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Monotonicity and Composition Analyses for Fractional Differences Involving Generalized Mittag-Leffler Kernels

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Abstract

This study establishes a rigorous monotonicity and composition analysis for fractional difference operators defined via the generalized Mittag-Leffler kernel. A fundamental criterion linking the sign of the discrete Atangana–Baleanu–Riemann (ABR) fractional difference to a new generalized form of monotonicity ($w(1 - w)$ –monotonicity) of the function is derived at first. Subsequently, a crucial composition rule for the ABR difference and its associated sum is proved, providing an explicit formula that elegantly incorporates initial conditions. The use of our theoretical findings is demonstrated practically by its application in solving linear discrete initial value problems, transforming them into explicit solutions by employing the composition theorem. The outcomes reveal the underlying dynamics of discrete models, significantly enhancing the theoretical frameworks and applications.

Keywords: nonsingular difference operators; Mittag-Leffler kernel; monotonicity; composition rule; dynamical models

MSC: 26D07; 26A48; 26D15; 26A33; 39A70

1. Introduction

In the simulation of discrete-time dynamical systems, discrete fractional operators have shown themselves to be an essential tool, offering a robust solution by introducing non-local operators that capture the system’s dependence on its entire past history [1–3]. The foundational work on fractional differences and sums [4–6] has paved the way for various operators, among which the Atangana–Baleanu (AB) types defined via the Mittag-Leffler kernel (MLK) [7,8] have gained prominence due to its superior potential in the modeling of memory and hereditary effects [9,10]. It is, however, absolutely essential that the overall underlying dynamics of the discrete fractional operators be well understood to take the theory and practice forward. In particular, how the operator’s sign dictates function monotonicity is an area that still needs much development [11–14]. To bridge this critical gap, our study provides a rigorous analysis of monotonicity for discrete operators defined via the generalized MLK, which has not been discussed in the literature.

In this article, we establish two pivotal theoretical contributions that build upon this framework. First, we derive a fundamental criterion that connects the sign of the



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nonsingular discrete operator (ABR) to a novel form of $w(1-w)$ -monotonicity (unlike classical monotonicity, this accommodates the non-local nature of fractional operators by requiring $y(t+1) \geq w(1-w)y(t)$, for $w \in (0, 0.5)$) for the underlying function itself. From the perspective of applications, the term $w(1-w)$ is interpreted as a damping factor that characterizes the dissipation of memory in applied models, like epidemiological and biological population models. In particular, the growth at each time step must be reduced due to the decaying impact of the history (hereditary damping). Remarkably, for $w \rightarrow 0$ or $w \rightarrow 1$, the nonlocal memory effect disappears and we obtain the usual dynamics of an integer order, and for $w = 0.5$, the scaling factor takes its smallest value of 0.25, meaning the maximum hereditary damping. Second, we prove a crucial composition rule for the ABR difference and its associated fractional sum, yielding an explicit formula that elegantly incorporates initial conditions. This is a powerful diagnostic tool for analyzing the behavior for solving linear discrete initial value problems (DIVPs) and it allows us to construct solutions by transforming the fractional equation into an explicit solution structure directly, a key challenge in the field [15,16].

The significance of these results lies in their dual contribution to both theory and application of discrete fractional calculus. Our findings provide essential analytical tools that advance the qualitative theory of discrete fractional systems, providing new analytical tools for systems governed by AB operators [17]. The composition theorem is applied to solve linear DIVPs as a practical utility of our results, which transforms complex fractional systems into explicit solutions and opens avenues for stability analysis [18,19]. Additionally, this research not only provides essential tools for current research but also lays the groundwork for future extensions, such as paving the way for exploring Ulam–Hyers stability [19], applying these techniques to other discrete operators with different kernel structures, and developing analogous frameworks for the corresponding delta fractional operators [20].

This study proceeds by first reviewing the theoretical background in Section 2, then Section 3 presents our core theorems on monotonicity and composition. Section 4 demonstrates their practical application through illustrative examples, and finally concludes with a summary and outlines future directions in Section 5.

2. Theoretical Background

To facilitate the development of our monotonicity analysis, this section first reviews the fundamental concepts and characteristics of nonsingular discrete operators and MLK functions, which form the theoretical backbone of our work.

2.1. MLK Functions and Their Corresponding Nonsingular Operators

Define $\mathbb{Z}_a := \{a, a+1, \dots\}$ for any a within real numbers. The v -th rising factorial serves as the discrete analogue of the continuous power function t^v and is expressed as (see [3,7,10,21]):

$$t^{\bar{v}} = \frac{\Gamma(t+v)}{\Gamma(t)}, \quad (1)$$

where $t, v \in \mathbb{R}$ and assuming $t+v$ to avoid the poles of the Gamma function. Note that this function is crucial because it can satisfy the discrete power rule:

$$\nabla(t^{\bar{v}}) = v t^{\bar{v}-1},$$

which directly leads to its monotonicity on $\mathbb{Z}_0$.

Now, the MLK can be defined based on (1) as its essential properties under the nabla difference operator are detailed in the next lemma; they are essential for its application in

discrete fractional calculus, and this function is particularly significant in the definition of discrete fractional operators, as demonstrated in the following definitions.

