73*, 2610–2619.
2. Li, L.; Cherouat, A.; Snoussi, H.; Wang, T. Grasping with Occlusion-Aware Ally Method in Complex Scenes. *IEEE Trans. Autom. Sci. Eng.* **2025**, *22*, 5944–5954.
3. Jin, J.; Qiu, L. A Robust Fast Convergence Zeroing Neural Network and Its Applications to Dynamic Sylvester Equation Solving and Robot Trajectory Tracking. *J. Frankl. Inst.* **2022**, *359*, 3183–3209. [\[CrossRef\]](#)
4. Wang, F.; Chen, K.; Zhen, S.; Chen, X.; Zheng, H.; Wang, Z. Udwadia Theory-Based Performance Guaranteed Adaptive Robust Constraint-Following Control for Uncertain Robotic Manipulators with Sensor Noise. *Nonlinear Dyn.* **2026**, *114*, 233.
5. Zhou, L.; Li, Z.; Li, Y.; Bai, S. Parallel MPPI with Gradient-Velocity Modulated SDF Cost for High-Performance Real-Time Dynamic Obstacle Avoidance by Robot Manipulators. *IEEE Trans. Robot.* **2025**, *41*, 5149–5168.
6. Ai-Fahed, A.; Panagiotopoulos, P. A linear complementarity approach to the articulated multifingered friction gripper. *J. Robot. Syst.* **1993**, *10*, 871–887. [\[CrossRef\]](#)
7. Carpentier, J.; Montaut, L.; Lidec, Q. From compliant to rigid contact simulation: A unified and efficient approach. *arXiv* **2024**, arXiv:2405.17020.
8. Lidec, Q.; Jallet, W.; Montaut, L.; Laptev, I.; Schmid, C.; Carpentier, J. Contact models in robotics: A comparative analysis. *IEEE Trans. Robot.* **2024**, *40*, 3740–3763. [\[CrossRef\]](#)
9. Ai-Fahed, A.; Panagiotopoulos, P. Multifingered frictional robot grippers: A new type of numerical implementation. *Comput. Struct.* **1992**, *42*, 555–562. [\[CrossRef\]](#)
10. Gholami, F.; Nasri, M.; Kövecses, J.; Teichmann, M. A linear complementarity formulation for contact problems with regularized friction. *Mech. Mach. Theory* **2016**, *105*, 568–582. [\[CrossRef\]](#)
11. Haas-Heger, M.; Iyengar, G.; Ciocarlie, M. Passive reaction analysis for grasp stability. *IEEE Trans. Autom. Sci. Eng.* **2018**, *15*, 955–966. [\[CrossRef\]](#)
12. Ali, R.; Akgul, A. A new matrix splitting generalized iteration method for linear complementarity problems. *Appl. Math. Comput.* **2024**, *464*, 128378.
13. Bai, Z. Modulus-based matrix splitting iteration methods for linear complementarity problems. *Numer. Linear Algebra Appl.* **2010**, *17*, 917–933. [\[CrossRef\]](#)
14. Cruz, J.B.; Ferreira, O.; Németh, S.; Prudente, L.F. A semi-smooth newton method for projection equations and linear complementarity problems with respect to the second order cone. *Linear Algebra Appl.* **2017**, *513*, 160–181. [\[CrossRef\]](#)
15. Ansoorge, R. Connections between the cimmino-method and the kaczmarz-method for the solution of singular and regular systems of equations. *Computing* **1984**, *33*, 367–375.
16. Strohmer, T.; Vershynin, R. A randomized kaczmarz algorithm with exponential convergence. *J. Fourier Anal. Appl.* **2009**, *15*, 262–278.
17. Nutini, J.; Sepehry, B.; Laradji, I.; Schmidt, M.; Koepke, H.; Virani, A. Convergence rates for greedy kaczmarz algorithms, and faster randomized kaczmarz rules using the orthogonality graph. *arXiv* **2016**, arXiv:1612.07838.
18. Bai, Z.; Wu, W. On greedy randomized kaczmarz method for solving large sparse linear systems. *SIAM J. Sci. Comput.* **2018**, *40*, A592–A606. [\[CrossRef\]](#)
19. Needell, D.; Tropp, J.A. Paved with good intentions: Analysis of a randomized block kaczmarz method. *Linear Algebra Appl.* **2014**, *441*, 199–221. [\[CrossRef\]](#)

20. Needell, D.; Zhao, R.; Zouzias, A. Randomized block kaczmarz method with projection for solving least squares. *Linear Algebra Appl.* **2015**, *484*, 322–343. [[CrossRef](#)]
21. Liu, Y.; Gu, C. On greedy randomized block kaczmarz method for consistent linear systems. *Linear Algebra Appl.* **2021**, *616*, 178–200. [[CrossRef](#)]
22. Jiang, X.; Zhang, K.; Yin, J. Randomized block kaczmarz methods with k-means clustering for solving large linear systems. *J. Comput. Appl. Math.* **2022**, *403*, 113828.
23. Jiang, X.-L.; Zhang, K. A randomized block extended kaczmarz method with hybrid partitions for solving large inconsistent linear systems. *Appl. Math. Lett.* **2024**, *152*, 109027. [[CrossRef](#)]
24. Xiao, A.; Yin, J. On averaging block kaczmarz methods for solving nonlinear systems of equations. *J. Comput. Appl. Math.* **2024**, *451*, 116041. [[CrossRef](#)]
25. Xiao, A.; Gao, X.; Yin, J. A fast block nonlinear bregman-kaczmarz method with averaging for nonlinear sparse signal recovery. *J. Sci. Comput.* **2025**, *104*, 71. [[CrossRef](#)]
26. Ai, Z.; Li, C. Calculation of robot multi-fingered grasping force and displacement based on the newton–subgradient non-smooth greedy randomized kaczmarz method for solving linear complementarity problem. *Sensors* **2025**, *25*, 2309. [[PubMed](#)]
27. Ai, Z.; Li, C. Accelerated computation of linear complementarity problem in dexterous robotic grasping via newton-subgradient non-smooth multi-step greedy kaczmarz method. *Actuators* **2025**, *14*, 307.
28. Peñuñuri, F.; Peón, R.; González-Sánchez, D.; Escalante Soberanis, M.A. Dual numbers and automatic differentiation to efficiently compute velocities and accelerations. *Acta Appl. Math.* **2020**, *170*, 649–659. [[CrossRef](#)]
29. Rockafellar, R.T. *Convex Analysis*; Princeton University Press: Princeton, NJ, USA, 1997; Volume 28.
30. Clarke, F.H. *Optimization and Nonsmooth Analysis*; SIAM: New Delhi, India, 1990.
31. Wang, Q.; Li, W.; Bao, W.; Gao, X. Nonlinear kaczmarz algorithms and their convergence. *J. Comput. Appl. Math.* **2022**, *399*, 113720. [[CrossRef](#)]
32. Chen, J.; Sun, D.; Sun, J. The sc1 property of the squared norm of the soc fischer–burmeister function. *Oper. Res. Lett.* **2008**, *36*, 385–392. [[CrossRef](#)]
33. Tan, L.; Guo, X. On nonlinear greedy randomized kaczmarz methods for solving large-scale systems of nonlinear equations. *Preprint* **2023**. [[CrossRef](#)] [[PubMed](#)]
34. Higham, N.J. *Accuracy and Stability of Numerical Algorithms*; SIAM: New Delhi, India, 2002.

**Disclaimer/Publisher’s Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.