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Time Series Forecasting with New Type-2 Fuzzy T-Norms and T-Conorms

Pablo Hernández-Varela ¹, Pedro Huidobro ², Francisco Javier Talavera ^{3,4}, Carmen Torres-Blanc ⁵,
Susana Cubillo ⁵ and Jorge Elorza ^{3,4,*}

¹ Facultad de Ingeniería, Universidad San Sebastian, Bellavista 7, Santiago 8420524, Chile; pablo.hernandez@uss.cl

² Departamento de Estadística e I.O. y Didáctica de la Matemática, Universidad de Oviedo, C. San Francisco, 3, 33003 Oviedo, Spain; huidobropedro@uniovi.es

³ Departamento de Física y Matemática Aplicada, Facultad de Ciencias, Universidad de Navarra, C. Irunlarrea 1, 31008 Pamplona, Spain; ftalaveraan@alumni.unav.es

⁴ Institute of Data Science and Artificial Intelligence (DATAI), Edificio Ismael Sánchez Bella, Universidad de Navarra, 31009 Pamplona, Spain

⁵ Departamento de Matemática Aplicada a las TIC, Universidad Politécnica de Madrid, Boadilla del Monte, 28660 Madrid, Spain; mariadelcarmen.torres@upm.es (C.T.-B.); scubillo@fi.upm.es (S.C.)

* Correspondence: jelorza@unav.es

Abstract

This paper presents new families of triangular norms (t-norms) and conorms (t-conorms) specifically constructed for type-2 fuzzy sets. Our approach considers several partially ordered sets of membership functions defined on the unit interval, each characterized by different structural properties and suited to representing particular kinds of uncertainty. For these settings, we define operators that consistently represent fuzzy intersection and union, filling gaps where no such type-2 operators were previously available. It is the first time that t-norms and t-conorms are obtained with respect to both usual partial orders in the set of normal membership functions. Building on this framework, we integrate the proposed operators into a type-2 fuzzy time series. A large-scale evaluation on multiple benchmark time series shows that the new operators consistently achieve the best predictive accuracy, in terms of MAPE, in comparison with classical type-2 and type-1 fuzzy time series baselines. These results demonstrate that the algebraic design of type-2 operators has a direct impact on forecasting performance and can substantially improve the modeling of uncertainty in time-dependent data.

Keywords: normal function; convex function; type-2 fuzzy set; interval type-2 fuzzy set; T-norm; T-conorm; time series

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1. Introduction

In recent years, it has become increasingly clear that different forms of fuzzy sets (FSs) are required to describe uncertainty with precision [1]. Among them, type-2 fuzzy sets (T2FSs) constitute a particularly relevant extension, whose use has expanded notably since their original introduction in [2,3]. In this regard, the value of the membership function of the T2FSs is itself a fuzzy set, enabling a more nuanced representation of uncertainty. That is, the degree, in which an element belongs to the set, is merely a label of the linguistic variable “TRUTH” (see [4–6]). A large body of work shows that T2FSs frequently

outperform both type-1 and interval-valued fuzzy sets, as they offer a richer framework for representing the uncertainty associated with linguistic terms. A comprehensive overview of this development can be found in [7].

Beyond these theoretical developments, fuzzy-logic-based models remain relevant in applied domains involving uncertainty, imprecision and complex decision processes. Recent examples include fuzzy semantic methods for object detection and tracking in sports competitions [8] and fuzzy-neural control strategies in electrical engineering automation [9]. These examples further motivate the development of robust algebraic tools for fuzzy reasoning.

The rapid evolution of this area has also led to multiple parallel definitions and notational conventions, a difficulty highlighted in [10]. To address this fragmentation, substantial effort over the last two decades has focused on providing T2FSs with a consistent algebraic foundation [5,11]. Within this unified viewpoint, several new operators have recently appeared, taking advantage of the underlying structure [6,12–14].

In the different practical applications, the characteristics of the membership degrees of T2FSs vary. In general, they are functions contained within the set M of all functions $f : [0, 1] \rightarrow [0, 1]$. However, it should be noted that many applications only consider the set C of convex functions. In fact, these functions are generally considered to be normal. In this particular case, the set of convex and normal functions is denoted by L . The occasions where normality is not necessary are studied in the recent works, for example, [14–16]. Furthermore, the set N where the functions are normal but not necessarily convex has attracted considerable interest. Indeed, the degrees of membership of the so-called interval type-2 fuzzy sets (IT2FSs) belong to the set K of functions mapping the unit interval into the set $\{0, 1\}$ (excluding the function $0(y) = 0$ for all $y \in [0, 1]$). These functions are normal but need not be convex. It is well established that IT2FSs coincide with set-valued fuzzy sets (SVFSs) and hesitant fuzzy sets (HFSs) (see Figure 1 and Section VII in [1]), which have been successfully applied to decision-making problems [17,18]. In certain contexts, it even suffices to consider the restricted class $K_c^F \subset K$, where the support consists of a finite union of closed intervals [19,20].

Given this diversity of functional settings, defining appropriate operators for each context becomes essential, particularly when modeling intersection and union within type-2 fuzzy reasoning systems [21]. Motivated by this need, and building on previous contributions such as [6,14,22–25], the present work develops new t-norms (triangular norms) and t-conorms (triangular conorms) for several subclasses of M where no suitable operators were previously available. These constructions provide new tools for modeling the logical connectors AND and OR under different structural assumptions on type-2 membership functions. Moreover, it should be noted that this is the first time that t-norms and t-conorms in N , K and K_c^F for both usual orders are determined.

This paper is organized as follows. Section 2 outlines the algebraic foundations of type-2 fuzzy sets, detailing the various subsets in which their membership functions may lie and the corresponding operations that can be performed on them. Section 3 introduces the proposed families of operators, which behave as t-norms and t-conorms on several relevant subsets of M , such as C , N , L , K and K_c^F (see Table 1). We also verify that these operators satisfy the axioms required for t-norms and t-conorms in each setting. In Section 4, we introduce a type-2 fuzzy time series forecasting method that incorporates the operators proposed in this work. We illustrate its practical applicability by evaluating its performance on several benchmark time series. Finally, Section 5 reviews the principal results of this work and discusses potential directions for further investigation.

Table 1. Some families of the set M of the functions from $[0, 1]$ to $[0, 1]$.

C	Convex functions in M
N	Normal functions in M
L	Convex and normal functions in M
K	Functions in N whose images are 0 or 1 (but not all 0)
K_c^F	Functions in K whose support is a finite union of closed sub-intervals in $[0, 1]$

2. Preliminaries

In this section, we introduce the definitions of type-2 fuzzy sets and interval type-2 fuzzy sets, together with several key properties and operations associated with them. Throughout the section, X denotes a non-empty universe of discourse, $\leq$ stands for the usual ordering on the real numbers, and $\wedge$ ($\vee$) represents the minimum (maximum) of two real values.

Definition 1 ([4]). A type-2 fuzzy set (T2FS) $\mathcal{A}$ is characterized by a membership function: $\mu_{\mathcal{A}} : X \rightarrow M$ where

$$M = [0, 1]^{[0, 1]} = \text{Map}([0, 1], [0, 1])$$

That is, $\mu_{\mathcal{A}}(x) : [0, 1] \rightarrow [0, 1]$ is a fuzzy set on the interval $[0, 1]$ and also the degree of membership of an element $x \in X$ to the set $\mathcal{A}$.

We now describe a number of relevant subsets of M that play a central role in the developments of this paper.

Definition 2. We say that a function $f \in M$ is normal if $\sup\{f(x) \mid x \in [0, 1]\} = 1$, and it is convex if, for any $x \leq y \leq z$, the inequality $f(y) \geq f(x) \wedge f(z)$ holds.

N denotes the set of all normal functions in M , C the set of convex functions and L the set of convex and normal functions.

From now on, the notation $|a, b|$, where $0 \leq a \leq b \leq 1$, will refer to any non-empty interval (closed, open or half-open interval) in $[0, 1]$, and its characteristic function $\overline{|a, b|} : [0, 1] \rightarrow [0, 1]$ is defined as:

$$\overline{|a, b|}(x) = \begin{cases} 1 & \text{if } x \in |a, b|, \\ 0 & \text{if } x \notin |a, b|. \end{cases}$$

Let us note that the characteristic function of any interval in $[0, 1]$ is an element of L . Moreover, we generally use the symbol $|$ when we want to indicate that an endpoint of the interval may or may not be included. However, to indicate that an endpoint is included, we use the symbol $[$ or $]$, as appropriate. Similarly, if an endpoint is not included, we use the symbol $($ or $)$, as appropriate. For example, in the interval $|0, 1|$, 1 is included, but 0 may or may not be included.