Definition 1 (See [10]). Consider $w, \beta, \eta \in \mathbb{C}$ and $\zeta \in \mathbb{R}$, where $\operatorname{Re}(w) > 0$ and $|\zeta| < 1$; the discrete MLK of two-parameter type can be formulated as

$$E_{w,\beta}(\zeta, \eta) := \sum_{\ell=0}^{\infty} \zeta^{\ell} \frac{\eta^{\overline{\ell w + \beta - 1}}}{\Gamma(\ell w + \beta)}. \quad (2)$$

while the discrete MLK function of one-parameter type is given as

$$E_{\overline{w}}(\zeta, \eta) := \sum_{\ell=0}^{\infty} \zeta^{\ell} \frac{\eta^{\overline{\ell w}}}{\Gamma(\ell w + 1)}. \quad (3)$$

Lemma 1 (See [10]). Consider the same parameters given in the above definition with $\operatorname{Re}(w), \operatorname{Re}(\beta), \operatorname{Re}(v) > 0$ and $n \in \mathbb{Z}_1$; one can see that

$$\nabla^n E_{w,\beta+n}(\zeta, \eta) = E_{w,\beta}(\zeta, \eta). \quad (4)$$

Also, for $\eta > a \in [0, \infty)$, we have

$${}^{RL}_a \nabla^v E_{w,\beta}(\zeta, \eta - a) = E_{w,\beta-v}(\zeta, \eta - a), \quad (5)$$

where ${}^{RL}_a \nabla^v$ denotes the nabla difference in the Riemann–Liouville sense (see [3,7,10,21]):

$$\left({}^{RL}_a \nabla^v y\right)(t) = \sum_{s=a+1}^t [t - \rho(s)]^{-v-1} y(s) / \Gamma(-v), \quad t \in \mathbb{Z}_{a+1}, \quad (6)$$

with $0 < v < 1$ and the backward jump operator $\rho(s) = s - 1$.

Based on the discrete MLK function of two-parameter type defined above, we now recall the nonsingular discrete operators of AB types for the fixed parameters $\gamma = 1$ and $\beta = w + 1$.

Definition 2 (See [10]). Let the parameter ζ be such that $\zeta = -\frac{w}{1-w}$ with $0 < w < 1/2$. Then for the specific case of the generalized operators where we fixed the parameters as above, we see that

- The nonsingular discrete operator of ABR type is formulated by applying the nabla difference operator to a discrete convolution and its expression is given by:

$$\left({}^{ABR}_a \nabla^{w,w+1} y\right)(t) = \frac{B(w)}{1-w} \nabla_t \sum_{s=a+1}^t E_{w,w+1}(\zeta, t - \rho(s)) y(s), \quad t \in \mathbb{Z}_a. \quad (7)$$

- The nonsingular discrete operator of ABC type is similarly formulated by applying the discrete convolution to the nabla difference of the function and its expression is given by:

$$\left({}^{ABC}_a \nabla^{w,w+1} y\right)(t) = \frac{B(w)}{1-w} \sum_{s=a+1}^t E_{w,w+1}(\zeta, t - \rho(s)) \nabla_s y(s), \quad t \in \mathbb{Z}_a. \quad (8)$$

- The Liouville–Caputo type AB can be expressed in terms of the Riemann–Liouville type AB through the following fundamental relationship, which highlights their difference by a term dependent on the initial condition $y(a)$:

$$\left({}^{ABC}_a\nabla^{w,w+1}y\right)(t)=\left({}^{ABR}_a\nabla^{w,w+1}y\right)(t)-\frac{B(w)}{1-w}y(a)E^1_{w,w+1}(\zeta,t-a). \quad (9)$$

In all expressions, $B(w)$ represents a normalization function satisfying $B(w) = 1$ at $w = 0, 1$.

Remark 1. The normalization $B(w)$ ensures the AB operators behave appropriately at the boundary values of w (where $w = 1$ represents the full fractional case and $w = 0$ corresponds to the standard integer-order case).

In this paper, we adopt the standard normalization function $B(w) = 1$ for the purpose of our theoretical analysis of monotonicity and composition results. In the literature, this is a common choice (e.g., [7,10]) since it is the simplest function meeting the required conditions $B(0) = B(1) = 1$. Thus, it allows us to focus on the core properties of the discrete MLK without introducing unnecessary complexity.

It is important to highlight that although the normalized value $B(w) = 1$ is chosen for convenience purposes, selecting a non-trivial smooth normalization function, such as $B(w) = 1 - w + \frac{w}{\Gamma(w)}$, will result in an inherent change to the bounds in Theorems 2 and 3. In particular, the $(1 - w)w$ bound would get scaled by a factor dependent on the value of $B(w)$, leading to a more complicated bound.

In the following definition, the fractional sum is constructed as an infinite series, where each term involves a scaled Riemann–Liouville fractional sum of order $wk - w$, which complements the ABR difference, as it is defined as its corresponding left-inverse operator and called the AB fractional sum.

