

Article

A Foundational Analysis of Local Kernel-Based Calculus

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Abstract

We introduce the local kernel-based calculus, a unifying framework for local differential and integral operators based on an arbitrary positive continuous kernel function. This framework encompasses conformable, non-conformable, and our newly introduced local Euler-kernel derivatives as special cases. The parameter of the kernel is unrestricted and may take negative values, reflecting its role as a genuine parameter rather than an order of fractional differentiation. Within this general setting, we rigorously prove a complete set of foundational theorems: linearity, the product rule, continuity, Rolle's theorem, the mean value theorem, and the fundamental theorem of calculus via the associated integral operator. We also derive a new formulation of the chain rule that expresses the chain rule entirely in terms of the kernel-based derivatives. While algebraically equivalent to the classical form, this representation preserves the intuitive structure of the chain rule without reference to the classical derivative. We further establish the Fundamental Theorem of Local Euler Calculus and its generalization, the Fundamental Theorem of Local Kernel-Based Calculus, confirming that the derivative and integral operators are genuine inverses, with the classical fundamental theorem recovered as special cases when the kernel reduces to unity. As an important illustration, we develop the local Euler calculus with the exponential kernel in full detail, providing explicit derivative and integral formulas for elementary functions. This special case demonstrates the simplicity and power of the functional approach. Overall, the local kernel-based calculus provides a solid, self-contained foundation that unifies a wide class of local operators and extends far beyond the traditional setting.

Keywords: mathematical logic and foundations; mathematical analysis; ODE; local kernel-based calculus; conformable calculus; Differentiation; mean value theorem; fundamental theorem of calculus

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1. Introduction

Developing foundational concepts in analysis and calculus has proven fruitful. The laconic mode of expression [1] has long been recognized as an admirable and enduring virtue in art, mathematics, science, logic, and philosophy. This perspective finds contemporary support in a recent study [2].

Recent years have witnessed a growing interest in extending classical differential theorems to *generalized local derivative frameworks*. Mohammad and Saadaoui [3] introduced a new *local derivative* based on an undefined kernel and proved Rolle's theorem and the mean value theorem (MVT) within that setting. Kamble and Kulkarni [4] defined an extended two-parameter *local derivative* and established analogous results, including

Lagrange’s MVT. Vivas-Cortez et al. [5] proposed a biparametric local derivative and derived generalized versions of Rolle’s theorem and the MVT. Chauhan and Chudasama [6] developed a q -analogue of the right local M -derivative (which they term “fractional” despite its local nature), obtaining q -versions of Rolle’s theorem, the MVT, and the fundamental theorem of calculus (FTC). Further contributions include the generalized conformable derivative applied to dynamical models [7]. Collectively, these works demonstrate a sustained effort to build a solid foundation for *local generalized calculi*; however, none of them addresses the general functional kernel and the specific exponential kernel, or provides the chain rule that we derive in this paper for the *local kernel-based calculus*.

We emphasize that the present work is strictly concerned with *local* generalizations of the standard derivative. Classical fractional derivatives—such as those of Riemann–Liouville, Caputo, and Grünwald–Letnikov—are inherently *non-local* operators involving convolution integrals or infinite series, and are therefore not subsumed by our framework. The term “fractional” in the references cited above is retained only when quoting the original authors’ terminology, and does not imply that our local kernel-based calculus encompasses genuinely non-local fractional operators.

On a broader scale, the quest for generalized operators continues through the development of generalized calculus in both local and non-local contexts. In a landmark contribution, Kiryakova [8] introduced fractional integral operators that use the Rathie I -function as a kernel, in the context of Generalized Fractional Calculus (GFC). This is a significant step beyond the classical GFC, which relies on Fox H - and Meijer G -functions, because the I -function can represent a wider array of special functions, including the multi-index Mittag-Leffler–Le Roy functions, and thus can serve as eigenfunctions for these new operators. While Kiryakova’s operators are inherently non-local integral transforms, our focus is complementary: we establish a rigorous local kernel-based differential calculus. Our local derivative $D_x^{F,\alpha}$ provides the fundamental building blocks for local differential structures—such as a revised chain rule and Rolle’s theorem—that are essential for any generalized local calculus. Thus, together with the non-local integral frameworks of Kiryakova, our work contributes to a broader, unified program of generalized analysis, where local and non-local approaches coexist as complementary tools.

To contextualize our study within the broader landscape of modern analysis, we note that our kernel-based calculus formalism provides a foundational framework for interpreting recent advances across several mathematical disciplines. Our analysis offers structural and asymptotic insights relevant to operator-based stabilization frameworks [9], contextualizes approximation-theoretic operator constructions, and informs modern perspectives on operator optimization and computational efficiency. Furthermore, the differential and computational operator methodologies central to our work may benefit optimization methods in finance and other applied fields.

Abel, Agratini, and Păltănea constructed Szász–Mirakjan–Durrmeyer operators using multiple Appell polynomials, thereby introducing a family of linear positive operators built upon specific polynomial kernels [10]; our unified kernel-based calculus formalism provides the foundational framework that naturally contextualizes such operator constructions, as it generalizes the very notion of differentiation and integration through arbitrary kernels, offering a rigorous mathematical setting within which kernel-defined operators—including those based on Appell polynomials—can be systematically formulated and studied. Our kernel-based calculus offers a comprehensive functional-analytic framework that naturally contextualizes this approach by generalizing the very concept of differentiation through kernel-defined operators, thereby bridging approximation theory, functional analysis, and local generalized calculus under a single, coherent mathematical framework.

We note that the kernel-based framework introduced here may also be viewed through the lens of functional analysis and measure theory. The operator $D_x^{F,\alpha}$ can be interpreted as a multiplication operator acting on the classical derivative, i.e., $D_x^{F,\alpha} f = F(x, \alpha) f'$, where $F(x, \alpha)$ serves as a pointwise multiplier. Similarly, the associated integral operator $I_x^{F,\alpha} f = \int f(x)/F(x, \alpha) dx$ can be understood as an integration with respect to a weighted measure $d\mu_{F,\alpha}(x) = dx/F(x, \alpha)$, provided $F(x, \alpha) > 0$ and is sufficiently regular. This perspective connects the present framework to the broader theory of weighted Sobolev spaces and Stieltjes integrals. However, a full exploration of these functional-analytic and measure-theoretic aspects would require a separate and substantial investigation. The present work focuses on establishing the elementary differential and integral structure of the local kernel-based calculus—including the chain rule, Rolle’s theorem, the mean value theorem, and the FTC—which serves as the necessary foundation for any such future extensions.

Our foundational kernel-based calculus establishes a rigorous mathematical platform that naturally extends to several domains of contemporary research. The tensor theories and manifold–metric pair frameworks developed by Ntelis [11,12] rely on differential structures on curved spaces; our kernel-based derivatives provide a generalized differential tool that could potentially reformulate or extend such constructions by replacing non-local fractional or classical derivatives with kernel-defined operators, thereby introducing local or weighted sensitivities on manifolds. Similarly, numerical studies of diffusion models employing fractional derivatives [13] motivate the need for flexible generalized operators; our kernel-based calculus offers a unified framework that encompasses and generalizes such approaches beyond the fractional paradigm, potentially enabling novel diffusion kernels with tailored memory or locality properties. More broadly, the kernel-based formalism may prove valuable in physical and cosmological applications [14–16], where modified differential structures offer new avenues for modeling non-standard evolution. While the present work focuses on establishing the foundational mathematical framework, these connections highlight the potential for future interdisciplinary impact, from diffusion theory to cosmology, once the kernel-based operators are integrated into corresponding dynamical systems.

We focus on the study of local kernel-based calculus, drawing on ideas from local fractional calculus [17–21]. Khalil et al. [17] developed a local conformable fractional derivative, establishing some foundational aspects of conformable fractional calculus. More recently, Fleitas et al. [20] and others [21] have proposed generalizations of conformable-type derivatives. The conformable derivative has been extended in several directions: Hyder and Soliman [19] introduced a generalized θ -conformable derivative with a broader class of kernels, while Martínez and Nápoles Valdés [18] developed a version for functions of several variables, further expanding its applicability.

We note that our framework differs from that of Fleitas et al. [20] in several essential respects. First, while their definition of the generalized conformable derivative is restricted to $\alpha > 0$, our analysis imposes no such restriction: the parameter α may take arbitrary real values, including negative ones, reflecting the fact that in our framework α is a genuine parameter of the kernel $F(x, \alpha)$ rather than an order of fractional differentiation. Second, although Fleitas et al. refer to their operators as “fractional derivatives,” this terminology has been critically examined in the literature. As noted by Abdelhakim [22], the conformable derivative is local by its very definition and fails to satisfy the essential property of non-locality that characterizes genuine fractional operators such as the Riemann–Liouville and Caputo derivatives. Indeed, the conformable derivative of a differentiable function reduces to a scaled ordinary derivative, and it is incapable of reproducing the fractional derivatives obtained from classical non-local operators [22,23]. Kiryakova has likewise emphasized

that conformable-type derivatives cannot be accepted as true fractional derivatives because they lack the fundamental non-local character required for fractional calculus [8]. Our framework, by contrast, makes no claim to fractional calculus; it is explicitly presented as a *local kernel-based calculus*, a unifying framework for local differential and integral operators that generalizes the classical derivative while remaining firmly within the realm of local analysis. Third, whereas Fleitas et al. define higher-order derivatives via finite differences for $\alpha > 1$, our derivative remains first-order in structure, with the kernel $F(x, \alpha)$ governing the local scaling of the independent variable. This makes our framework complementary to theirs: they provide a generalization in the direction of fractional-order local operators, while we provide a generalization in the direction of arbitrary kernel functions and unrestricted parameter values, together with a complete set of foundational theorems for the first-order case.

Although the two chain rule formulas are algebraically equivalent when $F(g(x), \alpha) \neq 0$, our formulation offers a structural advantage: it expresses the chain rule entirely in terms of the kernel-based derivatives, maintaining the intuitive structure

$$D_x^{F, \alpha}[f(g(x))] = D_g^{F, \alpha}[f(g)] \cdot D_x^{F, \alpha}[g(x)] \cdot F^{-1}(g(x), \alpha). \quad (1)$$

This form makes explicit the relationship between the three derivatives and requires no reference to the classical derivative f' , which is particularly useful when working with nested compositions or when the classical derivative is not readily available. By contrast, the chain rule in Fleitas et al. [20] is expressed as $G_T^\alpha(f \circ g)(t) = f'(g(t))G_T^\alpha g(t)$, which, while compact, obscures the role of the kernel in the composition and requires separate evaluation of the classical derivative.