The first rigorous definition of IT2FSs was introduced in [26]. However, this notion has sometimes been mistakenly interpreted as equivalent to interval-valued fuzzy sets. The distinction between the two was clarified by Bustince et al. in [27]. To prevent this misconception, a more precise formulation was later provided in [6], stated as follows:

Definition 3 ([6]). A type-2 fuzzy set is said to be an interval type-2 fuzzy set (IT2FS) if, for all $x \in X$, $\mu_{\mathcal{A}}(x) \in K = \text{Map}([0, 1], \{0, 1\}) \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is a constant function such that $\mathbf{0}(y) = 0$ for all $y \in [0, 1]$. That is, $\mu_{\mathcal{A}}(x)(y) \in \{0, 1\}$ for all $y \in [0, 1]$ and $\mu_{\mathcal{A}}(x) \neq \mathbf{0}$.

The support of any $f \in K$, denoted $Supp(f)$, may be any non-empty subset of $[0, 1]$; thus, it is not required to be convex. In fact, the support of a function in K can be expressed as a finite or infinite union of intervals, which may be closed, open, or half-open. We also consider the subset K_c^F , consisting of those functions in K whose support is a finite union of closed intervals. Clearly, the inclusions $K_c^F \subset K \subset N \subset M$ hold.

Zadeh's Extension Principle [2,3] provides a general mechanism for defining the join ($\sqcup$) and meet ($\sqcap$) of functions in M . Using this principle, these operations are given by:

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) \mid y \vee z = x\},$$

$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) \mid y \wedge z = x\}.$$

For any two T2FSs, their union and intersection are obtained pointwise. Concretely, for every $x \in X$: $\mu_{A \sqcup B}(x) = \mu_A(x) \sqcup \mu_B(x)$ and $\mu_{A \sqcap B}(x) = \mu_A(x) \sqcap \mu_B(x)$. At this point, we introduce the following two singleton functions, which will appear frequently:

$$\bar{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases} \quad \text{and} \quad \bar{1}(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases}$$

Some properties of the structure $(M, \sqcup, \sqcap, \bar{0}, \bar{1})$ are worth highlighting. As shown in [5], both $\sqcup$ and $\sqcap$ are idempotent; that is, for any $f \in M$ one has $f \sqcap f = f$ and $f \sqcup f = f$. Nevertheless, the absorption law does not hold, meaning that this structure does not form a lattice (see [28]). Even so, these operations allow us to define two natural partial orders on M .

Proposition 1 ([4,5]). *The relations $\sqsubseteq$ and $\preceq$ in M given by:*

$$f \sqsubseteq g \text{ if and only if } f \sqcap g = f \text{ and}$$

$$f \preceq g \text{ if and only if } f \sqcup g = g,$$

are partial orders. Furthermore, they are different in general.

These orders can be naturally restricted to the subsets C , N , L , K and K_c^F , giving rise to different partially ordered structures. Although each of these posets is bounded, their greatest and least elements do not necessarily coincide across all cases. Table 2 summarizes these bounds, using the notation $P^{\leq P}$ to represent each ordered pair $(P, \leq_P)$.

Table 2. Maximum and minimum in each poset of M (see [4–6,25]).

Poset	Min	Max	Poset	Min	Max
$M^{\sqsubseteq}$	$\mathbf{0}$	$\bar{1}$	$K^{\sqsubseteq}$ or $K^{\preceq}$	$\bar{0}$	$\bar{1}$
$M^{\preceq}$	$\bar{0}$	$\mathbf{0}$	$K_c^{F\sqsubseteq}$ or $K_c^{F\preceq}$	$\bar{0}$	$\bar{1}$
$N^{\sqsubseteq}$ or $N^{\preceq}$	$\bar{0}$	$\bar{1}$	$C^{\sqsubseteq}$	$\mathbf{0}$	$\bar{1}$
$L^{\sqsubseteq} \equiv L^{\preceq}$	$\bar{0}$	$\bar{1}$	$C^{\preceq}$	$\bar{0}$	$\mathbf{0}$

Computing the proposed operations is difficult in some situations and this is the reason why it is necessary to introduce the operators $f^L, f^R \in \mathbf{M}$ given by:

$$f^L(x) = \sup\{f(y) \mid y \leq x\} \text{ and } f^R(x) = \sup\{f(y) \mid y \geq x\}.$$

Note that f^L and f^R are, respectively, the non-decreasing and the non-increasing upper envelope of the function f . These functions enjoy a variety of useful properties, well documented in works such as [5,25,28,29], that make them particularly suitable for our analysis. Some key facts to be used later are:

$$f^L \text{ is non-decreasing and } f^R \text{ is non-increasing.} \quad (1)$$

$$f \leq f^L. \quad (2)$$

$$(f^L)^L = f^L \text{ and } (f^R)^R = f^R. \quad (3)$$

$$f \in \mathbf{C} \text{ if and only if } f = f^L \wedge f^R. \quad (4)$$

$$\text{If } f \in \mathbf{N} \text{ then } (f^L \wedge f^R) \in \mathbf{L}. \quad (5)$$

$$(f^L \wedge f^R)^R = f^R \text{ and } (f^L \wedge f^R)^L = f^L. \quad (6)$$

Note that the operations $\vee$ and $\wedge$ for functions in $\mathbf{M}$ are defined as $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in [0, 1]$, and the order $\leq$ as $f \leq g$ if and only if $f(x) \leq g(x)$, for all $x \in [0, 1]$.

A major advantage of these auxiliary functions is that they provide a simple way to characterize the partial orders $\sqsubseteq$ and $\preceq$.

Theorem 1 ([5]). *For any two functions $f, g \in \mathbf{M}$ we have:*

$$f \sqsubseteq g \text{ if and only if } (f^R \wedge g) \leq f \leq g^R \text{ and } f \preceq g \text{ if and only if } (g^L \wedge f) \leq g \leq f^L.$$

To close this section, we recall a useful characterization of these orders on the set $\mathbf{L}$ of convex and normal functions. In this setting, the two partial orders $\sqsubseteq$ and $\preceq$ are known to be identical.

Theorem 2 ([28,30]). *For any $f, g \in \mathbf{L}$, $f \sqsubseteq g$ (eq. $f \preceq g$) if and only if $g^L \leq f^L$ and $f^R \leq g^R$.*

When extending the analysis to all normal functions, only one implication remains valid.

Theorem 3 ([6]). *For any $f, g \in \mathbf{N}$ such that $f \sqsubseteq g$ or $f \preceq g$, we have $f^R \leq g^R$ and $g^L \leq f^L$.*

3. T-Norms and T-Conorms in the Different Subfamilies of $\mathbf{M}$

In this section, we will introduce new families of t-norms and t-conorms in the different posets of $\mathbf{M}$ discussed. We first need to recall some definitions and results about t-norms and t-conorms which will be used throughout the text.

Definition 4 ([31,32]). *Let $(P, \leq_P, 0_P, 1_P)$ be a bounded poset. The binary operation $T : P^2 \rightarrow P$ is a t-norm in P if:*

$$T1. \quad T(a, b) = T(b, a), \text{ for all } a, b \in P.$$

$$T2. \quad T(a, T(b, c)) = T(T(a, b), c), \text{ for all } a, b, c \in P.$$

$$T3. \quad T(a, 1_P) = a, \text{ for all } a \in P.$$

T4. Let $a, b, c \in P$ such that $b \leq_P c$, then $T(a, b) \leq_P T(a, c)$.

Definition 5 ([32]). Let $(P, \leq_P, 0_P, 1_P)$ be a bounded poset. The binary operation $S : P^2 \rightarrow P$ is a t -conorm in P if:

- S1. $S(a, b) = S(b, a)$, for all $a, b \in P$.
- S2. $S(a, S(b, c)) = S(S(a, b), c)$, for all $a, b, c \in P$.
- S3. $S(a, 0_P) = a$, for all $a \in P$.
- S4. Let $a, b, c \in P$ such that $b \leq_P c$, then $S(a, b) \leq_P S(a, c)$.

In the literature, there exist different examples of t -norms and t -conorms in some of the posets that we are studying. In [5], it was proven that $\sqcap$ and $\sqcup$ are t -norm and t -conorm, respectively, in L . Something similar occurs in C where $\sqcap$ ($\sqcup$) is a t -norm (t -conorm) with respect to $\sqsubseteq$ ($\sqsupseteq$), as it is established in [5,24,25]. Furthermore, in [25], we proved that $\sqcap$ ($\sqcup$) is a t -norm (t -conorm) in M , N , K and K_c^F with respect to the partial order $\sqsubseteq$ ($\sqsupseteq$). However, there are more operators that can act as t -norms and t -conorms. For example, in [23,24], the two following families of binary operations in M were proposed. These operations are extensions of those given in [5,33].