Definition 3 (See [10]). For the specific parameters $\gamma = 1$ and $\beta = w + 1$, the AB fractional sum is given by the following sum:

$$\left({}^{AB}_a\nabla^{-(w,w+1)}y\right)(t)=\sum_{j=0}^1\binom{1}{j}\frac{w^j}{B(w)(1-w)^{j-1}}\left({}_a\nabla^{-(wj-w)}y\right)(t), \quad (10)$$

where ${}_a\nabla^{-v}$ denotes the nabla sum in the Riemann–Liouville sense (see [3,7,10,21]):

$$\left({}_a\nabla^{-v}y\right)(t)=\sum_{d=a+1}^t\left[t-\rho(d)\right]^{\overline{v-1}}y(d)/\Gamma(v), \quad t\in\mathbb{Z}_{a+1},$$

with $0 < v$.

The ABR fractional difference (7) can be expressed in an alternative but mathematically equivalent form, which provides a series expansion that directly relates the ABR operator to the classical Riemann–Liouville fractional sums.

Theorem 1 (See [10]). For a function y defined on $\mathbb{Z}_a$, the ABR fractional difference admits the following convergent series representation:

$$\left({}^{ABR}_a\nabla^{w,w+1}y\right)(t)=\frac{B(w)}{1-w}\sum_{k=0}^{\infty}\zeta^k\frac{(1)_k}{k!}\left({}_a\nabla^{-(wk+w)}y\right)(t), \quad (11)$$

where $\zeta = -\frac{w}{1-w}$ with $0 < w < 1/2$, guaranteeing $|\zeta| < 1$ and ensuring convergence of (11) for each $t \in \mathbb{Z}_a$.

2.2. Properties of MLK Function and Monotonicity Concepts

A fundamental property for our analysis is the strict monotonicity of the 2-parameter MLK function $E_{w,w+1}$, which we now state.

Lemma 2. For any $0 < w < \frac{1}{2}, \eta \in \mathbb{C}$ and $\zeta = -\frac{w}{1-w}$, the two-parameter MLK function $E_{w,w+1}(\zeta, \eta)$ is strictly increasing.

Proof. From Lemma 1, we have

$$\nabla E_{w,\beta}(\zeta, \eta) = \nabla^2 E_{w,\beta+1}(\zeta, \eta) = \nabla^2 E_{w,(\beta-1)+2}(\zeta, \eta) = E_{w,\beta-1}(\zeta, \eta).$$

Following ([11] Remark 1), $E_{w,w}(\zeta, \eta)$ is strictly positive; it follows, for $\beta = w + 1$, that

$$\nabla E_{w,w+1}(\zeta, \eta) = E_{w,w}(\zeta, \eta) > 0. \quad (12)$$

This implies that $E_{w,w+1}(\zeta, \eta)$ is strictly increasing. Thus, the proof is done. $\square$

Remark 2. With the parameter ζ fixed as $\zeta = -\frac{w}{1-w}$, the initial evaluations of $E_{w,w+1}(\zeta, \eta)$ are determined explicitly. Specifically, we can find that

- $E_{w,w+1}(\zeta, 0) = 0.$
- $E_{w,w+1}(\zeta, 1) = 1 - w.$
- $E_{w,w+1}(\zeta, 2) = (1 - w)(1 + w - w^2).$
- $E_{w,w+1}(\zeta, 3) = \frac{1-w}{2}(2w^4 - 3w^3 - 2w^2 + 3w + 2).$

For functions defined on the discrete domain $\mathbb{Z}_a$, we recall the following generalized concept of monotonicity as defined in [11], which will serve as a fundamental tool for our analysis. It extends the classical ideas of increasing and decreasing behavior to the fractional setting.

Definition 4 (See [11]). Suppose that $y : \mathbb{Z}_a \rightarrow \mathbb{R}$ is a function. The concept of w -monotonicity based on the function's initial value is defined by:

- A w -increasing function requires $y(a) \geq 0$ and satisfies the condition

$$y(t+1) \geq w y(t), \quad \forall t \in \mathbb{Z}_a, \quad w \in (0, 1].$$

- A w -decreasing function requires $y(a) \leq 0$ and satisfies the condition

$$y(t+1) \leq w y(t), \quad \forall t \in \mathbb{Z}_a, \quad w \in (0, 1].$$

Example 1. Let us work on the domain $\mathbb{Z}_1$, and let the function be

$$y(t) = (0.7)^t + t.$$

Since the function is strictly increasing and $y(1) > 0$, it follows that it is 0.8-increasing. However, one can use the definition of 0.8-increasing directly to verify it. Figure 1 translates the abstract mathematical inequality, $y(t) \geq 0.8y(t-1)$, into a simple visual test.

- Blue stems: These represent the actual value of $y(t)$ at each time step.
- Red stems: These represent the minimum required value for the function to be considered 0.8-increasing. It is the value from the previous step, scaled down by 80%.

It clearly shows that the blue line stays consistently above the red line for all time steps plotted.

