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Nonexpansive Mappings and Fixed Point Theory in Fuzzy Normed GE-Algebras

Prashant Patel ¹, Amal S. Alali ^{2,*} and Ravi Kumar Bandaru ¹

¹ Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati 522241, Andhra Pradesh, India; prashant.patel9999@gmail.com (P.P.); ravimaths83@gmail.com (R.K.B.)

² Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

* Correspondence: asalali@pnu.edu.sa

Abstract

In this paper, we investigate the theory of nonexpansive mappings in the framework of fuzzy normed GE-algebras. After recalling the fundamental concepts of GE-algebras and fuzzy GE-norms, we introduce the notion of fuzzy nonexpansive mappings and examine their basic structural properties. We show that the composition of two fuzzy nonexpansive mappings remains fuzzy nonexpansive, and establish that every fuzzy nonexpansive mapping is sequentially continuous with respect to fuzzy convergence. Further, by employing the concept of a fuzzy GE-interpolation family, we define α -averaged mappings in fuzzy normed GE-algebras and prove that such averaged operators preserve nonexpansiveness. We also develop a demiclosedness-type principle and provide fixed point equivalence results between a mapping and its associated averaged mapping under suitable assumptions. Finally, we prove that the fixed point set of a fuzzy nonexpansive mapping is sequentially closed. These results extend classical ideas from metric fixed-point theory and Banach space theory to the algebraic setting of fuzzy normed GE-algebras.

Keywords: fuzzy GE-norm; fuzzy normed GE-algebra; nonexpansive mapping; α -averaged mapping; fixed point; convergence

MSC: 03G25; 06F35; 47H10; 54A20

1. Introduction

Fixed point theory plays a central role in nonlinear analysis due to its strong connections with optimization, equilibrium problems, nonlinear differential equations, and iterative numerical algorithms. Among the most important classes of operators in this theory are nonexpansive mappings, which preserve (or reduce) distances and serve as fundamental tools in approximation theory and nonlinear functional analysis. The classical foundation of this topic goes back to the celebrated Banach contraction principle [1], which initiated the development of fixed point theory in metric spaces. Later, the study of nonexpansive mappings in Banach spaces was significantly advanced by Browder [2], Takahashi [3], Ahmad [4] and Reich [5], leading to deep results on weak convergence and fixed point existence theorems.

In recent decades, fixed point theory has been extended beyond the classical framework of metric and normed spaces to various generalized distance structures. In this



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direction, completeness, convergence, topological properties, and fixed-point phenomena in generalized metric spaces have been widely investigated; see, for example, Cobzaş [6]. In the framework of fuzzy metric spaces, Moussaoui and Radenović [7] provided a comprehensive survey of fixed-point results, including contraction principles, simulation functions, and relevant topological aspects. Moreover, several generalizations and extensions of nonexpansive mappings have been developed and systematically studied in broader metric-type settings; see, for instance, the survey chapter by Pant, Shukla, and Patel [8]. A comprehensive survey of fixed-point results specifically in fuzzy metric spaces, covering contraction principles and simulation functions, can be found in [9,10]. Furthermore, weak and strong convergence results for nonexpansive mappings and their generalizations have been established in the different settings of spaces, for instance in [11–13].

On the other hand, algebraic systems arising from logic have become an active research area due to their strong applications in algebraic logic and information processing. One of the classical structures in this direction is the Hilbert algebra introduced by Diego [14]. Motivated by such logical frameworks, Bandaru et al. introduced the notion of GE-algebras [15], which form an important algebraic system equipped with a binary operation and a constant element satisfying axioms similar to implication-type algebras. Since their introduction, several algebraic and filter-theoretic aspects of GE-algebras have been investigated [16], including the notion of belligerent GE-filters and prominent GE-filters together with GE-morphisms [17]. These studies indicate that GE-algebras provide a flexible and useful structure for exploring various generalizations of logical and algebraic concepts.

To develop an analytic framework within GE-algebras, the concept of norms on GE-algebras was introduced by Bandaru and Jun [18], leading to the study of normed GE-algebras. Such norm structures allow one to define convergence, continuity, and stability in GE-algebras in a way analogous to normed linear spaces. More recently, Alali et al. [19] introduced the concept of fuzzy GE-norms and fuzzy normed GE-algebras, which combines the uncertainty modeling power of fuzzy theory with the algebraic framework of GE-algebras. This fuzzy setting naturally extends classical concepts and provides a broader platform for studying convergence and operator theory in non-classical algebraic environments. Despite these developments, the theory of nonexpansive mappings within fuzzy normed GE-algebras has not yet been investigated. In particular, three fundamental questions remain open: (i) Do fuzzy nonexpansive mappings preserve sequential convergence in the fuzzy GE-norm sense? (ii) Can the classical theory of averaged operators be meaningfully formulated in the GE-algebraic setting? (iii) Is the fixed-point set of a fuzzy nonexpansive mapping stable under fuzzy-convergent approximations? The present paper answers all three questions affirmatively. We compare our fuzzy GE-norm framework with classical fuzzy metric spaces in the sense of George–Veeramani, pointing out that a fuzzy metric is defined on pairs of points while a fuzzy GE-norm is defined on algebraic products $\rho * \sigma$, so the triangle inequality has the form $\mu(\rho * \tau, t_1 + t_2) \geq \min\{\mu(\rho * \sigma, t_1), \mu(\sigma * \tau, t_2)\}$ rather than the standard $M(x, z, t_1 + t_2) \geq \min\{M(x, y, t_1), M(y, z, t_2)\}$. We note that fuzzy normed linear spaces (Bag–Samanta, 2003) require a linear structure, whereas GE-algebras need not carry any linear or group operation, making our setting strictly more general in the algebraic direction. We mention that fixed-point results in b -metric and modular spaces do not cover our setting because they rely on a symmetric distance, while the GE-norm is intrinsically asymmetric. Since nonexpansive mappings are fundamental in classical fixed-point theory, it is natural to extend these ideas to fuzzy normed GE-algebras. However, the theory of nonexpansive mappings in fuzzy GE-normed structures is still at an early stage, and many important properties remain to be explored. The main contributions of this paper are as follows:

1. We introduce fuzzy nonexpansive mappings in fuzzy normed GE-algebras and prove that the composition of two such mappings is again fuzzy nonexpansive (Theorem 2).
2. We establish two-sided sequential continuity of fuzzy nonexpansive mappings with respect to fuzzy GE-norm convergence (Theorem 3).
3. By introducing a fuzzy GE-interpolation family, we define α -averaged mappings and prove that they inherit nonexpansiveness (Theorem 4).
4. We establish a demiclosedness-type principle (Theorem 5) and a fixed point equivalence theorem between a mapping and its α -averaged form (Theorem 6), and a convergence result showing that the averaged iteration converges to a fixed point of the original mapping (Theorem 7).
5. We prove sequential closedness of the fixed-point set (Theorem 8) and apply these results to the filter-theoretic structure of GE-algebras (Section 3).

These results extend several classical ideas from metric fixed point theory and nonexpansive operator theory to the algebraic framework of fuzzy normed GE-algebras and provide a basis for further research in this direction.

2. Preliminaries

In this section, we recall the fundamental notions used throughout the paper. We begin with GE-algebras, introduced in [15] as an implication-type algebraic structure generalizing Hilbert algebras. We then recall the GE-norm and its fuzzy counterpart, the fuzzy GE-norm introduced in [19], which equips a GE-algebra with a quantitative framework based on the binary operation $*$. The section concludes with the notion of fuzzy convergence in a fuzzy normed GE-algebra, which is the appropriate mode of convergence throughout Section 2.

Definition 1 ([15]). A GE-algebra is a nonempty set $\mathbf{X}$ with a constant 1 and a binary operation “ $*$ ” satisfying the following axioms:

$$(GE1) \quad q_1 * q_1 = 1,$$

$$(GE2) \quad 1 * q_1 = q_1,$$

$$(GE3) \quad q_1 * (q_2 * q_3) = q_1 * (q_2 * (q_1 * q_3))$$

for all $q_1, q_2, q_3 \in \mathbf{X}$.

In a GE-algebra $\mathbf{X}$, a binary relation “ $\leq_{\mathbf{X}}$ ” is defined by

$$(\forall q_1, q_2 \in \mathbf{X})(q_1 \leq_{\mathbf{X}} q_2 \Leftrightarrow q_1 * q_2 = 1). \quad (1)$$

Definition 2 ([15]). A GE-algebra $\mathbf{X}$ is called

- transitive if it satisfies:

$$(\forall q_1, q_2, q_3 \in \mathbf{X})(q_1 * q_2 \leq_{\mathbf{X}} (q_3 * q_1) * (q_3 * q_2)). \quad (2)$$

- commutative if it satisfies:

$$(\forall q_1, q_2 \in \mathbf{X})((q_1 * q_2) * q_2 = (q_2 * q_1) * q_1). \quad (3)$$

Proposition 1 ([15]). Every GE-algebra $\mathbf{X}$ satisfies the following properties:

$$q_1 * 1 = 1, \quad (4)$$

$$q_1 * (q_1 * q_2) = q_1 * q_2, \quad (5)$$

$$q_1 \leq_X q_2 * q_1, \quad (6)$$

$$q_1 * (q_2 * q_3) \leq_X q_2 * (q_1 * q_3), \quad (7)$$

$$1 \leq_X q_1 \Rightarrow q_1 = 1, \quad (8)$$

$$q_1 \leq_X (q_2 * q_1) * q_1, \quad (9)$$

$$q_1 \leq_X (q_1 * q_2) * q_2, \quad (10)$$

$$q_1 \leq_X q_2 * q_3 \Leftrightarrow q_2 \leq_X q_1 * q_3, \quad (11)$$

for all $q_1, q_2, q_3 \in \mathbf{X}$.

Definition 3 ([17]). Let $(\mathbf{X}, *_X, 1_X)$ and $(\mathbf{Y}, *_Y, 1_Y)$ be GE-algebras. A mapping $\psi_1 : \mathbf{X} \rightarrow \mathbf{Y}$ is called a GE-morphism if it satisfies:

$$(\forall \gamma_1, \gamma_2 \in \mathbf{X})(\psi_1(\gamma_1 *_X \gamma_2) = \psi_1(\gamma_1) *_Y \psi_1(\gamma_2)). \quad (12)$$

Definition 4 ([18]). A GE-norm on $\mathbf{X} := (\mathbf{X}, *, 1_X)$ is defined to be a mapping $\|\cdot\| : \mathbf{X} \rightarrow \mathbb{R}$ that satisfies:

$$(\forall \gamma_1 \in \mathbf{X})(\|\gamma_1\| \geq 0), \quad (13)$$

$$(\forall \gamma_1 \in \mathbf{X})(\|\gamma_1\| = 0 \Leftrightarrow \gamma_1 = 1_X), \quad (14)$$

$$(\forall \gamma_1, \gamma_2, \gamma_3 \in \mathbf{X})(\|\gamma_1 * \gamma_3\| \leq \|\gamma_1 * \gamma_2\| + \|\gamma_2 * \gamma_3\|). \quad (15)$$

A normed GE-algebra is a GE-algebra $\mathbf{X} := (\mathbf{X}, *, 1_X)$ equipped with a GE-norm $\|\cdot\| : \mathbf{X} \rightarrow \mathbb{R}$ and it is denoted by $(\mathbf{X}, \|\cdot\|)$.