In this paper, we consider and develop two new foundational calculus: the local Euler calculus, and the local kernel-based calculus. Our work is twofold. Firstly, we build upon and go beyond the previous studies, by redeveloping a new local kernel-based calculus, i.e., the local Euler calculus, and we study its properties. Then we generalize these calculi to what we call the local kernel-based calculus, and we study its properties. In particular, we propose a *local kernel-based derivative* $D_x^{F, \alpha}$, defined for a given positive continuous function $F(x, \alpha)$ by

$$D_x^{F, \alpha} f(x) = \lim_{h \rightarrow 0} \frac{f(x + \Delta x \cdot F(x, \alpha)) - f(x)}{\Delta x}, \quad (2)$$

which explicitly decouples the order from the scaling function and unifies several known fractional derivatives (including the conformable derivative and the local Euler derivative introduced here). Our main contributions are:

1. A chain rule for the local kernel-based derivative:

$$D_x^{F, \alpha}[f(g(x))] = D_g^{F, \alpha}[f(g)] \cdot D_x^{F, \alpha}[g(x)] \cdot F^{-1}(g(x), \alpha), \quad (3)$$

2. The introduction of a *local Euler calculus* as a special case with $F(x, \alpha) = e^{(1-\alpha)x}$, together with its FTC, Rolle's theorem, and the MVT.
3. A unified treatment of the corresponding integral operator and its properties.

From the definition, i.e., Equation (2), it is immediate that whenever f is ordinarily differentiable at x and $F(x, \alpha) \neq 0$, the limit reduces to

$$D_x^{F, \alpha} f(x) = F(x, \alpha) f'(x). \quad (4)$$

This identity is a useful consequence under these restrictive assumptions but does not exhaust the generality of the operator. The definition itself remains well-defined outside this restricted setting. In particular, if $F(x_0, \alpha) = 0$, the difference quotient vanishes identically for every function f , including discontinuous ones, yielding $D_x^{F, \alpha} f(x_0) = 0$ while the ordinary derivative $f'(x_0)$ need not exist. Consequently, differentiability in this degenerate sense does not imply continuity. To develop the full *local kernel-based calculus*—including Rolle’s theorem, the mean value theorem, the FTC, and the chain rule—we consequently adopt the assumption $F(x, \alpha) > 0$ on the interval under consideration. Under this assumption, the classical and weighted derivatives emerge as special subcases, while the kernel F , embedded inside the increment of the independent variable, governs the local scaling and gives the framework its name.

Finally, we establish the Fundamental Theorem of Local Euler Calculus (FTLEC) and its generalization, the Fundamental Theorem of Local Kernel-Based Calculus (FTLKBC). These theorems demonstrate that the local Euler and kernel-based derivatives and their associated integrals are genuine inverses, recovering the classical FTC as the special cases $\alpha = 1$ and $F(x, \alpha) = 1$, respectively. In this sense, the FTLEC and FTLKBC are not merely reformulations of the classical FTC, but rather genuine generalizations that validate the coherence of the new operator frameworks while reducing to the classical result when the kernel reduces to unity.

The presentation follows the historical order of our discovery. Section 2 first develops the *local Euler calculus* for the exponential kernel $F(x, \alpha) = e^{(1-\alpha)x}$. This special case is elementary yet already reveals the need for a chain rule. Section 3 then generalizes the entire framework to an arbitrary positive continuous function $F(x, \alpha)$, establishing the *local kernel-based calculus* as a unified theory. Readers primarily interested in the general result may skip directly to Section 3, where all theorems are proved in full generality; the Euler case serves as a concrete illustration and a consistency check.

This paper is organized as follows. In Section 2, we study the local Euler calculus. In Section 3, we generalize the previous results, and we study its properties in the framework of local kernel-based calculus. In Section 4, we conclude.

2. Local Euler Calculus

It is convenient to define the standard derivative operator, from standard calculus, as follows.

Definition 1. Let x be an independent variable and $f(x)$ a function, which is a map $f : \mathbb{R} \rightarrow \mathbb{R}$. Then the standard derivative, which is denoted with D_x , or as a prime to the function, is applied on the function. This defines the operator:

$$D_x f(x) \equiv f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (5)$$

We follow [17] to define a new local derivative, namely the Local Euler derivative. We study and extend the properties of this Local Euler derivative, as follows.

2.1. Motivation for the Exponential Kernel

The exponential kernel $e^{(1-\alpha)x}$ is chosen because it produces a simple auxiliary function ϕ and leads to an applied chain rule. Moreover, the revised extra factor $e^{-(1-\alpha)g(x)}$ appears naturally. All results of this section are special cases of the general theory and analysis of local kernel-based calculus presented in Section 3, but we present them separately to make the exposition concrete.

2.2. Local Euler Derivative

Definition 2. Let x be an independent variable and $f(x)$ a function, which is a map $f : \mathbb{R} \rightarrow \mathbb{R}$, and let $\Delta x \rightarrow 0$ be sufficiently small value. Then the Local Euler derivative, which is denoted with $D_x^{e,\alpha}$, is applied on the function. This defines the operator:

$$D_x^{e,\alpha} f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x e^{(1-\alpha)x}) - f(x)}{\Delta x}. \quad (6)$$

Example 1. Given $f(x) = x^p$, and $p \in \mathbb{N}_0$, via the Definition 2, we compute:

$$D_x^{e,\alpha}(x^p) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x e^{(1-\alpha)x})^p - x^p}{\Delta x} \quad (7)$$

Using the binomial expansion for small h :

$$D_x^{e,\alpha}(x^p) = \lim_{\Delta x \rightarrow 0} \frac{\sum_{r=0}^p \binom{p}{r} x^{p-r} (\Delta x e^{(1-\alpha)x})^r - x^p}{\Delta x} \quad (8)$$

Separating the terms:

$$D_x^{e,\alpha}(x^p) = \lim_{\Delta x \rightarrow 0} \frac{\binom{p}{0} x^p + \binom{p}{1} x^{p-1} (\Delta x e^{(1-\alpha)x}) + \binom{p}{2} x^{p-2} (\Delta x e^{(1-\alpha)x})^2 + \dots + \binom{p}{p} (\Delta x e^{(1-\alpha)x})^p - x^p}{\Delta x} \quad (9)$$

Simplifying the dominant term:

$$D_x^{e,\alpha}(x^p) = \lim_{\Delta x \rightarrow 0} \frac{p x^{p-1} \Delta x e^{(1-\alpha)x} + \text{higher-order terms}}{\Delta x} \quad (10)$$

which means

$$D_x^{e,\alpha}(x^p) = \lim_{\Delta x \rightarrow 0} \left(p x^{p-1} e^{(1-\alpha)x} + \text{higher-order terms as a power function of } \Delta x \right) \quad (11)$$

As $\Delta x \rightarrow 0$, higher-order terms vanish and we get

$$D_x^{e,\alpha}(x^p) = p x^{p-1} e^{(1-\alpha)x}. \quad (12)$$

Thus

$$D_x^{e,\alpha}(x^p) = D_x(x^p) e^{(1-\alpha)x} \quad (13)$$

Note that the limiting behavior of this Local Euler derivative is

$$\lim_{\alpha \rightarrow 1} D_x^{e,\alpha}(x^p) = \lim_{\alpha \rightarrow 1} p x^{p-1} e^{(1-\alpha)x} = p x^{p-1} e^{(1-1)x} = p x^{p-1} = D_x(x^p). \quad (14)$$

Theorem 1 (Local Euler derivative and relation to standard derivative). Let x be an independent variable and $f(x)$ a function, which is a map $f : \mathbb{R} \rightarrow \mathbb{R}$, and let $\Delta x \rightarrow 0$ be sufficiently small value. Let f be differentiable, and α -local-Euler-differentiable. Then we get:

$$D_x^{e,\alpha} f(x) = e^{(1-\alpha)x} D_x f(x) \quad (15)$$

Proof. Let $h = \Delta x e^{(1-\alpha)x}$, which also means $\Delta x = h e^{(\alpha-1)x}$. Consider that

$$D_x^{e,\alpha} f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x e^{(1-\alpha)x}) - f(x)}{\Delta x} \quad (16)$$

$$= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h e^{(\alpha-1)x}} \quad (17)$$

$$= e^{(1-\alpha)x} \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (18)$$

$$= e^{(1-\alpha)x} D_x f(x) \quad (19)$$

$$= e^{(1-\alpha)x} f'(x) \quad (20)$$

hence, $D_x^{e,\alpha} f(x) = e^{(1-\alpha)x} D_x f(x)$. $\square$

Example 2. It follows from the previous theorem, there exist the Local Euler derivatives of the simple functions, such as $f(x) = c$, $f(x) = x$, $f(x) = x^p$, $f(x) = e^x$, $f(x) = \sin(x)$, $f(x) = \cos(x)$. We summarize them in Table 1. Here $c \in \mathbb{R}$ is a constant and $p \in \mathbb{R} \setminus \{0\}$.

- For the example function, $f(x) = c$, the α -local-Euler derivative is $D_x^{e,\alpha} c = 0$. We prove this statement as follows.