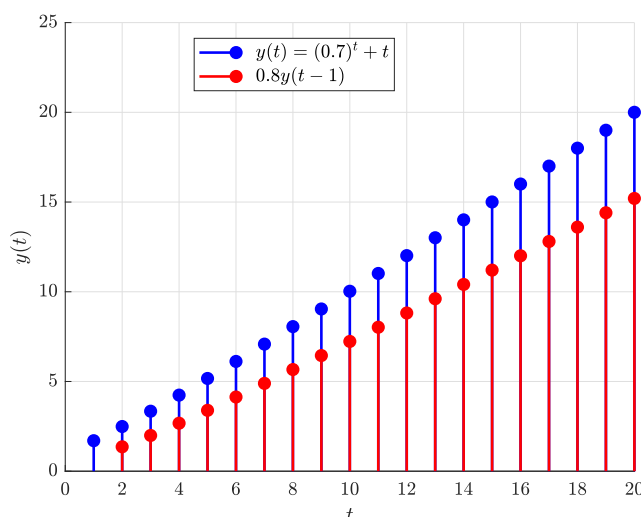


Figure 1. Verification of 0.8-increasing property for $y(t) = (0.7)^t + t$.

3. Monotonicity and Composition Theories

The core theoretical contributions of our study are presented here. We first establish the rigorous monotonicity results for the discrete operators introduced in Section 2. Specifically, we derive a fundamental link between the nonsingular operator's sign and the function's $w(1-w)$ -monotonicity. Following this analysis, we investigate the composition rule for the ABR operator with its associated fractional sum, a crucial tool for solving linear fractional initial value problems.

3.1. Monotonicity Results

Throughout this section, we suppose that $0 < w < \frac{1}{2}$.

Theorem 2. Suppose y is a mapping defined on $\mathbb{Z}_{a-1}$ taking values in $\mathbb{R}$. If the inequality

$$\left({}^{ABR}\nabla_{a-1}^{w,w+1}y \right)(t) \leq 2B(w)y(t), \quad t \in \mathbb{Z}_{a-1},$$

holds true, then it follows that y is $w(1-w)$ -increasing.

Proof. Rewrite $\left({}^{ABR}\nabla_{a-1}^{w,w+1}y \right)(t) = \frac{B(w)}{1-w} \nabla S(t)$, where $S(t) = \sum_{s=a}^t y(s) E_{w,w+1}(\zeta, t - \rho(s))$. From assumption and since $E_{w,w+1}(\zeta, 1) = 1 - w$, we have

$$\begin{aligned} & \frac{B(w)}{1-w} \nabla S(t) \\ &= \frac{B(w)}{1-w} [S(t) - S(t-1)] \\ &= \frac{B(w)}{1-w} \left[\sum_{j=a}^{t-1} y(j) \left(E_{w,w+1}(\zeta, t - \rho(j)) - E_{w,w+1}(\zeta, t-1 - \rho(j)) \right) + y(t) E_{w,w+1}(\zeta, 1) \right] \\ &= B(w)y(t) + B(w) \sum_{j=a}^{t-1} y(j) \frac{E_{w,w+1}(\zeta, t+1-j) - E_{w,w+1}(\zeta, t-j)}{1-w} \leq 2B(w)y(t). \end{aligned}$$

This yields

$$\nabla S(t) - 2(1-w)y(t) = \sum_{j=a}^{t-1} y(j) \left(E_{\overline{w}, \overline{w+1}}(\zeta, t+1-j) - E_{\overline{w}, \overline{w+1}}(\zeta, t-j) \right) - (1-w)y(t) \leq 0. \quad (13)$$

The proof continues by mathematical induction. Substituting $t = a$ into (13) yields $y(a) \geq 0$. Upon setting $t = a + 1$ in Equation (13), the sum on the right-hand side evaluates to only one term that corresponds to $s = a$; that is,

$$\begin{aligned} & \nabla S(a+1) - 2(1-w)y(a+1) \\ &= \sum_{j=a}^a y(j) \left(E_{\overline{w}, \overline{w+1}}(\zeta, a+2-j) - E_{\overline{w}, \overline{w+1}}(\zeta, a+1-j) \right) - (1-w)y(a+1) \leq 0. \end{aligned}$$

Next, applying Remark 2, it follows that

$$\begin{aligned} y(a+1) &\geq \frac{E_{\overline{w}, \overline{w+1}}(\zeta, 2) - E_{\overline{w}, \overline{w+1}}(\zeta, 1)}{1-w} y(a) \\ &= w(1-w)y(a). \end{aligned}$$

Now, we make the inductive hypothesis that

$$\begin{aligned} y(a+M) &\geq w(1-w)y(a+M-1) \geq w^2(1-w)^2y(a+M-2) \\ &\geq \dots \geq w^M(1-w)^My(a) \geq 0. \end{aligned}$$

We aim to demonstrate that $y(a+M+1) \geq w(1-w)y(a+M)$.