Definition 6 ([23,24]). Let $\star$ and Δ be continuous t -norms in $[0, 1]$, and ∇ a continuous t -conorm in $[0, 1]$. For each $f, g \in M$, we define the binary operations $\blacktriangle$ and $\blacktriangledown$ as:

$$\begin{aligned}(f \blacktriangle g)(x) &= \sup\{f(y) \star g(z) \mid y \Delta z = x\}, \\ (f \blacktriangledown g)(x) &= \sup\{f(y) \star g(z) \mid y \nabla z = x\}.\end{aligned}$$

Note that, taking $\star = \Delta = \wedge$ and $\nabla = \vee$, we have $\blacktriangle = \sqcap$ and $\blacktriangledown = \sqcup$. In [24], it was stated that $\blacktriangle$ ($\blacktriangledown$) is a t -norm (t -conorm) in L given the order $\sqsubseteq$ (in this case $\sqsubseteq \equiv \preceq$). Furthermore the fact that, taking $\star = \wedge$, $\blacktriangle$ ($\blacktriangledown$) is t -norm (t -conorm) in C with respect to the partial order $\sqsubseteq$ ($\sqsupseteq$) was proven in [5,24,25]. Additionally, some examples of functions for $\star$, Δ and ∇ , where $\blacktriangle$ and $\blacktriangledown$ are neither t -norm nor t -conorm in M , C , N , K and K_c^F with respect $\sqsubseteq$ or $\preceq$, were also provided. For a more concise summary of these previous results, see Table 3. From now on, we will focus on obtaining new families of t -norms and t -conorms in some of the studied posets.

As far as we know, no operators acting as t -norms (t -conorms) with respect to both partial orders $\sqsubseteq$ and $\preceq$ simultaneously have been found in any of the studied subsets of M excluding L . The objective of Section 3.2 is to propose such operators in N , K and K_c^F . In order to do that, it is necessary to recall some useful properties of $\blacktriangle$ and $\blacktriangledown$, whose proof is given in [24]. For any $f, g, h \in M$, we have the following:

$\blacktriangle$ and $\blacktriangledown$ are commutative and associative in M . (7)

$f \blacktriangle \bar{1} = f$, $f \blacktriangledown \bar{0} = f$, $f \blacktriangle \bar{0} = \bar{0}$ and $f \blacktriangledown \bar{1} = \bar{1}$. (8)

$f \blacktriangle \mathbf{1} = f^R$, $f \blacktriangledown \mathbf{1} = f^L$ where $\mathbf{1} = [\bar{0}, \bar{1}]$. (9)

$f^R \blacktriangle g^R = f \blacktriangle g^R = f^R \blacktriangle g = (f \blacktriangle g)^R$. (10)

$f^L \blacktriangledown g^L = f \blacktriangledown g^L = f^L \blacktriangledown g = (f \blacktriangledown g)^L$. (11)

$f^L \blacktriangle g^L = (f \blacktriangle g)^L$ and $f^R \blacktriangledown g^R = (f \blacktriangledown g)^R$. (12)

For all $f, g, h \in M$, such that $g \leq h : (f \blacktriangle g) \leq (f \blacktriangle h)$ and $(f \blacktriangledown g) \leq (f \blacktriangledown h)$. (13)

$f \blacktriangle \bar{0} = \bar{0}$, $f \blacktriangledown \bar{1} = \bar{1}$, for all $f \in N$. (14)

For all $a, b, c, d \in [0, 1]$ with $a \leq b$ and $c \leq d$:

$$[a, b] \blacktriangle [c, d] = [a \triangle c, b \triangle d] \text{ and } [a, b] \blacktriangledown [c, d] = [a \nabla c, b \nabla d].$$

If $\overline{a, b}, \overline{c, d} \neq \mathbf{0}$, then: $\overline{a, b} \blacktriangle \overline{c, d}, \overline{a, b} \blacktriangledown \overline{c, d} \in K$.

$\blacktriangle$ and $\blacktriangledown$ are closed on M, C, N and L .

$\blacktriangle$ ($\blacktriangledown$) is a t-norm (t-conorm) in the lattice $(L, \sqsubseteq)$.

These properties will be instrumental in showing that the operators proposed in the following are t-norms or t-conorms.

Table 3. Posets where the discussed operators are t-norms or t-conorms.

	T-Norms						T-Conorms			
	$\sqcap$	$\blacktriangle$ with $\star = \wedge$	$\blacktriangle$	$\odot$	$\sqcap$	$\sqcup$	$\blacktriangledown$ with $\star = \wedge$	$\blacktriangledown$	$\oplus$	$\sqcup$
$L^{\sqsubseteq}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$N^{\sqsubseteq}$	✓	✗	✗	✓	✓	✗	✗	✗	✗	✓
$N^{\preceq}$	✗	✗	✗	✗	✓	✓	✗	✗	✓	✓
$K^{\sqsubseteq}$	✓	✗	✗	✓	✓	✗	✗	✗	✗	✓
$K^{\preceq}$	✗	✗	✗	✗	✓	✓	✗	✗	✓	✓
$K_c^{F^{\sqsubseteq}}$	✓	✗	✗	✓	✓	✗	✗	✗	✗	✓
$K_c^{F^{\preceq}}$	✗	✗	✗	✗	✓	✓	✗	✗	✓	✓
$C^{\sqsubseteq}$	✓	✓	✗	✓	✗	✗	✗	✗	✗	✗
$C^{\preceq}$	✗	✗	✗	✗	✗	✓	✓	✗	✓	✗
$M^{\sqsubseteq}$	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗
$M^{\preceq}$	✗	✗	✗	✗	✗	✓	✗	✗	✓	✗

3.1. The Operations $\odot$ and $\oplus$

This subsection is devoted to introducing and analyzing two new operations ($\odot$ and $\oplus$), elucidating the posets where they are t-norms or t-conorms, respectively, in order to use them to model conjunction and disjunction in applied scenarios. They will be studied in M, C, N, L, K and K_c^F .

Definition 7. For each $f, g \in M$, we define the binary operations $\odot$ and $\oplus$ as:

$$f \odot g = \begin{cases} f & \text{if } g = \bar{1}, \\ g & \text{if } f = \bar{1}, \\ f^R \blacktriangle g^R & \text{otherwise,} \end{cases} \quad \text{and} \quad f \oplus g = \begin{cases} f & \text{if } g = \bar{0}, \\ g & \text{if } f = \bar{0}, \\ f^L \blacktriangledown g^L & \text{otherwise.} \end{cases}$$

The particular cases where the operators $\blacktriangle$ and $\blacktriangledown$ in M are replaced by the operator $\wedge$ in $[0, 1]$ were studied in [34]. In that case, $\odot$ ($\oplus$) is a t-norm (t-conorm) in L and M , with respect to $\sqsubseteq$ ($\preceq$).

Clearly, none of the operations in this definition is a particular case of $\blacktriangle$ or $\blacktriangledown$, as we can check in the following example.

Example 1. For each continuous t -norm (t -conorm) Δ (∇) in $[0, 1]$, there exist $a, b \in (0, 1)$ such that $a \Delta a = c$, $b \nabla b = d$, $b \Delta b = e$, $a \nabla a = j$ with $c, d, e, j \in (0, 1)$. In these cases, the computation of the identities $\bar{a} \odot \bar{a} = \overline{[0, c]}$ and $\bar{b} \oplus \bar{b} = \overline{[d, 1]}$ is easy using (15). Nevertheless:

$$\bar{a} \blacktriangle \bar{a} = \bar{c}, \quad \bar{b} \blacktriangle \bar{b} = \bar{e}, \quad \bar{a} \blacktriangledown \bar{a} = \bar{j} \quad \text{and} \quad \bar{b} \blacktriangledown \bar{b} = \bar{d}.$$

From now on, our main purpose is to prove that $\odot$ is a t -norm with respect to the partial order $\sqsubseteq$ and $\oplus$ is a t -conorm with respect to $\preceq$. In the following proposition, it is established that $\odot$ satisfies the axioms $T1$, $T2$ and $T3$ of t -norm, and that $\oplus$ satisfies the axioms $S1$, $S2$ and $S3$ of t -conorm in M .

Proposition 2. The operations $\odot$ and $\oplus$ are commutative and associative in M . Moreover, $f \odot \bar{1} = f$ and $f \oplus \bar{0} = f$ for all $f \in M$.