Given a GE-algebra $\mathbf{X} := (\mathbf{X}, *, 1_X)$, if there exists a function $\|\cdot\|$ mapping elements of $\mathbf{X}$ to non-negative real numbers satisfying the conditions (14) and (15), then $(\mathbf{X}, \|\cdot\|)$ is a normed GE-algebra.

Definition 5 ([19]). Let $(\mathbf{X}, *, 1_X)$ be a GE-algebra. A mapping $\mu : \mathbf{X} \times (0, \infty) \rightarrow [0, 1]$ is called a fuzzy GE-norm on $\mathbf{X}$ if, for all $\rho, \sigma, \tau \in \mathbf{X}$ and all $t_1, t_2 > 0$, the following conditions hold:

(1) (FN1) Zero property:

$$\mu(\rho, t_2) = 1 \quad \text{for all } t_2 > 0 \quad \Leftrightarrow \quad \rho = 1_X.$$

(2) (FN2) Monotonicity in scale:

$$0 < t_1 \leq t_2 \quad \Rightarrow \quad \mu(\rho, t_1) \leq \mu(\rho, t_2).$$

(3) (FN3) Limit at large scales:

$$\lim_{t_2 \rightarrow \infty} \mu(\rho, t_2) = 1 \quad \text{for every } \rho \in \mathbf{X}.$$

(4) (FN4) Triangle inequality: For all $t_1, t_2 > 0$,

$$\mu(\rho * \tau, t_1 + t_2) \geq \min\{\mu(\rho * \sigma, t_1), \mu(\sigma * \tau, t_2)\}.$$

Definition 6 ([19]). Let $(\mathbf{X}, *, 1_X, \mu)$ be a fuzzy normed GE-algebra. A sequence $\{x_n\}$ in $\mathbf{X}$ is said to converge to $x \in \mathbf{X}$ (denoted by $x_n \rightarrow_\mu x$) if for every $t > 0$,

$$\lim_{n \rightarrow \infty} \mu(x_n * x, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(x * x_n, t) = 1.$$

Theorem 1 ([19]). Let $(X, *, 1_X)$ be a commutative GE-algebra equipped with a fuzzy GE-norm μ satisfying (FN1)–(FN4). If a sequence $(\rho_n) \subset X$ converges (in the fuzzy sense) to both ρ and σ , then $\rho = \sigma$.

3. Main Results

The theory of nonexpansive mappings plays a fundamental role in classical fixed point theory and nonlinear functional analysis. In normed and metric spaces, such mappings are closely connected with the convergence of iterative schemes and stability of approximation algorithms. When these ideas are transferred to algebraic systems such as fuzzy normed GE-algebras, several new structural questions arise.

In particular, it is natural to ask whether the basic properties of nonexpansive operators remain valid in the fuzzy algebraic setting and how these mappings interact with fuzzy convergence. In this section, we introduce the notion of fuzzy nonexpansive mappings in fuzzy normed GE-algebras and investigate their fundamental structural and convergence properties.

Throughout this section, let $(X, *, 1_X, \mu)$ be a fuzzy normed GE-algebra, where $\mu : X \times (0, \infty) \rightarrow [0, 1]$ is a fuzzy GE-norm.

Definition 7 (Fuzzy nonexpansive mapping). Let $(X, *, 1_X, \mu)$ be a fuzzy normed GE-algebra. A mapping $T : X \rightarrow X$ is called fuzzy nonexpansive if for all $\tau_1, \tau_2 \in X$ and for every $t > 0$, the following inequalities hold:

$$\mu(T(\tau_1) * T(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t),$$

and

$$\mu(T(\tau_2) * T(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t).$$

Informally, T is fuzzy nonexpansive if it does not decrease the fuzzy closeness between any two elements: the membership value $\mu(T(\tau_1) * T(\tau_2), t)$ is at least as large as $\mu(\tau_1 * \tau_2, t)$ for every scale $t > 0$. The two-sided requirement (both orders of the $*$ -product) is essential because $*$ is not commutative in a general GE-algebra.

Example 1. (i). Let $(X, *, 1_X, \mu)$ be a fuzzy normed GE-algebra. Define a mapping $T : X \rightarrow X$ by $T(\tau_1) = 1$ for all $\tau_1 \in X$. Then T is fuzzy nonexpansive mapping on $(X, *, 1_X, \mu)$.

(ii). Let $(X, *, 1_X, \mu)$ be a fuzzy normed GE-algebra. Define a mapping $T : X \rightarrow X$ by $T(\tau_1) = \tau_1$ for all $\tau_1 \in X$. Then T is fuzzy nonexpansive mapping on $(X, *, 1_X, \mu)$.

(iii). Let

$$X = \{1_X, a_X, b_X, c_X, d_X\}$$

and define the binary operation $*_X$ on X by the following Cayley table:

$*_X$	1_X	a_X	b_X	c_X	d_X
1_X	1_X	a_X	b_X	c_X	d_X
a_X	1_X	1_X	c_X	c_X	c_X
b_X	1_X	a_X	1_X	d_X	d_X
c_X	1_X	a_X	1_X	1_X	1_X
d_X	1_X	a_X	1_X	1_X	1_X

Then, $(X, *_X, 1_X)$ is a GE-algebra.

Define a GE-norm $\|\cdot\| : \mathbf{X} \rightarrow [0, \infty)$ by

$$\|1_X\| = 0 \quad \text{and} \quad \|x\| = 1 \text{ for all } x \in \{a_X, b_X, c_X, d_X\}.$$

Now define $\mu_X : X \times (0, \infty) \rightarrow [0, 1]$ by

$$\mu_X(x, t) = \frac{t}{t + \|x\|}, \quad x \in X, t > 0.$$

Then,

$$\mu_X(1_X, t) = 1, \quad \mu_X(x, t) = \frac{t}{t + 1} \text{ for all } x \neq 1_X.$$

Hence, $(\mathbf{X}, *_X, 1_X, \mu_X)$ is a fuzzy normed GE-algebra.