Upon inserting $t = a + M + 1$ into (13) and utilizing (12), we get that

$$\begin{aligned} & \nabla S(a+M+1) - 2(1-w)y(a+M+1) \\ &= -(1-w)y(a+M+1) + \sum_{j=a}^{a+M} y(j) \left(E_{\overline{w}, \overline{w+1}}(\zeta, a+M+2-j) - E_{\overline{w}, \overline{w+1}}(\zeta, a+M+1-j) \right) \\ &= -(1-w)y(a+M+1) + \sum_{j=a}^{a+M} \nabla_k E_{\overline{w}, \overline{w+1}}(\zeta, a+M+2-j) y(j) \\ &= -(1-w)y(a+M+1) + \sum_{j=0}^M E_{\overline{w}, \overline{w}}(\zeta, M+2-j) y(j+a) \\ &= -(1-w)y(a+M+1) + \left[E_{\overline{w}, \overline{w}}(\zeta, M+2)y(a) + E_{\overline{w}, \overline{w}}(\zeta, M+1)y(a+1) \right. \\ &\quad \left. + \dots + E_{\overline{w}, \overline{w}}(\zeta, 3)y(M-1) + E_{\overline{w}, \overline{w}}(\zeta, 2)y(a+M) \right] \leq 0. \end{aligned}$$

Then by using (12) together with Remark 2 and the inductive hypothesis, it follows that

$$\begin{aligned} y(a+M+1) &\geq \frac{1}{1-w} \left[E_{\overline{w}, \overline{w}}(\zeta, M+2)y(a) + E_{\overline{w}, \overline{w}}(\zeta, M+1)y(a+1) \right. \\ &\quad \left. + \dots + E_{\overline{w}, \overline{w}}(\zeta, 3)y(a+M-1) + E_{\overline{w}, \overline{w}}(\zeta, 2)y(a+M) \right] \\ &\geq \frac{1}{1-w} E_{\overline{w}, \overline{w}}(\zeta, 2)y(a+M) \\ &= \frac{1}{1-w} \nabla E_{\overline{w}, \overline{w+1}}(\zeta, 2)y(a+M) \\ &= \frac{1}{1-w} \left[E_{\overline{w}, \overline{w+1}}(\zeta, 2) - E_{\overline{w}, \overline{w+1}}(\zeta, 1) \right] y(a+M) \\ &= w(1-w)y(a+M). \end{aligned}$$

This finishes the proof. $\square$

Theorem 3. Suppose $y : \mathbb{Z}_{a-1} \rightarrow \mathbb{R}$ satisfies the relation

$$\left({}^{ABR}_{a-1} \nabla^{w,w+1} y \right)(t) \geq 2B(w)y(t),$$

for all $t \in \mathbb{Z}_{a-1}$. Then, y possesses the property of being $w(1-w)$ -decreasing.

Proof. This proof follows a perturbation argument similar to Theorem 2. $\square$

Remark 3. This generalization of the classic monotonicity to $w(1-w)$ -monotonicity is a substantial improvement compared to the traditional monotonicity $y(t+1) \geq y(t)$. As the fractional operator exhibits non-local memory properties, positivity of the fractional difference does not guarantee that the solution will be monotonically increasing in the classic sense. Imposing an additional constraint such that $y(t+1) \geq w(1-w)y(t)$, where $w(1-w) < 1$ for $w \in (0, 0.5)$, allows for accounting for the decay and memory effects associated with the Mittag-Leffler kernel, thus capturing a broader and more physically relevant class of solutions that classical monotonicity would otherwise exclude.

Corollary 1. Consider a function $y : \mathbb{Z}_{a-1} \rightarrow \mathbb{R}$. If the inequality, at $t \in \mathbb{Z}_{a-1}$,

$$\left({}^{ABC}_{a-1} \nabla^{w,w+1} y \right)(t) \leq 2B(w)y(t) + \frac{B(w)}{1-w} E_{w,w+1}(\zeta, t-a+1)y(a-1),$$

holds true, then it yields that y is $w(1-w)$ -increasing.

Proof. This proof follows a perturbation argument similar to Theorem 2, utilizing (9) with $\beta = 1$. $\square$

Corollary 2. Consider a function $y : \mathbb{Z}_{a-1} \rightarrow \mathbb{R}$. Assume that for all $t \in \mathbb{Z}_{a-1}$,

$$\left({}^{ABC}_{a-1} \nabla^{w,w+1} y \right)(t) \geq 2B(w)y(t) + \frac{B(w)}{1-w} E_{w,w+1}(\zeta, t-a+1)y(a-1).$$

Under this condition, we conclude that y is $w(1-w)$ -decreasing.

Proof. By utilizing (9) with $\beta = 1$, the proof can be done with the same steps as in Theorem 3. $\square$

3.2. Composition Results

This section deals with a composition rule for the AB fractional difference with its associated fractional sum.

Lemma 3. For any $0 < w < \frac{1}{2}$, we have

$${}^{RL}_a \nabla^w \nabla E_{w,w+1}(\zeta, \tau) = \zeta E_{w,w}(\zeta, \tau) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)}, \quad (14)$$

for $\tau = 1 + t - a$ and $t \in \mathbb{Z}_a$.