Proof. The operations $\odot$ and $\oplus$ are commutative and associative because $\blacktriangle$ and $\blacktriangledown$ are commutative and associative by (7). In addition, $f \odot \bar{1} = f$ and $f \oplus \bar{0} = f$ by definition. $\square$

Remark 1. The boundary conditions of $\odot$ and $\oplus$ guarantee the fulfillment of the axioms $T3$ and $S3$. In fact, if they were defined as $f \odot g = f^R \blacktriangle g^R$ and $f \oplus g = f^L \blacktriangledown g^L$ for all $f, g \in M$, then the axioms would not necessarily hold. For example:

$$\begin{aligned} \bar{1} \odot \bar{1} &= \bar{1}^R \blacktriangle \bar{1}^R = \overline{[0, 1]} \blacktriangle \overline{[0, 1]} = \overline{[0, 1]} \neq \bar{1}, \\ \bar{0} \oplus \bar{0} &= \bar{0}^L \blacktriangledown \bar{0}^L = \overline{[0, 1]} \blacktriangledown \overline{[0, 1]} = \overline{[0, 1]} \neq \bar{0}. \end{aligned}$$

It is key for these operators to be closed in C , N , L , K and K_c^F in order to be t -norms and t -conorms. The next two propositions prove this fact.

Proposition 3. $\odot$ and $\oplus$ are closed in C , N and L . Moreover, if $f, g \in N$, then $(f^R \blacktriangle g^R), (f^L \blacktriangledown g^L) \in L = N \cap C$.

Proof. For all $f, g \in C$, we have $f \odot \bar{1} = f \in C$, $\bar{1} \odot g = g \in C$, $f \oplus \bar{0} = f \in C$ and $\bar{0} \oplus g = g \in C$. Consequently, suppose that $f \odot g = f^R \blacktriangle g^R$ and $f \oplus g = f^L \blacktriangledown g^L$. The functions f^R and g^R are non-increasing, and f^L and g^L are non-decreasing (see Equation (1)) and therefore they are all convex. Since the operations $\blacktriangle$ and $\blacktriangledown$ are closed in C by Equation (17), $f \odot g$ and $f \oplus g \in C$. Therefore, $\odot$ and $\oplus$ are closed in C .

If $f, g \in N$, then $f \odot \bar{1} = f \in N$, $\bar{1} \odot g = g \in N$, $f \oplus \bar{0} = f \in N$ and $\bar{0} \oplus g = g \in N$. Otherwise, we have $f \odot g = f^R \blacktriangle g^R$ and $f \oplus g = f^L \blacktriangledown g^L$. Since $f, g \in N$, the identities $f^R(0) = f^L(1) = g^R(0) = g^L(1) = 1$ hold. Thus, $f^R, f^L, g^R, g^L \in N$. As in the previous case, $\blacktriangle$ and $\blacktriangledown$ are closed in N , so it is clear that $f \odot g$ and $f \oplus g \in N$. As a consequence, $\odot$ and $\oplus$ are closed in N . Furthermore, taking into account all the previous discussion, we find that $\odot$ and $\oplus$ are closed in L .

In addition, we have proven that for all $f, g \in N$, the functions f^R, f^L, g^R and g^L belong to L . With this and recalling that $\blacktriangle$ and $\blacktriangledown$ are closed in L , we can state that $f^R \blacktriangle g^R, f^L \blacktriangledown g^L \in L$. $\square$

Proposition 4. $\odot$ and $\oplus$ are closed in K and K_c^F .

Proof. Let us take $v_i = \inf\{\text{Supp}(f)\}$, $w_i = \inf\{\text{Supp}(g)\}$, $v_s = \sup\{\text{Supp}(f)\}$ and $w_s = \sup\{\text{Supp}(g)\}$. If $f, g \in K_c^F$, it is easy to check that:

$$f^L = \overline{[v_i, 1]}, \quad f^R = \overline{[0, v_s]}, \quad g^L = \overline{[w_i, 1]} \quad \text{and} \quad g^R = \overline{[0, w_s]}.$$

Let us first deduce that $\odot$ is closed in K_c^F . We have that $f \odot \bar{1} = f \in K_c^F$ and $\bar{1} \odot g = g \in K_c^F$. Therefore, we need to consider only the case where $f \odot g = f^R \blacktriangle g^R = \overline{[0, v_s]} \blacktriangle \overline{[0, w_s]}$. According to Equation (15):

$$f \odot g = \overline{[0, v_s]} \blacktriangle \overline{[0, w_s]} = \overline{[0 \triangle 0, v_s \triangle w_s]} = \overline{[0, v_s \triangle w_s]} \in K_c^F.$$

Similarly, we can prove that $\oplus$ is closed in K_c^F .

Now, let us consider $f, g \in K$. In this situation:

$$f^L = \overline{[v_i, 1]}, \quad f^R = \overline{[0, v_s]}, \quad g^L = \overline{[w_i, 1]} \quad \text{and} \quad g^R = \overline{[0, w_s]}.$$

Note that all the previous functions are different from $\mathbf{0}$. By definition of $\odot$ and $\oplus$, we can assume that $f \odot \bar{1} = f \in K$ and $\bar{1} \odot g = g \in K$. Therefore, the only other possibility is that $f \odot g = f^R \blacktriangle g^R = \overline{[0, v_s]} \blacktriangle \overline{[0, w_s]} \in K$. This last statement is a consequence of Equation (16), since $\overline{[0, v_s]}, \overline{[0, w_s]} \neq \mathbf{0}$. We can show that $\oplus$ is closed in K in a similar way. $\square$

The operations that we are considering have different absorbent elements when we change the sets over which we define them.

Proposition 5. *The following statements hold:*

- (1) $\bar{0}$ is the absorbent element of the operation $\odot$ in N, L, K and K_c^F .
- (2) $\bar{1}$ is the absorbent element of $\oplus$ in N, L, K and K_c^F .
- (3) $\mathbf{0}$ is the absorbent element of $\odot$ and $\oplus$ in M and C .

Proof. We will first prove (1) and (2) together. From Equation (14), we know that $\bar{0}$ and $\bar{1}$ are the absorbent elements in N of $\blacktriangle$ and $\blacktriangledown$, respectively. Therefore, $f \odot \bar{0} = f^R \blacktriangle \bar{0}^R = f^R \blacktriangle \bar{0} = \bar{0}$ for all $f \in N$ with $f \neq \bar{1}$, and $f \oplus \bar{1} = f^L \blacktriangledown \bar{1}^L = f^L \blacktriangledown \bar{1} = \bar{1}$ for all $f \in N$ with $f \neq \bar{0}$. The remaining cases are direct since, by Definition 7, $\bar{1} \odot \bar{0} = \bar{0} \odot \bar{1} = \bar{0}$ and $\bar{1} \oplus \bar{0} = \bar{0} \oplus \bar{1} = \bar{1}$. Thus, $\bar{0}$ is the absorbent element of $\odot$, and $\bar{1}$ is the absorbent element of $\oplus$ on N . Furthermore, as $\bar{0}$ and $\bar{1}$ are contained in L, K and K_c^F and these are subsets of N , the absorbent elements are inherited.

Finally, to prove (3), we will take $f \in M$, such that $f \neq \bar{1}$. In this case:

$$(\mathbf{0} \odot f)(x) = (\mathbf{0}^R \blacktriangle f^R)(x) = (\mathbf{0} \blacktriangle f^R)(x) = \sup\{\mathbf{0}(y) \star f^R(z) \mid y \triangle z = x\} = 0$$

for all $x \in [0, 1]$. That is, $(\mathbf{0} \odot f) = \mathbf{0}$. In addition, whenever $f = \bar{1}$, $\mathbf{0} \odot \bar{1} = \mathbf{0}$. The proof for $\oplus$ is similar. $\square$

The next auxiliary result will help us to prove that $\odot$ is non-decreasing in each component respect to $\sqsubseteq$ and that $\oplus$ is non-decreasing in each component respect to $\preceq$.

Proposition 6. *The following properties hold:*

- (1) For all $f, g, h \in M$ with $g \sqsubseteq h$, we have:

$$(f^R \blacktriangle g^R) \sqsubseteq (f^R \blacktriangle h^R).$$

(2) For all $f, g, h \in \mathbf{M}$ with $g \preceq h$, we have:

$$(f^L \nabla g^L) \preceq (f^L \nabla h^L).$$

Proof. (1) Let $f, g, h \in \mathbf{M}$, with $g \sqsubseteq h$. From Theorem 1, we have that $(g^R \wedge h) \leq g \leq h^R$ and hence, $g^R \leq h^R$ using Equation (3). Let us check that:

$$(f^R \blacktriangle g^R)^R \wedge (f^R \blacktriangle h^R) \leq (f^R \blacktriangle g^R) \leq (f^R \blacktriangle h^R) = (f^R \blacktriangle h^R)^R \quad (19)$$

where the last equality comes from Equations (10) and (3). It is clear that $(f^R \blacktriangle g^R) \wedge (f^R \blacktriangle h^R) \leq (f^R \blacktriangle g^R)$. Moreover, since $g^R \leq h^R$, we can make use of Equation (13) to state that $(f^R \blacktriangle g^R) \leq (f^R \blacktriangle h^R)$. Therefore, the expression given by Equation (19) holds. Thanks to this chain of inequalities and using again Theorem 1, $(f^R \blacktriangle g^R) \sqsubseteq (f^R \blacktriangle h^R)$, for all $f, g, h \in \mathbf{M}$, such that $g \sqsubseteq h$.