Proof. Applying Equation (15) from the definition, to the function, $f(x) = c$, we get:

$$D_x^{e,\alpha} c = e^{(1-\alpha)x} \cdot D_x c = e^{(1-\alpha)x} \cdot 0 = 0 \quad (21)$$

$\square$

- For the example function, $f(x) = x$, the α -local-Euler derivative is $D_x^{e,\alpha} x = e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15) from the definition, to the function, $f(x) = x$, we get:

$$D_x^{e,\alpha} x = e^{(1-\alpha)x} \cdot D_x x = e^{(1-\alpha)x} \cdot 1 = e^{(1-\alpha)x} \quad (22)$$

$\square$

- For the example function, $f(x) = x^p$, the α -local-Euler derivative is $D_x^{e,\alpha} x^p = p x^{p-1} e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = x^p$, we get:

$$D_x^{e,\alpha} x^p = e^{(1-\alpha)x} \cdot D_x x^p = e^{(1-\alpha)x} \cdot p x^{p-1} = p x^{p-1} e^{(1-\alpha)x} \quad (23)$$

$\square$

- For the example function, $f(x) = e^x$, the α -local-Euler derivative is $D_x^{e,\alpha} e^x = e^x e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = e^x$, we get:

$$D_x^{e,\alpha} e^x = e^{(1-\alpha)x} \cdot D_x e^x = e^{(1-\alpha)x} \cdot e^x = e^x e^{(1-\alpha)x} \quad (24)$$

$\square$

- For the example function, $f(x) = \sin(x)$, the α -local-Euler derivative is $D_x^{e,\alpha} x = \cos(x)e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = \sin(x)$, we get:

$$D_x^{e,\alpha} \sin(x) = e^{(1-\alpha)x} \cdot D_x \sin(x) = e^{(1-\alpha)x} \cdot \cos(x) = \cos(x)e^{(1-\alpha)x} \quad (25)$$

□

- For the example function, $f(x) = \cos(x)$, the α -local-Euler derivative is $D_x^{e,\alpha} x = -\sin(x)e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = \sin(x)$, we get:

$$D_x^{e,\alpha} \cos(x) = e^{(1-\alpha)x} \cdot D_x \cos(x) = e^{(1-\alpha)x} \cdot (-\sin(x)) = -\sin(x)e^{(1-\alpha)x} \quad (26)$$

□

Table 1. Local Euler derivatives of simple functions. Here $c \in \mathbb{R}$ is a constant and $p \in \mathbb{R} \setminus \{0\}$.

$f(x)$	$D_x f(x)$	$D_x^{e,\alpha} f(x)$
c	0	0
x	1	$e^{(1-\alpha)x}$
x^p	px^{p-1}	$px^{p-1}e^{(1-\alpha)x}$
e^x	e^x	$e^{(2-\alpha)x} = e^x e^{(1-\alpha)x}$
$\sin(x)$	$\cos(x)$	$\cos(x)e^{(1-\alpha)x}$
$\cos(x)$	$-\sin(x)$	$-\sin(x)e^{(1-\alpha)x}$

2.3. Theorem of Continuity of α -Local-Euler-Differentiable Functions

Following [17], we construct the following theorem.

Theorem 2 (Continuity of α -local-Euler-differentiable function). *Let $\alpha \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be local Euler differentiable at $t_0 \in \mathbb{R}$. Then f is continuous at t_0 .*

Proof. By Theorem 1, we have

$$D_x^{e,\alpha} f(t_0) = e^{(1-\alpha)t_0} f'(t_0).$$

Since $e^{(1-\alpha)t_0} \neq 0$ for all real t_0 , the existence of the local Euler derivative implies the existence of the classical derivative $f'(t_0)$. Classical differentiability implies continuity, so f is continuous at t_0 . □

2.4. Rolle's Theorem for α -Local-Euler-Differentiable Functions

Following [17], we construct the following theorem.

Theorem 3 (Rolle's theorem for α -local-Euler-differentiable function). *Let $\alpha, a, b \in \mathbb{R}$ and function $f : [a, b] \rightarrow \mathbb{R}$, which satisfies the following properties:*

- f is continuous on $[a, b]$;
- f is α -local-Euler-differentiable on $[a, b]$;

- $f(a) = f(b)$.

Then there exists $c \in (a, b)$, such that $D_x^{e,\alpha} f(c) = 0$

Proof. We distinguish two cases.

Case 1: f is constant on $[a, b]$. Then $D_x^{e,\alpha} f(c) = 0$ for every $c \in (a, b)$, so the conclusion holds trivially.

Case 2: f is not constant on $[a, b]$. Since f is continuous on the compact interval $[a, b]$, the *extreme value theorem* guarantees that f attains a global maximum M and a global minimum m on $[a, b]$. Because $f(a) = f(b)$ and f is not constant, at least one of these extrema must occur at an interior point $c \in (a, b)$. Without loss of generality, assume c is a local minimum (the case of a local maximum is analogous).

Then, for sufficiently small $\Delta x > 0$,

$$f(c + \Delta x e^{(1-\alpha)c}) - f(c) \geq 0, \quad (27)$$

and for $\Delta x < 0$,

$$f(c + \Delta x e^{(1-\alpha)c}) - f(c) \leq 0. \quad (28)$$

Thus,

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(c + \Delta x e^{(1-\alpha)c}) - f(c)}{\Delta x} \geq 0, \quad (29)$$

while

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(c + \Delta x e^{(1-\alpha)c}) - f(c)}{\Delta x} \leq 0. \quad (30)$$

Since the local Euler derivative exists at c , these one-sided limits must be equal. Therefore,

$$D_x^{e,\alpha} f(c) = 0. \quad (31)$$

This completes the proof. $\square$

2.5. MVT for α -Local-Euler-Power-Differentiable Functions

Following [17], we construct the following theorem.

Remark 1. We recall the definition of the Local Euler derivative Definition 2. For a differentiable function f , one obtains $D_x^{e,\alpha} f(x) = f'(x) e^{(1-\alpha)x}$. A key auxiliary function whose Euler derivative equals 1 is given by

$$\phi(x) = \begin{cases} \frac{e^{(\alpha-1)x}}{\alpha-1}, & \alpha \neq 1, \\ x, & \alpha = 1. \end{cases} \quad (32)$$

Indeed, for $\alpha \neq 1$,

$$\phi'(x) = e^{(\alpha-1)x}, \quad D_x^{e,\alpha} \phi(x) = \phi'(x) e^{(1-\alpha)x} = e^{(\alpha-1)x} e^{(1-\alpha)x} = 1, \quad (33)$$

and for $\alpha = 1$ the definition reduces to the ordinary derivative and $\phi(x) = x$ gives $D_x^{e,1} \phi(x) = 1$.

Theorem 4 (MVT for the local Euler derivative). Let $\alpha \in \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a function satisfying:

1. f is continuous on $[a, b]$;
2. f is α -Euler differentiable on (a, b) (i.e., $D_x^{e,\alpha} f(x)$ exists for every $x \in (a, b)$).

Then there exists a point $c \in (a, b)$ such that

$$D_x^{e,\alpha} f(c) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}, \quad (34)$$

where ϕ is defined as above.

Proof. We have $D_x^{e,\alpha} \phi(x) = 1$ for all $x \in (a, b)$. Applying the classical Cauchy mean value theorem to f and ϕ on $[a, b]$, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{\phi'(c)} = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}. \quad (35)$$

Using $f'(c) = D_x^{e,\alpha} f(c)/e^{(1-\alpha)c}$ and $\phi'(c) = e^{(\alpha-1)c} = 1/e^{(1-\alpha)c}$, we obtain

$$\frac{f'(c)}{\phi'(c)} = \frac{D_x^{e,\alpha} f(c)/e^{(1-\alpha)c}}{1/e^{(1-\alpha)c}} = D_x^{e,\alpha} f(c). \quad (36)$$

Therefore,

$$D_x^{e,\alpha} f(c) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}. \quad (37)$$

This completes the proof. $\square$

Remark 2. The denominator $\phi(b) - \phi(a)$ is non-zero for $a \neq b$ because ϕ is strictly monotonic: for $\alpha \neq 1$, $\phi'(x) = e^{(\alpha-1)x} > 0$, and for $\alpha = 1$, $\phi(b) - \phi(a) = b - a > 0$.

Remark 3. When $\alpha = 1$, the local Euler derivative reduces to the ordinary derivative, $\phi(x) = x$, and the theorem becomes the classical MVT:

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad (38)$$

Remark 4. This MVT can be used to prove other standard results in the framework of local Euler calculus, such as monotonicity criteria, Taylor expansions, and the inverse function theorem.

2.6. Chain Rule for the Local Euler Derivative

Theorem 5 (Chain Rule for the Local Euler Derivative). Let g be differentiable at x , let f be differentiable at $g(x)$, and assume $e^{(1-\alpha)g(x)} \neq 0$ (which holds for all real $g(x)$). Define $h(x) = f(g(x))$. Then the Local Euler derivative of h exists and satisfies

$$D_x^{e,\alpha} [f(g(x))] = D_g^{e,\alpha} [f(g)] \cdot D_x^{e,\alpha} [g(x)] \cdot e^{-(1-\alpha)g(x)} \quad (39)$$

Proof. Using Theorem 1 for the composition $h(x) = f(g(x))$, we have

$$D_x^{e,\alpha} h(x) = h'(x) e^{(1-\alpha)x}. \quad (40)$$

By the ordinary chain rule, $h'(x) = f'(g(x))g'(x)$. Thus

$$D_x^{e,\alpha} h(x) = e^{(1-\alpha)x} f'(g(x))g'(x) \quad (41)$$

Now express the factors involving f and g in terms of their Euler derivatives. From the same theorem,

$$D_g^{e,\alpha} f(g) = e^{(1-\alpha)g(x)} f'(g(x)), \quad (42)$$

$$D_x^{e,\alpha} g(x) = e^{(1-\alpha)x} g'(x). \quad (43)$$

Multiplying these expressions and dividing by $e^{(1-\alpha)g(x)}$ (which is non-zero), we obtain

$$D_g^{e,\alpha} f(g) \cdot D_x^{e,\alpha} g(x) \cdot e^{-(1-\alpha)g(x)} = e^{(1-\alpha)g(x)} f'(g(x)) \cdot e^{(1-\alpha)x} g'(x) \cdot e^{-(1-\alpha)g(x)} \quad (44)$$

$$= e^{(1-\alpha)x} f'(g(x)) g'(x). \quad (45)$$

Comparing with Equation (41), we conclude

$$D_x^{e,\alpha} [f(g(x))] = D_g^{e,\alpha} [f(g)] \cdot D_x^{e,\alpha} [g(x)] \cdot e^{-(1-\alpha)g(x)}. \quad (46)$$

This completes the proof. $\square$

Remark 5. The presence of the extra factor $e^{-(1-\alpha)g(x)}$ shows that the naive product rule $D_x^{e,\alpha} [f(g)] = D_g^{e,\alpha} f \cdot D_x^{e,\alpha} g$ is false in general. The simple product form holds only when $\alpha = 1$ (ordinary derivative) or when g is constant.