Define a mapping $T : \mathbf{X} \rightarrow \mathbf{X}$ by

$$T(1_X) = 1_X, \quad T(a_X) = a_X, \quad T(b_X) = 1_X, \quad T(c_X) = c_X, \quad T(d_X) = d_X.$$

It is easy to verify that T is a fuzzy nonexpansive mapping on $(\mathbf{X}, *_X, 1_X, \mu_X)$.

Averaged mappings are widely used in classical fixed-point theory and iterative algorithms for approximating fixed points of nonexpansive operators. These mappings often provide improved convergence behavior compared with the original operators. In order to extend this idea to fuzzy normed GE-algebras, we introduce the concept of a fuzzy GE-interpolation family, which serves as a tool for constructing averaged mappings in the fuzzy algebraic framework.

Definition 8 (Fuzzy GE-interpolation family). Let $(\mathbf{X}, *, 1_X, \mu)$ be a fuzzy normed GE-algebra. A family of binary operations $\{*_\alpha\}_{\alpha \in [0,1]}$ on $\mathbf{X}$ is called a fuzzy GE-interpolation family if for all $\tau_1, \tau_2, u, v, w, z \in \mathbf{X}$, all $\alpha \in [0, 1]$, and all $t > 0$, the following conditions are satisfied:

(I1) (Endpoint condition)

$$\tau_1 *_0 \tau_2 = \tau_1 \quad \text{and} \quad \tau_1 *_1 \tau_2 = \tau_2.$$

(I2) (Fuzzy interpolation inequality)

$$\mu((u *_\alpha v) * (w *_\alpha z), t) \geq \min \left\{ \mu(u * w, t), \mu(v * z, t) \right\}.$$

Definition 9 (α -averaged mapping). Let $(\mathbf{X}, *, 1_X, \mu)$ be a fuzzy normed GE-algebra and let $\{*_\alpha\}_{\alpha \in [0,1]}$ be a fuzzy GE-interpolation family on $\mathbf{X}$. Let $T : \mathbf{X} \rightarrow \mathbf{X}$ be a self-mapping and fix $\alpha \in (0, 1)$. The mapping $T_\alpha : \mathbf{X} \rightarrow \mathbf{X}$ defined by

$$T_\alpha(\tau) := \tau *_\alpha T(\tau), \quad \tau \in \mathbf{X},$$

is called the α -averaged mapping associated with T .

Fix $\alpha \in (0, 1)$. Let $T : \mathbf{X} \rightarrow \mathbf{X}$ be a mapping and let $T_\alpha : \mathbf{X} \rightarrow \mathbf{X}$ be the associated α -averaged mapping. Assume that the following condition holds for every $\tau_1 \in \mathbf{X}$ and every $t > 0$:

(A1) (Displacement scaling)

$$\mu(T_\alpha(\tau_1) * \tau_1, t) = \mu(T(\tau_1) * \tau_1, \frac{t}{\alpha}), \quad \mu(\tau_1 * T_\alpha(\tau_1), t) = \mu(\tau_1 * T(\tau_1), \frac{t}{\alpha}).$$

An important structural property of operator classes in fixed point theory is their stability under composition. In classical Banach space theory, the composition of nonexpansive

mappings remains nonexpansive. It is therefore natural to investigate whether a similar property holds in fuzzy normed GE-algebras. The following theorem shows that the class of fuzzy nonexpansive mappings is closed under composition.

Theorem 2 (Composition of fuzzy nonexpansive mappings). *Let $(\mathbf{X}, *, 1_{\mathbf{X}}, \mu)$ be a fuzzy normed commutative GE-algebra. Suppose $S, T : \mathbf{X} \rightarrow \mathbf{X}$ are fuzzy nonexpansive mappings. Define $U := S \circ T : \mathbf{X} \rightarrow \mathbf{X}$ by $U(\tau_1) = S(T(\tau_1))$. Then, U is fuzzy nonexpansive, that is,*

$$\mu(U(\tau_1) * U(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t) \quad \text{and} \quad \mu(U(\tau_2) * U(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t)$$

for all $\tau_1, \tau_2 \in \mathbf{X}$.

Proof. Let $\tau_1, \tau_2 \in \mathbf{X}$ be arbitrary, $t > 0$, and set $U = S \circ T$.

We have

$$\mu(U(\tau_1) * U(\tau_2), t) = \mu(S(T(\tau_1)) * S(T(\tau_2)), t).$$

Since S is fuzzy nonexpansive,

$$\mu(S(T(\tau_1)) * S(T(\tau_2)), t) \geq \mu(T(\tau_1) * T(\tau_2), t).$$

Since T is also fuzzy nonexpansive,

$$\mu(T(\tau_1) * T(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t).$$

Combining these two inequalities gives

$$\mu(U(\tau_1) * U(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t).$$

Similarly,

$$\mu(U(\tau_2) * U(\tau_1), t) = \mu(S(T(\tau_2)) * S(T(\tau_1)), t) \geq \mu(T(\tau_2) * T(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t),$$

where we again used fuzzy nonexpansiveness of S (with arguments $T(\tau_2)$ and $T(\tau_1)$), followed by fuzzy nonexpansiveness of T (with arguments τ_2 and τ_1).

Thus, for all $\tau_1, \tau_2 \in \mathbf{X}$,

$$\mu(U(\tau_1) * U(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t) \quad \text{and} \quad \mu(U(\tau_2) * U(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t),$$

so U is fuzzy nonexpansive. $\square$

Continuity properties of operators play a crucial role in the study of convergence and stability of iterative processes. In the fuzzy setting, sequential convergence provides a natural notion of limit in fuzzy normed GE-algebras. Therefore, it is important to examine whether fuzzy nonexpansive mappings preserve such convergence behavior. The following result establishes sequential continuity of fuzzy nonexpansive mappings with respect to fuzzy convergence.