(2) The proof is analogous to the previous one using Equation (11) instead of Equation (10). $\square$

Finally, we can prove the fulfillment of the last requirement for $\odot$ to be t-norm and $\oplus$ to be t-conorm, the monotonicity in each argument.

Proposition 7. The following properties hold:

- (1) The operation $\odot$ is non-decreasing in each argument in $\mathbf{M}$ respect to the partial order $\sqsubseteq$.
- (2) The operation $\oplus$ is non-decreasing in each argument in $\mathbf{M}$ respect to the partial order $\preceq$.

Proof. (1) Let $f, g, h \in \mathbf{M}$, such that $g \sqsubseteq h$. We have to study four cases separately:

- (a) If all arguments are different from $\bar{1}$, $f \odot g = f^R \blacktriangle g^R$ and $f \odot h = f^R \blacktriangle h^R$ by definition. Thus, from Proposition 6, we have that $(f \odot g) \sqsubseteq (f \odot h)$.
- (b) If $g = \bar{1}$, then $h = \bar{1}$ since $\bar{1}$ is the maximum in $\mathbf{M}^\sqsubseteq$. Therefore, by definition, $f \odot g = f = f \odot h$.
- (c) If $f = \bar{1}$, since $g \sqsubseteq h$, we have $f \odot g = g \sqsubseteq h = f \odot h$.
- (d) Finally, if $f \neq \bar{1}$ and $\bar{1} \neq g \sqsubseteq h = \bar{1}$, it must be checked that $(f \odot g) \sqsubseteq (f \odot h)$. In this case, $f \odot g = f^R \blacktriangle g^R$ and $f \odot h = f \odot \bar{1} = f$. Consequently, it suffices to prove that:

$$(f^R \blacktriangle g^R) \sqsubseteq f.$$

Since $g \sqsubseteq \bar{1}$, then $(f^R \blacktriangle g^R) \sqsubseteq (f^R \blacktriangle \bar{1}^R)$ by Proposition 6. Let us check that $(f^R \blacktriangle \bar{1}^R) \sqsubseteq f$. Using Equations (9) and (3):

$$f^R \blacktriangle \bar{1}^R = f^R \blacktriangle [0, 1] = (f^R)^R = f^R.$$

Additionally, the inequality $((f^R)^R \wedge f) \leq f^R \leq f^R$ holds as a consequence of Equation (2), and thus $f^R \sqsubseteq f$ because of Theorem 1. In summary,

$$f \odot g = (f^R \blacktriangle g^R) \sqsubseteq (f^R \blacktriangle \bar{1}^R) = f^R \sqsubseteq f = f \odot h.$$

With this discussion, we have shown that $\odot$ is non-decreasing in each argument in $\mathbf{M}^\sqsubseteq$. Consequently, it is also non-decreasing in the posets $\mathbf{C}^\sqsubseteq$, $\mathbf{N}^\sqsubseteq$, $\mathbf{L}^\sqsubseteq$, $\mathbf{K}^\sqsubseteq$ and $\mathbf{K}_c^F$ as they are subsets of $\mathbf{M}$.

(2) To prove that $\oplus$ is non-decreasing in each argument in $\mathbf{M}^\preceq$, $\mathbf{C}^\preceq$, $\mathbf{N}^\preceq$, $\mathbf{L}^\preceq$, $\mathbf{K}^\preceq$ and $\mathbf{K}_c^F$, the procedure is similar. $\square$

Based on the results in this subsection, we can conclude that the new operators we presented are t-norms and t-conorms, respectively, with respect to the corresponding orders.

Corollary 1. *The two following statements hold:*

- (1) $\odot$ is a t -norm in $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{L}, \mathbf{K}$ and $\mathbf{K}_c^F$ with respect to the partial order $\sqsubseteq$. $\bar{1}$ is the neutral element, $\bar{0}$ is the absorbent element in $\mathbf{N}, \mathbf{L}, \mathbf{K}$ and $\mathbf{K}_c^F$, and $\mathbf{0}$ is the absorbent element in $\mathbf{M}$ and $\mathbf{C}$.
- (2) $\oplus$ is a t -conorm in $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{L}, \mathbf{K}$ and $\mathbf{K}_c^F$ with respect to the partial order $\preceq$. $\bar{0}$ is the neutral element; $\bar{1}$ is the absorbent element on $\mathbf{N}, \mathbf{L}, \mathbf{K}$ and $\mathbf{K}_c^F$; and $\mathbf{0}$ is the absorbent element in $\mathbf{M}$ and $\mathbf{C}$.

Remark 2. Note that $\odot$ and $\oplus$ are, respectively, families of t -norms and t -conorms in $\mathbf{L}$ that are different from those obtained in previous works.

The following example shows that the operations $\odot$ and $\oplus$ are not monotonically increasing with respect to the partial orders $\preceq$ and $\sqsubseteq$, respectively.

Example 2. Let us consider:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, 1\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g = \overline{0.3}.$$

If we set $\blacktriangle = \sqcap$, it is easy to check that $g \preceq \bar{1}$. Nevertheless:

$$f \odot g = f^R \blacktriangle g^R = \overline{[0, 1]} \sqcap \overline{[0, 0.3]} = \overline{[0, 0.3]} \not\sqsubseteq f = f \odot \bar{1}$$

(see Theorem 1). Therefore, $\odot$ is not monotonically increasing with respect to $\preceq$, and therefore it is not a t -norm respect to $\preceq$.

Now, let $\blacktriangle = \sqcup$. We have $\bar{0} \sqsubseteq g$. Using Theorem 1 again, we have:

$$f \oplus \bar{0} = f \not\sqsubseteq \overline{[0.3, 1]} = \overline{[0, 1]} \sqcup \overline{[0.3, 1]} = f^L \blacktriangledown g^L = f \oplus g.$$

It is clear then that $\oplus$ is not a t -conorm with respect to the order $\sqsubseteq$.

3.2. The Operations $\mathbb{M}$ and $\mathbb{U}$

In the previous subsection, given the families $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{K}$ and $\mathbf{K}_c^F$, we have only obtained t -norms with respect to $\sqsubseteq$ and t -conorms with respect to $\preceq$. It is also important to define some t -norms (t -conorms) on such sets, with respect to $\preceq$ ($\sqsubseteq$). In this subsection, we address this issue. We will find these kind of operators in $\mathbf{N}, \mathbf{L}, \mathbf{K}$ and $\mathbf{K}_c^F$. Nevertheless, they will not work for $\mathbf{M}$ or $\mathbf{C}$, which will become evident in the following (see Remark 3 and Example 4).

Definition 8. Let $f, g \in \mathbf{M} \setminus \{\mathbf{0}\}$ be two functions with $v_i = \inf\{\text{Supp}(f)\}$, $v_s = \sup\{\text{Supp}(f)\}$, $w_i = \inf\{\text{Supp}(g)\}$ and $w_s = \sup\{\text{Supp}(g)\}$. Let $\Delta, \bar{\wedge}$ be t -norms in $[0, 1]$, and $\nabla, \bar{\vee}$ t -conorms in $[0, 1]$, such that $(x \Delta y) \leq (x \bar{\wedge} y)$ and $(x \nabla y) \leq (x \bar{\vee} y)$ for all $x, y \in [0, 1]$. We define the binary operations $\mathbb{M}$ and $\mathbb{U}$ on $\mathbf{M}$ as follows:

$$f \mathbb{M} g = \begin{cases} f & \text{if } g = \bar{1}, \\ g & \text{if } f = \bar{1}, \\ \overline{[v_i \Delta v_s \Delta w_i \Delta w_s, v_i \bar{\wedge} v_s \bar{\wedge} w_i \bar{\wedge} w_s]} & \text{otherwise,} \end{cases}$$

$$f \mathbb{U} g = \begin{cases} f & \text{if } g = \bar{0}, \\ g & \text{if } f = \bar{0}, \\ \overline{[v_i \nabla v_s \nabla w_i \nabla w_s, v_i \bar{\vee} v_s \bar{\vee} w_i \bar{\vee} w_s]} & \text{otherwise.} \end{cases}$$

Below, we present an illustrative example in order to show how we operate with these new functions in a simple case.