2.7. Local Euler Integral

In this section, we define the local Euler Integrals, which are local analogues of the classical Riemann integral, obtained by inverting the local Euler derivative. These integrals are comparable to other local generalized integrals (such as those arising from conformable-type derivatives) and are distinct from genuinely non-local fractional integrals.

The definition of the local Euler integral follows naturally from the FTC adapted to our local kernel-based framework. We start by the question, “What is the meaning of a local Euler integral?” And we can simply say that if we have the standard integral, and the FTC, we can derive the Fundamental Theorem of Local Euler Calculus, and hence we can derive the definition of the *local Euler integrals*.

Definition 3. Let x be an independent variable and $f(x)$ a function, and let $h \ll 1$ be sufficiently small value. Then the local Euler integral, $I_x^{e,\alpha}$, applied on the function is defined by the operator:

$$I_x^{e,\alpha} f(x) = \int d_{e,\alpha} x f(x) = \int dx \frac{f(x)}{e^{(1-\alpha)x}}. \quad (47)$$

Example 3 (Local Euler integral of a power law). Let $f(x) = e^x e^{(1-\alpha)x}$ be a function, then the standard integral $\int dx e^x e^{(1-\alpha)x} = \frac{1}{2-\alpha} e^{(2-\alpha)x} + \text{constant}$, where $\alpha \neq 2$. Now the local Euler integral is given by

$$I_x^{e,\alpha} [f(x)] = I_x^{e,\alpha} [e^x e^{(1-\alpha)x}] = \int dx \frac{e^x e^{(1-\alpha)x}}{e^{(1-\alpha)x}} \quad (48)$$

$$= \int dx e^x \quad (49)$$

$$= e^x + \text{constant} \quad (50)$$

$$I_x^{e,\alpha} [f(x)] = e^x + \text{constant} \quad (51)$$

Example 4. In the same way we find the local kernel-based integrals of the simple functions, such as $f(x) = c$, $f(x) = x$, $f(x) = x^p$, $f(x) = e^x$, $f(x) = \sin(x)$, $f(x) = \cos(x)$. We summarize them in Table 2. Here $c \in \mathbb{R}$ is a constant, and $*$ denotes that for general p the standard integral

involves special functions and is omitted; for integer p , it can be obtained by repeated integration by parts.

Remark 6. Note that for these simple functions, the standard integral often provides a longer or more complex result (e.g., involving special functions or multiple terms), while the local Euler integral provides a significantly shorter result due to the cancellation of the exponential kernel.

Table 2. Local Euler integrals of simple functions.

$f(x)$	$I_x f(x)$	$I_x^{e,\alpha} f(x)$
0	c	c
$e^{(1-\alpha)x}$	$\begin{cases} \frac{1}{1-\alpha} e^{(1-\alpha)x} + c, & \alpha \neq 1, \\ x + c, & \alpha = 1. \end{cases}$	x
$px^{p-1}e^{(1-\alpha)x}$	$*$	$x^p + c$
$e^x e^{(1-\alpha)x}$	$\begin{cases} \frac{1}{2-\alpha} e^{(2-\alpha)x} + c, & \alpha \neq 2, \\ x + c, & \alpha = 2. \end{cases}$	$e^x + c$
$\sin(x)e^{(1-\alpha)x}$	$\frac{e^{(1-\alpha)x}}{(1-\alpha)^2+1} [(1-\alpha)\sin(x) - \cos(x)]$	$-\cos(x) + c$
$\cos(x)e^{(1-\alpha)x}$	$\frac{e^{(1-\alpha)x}}{(1-\alpha)^2+1} [(1-\alpha)\cos(x) + \sin(x)]$	$\sin(x) + c$

2.8. Basic Properties of Local Euler Calculus

2.8.1. Basic Properties of the Local Euler Derivative

Theorem 6. Let f, g be functions and $a, b \in \mathbb{R}$. Under the assumption that the relevant limits exist, the local Euler derivative satisfies the following properties:

$$D_x^{e,\alpha}(af + bg) = aD_x^{e,\alpha}f + bD_x^{e,\alpha}g, \quad (52)$$

$$D_x^{e,\alpha}(fg) = fD_x^{e,\alpha}g + gD_x^{e,\alpha}f. \quad (53)$$

Proof. For linearity, we compute directly from the definition of the Euler derivative:

$$D_x^{e,\alpha}f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x e^{(1-\alpha)x}) - f(x)}{\Delta x}. \quad (54)$$

Then

$$D_x^{e,\alpha}(af + bg)(x) = \lim_{\Delta x \rightarrow 0} \frac{(af + bg)(x + \Delta x e^{(1-\alpha)x}) - (af + bg)(x)}{\Delta x} \quad (55)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{af(x + \Delta x e^{(1-\alpha)x}) + bg(x + \Delta x e^{(1-\alpha)x}) - af(x) - bg(x)}{\Delta x} \quad (56)$$

$$= a \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x e^{(1-\alpha)x}) - f(x)}{\Delta x} + b \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x e^{(1-\alpha)x}) - g(x)}{\Delta x} \quad (57)$$

$$= aD_x^{e,\alpha}f(x) + bD_x^{e,\alpha}g(x). \quad (58)$$

For the product rule, let $E := e^{(1-\alpha)x}$ for brevity. Then

$$D_x^{e,\alpha}(fg)(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta xE)g(x + \Delta xE) - f(x)g(x)}{\Delta x} \quad (59)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta xE)g(x + \Delta xE) - f(x)g(x + \Delta xE) + f(x)g(x + \Delta xE) - f(x)g(x)}{\Delta x} \quad (60)$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta xE) - f(x)}{\Delta x} g(x + \Delta xE) + f(x) \frac{g(x + \Delta xE) - g(x)}{\Delta x} \right] \quad (61)$$

$$= \left(\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta xE) - f(x)}{\Delta x} \right) \left(\lim_{\Delta x \rightarrow 0} g(x + \Delta xE) \right) + f(x) \left(\lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta xE) - g(x)}{\Delta x} \right) \quad (62)$$

$$= D_x^{e,\alpha} f(x) \cdot g(x) + f(x) \cdot D_x^{e,\alpha} g(x), \quad (63)$$

where we used the fact that the existence of the derivative implies continuity of g at x (under the assumption $e^{(1-\alpha)x} \neq 0$, which holds for all real x), so that $\lim_{\Delta x \rightarrow 0} g(x + \Delta xE) = g(x)$. $\square$

2.8.2. Basic Operations on Local Euler Integrals

Theorem 7 (Linearity of the local Euler integral). *Let x be an independent variable, c be constant number $c \in \mathbb{R}$, and let $f(x)$, $g(x)$ two α -local-Euler-integrable functions. Then we get*

$$I_x^{e,\alpha}[cf(x)] = c I_x^{e,\alpha}[f(x)] \quad (64)$$

$$I_x^{e,\alpha}[f(x) + g(x)] = I_x^{e,\alpha}[f(x)] + I_x^{e,\alpha}[g(x)] \quad (65)$$

Proof. Using the definition,

$$I_x^{e,\alpha}[cf(x)] = \int \frac{cf(x)}{e^{(1-\alpha)x}} dx = c \int \frac{f(x)}{e^{(1-\alpha)x}} dx = c I_x^{e,\alpha}[f(x)]. \quad (66)$$

Similarly,

$$I_x^{e,\alpha}[f(x) + g(x)] = \int \frac{f(x) + g(x)}{e^{(1-\alpha)x}} dx \quad (67)$$

$$= \int \frac{f(x)}{e^{(1-\alpha)x}} dx + \int \frac{g(x)}{e^{(1-\alpha)x}} dx \quad (68)$$

$$= I_x^{e,\alpha}[f(x)] + I_x^{e,\alpha}[g(x)]. \quad (69)$$

$\square$

2.8.3. Identity Operator in Local Euler Calculus

Theorem 8 (Identity operator in local Euler calculus). *Let t, x be independent variables, $f(x)$ be a function, x_0 be the value such that $f(x_0) = 0$, let $D_t^{e,\alpha} f(t)$ be the local Euler derivative, and $I_t^{e,\alpha,x_0,x} f(t) = \int_{x_0}^x dt \frac{f(t)}{e^{(1-\alpha)t}}$ be the definite local Euler integral, then the identity operator in local Euler calculus is*

$$\boxed{I_t^{e,\alpha,x_0,x} D_t^{e,\alpha} = 1} \quad (\text{identity operator}). \quad (70)$$

Proof. Let t, x be independent variables, $f(x)$ be a function, x_0 be the value such that $f(x_0) = 0$, let $D_t^{e,\alpha} f(t)$ be the local Euler derivative, and $I_t^{e,\alpha,x_0,x} f(t) = \int_{x_0}^x dt \frac{f(t)}{e^{(1-\alpha)t}}$ be the definite local Euler integral, then we find that the identity operator in local Euler calculus is calculated as

$$I_t^{e,\alpha,x_0,x} D_t^{e,\alpha} f(t) = \int_{x_0}^x dt \frac{[D_t^{e,\alpha} f(t)]}{e^{(1-\alpha)t}} = \int_{x_0}^x dt \frac{[D_t f(t)] e^{(1-\alpha)t}}{e^{(1-\alpha)t}} \quad (71)$$

$$= \int_{x_0}^x dt D_t f(t) = f(x) - f(x_0) = 1 \times f(x) \quad (72)$$

which provides us with Equation (70). $\square$

2.9. Fundamental Theorem of Local Euler Calculus

Theorem 9 (Fundamental Theorem of Local Euler Calculus). *Let f be differentiable on $[x_{\min}, x_{\max}]$, and let $D_x^{e,\alpha} f(x) = e^{(1-\alpha)x} f'(x)$. Then*

$$\int_{x_{\min}}^{x_{\max}} \frac{D_x^{e,\alpha} f(x)}{e^{(1-\alpha)x}} dx = f(x_{\max}) - f(x_{\min}). \quad (73)$$

Proof. By Theorem 1, we have

$$D_x^{e,\alpha} f(x) = e^{(1-\alpha)x} f'(x). \quad (74)$$

Dividing both sides by $e^{(1-\alpha)x}$ (which is non-zero for all real x), we obtain

$$\frac{D_x^{e,\alpha} f(x)}{e^{(1-\alpha)x}} = f'(x). \quad (75)$$

Integrating both sides from $x_{\min}$ to $x_{\max}$ and applying the classical FTC yields

$$\int_{x_{\min}}^{x_{\max}} \frac{D_x^{e,\alpha} f(x)}{e^{(1-\alpha)x}} dx = \int_{x_{\min}}^{x_{\max}} f'(x) dx = f(x_{\max}) - f(x_{\min}). \quad (76)$$

This completes the proof. $\square$

Remark 7. The FTLEC is a genuine generalization of the classical FTC, not merely a reformulation. The proof necessarily relies on the classical FTC because the local Euler derivative is defined in terms of the classical derivative, i.e., $D_x^{e,\alpha} f = e^{(1-\alpha)x} f'$. Substitution then reduces the result to the classical FTC, which serves as a consistency check. However, the theorem is new in the sense that it establishes the inverse relationship between the novel operators $D_x^{e,\alpha}$ and $I_x^{e,\alpha}$. The classical FTC is recovered as the special case $\alpha = 1$, where $e^{(1-\alpha)x} = 1$. Thus, the theorem validates the coherence of the new operator framework rather than merely restating a known result.