Theorem 3 (Sequential continuity of fuzzy nonexpansive mappings). *Let $(\mathbf{X}, *, 1_{\mathbf{X}}, \mu)$ be a fuzzy normed commutative GE-algebra. Suppose that $T : \mathbf{X} \rightarrow \mathbf{X}$ is fuzzy nonexpansive. If $\{\tau_{1n}\}$ is a sequence in $\mathbf{X}$ such that*

$$\tau_{1n} \rightarrow \tau_1 \quad \text{in the sense of Definition 6,}$$

then,

$$T(\tau_{1n}) \rightarrow T(\tau_1)$$

in the same sense. In particular, every fuzzy nonexpansive mapping is two-sided sequentially continuous.

Proof. Assume that $\tau_{1n} \rightarrow \tau_1$. We have to show that

$$\lim_{n \rightarrow \infty} \mu(T(\tau_{1n}) * T(\tau_1), t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(T(\tau_1) * T(\tau_{1n}), t) = 1.$$

By the fuzzy nonexpansiveness of T , we have for each $n \in \mathbb{N}$,

$$\mu(T(\tau_{1n}) * T(\tau_1), t) \geq \mu(\tau_{1n} * \tau_1, t).$$

Since by hypothesis $\lim_{n \rightarrow \infty} \mu(\tau_{1n} * \tau_1, t) = 1$, it follows from this inequality that

$$\lim_{n \rightarrow \infty} \mu(T(\tau_{1n}) * T(\tau_1), t) \geq 1.$$

We also have

$$\mu(T(\tau_{1n}) * T(\tau_1), t) \leq 1.$$

Thus, we get

$$1 \leq \lim_{n \rightarrow \infty} \mu(T(\tau_{1n}) * T(\tau_1), t) \leq 1.$$

It gives

$$\lim_{n \rightarrow \infty} \mu(T(\tau_{1n}) * T(\tau_1), t) = 1.$$

Similarly, using the fuzzy nonexpansiveness of T , we obtain for each $n \in \mathbb{N}$,

$$\mu(T(\tau_1) * T(\tau_{1n}), t) \geq \mu(\tau_1 * \tau_{1n}, t).$$

Since $\tau_{1n} \rightarrow \tau_1$, thus, using $\lim_{n \rightarrow \infty} \mu(\tau_1 * \tau_{1n}, t) = 1$, we conclude that

$$\lim_{n \rightarrow \infty} \mu(T(\tau_1) * T(\tau_{1n}), t) = 1.$$

Therefore,

$$\lim_{n \rightarrow \infty} \mu(T(\tau_{1n}) * T(\tau_1), t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(T(\tau_1) * T(\tau_{1n}), t) = 1.$$

Which, by Definition 6, means that $T(\tau_{1n}) \rightarrow T(\tau_1)$ in $\mathbf{X}$.

Thus, T is two-sided sequentially continuous. $\square$

Averaged operators play an essential role in the analysis of iterative methods for fixed-point problems. In classical settings, averaged mappings preserve fuzzy nonexpansiveness and often lead to improved convergence properties. The following theorem shows that the α -averaged mappings constructed via a fuzzy GE-interpolation family also preserve fuzzy nonexpansiveness in fuzzy normed GE-algebras.

Theorem 4 (Fuzzy nonexpansiveness of averaged mappings). *Let $(\mathbf{X}, *, 1_{\mathbf{X}}, \mu)$ be a fuzzy normed commutative GE-algebra. Let $\{\ast_{\alpha}\}_{\alpha \in [0,1]}$ be a GE-interpolation family on $\mathbf{X}$ satisfying (I1)–(I2). Assume that $T : \mathbf{X} \rightarrow \mathbf{X}$ is fuzzy nonexpansive, for a fixed $\alpha \in (0, 1)$, define the α -averaged mapping*

$$T_{\alpha}(\tau_1) := \tau_1 \ast_{\alpha} T(\tau_1), \text{ for all } \tau_1 \in \mathbf{X}.$$

Then, T_{α} is also nonexpansive, that is,

$$\mu(T_{\alpha}(\tau_1) * T_{\alpha}(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t) \quad \text{and} \quad \mu(T_{\alpha}(\tau_2) * T_{\alpha}(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t)$$

for all $\tau_1, \tau_2 \in \mathbf{X}$.

Proof. Fix $\alpha \in (0, 1)$ and arbitrary $\tau_1, \tau_2 \in \mathbf{X}$.

By Definition of T_α ,

$$T_\alpha(\tau_1) = \tau_1 *_\alpha T(\tau_1), \quad T_\alpha(\tau_2) = \tau_2 *_\alpha T(\tau_2).$$

By (I2), we get

$$\mu((\tau_1 *_\alpha T(\tau_1)) * (\tau_2 *_\alpha T(\tau_2)), t) \geq \min\{\mu(\tau_1 * \tau_2, t), \mu(T(\tau_1) * T(\tau_2), t)\}. \quad (16)$$

Using the fuzzy nonexpansiveness of T , we have

$$\mu(\tau_1 *_\alpha T(\tau_1) * \tau_2 *_\alpha T(\tau_2), t) \geq \min\{\mu(\tau_1 * \tau_2, t), \mu(\tau_1 * \tau_2, t)\}.$$

Thus, we get

$$\mu(T_\alpha(\tau_1) * T_\alpha(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t).$$

By (I2) again, we get

$$\mu((\tau_2 *_\alpha T(\tau_2)) * (\tau_1 *_\alpha T(\tau_1)), t) \geq \min\{\mu(\tau_2 * \tau_1, t), \mu(T(\tau_2) * T(\tau_1), t)\}. \quad (17)$$