Example 3. Let us consider $f = \overline{[0.1, 0.2]}$ and $g = \overline{[0.5, 0.7]}$. Moreover, let us set $\Delta = T_P$ (product t-norm), $\bar{\wedge} = \wedge$, $\bar{\vee} = \vee$, and $\bar{\vee} = S_D$ (drastic t-conorm). In this case:

$$\begin{aligned} f \bar{\cap} g &= \overline{[0.1 \cdot 0.2 \cdot 0.5 \cdot 0.7, 0.1 \wedge 0.2 \wedge 0.5 \wedge 0.7]} = \overline{[0.007, 0.1]} \\ f \bar{\cup} g &= \overline{[0.1 \vee 0.2 \vee 0.5 \vee 0.7, 0.1 \vee 0.2 \vee 0.5 \vee 0.7]} = \overline{[0.7, 1]}. \end{aligned}$$

Remark 3. Note that Definition 8 is given for $f, g \in \mathbf{M} \setminus \{0\}$ because $\text{Supp}(0) = \emptyset$, and thus the operations $\bar{\cap}$ and $\bar{\cup}$ are not defined when one of the arguments is 0 . Additionally, Table 2 establishes 0 as one of the bounds in all the posets $\mathbf{M}^{\bar{\cap}}, \mathbf{M}^{\bar{\cup}}, \mathbf{C}^{\bar{\cap}}$ and $\mathbf{C}^{\bar{\cup}}$. Therefore, using $\bar{\cap}$ or $\bar{\cup}$ as t-norms or t-conorms in the aforementioned posets is not possible since the bounds should be either a neutral or an absorbing element. For this reason, we exclude the study of these function in the sets $\mathbf{M}$ and $\mathbf{C}$.

The next example gives another justification for setting the sets $\mathbf{M}$ and $\mathbf{C}$ apart in the following discussion.

Example 4. Let us show that, generally, property T4 does not hold for $\bar{\cap}$ and $\bar{\cup}$ in $\mathbf{M}^{\bar{\cap}}, \mathbf{M}^{\bar{\cup}}, \mathbf{C}^{\bar{\cap}}$ or $\mathbf{C}^{\bar{\cup}}$. Let $f, g, s, \bar{0}, \bar{1} \in \mathbf{C}$ with:

$$f(x) = g(x) = \begin{cases} 0.5 & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad s(x) = \begin{cases} 0.5 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $g \subseteq \bar{1}$. Nevertheless, $f \bar{\cap} g = \bar{1} \not\subseteq f = f \bar{\cap} \bar{1}$. Moreover, by Theorem 1, $\bar{1} \preceq g$ but $f \bar{\cap} \bar{1} = f \not\subseteq \bar{1} = f \bar{\cap} g$. In addition, $\bar{0} \preceq \bar{1}$ and $f \bar{\cup} \bar{0} = f \not\subseteq \bar{1} = f \bar{\cup} \bar{1}$. Finally, $s \subseteq \bar{0}$, but $f \bar{\cup} s = \bar{1} \not\subseteq f = f \bar{\cup} \bar{0}$.

We can draw some other conclusions from the definition:

- $\bar{\cap}$ and $\bar{\cup}$ are closed in $\mathbf{M} \setminus \{0\}$, $\mathbf{C} \setminus \{0\}$, $\mathbf{N}$, $\mathbf{L}$, $\mathbf{K}$ and $\mathbf{K}_c^F$.
- $\bar{\cap}$ and $\bar{\cup}$ are commutative and associative because Δ , $\bar{\nabla}$, $\bar{\wedge}$ and $\bar{\vee}$ are commutative and associative.
- $\bar{1}$ and $\bar{0}$ are the neutral elements of $\bar{\cap}$ and $\bar{\cup}$, respectively.
- The boundary conditions of the operations $\bar{\cap}$ and $\bar{\cup}$ in Definition 8 ensure that $\bar{\cap}$ fulfills t-norm axiom T3, and $\bar{\cup}$ satisfies t-conorm axiom S3. Note that, if those boundary conditions were eliminated, these properties may not be true as the next example proves.

Example 5. Let us suppose that we omit in Definition 8 the special cases where $f, g \in \{\bar{0}, \bar{1}\}$. If $\Delta = \bar{\wedge} = T_P$ and $\bar{\nabla} = \bar{\vee} = S_P$ (probabilistic sum t-conorm), then:

$$\begin{aligned} \bar{0.5} \bar{\cap} \bar{1} &= \overline{[0.5 \cdot 0.5 \cdot 1 \cdot 1, 0.5 \cdot 0.5 \cdot 1 \cdot 1]} = \overline{0.25} \neq \bar{0.5}, \\ \bar{0.5} \bar{\cup} \bar{0} &= \overline{[0.5 \bar{\nabla} 0.5 \bar{\nabla} 0 \bar{\nabla} 0, 0.5 \bar{\vee} 0.5 \bar{\vee} 0 \bar{\vee} 0]} = \overline{0.75} \neq \bar{0.5}. \end{aligned}$$

Therefore, without the boundary conditions, $\bar{1}$ would not be the neutral element of $\bar{\cap}$, and $\bar{0}$ would not be the neutral element of $\bar{\cup}$.

It is important to assess with Example 6 that $\bar{\cap}$ and $\bar{\cup}$ are not obtained as particular cases of $\blacktriangle$, $\blacktriangledown$, $\odot$ or $\oplus$, and hence to make clear that they are new operators.

Example 6. It is easy to check that $\overline{[0, 1]} \bar{\cap} \overline{[0, 1]} = \bar{0}$ and $\overline{[0, 1]} \bar{\cup} \overline{[0, 1]} = \bar{1}$. Nevertheless:

$$\begin{aligned}\overline{[0,1]} \blacktriangle \overline{[0,1]} &= \overline{[0,1]}, & \overline{[0,1]} \blacktriangledown \overline{[0,1]} &= \overline{[0,1]}, \\ \overline{[0,1]} \odot \overline{[0,1]} &= \overline{[0,1]}^R \blacktriangle \overline{[0,1]}^R = \overline{[0,1]}, \\ \overline{[0,1]} \oplus \overline{[0,1]} &= \overline{[0,1]}^L \blacktriangledown \overline{[0,1]}^L = \overline{[0,1]}.\end{aligned}$$

Since $\overline{[0,1]} \in K_c^F, K, L, N, C, M$, the operations $\mathfrak{M}$ and $\mathfrak{U}$ are always different from $\blacktriangle, \blacktriangledown, \odot$ and $\oplus$.

Let us determine the absorbent elements of $\mathfrak{M}$ and $\mathfrak{U}$.

Proposition 8. $\bar{0}$ is the absorbent element of $\mathfrak{M}$, and $\bar{1}$ is the absorbent element of $\mathfrak{U}$ in $M \setminus \{0\}$.

Proof. Let us prove that $f \mathfrak{M} \bar{0} = \bar{0}$ for all $f \in M \setminus \{0\}$. Clearly, $\bar{1} \mathfrak{M} \bar{0} = \bar{0} \mathfrak{M} \bar{1} = \bar{0}$. Let us suppose that $f \neq \bar{1}$. Since 0 is the absorbent element of any of the t-norms Δ and $\bar{\wedge}$, then $f \mathfrak{M} \bar{0} = \overline{[v_i \Delta v_s \Delta 0 \Delta 0, v_i \bar{\wedge} v_s \bar{\wedge} 0 \bar{\wedge} 0]} = \bar{0}$.

The proof that $\bar{1}$ is the absorbent element of $\mathfrak{U}$ is analogous. $\square$

The last step to prove that $\mathfrak{M}$ is t-norm and $\mathfrak{U}$ is t-conorm in N , with respect to both partial orders, is to show that they are non-decreasing in each component.

Proposition 9. $\mathfrak{M}$ and $\mathfrak{U}$ are non-decreasing in each argument in N, L, K and K_c^F with respect to each partial order.