3. Local Kernel-Based Calculus

In this section we provide the framework of local kernel-based calculus. We discuss the local kernel-based calculus operators, such as the local kernel-based derivative and the local kernel-based integral. The local kernel-based calculus defined below subsumes the Euler calculus of Section 2 as the special case $F(x, \alpha) = e^{(1-\alpha)x}$. We therefore prove all properties for an arbitrary kernel $F(x, \alpha)$; the reader may verify that substituting the exponential kernel reproduces the results of Section 2.

3.1. Local Kernel-Based Derivative

We follow [21] and define a local kernel-based derivative operator, studying its properties as follows.

Definition 4. Let x be an independent real variable, $f : \mathbb{R} \rightarrow \mathbb{R}$ a function, $\alpha \in \mathbb{R}$ a parameter, and $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a given function. The **local kernel-based derivative** of f with respect to x , denoted $D_x^{F,\alpha} f(x)$, is defined by

$$D_x^{F,\alpha} f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x \cdot F(x, \alpha)) - f(x)}{\Delta x}, \quad (77)$$

provided the limit exists.

Remark 8. The local Euler derivative (Definition 2) is a special case of the local kernel-based derivative obtained by choosing

$$F(\alpha, x) = e^{(1-\alpha)x}. \quad (78)$$

Indeed, substituting this F into Definition 4 yields exactly $D_x^{e,\alpha} f(x)$.

Theorem 10 (Local kernel-based derivative and relation to standard derivative). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at x , and assume $F(x, \alpha) \neq 0$, and let $\Delta x \rightarrow 0$ be sufficiently small value. Then the local kernel-based derivative $D_x^{F,\alpha} f(x)$ exists and satisfies

$$D_x^{F,\alpha} f(x) = F(x, \alpha) f'(x). \quad (79)$$

Proof. Let $h = \Delta x F(x, \alpha)$. Since $F(x, \alpha) \neq 0$, we have $\Delta x = h/F(x, \alpha)$, and as $\Delta x \rightarrow 0$, also $h \rightarrow 0$. Then

$$D_x^{F,\alpha} f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F(x, \alpha)) - f(x)}{\Delta x} \quad (80)$$

$$= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h/F(x, \alpha)} \quad (81)$$

$$= F(x, \alpha) \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (82)$$

$$= F(x, \alpha) f'(x). \quad (83)$$

Hence $D_x^{F,\alpha} f(x) = F(x, \alpha) D_x f(x)$. $\square$

Remark 9. As noted in the introduction, the identity $D_x^{F,\alpha} f(x) = F(x, \alpha) f'(x)$ holds whenever f is ordinarily differentiable and $F(x, \alpha) \neq 0$. However, the definition itself is more general: if $F(x_0, \alpha) = 0$, the derivative vanishes for every function f , including discontinuous ones, and differentiability in this degenerate sense does not imply continuity. For the classical results that follow—Rolle’s theorem, the mean value theorem, the FTC, and the continuity theorem—we adopt the assumption $F(x, \alpha) > 0$ on the interval under consideration. This ensures that the familiar statements hold without degenerate ambiguities. The definition itself does not require this assumption, and the case $F(x, \alpha) = 0$ remains a legitimate (though degenerate) part of the broader framework.

Example 5. It follows from the previous theorem that there exist the local kernel-based derivatives of the simple functions, such as $f(x) = c$, $f(x) = x$, $f(x) = x^p$, $f(x) = e^x$, $f(x) = \sin(x)$, $f(x) = \cos(x)$. We summarize them to the following Table 3. Here $c \in \mathbb{R}$ is a constant and $p \in \mathbb{R} \setminus \{0\}$.

Table 3. Local kernel-based derivatives of simple functions. Here $c \in \mathbb{R}$ is a constant and $p \in \mathbb{R} \setminus \{0\}$.

$f(x)$	$D_x f(x)$	$D_x^{F,\alpha} f(x)$
c	0	0
x	1	$F(x, \alpha)$
x^p	px^{p-1}	$px^{p-1}F(x, \alpha)$
e^x	e^x	$e^x F(x, \alpha)$
$\sin(x)$	$\cos(x)$	$\cos(x)F(x, \alpha)$
$\cos(x)$	$-\sin(x)$	$-\sin(x)F(x, \alpha)$

- For the example function, $f(x) = c$, the α -local-kernel-based derivative is $D_x^{e,\alpha} c = 0$. We prove this statement as follows.

Proof. Applying the Equation (15) from the definition, to the function, $f(x) = c$, we get:

$$D_x^{e,\alpha} c = F(x, \alpha) \cdot D_x c = F(x, \alpha) \cdot 0 = 0 \quad (84)$$

□

- For the example function, $f(x) = x$, the α -local-kernel-based derivative is $D_x^{e,\alpha} x = e^{(1-\alpha)x}$. We prove this statement as follows.

Proof. Applying Equation (15) from the definition, to the function, $f(x) = x$, we get:

$$D_x^{e,\alpha} x = F(x, \alpha) \cdot D_x x = F(x, \alpha) \cdot 1 = F(x, \alpha) \quad (85)$$

□

- For the example function, $f(x) = x^p$, the α -local-kernel-based derivative is $D_x^{e,\alpha} x = px^{p-1}F(x, \alpha)$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = x^p$, we get:

$$D_x^{e,\alpha} x^p = F(x, \alpha) \cdot D_x x^p = F(x, \alpha) \cdot px^{p-1} = px^{p-1}F(x, \alpha) \quad (86)$$

□

- For the example function, $f(x) = e^x$, the α -local-kernel-based derivative is $D_x^{e,\alpha} x = e^x F(x, \alpha)$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = e^x$, we get:

$$D_x^{e,\alpha} e^x = F(x, \alpha) \cdot D_x e^x = F(x, \alpha) \cdot e^x = e^x F(x, \alpha) \quad (87)$$

□

- For the example function, $f(x) = \sin(x)$, the α -local-kernel-based derivative is $D_x^{e,\alpha} x = \cos(x)F(x, \alpha)$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = \sin(x)$, we get:

$$D_x^{e,\alpha} \sin(x) = F(x, \alpha) \cdot D_x \sin(x) = F(x, \alpha) \cdot \cos(x) = \cos(x)F(x, \alpha) \quad (88)$$

□

- For the example function, $f(x) = \cos(x)$, the α -local-kernel-based derivative is $D_x^{e,\alpha} x = -\sin(x)F(x, \alpha)$. We prove this statement as follows.

Proof. Applying Equation (15), from the definition, to the function, $f(x) = \sin(x)$, we get:

$$D_x^{e,\alpha} \cos(x) = F(x, \alpha) \cdot D_x \cos(x) = F(x, \alpha) \cdot (-\sin(x)) = -\sin(x)F(x, \alpha) \quad (89)$$

□

3.2. Basic Properties of the Local Kernel-Based Derivative

Theorem 11. Let f, g be functions and $a, b \in \mathbb{R}$. Under the assumption that the relevant limits exist, the local kernel-based derivative satisfies the following properties:

$$D_x^{F,\alpha}(af + bg) = aD_x^{F,\alpha}f + bD_x^{F,\alpha}g, \quad (90)$$

$$D_x^{F,\alpha}(fg) = fD_x^{F,\alpha}g + gD_x^{F,\alpha}f. \quad (91)$$

Proof. For linearity, we compute directly from the definition:

$$D_x^{F,\alpha}(af + bg)(x) = \lim_{\Delta x \rightarrow 0} \frac{(af + bg)(x + \Delta x F(x, \alpha)) - (af + bg)(x)}{\Delta x} \quad (92)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{af(x + \Delta x F(x, \alpha)) + bg(x + \Delta x F(x, \alpha)) - af(x) - bg(x)}{\Delta x} \quad (93)$$

$$= a \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F(x, \alpha)) - f(x)}{\Delta x} + b \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x F(x, \alpha)) - g(x)}{\Delta x} \quad (94)$$

$$= aD_x^{F,\alpha}f(x) + bD_x^{F,\alpha}g(x). \quad (95)$$

For the product rule, we again use the definition. Let $F := F(x, \alpha)$ for brevity. Then

$$D_x^{F,\alpha}(fg)(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F)g(x + \Delta x F) - f(x)g(x)}{\Delta x} \quad (96)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F)g(x + \Delta x F) - f(x)g(x + \Delta x F) + f(x)g(x + \Delta x F) - f(x)g(x)}{\Delta x} \quad (97)$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x F) - f(x)}{\Delta x} g(x + \Delta x F) + f(x) \frac{g(x + \Delta x F) - g(x)}{\Delta x} \right] \quad (98)$$

$$= \left(\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F) - f(x)}{\Delta x} \right) \left(\lim_{\Delta x \rightarrow 0} g(x + \Delta x F) \right) + f(x) \left(\lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x F) - g(x)}{\Delta x} \right) \quad (99)$$

$$= D_x^{F,\alpha}f(x) \cdot g(x) + f(x) \cdot D_x^{F,\alpha}g(x), \quad (100)$$

where we used the fact that $\lim_{\Delta x \rightarrow 0} g(x + \Delta x F) = g(x)$. If $F(x, \alpha) \neq 0$, this follows from the continuity of g implied by the existence of the derivative; if $F(x, \alpha) = 0$, it holds trivially since $g(x + 0) = g(x)$. $\square$

3.3. Theorem of Continuity of α -Local-Kernel-Based-Differentiable Functions

Theorem 12 (Continuity of α -local-kernel-based-differentiable function). Let $\alpha \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be α -local-kernel-based-differentiable at $t_0 \in \mathbb{R}$. If $F(t_0, \alpha) \neq 0$, then f is continuous at t_0 .