Using the fuzzy nonexpansiveness of T , we have

$$\mu(\tau_2 *_\alpha T(\tau_2) * \tau_1 *_\alpha T(\tau_1), t) \geq \min\{\mu(\tau_2 * \tau_1, t), \mu(\tau_2 * \tau_1, t)\}.$$

Thus, we get

$$\mu(T_\alpha(\tau_2) * T_\alpha(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t).$$

We have shown that for all $\tau_1, \tau_2 \in \mathbf{X}$,

$$\mu(T_\alpha(\tau_1) * T_\alpha(\tau_2), t) \geq \mu(\tau_1 * \tau_2, t) \quad \text{and} \quad \mu(T_\alpha(\tau_2) * T_\alpha(\tau_1), t) \geq \mu(\tau_2 * \tau_1, t),$$

so T_α is fuzzy nonexpansive. $\square$

The demiclosedness principle is a fundamental tool in fixed-point theory and nonlinear operator analysis. It provides an important criterion for identifying fixed points through limit processes. Extending this principle to fuzzy normed GE-algebras helps establish the stability of fixed point structures under fuzzy convergence.

Theorem 5 (Demiclosedness Principle). *Suppose $\mathbf{X}$ is a fuzzy normed commutative GE-algebra and $T : \mathbf{X} \rightarrow \mathbf{X}$ is a fuzzy nonexpansive mapping. If $\tau_{1n} \rightarrow \tau_1$ and $T(\tau_{1n}) \rightarrow \tau_1$ then $T(\tau_1) = \tau_1$.*

Proof. Let $\tau_{1n} \rightarrow \tau_1$ and $T(\tau_{1n}) \rightarrow \tau_1$. Applying sequential continuity of T from Theorem 3, we get $T(\tau_{1n}) \rightarrow T(\tau_1)$. Hence, by uniqueness of the limit of a sequence (Theorem 1), we get $T(\tau_1) = \tau_1$. $\square$

Understanding the relationship between a mapping and its associated averaged operator is important in the analysis of iterative fixed point methods. In many classical settings, averaged operators share the same fixed points as the original mapping. The following theorem shows that a similar equivalence holds for fuzzy nonexpansive mappings and their α -averaged counterparts in fuzzy normed GE-algebras.

Theorem 6 (GE-Fixed point equivalence). *Suppose $\mathbf{X}$ is a fuzzy normed commutative GE-algebra, $T : \mathbf{X} \rightarrow \mathbf{X}$ is a fuzzy nonexpansive mapping, and T_α is an α -averaged mapping satisfying*

(A1). Assume that the interpolation family $\{*_\alpha\}$ satisfies: $\tau_1 *_\alpha \tau_1 = \tau_1$ for all $\tau_1 \in \mathbf{X}$. Then, the set of fixed points of T coincides with that of T_α .

Proof. Suppose $F(T)$ is the set of fixed points of the mapping T and $F(T_\alpha)$ is the set of fixed points of the α -averaged mapping T_α . Then, for each $\tau_1 \in F(T)$

$$T_\alpha(\tau_1) = \tau_1 *_\alpha T(\tau_1) = \tau_1 *_\alpha \tau_1 = \tau_1.$$

Thus, $T_\alpha(\tau_1) = \tau_1$, i.e., $\tau_1 \in F(T_\alpha)$.

Conversely, let $\tau_1 \in F(T_\alpha)$, then $T_\alpha(\tau_1) = \tau_1$. Now using (A1), we get

$$\mu(T_\alpha(\tau_1) * \tau_1, t) = \mu\left(T(\tau_1) * \tau_1, \frac{t}{\alpha}\right).$$

Since $T_\alpha(\tau_1) = \tau_1$, the left-hand side is

$$\mu(T_\alpha(\tau_1) * \tau_1, t) = \mu(\tau_1 * \tau_1, t) = 1.$$

Thus,

$$1 = \mu(T_\alpha(\tau_1) * \tau_1, t) = \mu\left(T(\tau_1) * \tau_1, \frac{t}{\alpha}\right),$$

and since $\alpha \in (0, 1)$ so $\frac{t}{\alpha} > 0$, we obtain

$$\mu\left(T(\tau_1) * \tau_1, \frac{t}{\alpha}\right) = 1.$$

Using the zero property, we get

$$T(\tau_1) * \tau_1 = 1.$$

Similarly, we can show that

$$\tau_1 * T(\tau_1) = 1.$$

Since $\mathbf{X}$ is commutative, we have $T(\tau_1) = \tau_1$, so $\tau_1 \in F(T)$. $\square$

Iterative processes are commonly used to approximate fixed points of fuzzy non-expansive mappings. In fuzzy normed GE-algebras, such iterative schemes must be analyzed with respect to fuzzy convergence. The following result shows that the limit of the iterative sequence generated by the averaged mapping is indeed a fixed point of the original mapping.

Theorem 7. (Convergence of the averaged iteration). Let $(\mathbf{X}, *, 1_{\mathbf{X}}, \mu_{\mathbf{X}})$ be a fuzzy normed commutative GE-algebra. Let $T : \mathbf{X} \rightarrow \mathbf{X}$ be a fuzzy nonexpansive mapping and let T_α be the fuzzy α -averaged mapping defined by

$$T_\alpha(\tau) = \tau *_\alpha T(\tau), \quad \tau \in \mathbf{X},$$

for some fixed $\alpha \in (0, 1)$. Assume that T_α is two-sided sequentially continuous and (A1) holds. Let $\{\tau_n\}$ be the sequence defined by

$$\tau_{n+1} = T_\alpha(\tau_n), \quad n \geq 1.$$

If $\tau_n \rightarrow_\mu \tau^\dagger$, then, $\tau^\dagger \in F(T)$.