Proof. In the first place, let us study the monotonicity of $\mathfrak{M}$ respect to the partial order $\preceq$. Let $f, g, h \in N$, such that $g \preceq h$. By Theorem 3, we have that $g^R \leq h^R$ and $h^L \leq g^L$. Let us take $v_i = \inf\{Supp(f)\}$, $v_s = \sup\{Supp(f)\}$, $w_i = \inf\{Supp(g)\}$, $w_s = \sup\{Supp(g)\}$, $z_i = \inf\{Supp(h)\}$ and $z_s = \sup\{Supp(h)\}$. At this point, we need to distinguish four cases:

1. If all arguments are different from $\bar{1}$, we have that:

$$\begin{aligned}f \mathfrak{M} g &= \overline{[v_i \Delta v_s \Delta w_i \Delta w_s, v_i \bar{\wedge} v_s \bar{\wedge} w_i \bar{\wedge} w_s]} = \overline{[a, b]}, \\ f \mathfrak{M} h &= \overline{[v_i \Delta v_s \Delta z_i \Delta z_s, v_i \bar{\wedge} v_s \bar{\wedge} z_i \bar{\wedge} z_s]} = \overline{[c, d]}.\end{aligned}$$

with $a, b, c, d \in [0, 1]$. Since $h^L \leq g^L$ and $g^R \leq h^R$, then $w_i \leq z_i$ and $w_s \leq z_s$. Moreover, Δ and $\bar{\wedge}$ are non-decreasing in each argument, so we can conclude that:

$$\begin{aligned}a &= (v_i \Delta v_s \Delta w_i \Delta w_s) \leq (v_i \Delta v_s \Delta z_i \Delta z_s) = c, \\ b &= (v_i \bar{\wedge} v_s \bar{\wedge} w_i \bar{\wedge} w_s) \leq (v_i \bar{\wedge} v_s \bar{\wedge} z_i \bar{\wedge} z_s) = d.\end{aligned}$$

As a consequence, $\overline{[c, d]}^L \leq \overline{[a, b]}^L$ and $\overline{[a, b]}^R \leq \overline{[c, d]}^R$. Taking into account that $(f \mathfrak{M} g), (f \mathfrak{M} h) \in L$ (they are the characteristic functions of closed intervals) and by Theorem 2:

$$f \mathfrak{M} g = \overline{[a, b]} \preceq \overline{[c, d]} = f \mathfrak{M} h.$$

2. Let us now consider the case in which $g = \bar{1}$. Since $g \preceq h$, and according to Table 2, $h = \bar{1}$, we have $f \mathfrak{M} g = f = f \mathfrak{M} h$.
3. If $f = \bar{1}$, then $f \mathfrak{M} g = g \preceq h = f \mathfrak{M} h$.
4. The only remaining case is when $f \neq \bar{1}$ and $\bar{1} \neq g \preceq h = \bar{1}$. It is necessary to prove that $(f \mathfrak{M} g) \preceq (f \mathfrak{M} h)$. In this case:

$$f \mathfrak{M} g = \overline{[v_i \Delta v_s \Delta w_i \Delta w_s, v_i \bar{\wedge} v_s \bar{\wedge} w_i \bar{\wedge} w_s]} = \overline{[a, b]}$$

and $f \mathbin{\mathbb{M}} h = f \mathbin{\mathbb{M}} \bar{1} = f$. Therefore, it suffices to prove that $\overline{[a, b]} \preceq f$. By Theorem 1, this is the same as proving that:

$$(\overline{[a, b]} \wedge f^L) \leq f \leq \overline{[a, b]}^L. \quad (20)$$

Let us recall that if $\bar{\wedge}$ is a t-norm, then $(x \bar{\wedge} y) \leq x$ and $(x \bar{\wedge} y) \leq y$. Hence,

$$a \leq b = (v_i \bar{\wedge} v_s \bar{\wedge} w_i \bar{\wedge} w_s) \leq v_i \leq v_s$$

and it is clear that $f \leq \overline{[v_i, v_s]} \leq \overline{[a, 1]} = \overline{[a, b]}^L$. Moreover, $\overline{[a, b]}(x) = 0$ for all $x \in [0, 1] \setminus [a, b]$ and, for those same points,

$$(\overline{[a, b]}(x) \wedge f^L(x)) = (0 \wedge f^L(x)) = 0 \leq f(x).$$

In addition, if $x \in [a, b]$ and because $b \leq v_i$, then $f(x) = f^L(x)$. Therefore,

$$(\overline{[a, b]}(x) \wedge f(x)^L) = 1 \wedge f(x) = f(x).$$

As a consequence, the inequality in Equation (20) is proven, and the monotony is given in this case.

We can now state that $\mathbb{M}$ is non-decreasing in each argument in N and its subsets K_c^F , K and L , with respect to the partial order $\preceq$. The proof for the rest of the cases is similar. $\square$

Corollary 2. *The following statements regarding the operations $\mathbb{M}$ and $\mathbb{W}$ hold:*

- (1) $\mathbb{M}$ is a t-norm in N , L , K and K_c^F respect to both partial orders, $\preceq$ and $\sqsubseteq$. The neutral element is the function $\bar{1}$ and the absorbent element the function $\bar{0}$.
- (2) $\mathbb{W}$ is a t-conorm in N , L , K and K_c^F respect to both partial orders, $\preceq$ and $\sqsubseteq$. The neutral element is the function $\bar{0}$ and the absorbent element the function $\bar{1}$.

Remark 4. *Let us note that, as a consequence of the previous corollary, $\mathbb{M}$ and $\mathbb{W}$ are, respectively, new families of t-norms and t-conorms in L that are different from all those given in previous research.*

Table 3 notes all the general operators discussed in this section, indicating in which posets of M they are always t-norm or t-conorm. The table also includes the cases where there exists a counterexample.

4. Type-2 Fuzzy Operators for Time Series Forecasting

Time series forecasting is a key tool in many scientific and decision-making contexts. A time series is an ordered sequence of numerical observations $y_1, \dots, y_n \in \mathbb{R}$, where each y_t corresponds to time t . The goal is to understand the underlying dynamics of the phenomenon and to produce forecasts of future values from past observations.

Classical time series analysis relies on statistical models such as AR, MA, ARMA and ARIMA, as well as decomposition, exponential smoothing and long-term seasonal models [35]. However, these approaches often require strong assumptions (linearity, normality, stationarity) and a sufficiently large number of observations to be reliable, which may be restrictive in noisy, highly uncertain or data-poor settings [36].

Fuzzy Time Series (FTS) models, introduced by Song and Chissom [37] and popularized by Chen [38], provide an alternative framework. They transform the original numerical series into a sequence of fuzzy linguistic terms, allowing one to deal explicitly

with imprecision, capture nonlinear behavior and build interpretable, rule-based models [36].

In this work we combine the FTS structure with our type-2 fuzzy operators to construct forecasting models that can handle additional uncertainty in the membership functions.

4.1. Data Structure and Temporal Split

We use a broad collection of real series from the fma package [39]. For each series we require: (i) a minimum length to allow for a meaningful train–test split (at least 30 observations), (ii) strictly positive, finite observations, and (iii) no missing values.

Given a series $y_1, \dots, y_n$, we define the training set as $y_1, \dots, y_{n-h}$, and the test set of length h as $y_{n-h+1}, \dots, y_n$. We set h so that approximately 80% of the data are used for training and 20% for testing. Forecasts are generated in a rolling one-step-ahead fashion: at each time t , we use information up to y_t to predict y_{t+1} .

Table 4 summarizes the datasets used in the experiments, grouped by application domain. The classification is indicative, as some series may belong to multiple categories.

Table 4. Classification of the time series datasets used.

Category	Time Series
Economic/financial	advsales (advert), advsales (sales), capital, cpimel, dowjones, dj, wagesuk, ustreas
Sales/commercial	beer, books (Hardcover), books (Paperback), chicken, eggs, milk, shampoo
Industrial/production	bricksq, copper, plastics, nail, bicoal
Energy/resources	oilprice, petrol (Chemicals), petrol (Coal), petrol (Petrol), petrol (Vehicles), elec
Environmental/demographic	lynx, pollution, ukdeaths, usdeaths, cowtemp
Transport/tourism	airpass, motel (Roomnights), motel (Takings)
Other	internet, strikes, writing, sheep

4.2. Fuzzy Partition of the Universe of Discourse

From the training data we define an expanded universe of discourse

$$[\min(y_i) - r|\min(y_i)|, \max(y_i) + r|\max(y_i)|],$$

with a stretching factor $r = 0.1$ [40].

On this interval, we build k equally spaced triangular FSs

$$A_j = (a_j, b_j, c_j), \quad j = 1, \dots, k,$$

forming the type-1 partition used to learn the fuzzy rules. These triangles also act as the upper membership functions (UMFs) of the IT2FSs.

Since the performance of an FTS model is highly influenced by the number of linguistic labels, we first perform a preliminary search with Chen’s method [38] for $k \in \{7, 8, 9, 10, 11, 12\}$. The value $k = 8$ provides the best trade-off between overall accuracy and stability across the series, and is therefore used in all subsequent experiments.