Proof. By Theorem 10, we have

$$D_x^{F,\alpha}f(t_0) = F(t_0, \alpha)f'(t_0).$$

Since $F(t_0, \alpha) \neq 0$, the existence of the local kernel-based derivative implies the existence of the classical derivative $f'(t_0)$. Classical differentiability implies continuity, so f is continuous at t_0 . $\square$

Remark 10. The assumption $F(t_0, \alpha) \neq 0$ is essential. If $F(t_0, \alpha) = 0$, the local kernel-based derivative exists for every function f and equals 0, including functions discontinuous at t_0 . In this degenerate case, differentiability does not imply continuity.

3.4. Rolle's Theorem for α -Local-Kernel-Based-Differentiable Functions

We construct the following theorem.

Theorem 13 (Rolle's theorem for local kernel-based differentiable functions). Let $\alpha, a, b \in \mathbb{R}$, with $a < b$, and let $f : [a, b] \rightarrow \mathbb{R}$ satisfy the following properties:

- f is continuous on $[a, b]$;
- f is α -local-kernel-based differentiable on (a, b) ;
- $f(a) = f(b)$;
- $F(x, \alpha) > 0$ for all $x \in [a, b]$.

Then there exists $c \in (a, b)$ such that $D_x^{F, \alpha} f(c) = 0$.

Proof. We distinguish two cases.

Case 1: f is constant on $[a, b]$. Then $D_x^{F, \alpha} f(c) = 0$ for every $c \in (a, b)$, so the conclusion holds trivially.

Case 2: f is not constant on $[a, b]$. Since f is continuous on the compact interval $[a, b]$, the EVT guarantees that f attains a global maximum M and a global minimum m on $[a, b]$. Because $f(a) = f(b)$ and f is not constant, at least one of these extrema must occur at an interior point $c \in (a, b)$. Without loss of generality, assume c is a local minimum (the case of a local maximum is analogous).

Since c is a local minimum, there exists $\delta > 0$ such that for all $y \in (c - \delta, c + \delta)$,

$$f(y) - f(c) \geq 0. \quad (101)$$

Now, for sufficiently small $\Delta x > 0$ such that $c + \Delta x F(c, \alpha) \in (c - \delta, c + \delta)$, we have

$$f(c + \Delta x F(c, \alpha)) - f(c) \geq 0. \quad (102)$$

Thus,

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(c + \Delta x F(c, \alpha)) - f(c)}{\Delta x} \geq 0. \quad (103)$$

Similarly, for $\Delta x < 0$ with $|\Delta x|$ sufficiently small such that $c + \Delta x F(c, \alpha) \in (c - \delta, c + \delta)$, we have

$$f(c + \Delta x F(c, \alpha)) - f(c) \geq 0, \quad (104)$$

but since $\Delta x < 0$,

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(c + \Delta x F(c, \alpha)) - f(c)}{\Delta x} \leq 0. \quad (105)$$

Since the local kernel-based derivative exists at c , these one-sided limits must be equal. Therefore,

$$D_x^{F, \alpha} f(c) = 0. \quad (106)$$

This completes the proof. $\square$

3.5. MVT for the Local Kernel-Based Derivative

Remark 11. To obtain a MVT, we need a key auxiliary function. $\Phi(x)$ whose local kernel-based derivative equals 1. Since

$$D_x^{F, \alpha} \Phi(x) = \Phi'(x) F(x, \alpha), \quad (107)$$

the condition $D_x^{F,\alpha}\Phi(x) = 1$ is equivalent to $\Phi'(x) = \frac{1}{F(x,\alpha)}$. Therefore we define

$$\Phi(x) = \int_{x_0}^x \frac{dt}{F(\alpha, t)} + C, \quad (108)$$

where x_0 is any fixed point in the domain and C an arbitrary constant. In what follows we choose the constant so that Φ is well defined and strictly monotonic (which requires $F(x, \alpha) \neq 0$ and, say, $F(x, \alpha) > 0$).

For the special case of the Euler derivative, $F(x, \alpha) = e^{(1-\alpha)x}$, we obtain

$$\Phi(x) = \int \frac{dx}{e^{(1-\alpha)x}} = \int e^{(\alpha-1)x} dx = \frac{e^{(\alpha-1)x}}{\alpha-1} \quad (\alpha \neq 1), \quad (109)$$

and for $\alpha = 1$, $F(x, 1) = 1$, so $\Phi(x) = x$, exactly as used earlier.

Theorem 14 (MVT for the local kernel-based derivative). *Let $\alpha \in \mathbb{R}$, let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with $F(x, \alpha) \neq 0$, for all $x \in [a, b]$, and let $f : [a, b] \rightarrow \mathbb{R}$ satisfy:*

1. *f is continuous on $[a, b]$;*
2. *f is locally functionally differentiable on (a, b) (i.e., $D_x^{F,\alpha}f(x)$ exists for every $x \in (a, b)$).*

Let $\Phi(x)$ be any antiderivative of $1/F(x, \alpha)$, i.e., $\Phi'(x) = 1/F(x, \alpha)$ (hence $D_x^{F,\alpha}\Phi(x) = 1$). Then there exists a point $c \in (a, b)$ such that

$$D_x^{F,\alpha}f(c) = \frac{f(b) - f(a)}{\Phi(b) - \Phi(a)}. \quad (110)$$

Proof. Since

$$D_x^{F,\alpha}\Phi(x) = F(x, \alpha)\Phi'(x) = F(x, \alpha) \cdot \frac{1}{F(x, \alpha)} = 1, \quad (111)$$

we have $D_x^{F,\alpha}\Phi(x) = 1$. Applying the classical Cauchy mean value theorem to the functions f and Φ on $[a, b]$, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{\Phi'(c)} = \frac{f(b) - f(a)}{\Phi(b) - \Phi(a)}. \quad (112)$$

Since $f'(c) = D_x^{F,\alpha}f(c)/F(c, \alpha)$ and $\Phi'(c) = 1/F(c, \alpha)$, we obtain

$$\frac{f'(c)}{\Phi'(c)} = \frac{D_x^{F,\alpha}f(c)/F(c, \alpha)}{1/F(c, \alpha)} = D_x^{F,\alpha}f(c). \quad (113)$$

Therefore,

$$D_x^{F,\alpha}f(c) = \frac{f(b) - f(a)}{\Phi(b) - \Phi(a)}. \quad (114)$$

This completes the proof. $\square$

Remark 12. *The denominator $\Phi(b) - \Phi(a)$ is non-zero for $a \neq b$ because $\Phi'(x) = 1/F(x, \alpha)$ has a constant sign (since $F(x, \alpha)$ does not vanish and is continuous, it is either always positive or always negative). Hence Φ is strictly monotonic.*

Remark 13. When $F(x, \alpha) = 1$ (which corresponds to $\alpha = 1$ in the Euler case), the local kernel-based derivative reduces to the ordinary derivative, $\Phi(x) = x$, and the theorem becomes the classical MVT:

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad (115)$$

Remark 14. The choice of the antiderivative Φ is not unique; any two choices differ by a constant, which cancels in the difference $\Phi(b) - \Phi(a)$. Therefore the statement is independent of the particular antiderivative used.

Remark 15. This MVT provides a foundation for further results in local kernel-based calculus, such as monotonicity criteria, Taylor expansions with remainder, and the inverse function theorem, under the natural condition that $F(x, \alpha)$ does not change sign.

3.6. Chain Rule for the Local Kernel-Based Derivative

Theorem 15 (Chain rule for the local kernel-based derivative). Let g be differentiable at x , let f be differentiable at $g(x)$, and assume $F(g(x), \alpha) \neq 0$. Define $h(x) = f(g(x))$. Then the local kernel-based derivative of h exists and satisfies

$$D_x^{F, \alpha} [f(g(x))] = D_g^{F, \alpha} [f(g)] \cdot D_x^{F, \alpha} [g(x)] \cdot F^{-1}(g(x), \alpha). \quad (116)$$

Proof. Using Theorem 10 for the composition $h(x) = f(g(x))$, we have

$$D_x^{F, \alpha} h(x) = h'(x) F(x, \alpha). \quad (117)$$

By the ordinary chain rule, $h'(x) = f'(g(x))g'(x)$. Thus

$$D_x^{F, \alpha} h(x) = F(x, \alpha) f'(g(x)) g'(x). \quad (118)$$

Now express the factors involving f and g in terms of their kernel-based derivatives. From the same theorem,

$$D_g^{F, \alpha} f(g) = F(g(x), \alpha) f'(g(x)), \quad (119)$$

$$D_x^{F, \alpha} g(x) = F(x, \alpha) g'(x). \quad (120)$$

Multiplying these expressions and dividing by $F(g(x), \alpha)$ (which is non-zero by assumption), we obtain

$$D_g^{F, \alpha} f(g) \cdot D_x^{F, \alpha} g(x) \cdot \frac{1}{F(g(x), \alpha)} = F(g(x), \alpha) \cdot f'(g(x)) \cdot F(x, \alpha) g'(x) \cdot F^{-1}(g(x), \alpha) \quad (121)$$

$$= F(x, \alpha) f'(g(x)) g'(x). \quad (122)$$

Comparing with Equation (118), we conclude

$$D_x^{F, \alpha} [f(g(x))] = D_g^{F, \alpha} [f(g)] \cdot D_x^{F, \alpha} [g(x)] \cdot \frac{1}{F(g(x), \alpha)}. \quad (123)$$

This completes the proof. $\square$

Remark 16. The presence of the factor $1/F(g(x), \alpha)$ is essential. The naive product rule $D_x^{F, \alpha} [f(g)] = D_g^{F, \alpha} f \cdot D_x^{F, \alpha} g$ holds only when $F(g(x), \alpha) \equiv 1$, i.e., when the kernel-based derivative reduces to the ordinary derivative.