Proof. In light of (6), we have

$$\begin{aligned} {}^{RL}_a \nabla^w y(t) &= {}^{RL}_{a-1} \nabla^w y(t) - \frac{(t-a+1)^{-w-1}}{\Gamma(-w)} y(a) \\ &= {}^{RL}_{a-1} \nabla^w y(t) - \frac{\tau^{-w-1}}{\Gamma(-w)} y(a). \end{aligned}$$

For $y(t) = E_{\overline{w}, \overline{w}}(\zeta, \tau)$, it follows that

$$\begin{aligned} {}_a \nabla^w E_{\overline{w}, \overline{w}}(\zeta, \tau) &= {}^{RL}_{a-1} \nabla^w E_{\overline{w}, \overline{w}}(\zeta, t - (a-1)) - \frac{\tau^{-w-1}}{\Gamma(-w)} E_{\overline{w}, \overline{w}+1}(\zeta, 1) \\ &= {}^{RL}_{a-1} \nabla^w E_{\overline{w}, \overline{w}}(\zeta, \tau) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)}. \end{aligned} \quad (15)$$

Then by utilizing Lemma 1 and identity (15), one can have

$$\begin{aligned} {}^{RL}_a \nabla^w \nabla E_{\overline{w}, \overline{w}+1}(\zeta, \tau) &= {}^{RL}_a \nabla^w E_{\overline{w}, \overline{w}}(\zeta, \tau) \\ &= {}^{RL}_{a-1} \nabla^w E_{\overline{w}, \overline{w}}(\zeta, t - (a-1)) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)} \\ &= E_{\overline{w}, 0}(\zeta, \tau) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)} \\ &= \zeta E_{\overline{w}, \overline{w}}(\zeta, \tau) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)}, \end{aligned}$$

which completes the proof. $\square$

Based on [10], it follows that

$$\left({}^{AB}_a \nabla^{-(w, w+1)} {}^{ABR}_a \nabla^{w, w+1} y \right)(t) = y(t), \quad (16)$$

with absence of $y(a)$. However, the following theorem explicitly incorporates the initial condition $y(a)$, providing an essential framework for solving linear fractional difference equations.

Theorem 4. Suppose y is a function defined on $\mathbb{Z}_{a-1}$. Then for $0 < w < \frac{1}{2}$, we have

$$\left({}^{AB}_a \nabla^{-(w, w+1)} {}^{ABR}_{a-1} \nabla^{w, w+1} y \right)(t) = y(t) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)} y(a). \quad (17)$$

Proof. In light of (7), we have

$$\begin{aligned} &\left({}^{AB}_a \nabla^{-(w, w+1)} {}^{ABR}_{a-1} \nabla^{w, w+1} y \right)(t) \\ &= \frac{B(w)}{1-w} {}^{AB}_a \nabla^{-(w, w+1)} \nabla_t \left[\sum_{j=a}^t E_{\overline{w}, \overline{w}+1}(\zeta, t - \rho(j)) y(j) \right] \\ &= \frac{B(w)}{1-w} {}^{AB}_a \nabla^{-(w, w+1)} \nabla_t \left[\sum_{j=a+1}^t E_{\overline{w}, \overline{w}+1}(\zeta, t - \rho(j)) y(j) + y(a) E_{\overline{w}, \overline{w}+1}(\zeta, \tau) \right] \\ &= {}^{AB}_a \nabla^{-(w, w+1)} {}^{ABR}_a \nabla^{w, \gamma} y(t) + \frac{B(w)}{1-w} y(a) {}^{AB}_a \nabla^{-(w, w+1)} \nabla_t E_{\overline{w}, \overline{w}+1}(\zeta, \tau) \\ &\stackrel{\text{by (16)}}{=} y(t) + \frac{B(w)}{1-w} y(a) {}^{AB}_a \nabla^{-(w, w+1)} \nabla_t E_{\overline{w}, \overline{w}+1}(\zeta, \tau). \end{aligned}$$

Applying (10) with $\gamma = 1$ and $\beta = w + 1$, we deduce

$$\begin{aligned} & \left({}^{AB}_a \nabla^{-(w,w+1)} {}^{ABR}_{a-1} \nabla^{w,w+1} y \right) (t) \\ &= y(t) + \frac{B(w)}{1-w} y(a) \sum_{i=0}^1 \binom{1}{i} \frac{w^i}{B(w)(1-w)^{i-1}} {}^a \nabla^{-(w,w+1)} E_{w,w+1}(\zeta, \tau) \\ &= y(t) + y(a) \sum_{i=0}^1 \frac{1}{\Gamma(2-i)!} (-1)^i \zeta^i {}^a \nabla^{-(w,w+1)} E_{w,w+1}(\zeta, \tau) \\ &= y(t) + y(a) {}^a \nabla^w E_{w,w+1}(\zeta, \tau) - \zeta y(a) {}^a \nabla^0 E_{w,w+1}(\zeta, \tau). \end{aligned}$$

By employing Lemmas 1 and 3, we obtain

$$\begin{aligned} \left({}^{AB}_a \nabla^{-(w,w+1)} {}^{ABR}_{a-1} \nabla^{w,w+1} y \right) (t) &= y(t) - \zeta E_{w,w}(\zeta, \tau) y(a) \\ &\quad + \zeta E_{w,w}(\zeta, \tau) y(a) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)} y(a) \\ &= y(t) - (1-w) \frac{\tau^{-w-1}}{\Gamma(-w)} y(a), \end{aligned}$$

which completes the required result. $\square$

4. Applications

This section validates the core theorems in the previous section and, more significantly, proves their value as essential tools for resolving discrete fractional initial-value problems.