Proof. Assume that $\tau_n \rightarrow_\mu \tau^\dagger$. Since T_α is two-sided sequentially continuous, we have $T_\alpha(\tau_n) \rightarrow_\mu T_\alpha(\tau^\dagger)$. But $T_\alpha(\tau_n) = \tau_{n+1}$, hence $\tau_{n+1} \rightarrow_\mu T_\alpha(\tau^\dagger)$. Also, $\tau_{n+1} \rightarrow_\mu \tau^\dagger$. Therefore, $T_\alpha(\tau^\dagger) = \tau^\dagger$.

Thus,

$$T_\alpha(\tau^\dagger) * \tau^\dagger = \tau^\dagger * \tau^\dagger = 1_X,$$

and hence,

$$\mu_X(T_\alpha(\tau^\dagger) * \tau^\dagger, t) = \mu_X(1_X, t) = 1, \quad \forall t > 0.$$

Using (A1), we obtain

$$\mu_X(T(\tau^\dagger) * \tau^\dagger, \frac{t}{\alpha}) = 1, \quad \forall t > 0.$$

Following the same steps again, we can also get

$$\mu_X(\tau^\dagger * T(\tau^\dagger), \frac{t}{\alpha}) = 1, \quad \forall t > 0.$$

Thus, $\mu_X(T(\tau^\dagger) * \tau^\dagger, s) = 1$ and $\mu_X(\tau^\dagger * T(\tau^\dagger), s) = 1$ for all $s > 0$, and by (FN1),

$$T(\tau^\dagger) * \tau^\dagger = 1_X \text{ and } \tau^\dagger * T(\tau^\dagger) = 1_X$$

Since X is commutative, we conclude $T(\tau^\dagger) = \tau^\dagger$. Hence, $\tau^\dagger \in F(T)$. $\square$

The topological structure of fixed point sets is important in the analysis of operator equations and iterative methods. In particular, closedness of the fixed point set guarantees stability of solutions under limiting processes. The following theorem establishes that the fixed point set of a fuzzy nonexpansive mapping is sequentially closed in fuzzy normed GE-algebras.

Theorem 8 (Closedness of the fixed point set). *Let $(X, *, 1_X, \mu)$ be a fuzzy normed commutative GE-algebra. Let $T : X \rightarrow X$ be a fuzzy nonexpansive mapping. Then, the fixed-point set*

$$F(T) := \{\tau_1 \in X : T(\tau_1) = \tau_1\},$$

is sequentially closed with respect to the convergence induced by μ , i.e., if $\{\tau_{1n}\} \subseteq F(T)$ and $\tau_{1n} \rightarrow \tau_1$ in X , then $\tau_1 \in F(T)$.

Proof. Let $\{\tau_{1n}\}$ be a sequence in $F(T)$ such that $\tau_{1n} \rightarrow \tau_1$ in X . By Definition of $F(T)$, we have

$$T(\tau_{1n}) = \tau_{1n} \quad \text{for all } n \in \mathbb{N}.$$

Since T is two-sided sequentially continuous, the convergence $\tau_{1n} \rightarrow \tau_1$ implies

$$T(\tau_{1n}) \rightarrow T(\tau_1)$$

in X . On the other hand, as $T(\tau_{1n}) = \tau_{1n}$ for all n , we also have $T(\tau_{1n}) \rightarrow \tau_1$. By the uniqueness of limits in a fuzzy normed commutative GE-algebra (see Theorem 1), it follows that

$$T(\tau_1) = \tau_1,$$

i.e., $\tau_1 \in F(T)$. Thus $F(T)$ is sequentially closed. $\square$

4. Application to GE-Filters

In this section, we demonstrate how the fixed-point results established in Section 2 interact naturally with the filter-theoretic structure of GE-algebras. Recall that GE-filters are fundamental objects in the study of GE-algebras [15–17], playing a role analogous to ideals

in ring theory and open sets in topology. We show that fixed points of fuzzy nonexpansive mappings give rise to GE-filters, and that the fixed point set itself induces a filter-preserving structure under fuzzy convergence.

Definition 10 ([15]). Let $(X, *, 1_X)$ be a GE-algebra. A nonempty subset $\mathcal{F} \subseteq X$ is called a GE-filter of X if it satisfies:

- (F1) $1_X \in \mathcal{F}$,
- (F2) $(\forall x, y \in X)(x \in \mathcal{F}, x * y \in \mathcal{F} \Rightarrow y \in \mathcal{F})$.

Definition 11. Let $(X, *, 1_X)$ be a transitive GE-algebra and let $\tau \in X$. The principal filter generated by τ is defined as

$$\langle \tau \rangle := \{x \in X : \tau \leq_X x\} = \{x \in X : \tau * x = 1_X\}.$$

Remark 8. It is straightforward to verify that $\langle \tau \rangle$ is always a GE-filter of a transitive GE-algebra X for any $\tau \in X$.

Proposition 2. Let $(X, *, 1_X, \mu)$ be a fuzzy normed commutative GE-algebra and let $T : X \rightarrow X$ be a fuzzy nonexpansive mapping. If $\tau^+ \in F(T)$ is any fixed point of T , then the principal filter $\langle \tau^+ \rangle$ is a GE-filter of X that is invariant under T , in the sense that

$$x \in \langle \tau^+ \rangle \implies T(x) \in \langle \tau^+ \rangle.$$

Proof. Let $x \in \langle \tau^+ \rangle$, so that $\tau^+ * x = 1_X$, i.e., $\tau^+ \leq_X x$. We need to show $\tau^+ \leq_X T(x)$, i.e., $\tau^+ * T(x) = 1_X$. Since $\tau^+ \in F(T)$, we have $T(\tau^+) = \tau^+$. The fuzzy nonexpansiveness of T gives, for all $t > 0$,

$$\mu(T(\tau^+) * T(x), t) \geq \mu(\tau^+ * x, t) = \mu(1_X, t) = 1,$$

where the last equality uses (FN1). Since μ takes values in $[0, 1]$, we obtain

$$\mu(T(\tau^+) * T(x), t) = 1 \quad \text{for all } t > 0.$$

Applying (FN1) again gives $T(\tau^+) * T(x) = 1_X$, i.e., $\tau^+ * T(x) = 1_X$ (since $T(\tau^+) = \tau^+$). Therefore, $T(x) \in \langle \tau^+ \rangle$. $\square$