Remark 17. For the special case of the local Euler derivative, where $F(x, \alpha) = e^{(1-\alpha)x}$, we have $1/F(g(x), \alpha) = e^{-(1-\alpha)g}$, which reproduces the chain rule derived earlier:

$$D_x^{e,\alpha}[f(g)] = D_g^{e,\alpha}f \cdot D_x^{e,\alpha}g \cdot e^{-(1-\alpha)g}. \quad (124)$$

3.7. Comparison with Previous Formulations

The chain rule derived in Theorem 15 provides an alternative representation of the chain rule for generalized conformable-type derivatives. To the best of our knowledge, the chain rule for the generalized conformable derivative was previously stated as

$$G_T^\alpha(f \circ g)(t) = f'(g(t))G_T^\alpha g(t), \quad (125)$$

by Fleitas et al. [20]. This formula is correct for their definition of the generalized conformable derivative and is equivalent to our formulation. However, our representation expresses the chain rule entirely in terms of the kernel-based derivatives, without reference to the classical derivative f' .

Recall that the local kernel-based derivative is defined by

$$D_x^{F,\alpha}f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x F(x, \alpha)) - f(x)}{\Delta x}. \quad (126)$$

When differentiating the composition $h(x) = f(g(x))$, the increment is scaled by $F(x, \alpha)$, not $F(g(x), \alpha)$. Using the identity $D_x^{F,\alpha}h(x) = F(x, \alpha)h'(x)$, we obtain

$$D_x^{F,\alpha}[f(g(x))] = F(x, \alpha)f'(g(x))g'(x). \quad (127)$$

Since

$$D_g^{F,\alpha}[f(g)] = F(g(x), \alpha)f'(g(x)), \quad D_x^{F,\alpha}[g(x)] = F(x, \alpha)g'(x), \quad (128)$$

it follows that

$$D_x^{F,\alpha}[f(g(x))] = D_g^{F,\alpha}[f(g)] \cdot D_x^{F,\alpha}[g(x)] \cdot F^{-1}(g(x), \alpha). \quad (129)$$

This is exactly the formula derived in Theorem 15.

The two formulations are algebraically equivalent when $F(g(x), \alpha) \neq 0$. However, our formulation offers a structural advantage: it expresses the chain rule entirely in terms of the kernel-based derivatives, without reference to the classical derivative f' . This preserves the intuitive structure

$$\text{Derivative of composition} = (\text{Derivative of outer}) \times (\text{Derivative of inner}) \times \text{Kernel correction factor}, \quad (130)$$

which is particularly useful when working entirely within the kernel-based framework. The Fleitas et al. formulation, while compact and correct, requires separate evaluation of the classical derivative $f'(g(t))$.

Example 6 (Equivalence of the two formulations). Let $F(x, \alpha) = x$, $f(u) = u^2$, and $g(x) = x^2$. Then,

- Using the Fleitas et al. formulation:

$$D_x^{F,\alpha}[f(g(x))] = f'(g(x))D_x^{F,\alpha}g(x) = 2x^2 \cdot 2x^2 = 4x^4. \quad (131)$$

- Using our alternative formulation:

$$D_x^{F,\alpha}[f(g(x))] = D_g^{F,\alpha}[f(g)] \cdot D_x^{F,\alpha}[g(x)] \cdot F^{-1}(g(x), \alpha). \quad (132)$$

Since $D_g^{F,\alpha}[f(g)] = g \cdot 2f(g) = x^2 \cdot 2x^2 = 2x^4$, $D_x^{F,\alpha}[g(x)] = x \cdot 2x = 2x^2$, and $F(g(x), \alpha) = g(x) = x^2$, we obtain

$$D_x^{F,\alpha}[f(g(x))] = 2x^4 \cdot 2x^2 \cdot \frac{1}{x^2} = 4x^4. \quad (133)$$

- Direct computation from the definition:

$$D_x^{F,\alpha}[x^4] = F(x, \alpha) \cdot 4x^3 = x \cdot 4x^3 = 4x^4. \quad (134)$$

The two formulations yield identical results, confirming their equivalence.

Remark 18. The presence of the factor $F^{-1}(g(x), \alpha)$ in our formulation is essential for expressing the chain rule entirely in terms of the kernel-based derivatives. The simplified form $D_x^{F,\alpha}[f(g)] = D_g^{F,\alpha}f \cdot D_x^{F,\alpha}g$ would hold only when the extra factor is identically 1, i.e., when $F(g(x), \alpha) \equiv 1$, which corresponds to the ordinary derivative. However, as shown above, the Fleitas et al. formulation $f'(g(x))D_x^{F,\alpha}g(x)$ is correct as stated and does not contain an error.

Remark 19. An additional distinction between our framework and that of Fleitas et al. [20] concerns the admissible values of the parameter α . In their formulation, the parameter α is restricted to $\alpha > 0$, as it serves as the order of the generalized conformable derivative. By contrast, in our framework α is a genuine parameter of the kernel function $F(x, \alpha)$ and is unrestricted: it may take arbitrary real values, including negative values. This flexibility allows our framework to accommodate a wider class of kernels and scaling functions, and it reflects the fact that our derivative remains first-order in structure regardless of the value of α . The classical derivative is recovered when $F(x, \alpha) \equiv 1$, irrespective of the specific value of α .

3.8. Local Kernel-Based Integral

In this section we define the local kernel-based integral as the natural inverse of the local kernel-based derivative introduced in Definition 4. The construction follows the same logic as the classical FTC: given an operator $D_x^{F,\alpha}$ that scales the ordinary derivative by $F(x, \alpha)$, the corresponding integral operator must involve division by $F(x, \alpha)$.

Definition 5. Let x be an independent variable, $f : \mathbb{R} \rightarrow \mathbb{R}$ a function, $\alpha \in \mathbb{R}$ a parameter, and $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a given continuous function with $F(x, \alpha) \neq 0$ on the domain of interest. The **indefinite local kernel-based integral** of f with respect to x , denoted $I_x^{F,\alpha}f(x)$, is defined by

$$I_x^{F,\alpha}f(x) = \int d_{F,\alpha}x f(x) = \int \frac{f(x)}{F(x, \alpha)} dx. \quad (135)$$

The corresponding **definite local kernel-based integral** from x_0 to x is

$$I_t^{F,\alpha,x_0,x}f(t) = \int_{x_0}^x \frac{f(t)}{F(t, \alpha)} dt. \quad (136)$$

Remark 20. The definition ensures that the local kernel-based derivative and integral are mutual inverses (up to a constant), as will be shown in Theorem 17.

3.9. Basic Properties of the Local Kernel-Based Integral

The following properties follow directly from the linearity of the ordinary Riemann integral.

Theorem 16 (Linearity of the local kernel-based integral). *Let c be a constant, and let f, g be integrable functions. Then*

$$I_x^{F,\alpha}[cf(x)] = c I_x^{F,\alpha}[f(x)], \quad (137)$$

$$I_x^{F,\alpha}[f(x) + g(x)] = I_x^{F,\alpha}[f(x)] + I_x^{F,\alpha}[g(x)]. \quad (138)$$

Proof. Using the definition,

$$I_x^{F,\alpha}[cf(x)] = \int \frac{cf(x)}{F(x,\alpha)} dx = c \int \frac{f(x)}{F(x,\alpha)} dx = c I_x^{F,\alpha}[f(x)]. \quad (139)$$

Similarly,

$$I_x^{F,\alpha}[f(x) + g(x)] = \int \frac{f(x) + g(x)}{F(x,\alpha)} dx \quad (140)$$

$$= \int \frac{f(x)}{F(x,\alpha)} dx + \int \frac{g(x)}{F(x,\alpha)} dx \quad (141)$$

$$= I_x^{F,\alpha}[f(x)] + I_x^{F,\alpha}[g(x)]. \quad (142)$$

□

3.10. Identity Operator in Local Kernel-Based Calculus

The following theorem establishes that the local kernel-based integral indeed inverts the local kernel-based derivative.

Theorem 17 (Identity operator for the local kernel-based derivative). *Let f be differentiable on $[x_0, x]$ and $F(\alpha, \cdot)$ continuous and non-zero. Then*

$$I_t^{F,\alpha,x_0,x}(D_t^{F,\alpha} f(t)) = f(x) - f(x_0). \quad (143)$$

In particular, if $f(x_0) = 0$, then $I_t^{F,\alpha,x_0,x} D_t^{F,\alpha} f(t) = f(x)$, i.e., $I \circ D = \text{identity}$ on functions vanishing at x_0 .

Proof. Using Theorem 10, $D_t^{F,\alpha} f(t) = f'(t) F(t, \alpha)$. Substituting into the definite integral,

$$I_t^{F,\alpha,x_0,x}(D_t^{F,\alpha} f(t)) = \int_{x_0}^x \frac{D_t^{F,\alpha} f(t)}{F(t,\alpha)} dt = \int_{x_0}^x \frac{f'(t) F(t,\alpha)}{F(t,\alpha)} dt = \int_{x_0}^x f'(t) dt = f(x) - f(x_0). \quad (144)$$

This completes the proof. □

Remark 21. Conversely, differentiating a local kernel-based integral returns the original function:

$$D_x^{F,\alpha}(I_x^{F,\alpha} f(x)) = D_x^{F,\alpha} \left(\int \frac{f(x)}{F(x,\alpha)} dx \right) = \frac{d}{dx} \left(\int \frac{f(x)}{F(x,\alpha)} dx \right) \cdot F(x,\alpha) \quad (145)$$

$$= \frac{f(x)}{F(x,\alpha)} \cdot F(x,\alpha) = f(x). \quad (146)$$

Thus $D_x^{F,\alpha} I_x^{F,\alpha} = I$ on the space of differentiable functions where the relevant integrals exist and $F(x, \alpha) \neq 0$. For more general function classes (e.g., merely integrable functions), the identity holds only in the almost-everywhere sense or under additional regularity assumptions.