Example 2. Let us analyze the function

$$\begin{aligned} y(x) &= \frac{1}{1-w} \sum_{s=1}^{x-1} \nabla_x E_{w,w+1}(\zeta, x-s+1) y(s), \quad \forall x > 1, \\ y(1) &= 1 \quad (\text{base case}), \end{aligned}$$

with $a = 1$. Then we need to compute the ABR fractional difference operator and verify if it satisfies

$$\left({}^{ABR}_{a-1} \nabla^{w,w+1} y \right) (x) \leq 2B(w)y(x). \quad (18)$$

From the proof of Theorem 2 (see Equation (13)) with our given $y(t)$, we have

$$\begin{aligned} \left({}^{ABR}_{a-1} \nabla^{w,w+1} y \right) (x) &= B(w) \left[y(x) + \frac{1}{1-w} \sum_{s=1}^{x-1} \nabla_x E_{w,w+1}(\zeta, x-s+1) y(s) \right] \\ &= B(w) [y(x) + y(x)] \leq 2B(w)y(x). \end{aligned}$$

Therefore, the condition (18) is satisfied with equality for each $x > 1$. As a result, according to Theorem 2, this function must also be $w(1-w)$ -increasing, meaning it verifies $y(x+1) \geq w(1-w)y(x)$ for each $x \in \mathbb{Z}_a$. For this specific function $y(x)$, we can explicitly verify this property by following the induction steps in the proof of Theorem 2.

On the other hand, we see that (18) is equivalent to

$$g(t) := \sum_{s=1}^{x-1} \nabla_x E_{w,w+1}(\zeta, x-s+1) y(s) - (1-w)y(x) \leq 0.$$

As one can see from the bottom of Figure 2, the green line is exactly at 0 (within numerical precision). This is because we constructed $y(x)$ to satisfy the condition with equality. Since the value is 0, it is certainly less than or equal to 0, which confirms that the condition of Theorem 2 is met for all x .

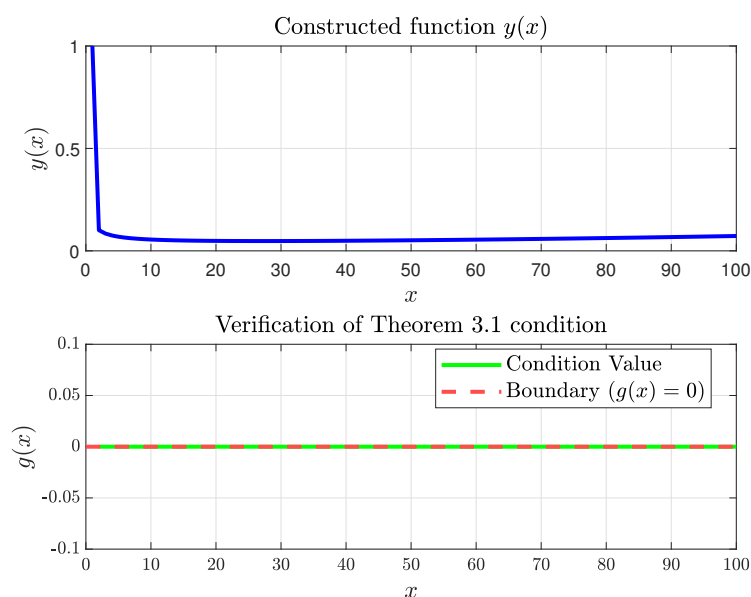


Figure 2. Verification of Theorem 2 for the constructed function $y(x)$.

All in all, Figure 2 demonstrates a function $y(t)$ that was recursively constructed to satisfy the condition of Theorem 2. The top figure shows the resulting positive, $w(1-w)$ -increasing function, while the bottom figure provides the numerical proof, confirming that the condition $g(t) \leq 0$ from Theorem 2. The green line shows the value of $g(t)$, which is identically 0 (within numerical precision), satisfying the inequality (18) (or $g(t) \leq 0$) with equality. The red dashed line represents the boundary value of 0.

Our last example demonstrates that Theorem 4 is essential for solving linear fractional difference equations. This provides the direct link to transform an equation involving a fractional difference into an explicit formula for the solution. Finding the solution without this theorem would be significantly more difficult. It makes it a cornerstone for analyzing discrete-time systems with memory modeled by the AB fractional operators.

Example 3. Consider the following discrete initial value problem (DIVP):

$$\left({}^{ABR}\nabla_{a-1}^{w,w+1}y\right)(t) = f(t), \quad t \in \mathbb{Z}_a, \quad (19)$$

with $y(a) = y_0$, where $0 < w < 0.5$.