Proposition 3. Let $(X, *, 1_X, \mu)$ be a fuzzy normed commutative GE-algebra and let $T : X \rightarrow X$ be a fuzzy nonexpansive mapping. Let $\{\tau_{1n}\} \subseteq F(T)$ be a sequence of fixed points such that $\tau_{1n} \rightarrow_\mu \tau$ for some $\tau \in X$. Then:

- (i) $\tau \in F(T)$ (by Theorem 8), and
- (ii) for every $n \in \mathbb{N}$,

$$\langle \tau \rangle \subseteq \langle \tau_{1n} \rangle,$$

provided $\tau_{1n} \leq_X \tau$ for all n .

In particular, the family $\{\langle \tau_{1n} \rangle\}$ of principal filters is eventually nested, and the limit filter $\langle \tau \rangle$ is itself a GE-filter of X .

Proof. Part (i) follows directly from Theorem 8 (sequential closedness of $F(T)$). For part (ii), suppose $\tau_{1n} \leq_X \tau$ for all n . Let $x \in \langle \tau \rangle$, so $\tau * x = 1_X$, i.e., $\tau \leq_X x$. By the transitivity of $\leq_X$ (Definition 2) and the assumption $\tau_{1n} \leq_X \tau$, we get $\tau_{1n} \leq_X x$, i.e., $\tau_{1n} * x = 1_X$, so $x \in \langle \tau_{1n} \rangle$. $\square$

Example 2. Consider the GE-algebra $X = \{1_X, a_X, b_X, c_X, d_X\}$ with the Cayley table, GE-norm $\|\cdot\|$, and fuzzy GE-norm μ_X defined in Example 1(iii), and the fuzzy nonexpansive mapping

$$T(1_X) = 1_X, \quad T(a_X) = a_X, \quad T(b_X) = 1_X, \quad T(c_X) = c_X, \quad T(d_X) = d_X.$$

The fixed point set is $F(T) = \{1_X, a_X, c_X, d_X\}$.

Step 1: Fixed points generate GE-filters. By Proposition 2, each fixed point generates an invariant GE-filter. For instance:

$$\begin{aligned} \langle a_X \rangle &= \{x \in X : a_X * x = 1_X\} = \{1_X, a_X\}, \\ \langle c_X \rangle &= \{x \in X : c_X * x = 1_X\} = \{1_X, a_X, b_X, c_X, d_X\} = X. \end{aligned}$$

One verifies from the Cayley table that each of these is indeed a GE-filter satisfying (F1)–(F2), and that T maps each filter to itself.

Step 2: A convergent sequence of fixed points. Define the constant sequence $\tau_{1n} = a_X$ for all $n \in \mathbb{N}$. Then, $\tau_{1n} \rightarrow_\mu a_X$ trivially (since $\mu_X(a_X * a_X, t) = \mu_X(1_X, t) = 1$ for all $t > 0$), and by Theorem 8, $a_X \in F(T)$, which is confirmed directly. The principal filter $\langle a_X \rangle = \{1_X, a_X\}$ is the limit filter, and it is preserved by T (Proposition 2).

Step 3: Interpretation. This example shows concretely how the abstract fixed-point results of Section 2 produce tangible filter-theoretic conclusions: the sequential closure of $F(T)$ (Theorem 8) guarantees that the filter structure is stable under fuzzy-convergent approximation sequences, which is meaningful in the context of fuzzy algebraic reasoning systems where $\mathcal{F}$ -filters model sets of “provable” or “accepted” propositions.

Remark 8. The interaction between fixed point sets and GE-filters described above opens several directions for further study. For instance, one may investigate conditions under which the fixed point set $F(T)$ is itself a GE-filter of X , or study the relationship between belligerent GE-filters [17] and fixed points of specific classes of fuzzy nonexpansive mappings. These questions are left as topics for future research.

5. Conclusions

In this paper, we investigated nonexpansive mappings in the framework of fuzzy normed GE-algebras. We introduced the concept of fuzzy nonexpansive mappings and studied several of their fundamental properties. In particular, we proved that the composition of two fuzzy nonexpansive mappings is again fuzzy nonexpansive and established that every fuzzy nonexpansive mapping is two-sided sequentially continuous with respect to fuzzy convergence.

Moreover, by using a fuzzy GE-interpolation family, we defined the notion of α -averaged mappings in fuzzy normed GE-algebras and showed that such averaged mappings preserve fuzzy nonexpansiveness. We also obtained a demiclosedness type principle and proved fixed point equivalence results between a mapping and its associated α -averaged mapping under suitable assumptions. Finally, we established that the fixed-point set of a fuzzy nonexpansive mapping is sequentially closed.

These results extend several classical ideas of fixed-point theory and nonexpansive operator theory from Banach spaces to fuzzy normed GE-algebras. The obtained framework may be useful for developing iterative algorithms and for further studies on convergence theorems and fixed point approximation in fuzzy algebraic structures.

Open Problems

- (1) Can we define iterative algorithms in the setting of Fuzzy normed GE-algebras?

- (2) Can we extend and generalize the results for a more general class of nonexpansive type of mappings, such as Suzuki type mappings and Mappings satisfying condition (E) in the setting of Fuzzy normed GE-algebras?

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