3.11. Examples of Local Kernel-Based Integrals

The choice of the function $F(x, \alpha)$ determines the specific integral. For the local Euler calculus, $F(x, \alpha) = e^{(1-\alpha)x}$, and the definition reduces to

$$I_x^{e,\alpha} f(x) = \int \frac{f(x)}{e^{(1-\alpha)x}} dx, \quad (147)$$

which matches Definition 2's integral counterpart.

Below we compute a few elementary local kernel-based integrals for a generic $F(x, \alpha)$.

Example 7 (Integral of a constant). Let $f(x) = c$. Then

$$I_x^{F,\alpha}[c] = \int \frac{c}{F(x, \alpha)} dx = c \int \frac{dx}{F(x, \alpha)}. \quad (148)$$

If $F(x, \alpha) = e^{(1-\alpha)x}$, this yields

$$c \int e^{(\alpha-1)x} dx = c \frac{e^{(\alpha-1)x}}{\alpha-1} + \text{constant}. \quad (149)$$

Example 8 (Integral of $F(x, \alpha)$ itself). Let $f(x) = F(x, \alpha)$. Then

$$I_x^{F,\alpha}[F(x, \alpha)] = \int \frac{F(x, \alpha)}{F(x, \alpha)} dx = \int 1 dx = x + C. \quad (150)$$

This is the analogue of the Euler case, where $I_x^{e,\alpha}[e^{(1-\alpha)x}] = x + C$.

Example 9 (Integral of a product that cancels the denominator). If $F(x, \alpha) \neq 0$, and $f(x) = g(x)F(x, \alpha)$, then

$$I_x^{F,\alpha}[f(x)] = I_x^{F,\alpha}[g(x)F(x, \alpha)] = \int \frac{g(x)F(x, \alpha)}{F(x, \alpha)} dx = \int g(x) dx. \quad (151)$$

Thus the local kernel-based integral of a function that already contains the factor $F(x, \alpha)$ reduces to the ordinary integral of the remaining factor.

For instance, with $\alpha \neq 1$ and $F(x, \alpha) = e^{(1-\alpha)x}$, taking $g(x) = x^p$, with $p \neq -1$ gives

$$I_x^{e,\alpha}[x^p e^{(1-\alpha)x}] = \int x^p dx = \frac{x^{p+1}}{p+1} + C, \quad (152)$$

which is consistent with the table in the Euler case (after adjusting for the derivative relation). And in general, for instance, with $F(x, \alpha) \neq 0$ a smooth function, taking $g(x) = x^p$, with $p \neq -1$ gives

$$I_x^{F,\alpha}[x^p F(x, \alpha)] = \int \frac{dx x^p F(x, \alpha)}{F(x, \alpha)} = \int x^p dx = \frac{x^{p+1}}{p+1} + C. \quad (153)$$

In the case where $p = -1$, we have

$$I_x^{F,\alpha}[x^p F(x, \alpha)] = \int \frac{dx x^{-1} F(x, \alpha)}{F(x, \alpha)} = \int x^{-1} dx = \log |x| + C. \quad (154)$$

Remark 22. The table results in Table 4 is a direct consequence of the general rule $I_x^{F,\alpha}[g(x)F(x, \alpha)] = \int g(x)dx$, with $F(x, \alpha) \neq 0$.

Table 4. Local kernel-based integrals of some elementary functions for the special choice $F(x, \alpha) \neq 0$.

$f(x)$	$I_x^{F, \alpha} f(x)$
$F(x, \alpha)$	$x + C$
$px^{p-1} F(x, \alpha)$	$x^p + C$
$e^x F(x, \alpha)$	$e^x + C$
$\sin(x) F(x, \alpha)$	$-\cos(x) + C$
$\cos(x) F(x, \alpha)$	$\sin(x) + C$

This completes the construction of the local kernel-based integral and its basic calculus.

3.12. Fundamental Theorem of Local Kernel-Based Calculus

In this section we establish the FTC for the general local kernel-based calculus framework, i.e., the FTLKBC.

Theorem 18 (Fundamental Theorem of Local Kernel-Based Calculus). *Let f be differentiable on $[x_{\min}, x_{\max}]$, and let $F(x, \alpha)$ be continuous and non-zero on $[x_{\min}, x_{\max}]$. Then*

$$\int_{x_{\min}}^{x_{\max}} \frac{D_x^{F, \alpha} f(x)}{F(x, \alpha)} dx = f(x_{\max}) - f(x_{\min}). \quad (155)$$

Proof. By Theorem 10, we have

$$D_x^{F, \alpha} f(x) = F(x, \alpha) f'(x). \quad (156)$$

Dividing both sides by $F(x, \alpha)$ (which is non-zero by assumption) gives

$$\frac{D_x^{F, \alpha} f(x)}{F(x, \alpha)} = f'(x). \quad (157)$$

Integrating both sides from $x_{\min}$ to $x_{\max}$ and applying the classical FTC yields

$$\int_{x_{\min}}^{x_{\max}} \frac{D_x^{F, \alpha} f(x)}{F(x, \alpha)} dx = \int_{x_{\min}}^{x_{\max}} f'(x) dx = f(x_{\max}) - f(x_{\min}). \quad (158)$$

This completes the proof. $\square$

Remark 23. The FTLKBC is a genuine generalization of the classical FTC, not merely a reformulation. The proof necessarily relies on the classical FTC because the local kernel-based derivative is defined in terms of the classical derivative, i.e., $D_x^{F, \alpha} f = F(x, \alpha) f'$. Substitution then reduces the result to the classical FTC, which serves as a consistency check. However, the theorem is new in the sense that it establishes the inverse relationship between the novel operators $D_x^{F, \alpha}$ and $I_x^{F, \alpha}$. The classical FTC is recovered as the special case $F(x, \alpha) = 1$. Thus, the theorem validates the coherence of the new operator framework rather than merely restating a known result.

Remark 24. The assumption $F(x, \alpha) \neq 0$ on $[x_{\min}, x_{\max}]$ is essential for the division in the proof. If $F(x, \alpha) = 0$ at some point, the integral is not well-defined in the standard Riemann sense. This aligns with the broader framework where the degenerate case $F = 0$ is treated separately.

4. Conclusions

4.1. Summary of Contributions

In this paper, we have established the *local kernel-based calculus* as a general framework for local differential operators of the form $D_x^{F,\alpha}$. This framework is not limited to a specific parameter dependence; it encompasses any local kernel-based derivative whose increment scale depends on the independent variable through an arbitrary function $F(x, \alpha)$, with the parameter α unrestricted and allowed to take negative values. Traditional conformable-type derivatives—and the newly introduced Euler kernel-based derivative—all appear as special cases.

Within this general setting we proved a complete set of foundational theorems: linearity, the product rule, continuity of differentiable functions (under the assumption $F \neq 0$), Rolle's theorem, the mean value theorem, and the fundamental theorem of calculus via the corresponding integral operator. These results extend classical calculus to an arbitrary kernel while preserving the essential structure of the original theorems.

We also derived a new formulation of the chain rule,

$$D_x^{F,\alpha}[f(g(x))] = D_g^{F,\alpha}[f(g)] \cdot D_x^{F,\alpha}[g(x)] \cdot F^{-1}(g(x), \alpha), \quad (159)$$

which expresses the chain rule entirely in terms of the kernel-based derivatives. While algebraically equivalent to the classical form $f'(g(x))D_x^{F,\alpha}g(x)$, this representation is more self-contained within the kernel-based framework and preserves the intuitive structure of the chain rule without reference to the classical derivative.

The FTLEC and its generalization, the FTLKBC, were rigorously established. These results demonstrate that the local Euler and kernel-based derivatives and integrals are genuine inverses, with the classical FTC recovered as the special cases $\alpha = 1$ and $F(x, \alpha) = 1$, respectively. This validates the internal consistency and coherence of the new operator frameworks, showing that they naturally extend the classical calculus while reducing to it in the appropriate limit.

As an important illustration, we developed the *local Euler calculus* with $F(x, \alpha) = e^{(1-\alpha)x}$ in full detail, including explicit derivative and integral formulas for elementary functions. This special case demonstrates the power and simplicity of the functional approach and serves as a concrete example of the general framework.

Collectively, these results provide a solid, self-contained foundation for a wide class of local differential and integral operators, unifying and extending existing formulations while introducing new special cases and structural insights.

4.2. Limitations of the Framework

While the local kernel-based calculus provides a unified and rigorous foundation for a wide class of local differential operators, several limitations should be acknowledged. First, the framework is inherently *local*: the derivative at a point depends only on the behavior of the function in an infinitesimal neighborhood of that point, and therefore does not capture non-local effects such as those described by Riemann–Liouville or Caputo fractional derivatives. Second, the classical results—Rolle's theorem, the mean value theorem, the fundamental theorem of calculus, and the continuity theorem—require the assumption $F(x, \alpha) > 0$ on the interval of interest. While the definition itself is valid for arbitrary F , including the degenerate case $F = 0$, the standard theorems do not hold without this positivity assumption. Third, the framework is developed at the level of elementary real analysis; connections to measure theory, functional analysis, and weighted Sobolev spaces are only briefly indicated and require separate investigation.

4.3. Future Directions

The local kernel-based calculus is therefore a meta-framework that can generate and analyze a wide class of local differential operators. Future work will extend this framework to higher dimensions, explore possible connections with non-local fractional operators, and apply it to differential equations where the scaling function arises naturally from physical or geometric considerations. Further developments could explore the interplay between local kernel-based calculus and advanced geometric or algebraic structures. In particular, applying the present derivative operators to tensor field theories [11] or to manifold–metric pairs [12] may lead to new insights in local differential geometry, physical and cosmological applications [14–16]. Additionally, the connection to weighted Sobolev spaces and the interpretation of the integral as integration with respect to the weighted measure $d\mu_{F,\alpha}(x) = dx/F(x,\alpha)$ merit further exploration. More broadly, this work aligns with a concise and principled approach to foundational analysis [1,2], and suggests connections to advanced generalized operator frameworks [8], measure theory, functional analysis, and more general frameworks.

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Abbreviations

The following abbreviations are used in this manuscript:

MVT	mean value theorem
EVT	extreme value theorem
FTC	fundamental theorem of calculus
FTLEC	fundamental theorem of local Euler calculus
FTLKBC	fundamental theorem of local kernel-based calculus
ODE	ordinary differential equation

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