Apply the fractional sum operator ${}^{AB}\nabla_a^{-(w,w+1)}$ to both sides of the equation:

$$\left({}^{AB}\nabla_a^{-(w,w+1)} {}^{ABR}\nabla_{a-1}^{w,w+1}y\right)(t) = \left({}^{AB}\nabla_a^{-(w,w+1)} f\right)(t).$$

Applying Theorem 4 to the left-hand side and solve for y with the initial condition, we can write the explicit solution:

$$y(t) = (1-w) \frac{(t-a+1)^{\overline{-w-1}}}{\Gamma(-w)} y_0 + \left({}^{AB}\nabla_a^{-(w,w+1)} f\right)(t).$$

The solution is composed of two distinct parts:

- **Particular Solution:** The term $\left({}^{AB}_a \nabla^{-(w,w+1)} f\right)(t)$ is the response of the system (19) to the external input or forcing function $f(t)$.
- **Homogeneous Solution:** The term $(1-w) \frac{(t-a+1)^{-w-1}}{\Gamma(-w)} y_0$ is the solution to the corresponding homogeneous equation, where $f(t) = 0$. This represents the natural behavior of the system (19), which is determined entirely by its initial state y_0 .

Particularly, if we consider the following specific DIVP:

$$\begin{aligned} \left({}^{ABR}_a \nabla^{w,w+1} y\right)(t) &= 1, \quad t \in \mathbb{Z}_a, \\ y(1) &= 1. \end{aligned}$$

As shown in the previous example, it gives the solution:

$$y(t) = \underbrace{(1-w) \frac{(t-a+1)^{-w-1}}{\Gamma(-w)}}_{y_p = \text{particular solution}} + \underbrace{\left({}^{AB}_a \nabla^{-(w,w+1)} 1\right)(t)}_{y_h = \text{homogeneous solution}}.$$

In Figure 3, we have demonstrated the total solution $y(t)$ for t from 1 to 20 with $a = 1$. The y_h (red line) follows a power form t^{-w-1} and is not identically zero, though it appears to be so when plotted on the same scale as the linearly growing y_p (blue dashed line). The sum of these two components is the total solution $y(t)$ (black line), with the contribution of the solution y_h being most significant for small values of t and diminishing as t increases.

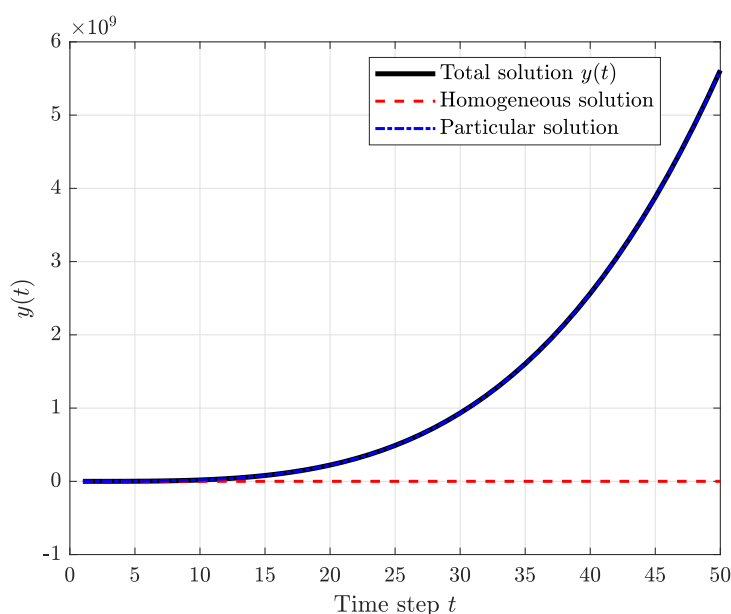


Figure 3. Verification of Theorem 4 for a constructed function: The total solution $y(t)$ (black) is the sum of y_h (red) and y_p (blue).

Figure 3 provides an intuitive confirmation of the analytical solution derived using Theorem 4. This visually separates the contributions of the initial condition and the external forcing, showing how the system's “memory” of its starting state fades away, leaving only the response to the ongoing input. The initial dip and the eventual linear growth are key signatures of the dynamics of this specific DIVP.

5. Conclusions and Future Directions

This paper has successfully established rigorous analytical tools for discrete fractional operators with generalized Mittag-Leffler kernels. We derived a novel monotonicity cri-

terion that directly connects the sign of the ABR fractional difference to a new $w(1 - w)$ monotonicity of the function. Furthermore, we proved a fundamental composition rule for the ABR difference and its associated sum, providing an explicit solution structure for linear initial value problems. These results significantly advance the qualitative theory of discrete fractional calculus.

Future work will focus on and immediately pursue four key directions:

- Extending these analyses to other fractional operators with different kernel structures.
- Investigating the Ulam–Hyers stability of fractional difference systems using these new tools (see [22,23]).
- Applying the theory to more complex real-world phenomena, nonlinear dynamical models (see [24]).
- Establishing the analogous framework for the corresponding delta fractional operators (see [20]).

Moreover, there is an unresolved issue on how to extend the existing monotonicity and composition results to the $0.5 \leq w < 1$ range of parameters. As the inequality $|\zeta| < 1$ does not hold in this case, future research can consider some different approaches, for example, sequential fractional difference or analytic continuation methods, which are required due to divergence of the corresponding infinite sums.

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