4.3. Construction of IT2FSs

Each triangular set (a_j, b_j, c_j) is converted into an IT2FS. We use this triangle as the UMF, denoted as (a_j^U, b_j^U, c_j^U) . The lower membership function is obtained by a controlled contraction governed by a parameter $u \in (0, 1)$:

$$a_j^L = a_j^U + u(b_j^U - a_j^U), \quad c_j^L = c_j^U - u(c_j^U - b_j^U), \quad b_j^L = b_j^U.$$

Thus, each label is described by a pair of triangles $\mu_j^U(x)$ and $\mu_j^L(x)$ that bound the type-2 footprint of uncertainty:

$$\mu_j^U(x) = \mu_{\text{tri}}(x; a_j^U, b_j^U, c_j^U), \quad \mu_j^L(x) = \mu_{\text{tri}}(x; a_j^L, b_j^L, c_j^L),$$

where $\mu_{\text{tri}}(\cdot; a, b, c)$ denotes the standard triangular membership function with support $[a, c]$ and peak at b . Clearly, these IT2FSs are convex and normal, which ensures compatibility with the operators introduced in Section 3.

4.4. Learning Type-1 Fuzzy Rules

Rule learning is carried out solely on the type-1 partition, as is standard in FTS [36]. Let $\mu_i(y_t)$ denote the degree of membership of y_t to A_i . The strength of the transition from A_i to A_j is given by

$$R_{ij} = \max_t (\min(\mu_i(y_t), \mu_j(y_{t+1}))),$$

i.e., using the minimum t-norm on the temporal co-activation of labels.

For the IT2-based models, the resulting matrix R is normalized row-wise, yielding stable fuzzy transition relations. The same normalized relation matrix is used for all IT2 configurations, independently of the type-2 operators and the reduction method.

For comparison purposes, the Song–Chissom model [37] employs the same definition of R but without row normalization, while Chen’s method relies on fuzzy logical relationship groups rather than on a numerical relation matrix.

4.5. Type-2 Operators Under Comparison

The main contribution of this method lies in the use of alternative type-2 fuzzy operators at the inference stage of fuzzy time series models. More precisely, once the fuzzy rules have been learned and the interval-valued firing strengths are obtained, the aggregation and composition of interval information are carried out using different families of type-2 operators. In the type-2 inference stage, we evaluate three families of operators on intervals:

1. **Classic type-2 (interval Mamdani):** Extension of Mamdani inference to interval type-2 sets, using the minimum t-norm and maximum t-conorm on the lower and upper bounds of the intervals [41].
2. **Operators $\odot$ and $\oplus$:** Our proposed pair of type-2 t-norm/t-conorm operators, denoted as $\odot$ in the methods.
3. **Operators $\boxtimes$ and $\boxplus$:** The second family of operators introduced in Section 3. In the experiments we refer to this configuration simply as Cap.

As type-1 benchmarks, we include two classical fuzzy time series models. First, Chen’s method [38] is considered in two variants: the standard frequency-based version (*Chen-freq*) and a variant that only accounts for distinct labels in each fuzzy logical relationship (*Chen-unique*). Second, we include the original Song–Chissom model [37], based on max–min composition of fuzzy relations.

4.6. Type-2 Reduction: Nie–Tan and Karnik–Mendel

We consider the two most used type-2 reduction methods.

The Nie–Tan method [42] approximates the type-2 set by an equivalent type-1 membership function,

$$\mu_{NT}(x) = \frac{1}{2}(\mu_U(x) + \mu_L(x)),$$

and then applies standard defuzzification (centroid) to obtain a crisp output with low computational cost.

The Karnik–Mendel method [43] computes the extreme values y_L and y_R of the type-2 centroid via an iterative algorithm, and the final prediction is

$$\hat{y} = \frac{1}{2}(y_L + y_R),$$

preserving the interval structure.

4.7. Model Configurations

The experimental framework includes all combinations of structure, reduction and operator, as shown in Table 5.

Table 5. Types of structures, reductions and operators analyzed.

Structure	Reduction	Operator
IT2	NT, KM	Classic T2, odot, Cap
T1 (FTS)	–	Chen-freq, Chen-unique, Song

This yields families of IT2 models sharing the same linguistic partition and the same type-1 fuzzy relation matrix, differing only in the interval-valued operators used in the inference stage and in the type-reduction method.

Each combination of structure, operator and reduction method is evaluated on all series using the same protocol:

1. Generation of rolling one-step-ahead forecasts on the test set;
2. Computation of the mean absolute percentage error (MAPE), a widely used scale-free metric in forecasting studies [44];
3. Recording the computation time associated with each configuration.

4.8. Results

We first examine a representative series, *advsales* (*advert*), to illustrate the behavior of the different methods. The results are shown in Table 6.

Table 6. Results for series *advsales* (*advert*).

Method	RMSE	MAE	MAPE	Time (s)
IT2-NT-Cap	26.91	23.44	343.74	1.23
IT2-KM-odot	26.29	22.52	345.17	0.02
IT2-KM-Cap	27.13	23.72	345.36	0.02
IT2-KM-classic	27.28	23.94	353.53	0.00
IT2-NT-classic	28.09	25.28	437.40	0.92
FTS-CHEN-freq	28.30	25.53	438.87	0.00
FTS-CHEN-unique	28.30	25.53	438.87	0.02
FTS-SONG	28.30	25.53	438.87	0.02
IT2-NT-odot	30.69	29.04	714.33	0.64

The MAPE values in Table 6 are unusually large for all methods, and especially for the IT2-NT-odot configuration. This behavior is mainly due to the well-known sensitivity of MAPE to small observed values, since the absolute forecasting error is divided by the actual

value. Thus, moderate differences in mean absolute error (MAE) or root mean squared error (RMSE) may lead to very large percentage errors when some observations in the test set are close to zero. In this example, IT2-NT-odot has the largest MAPE, but its RMSE and MAE remain of the same order of magnitude as the remaining methods. Therefore, this case should be interpreted as an outlier effect of the evaluation metric rather than as a proportionally large absolute forecasting error.

For this particular series, the best MAPE is obtained by IT2-NT-Cap, slightly improving upon IT2-KM-odot and clearly outperforming both the classical IT2 baseline and the type-1 FTS models. Table 7 summarizes, for all series, the method with the smallest MAPE.

Table 7. Best method per series according to MAPE.

Series	Method	MAPE	Time (s)	Series	Method	MAPE	Time (s)
advsales (advert)	IT2-NT-Cap	343.74	1.11	advsales (sales)	IT2-KM-odot	10.95	0.02
airpass	IT2-KM-CLASSIC	9.60	0.06	beer	IT2-KM-odot	11.39	0.03
bicoal	IT2-NT-Cap	6.36	10.54	books (Hardcover)	IT2-NT-Cap	14.06	3.57
books (Paperback)	IT2-NT-odot	10.82	1.01	boston (bse)	IT2-KM-odot	23.66	0.01
boston (nyase)	IT2-KM-odot	10.47	0.01	bricksq	FTS-CHEN-freq	7.53	0.04
capital (appropriations)	IT2-NT-Cap	8.45	4.72	capital (capital)	IT2-KM-Cap	4.99	0.03
chicken	IT2-NT-CLASSIC	16.21	0.02	condmilk	FTS-CHEN-freq	14.53	0.06
copper	IT2-KM-Cap	17.99	0.12	copper3	FTS-CHEN-unique	8.44	0.01
cowtemp	IT2-KM-odot	13.94	0.03	cpimel	IT2-NT-Cap	2.13	3.45
dj	IT2-KM-Cap	0.87	0.22	dole	IT2-NT-Cap	5.63	37.89
dowjones	IT2-NT-Cap	0.40	0.62	eggs	IT2-KM-odot	10.26	0.03
elec	FTS-CHEN-freq	6.55	0.18	fancy	IT2-NT-odot	71.54	1.59
housing (construction)	FTS-CHEN-freq	7.71	0.04	housing (hstarts)	IT2-NT-Cap	9.09	15.48
housing (interest)	IT2-KM-Cap	1.98	0.03	hsales	FTS-CHEN-freq	9.09	0.09
hsales2	IT2-NT-CLASSIC	10.17	8.00	huron	IT2-NT-Cap	7.99	14.39
ibm (Sales)	IT2-NT-CLASSIC	6.51	0.15	ibmclose	IT2-KM-Cap	1.89	0.14
input	IT2-KM-CLASSIC	31.98	0.02	internet	FTS-CHEN-freq	4.20	0.11
jcars	IT2-NT-Cap	3.69	0.88	labour	IT2-KM-Cap	1.83	0.18
lynx	IT2-NT-Cap	95.53	11.43	milk	FTS-CHEN-unique	4.40	0.06
mink	IT2-KM-odot	21.51	0.03	motel (Roomnights)	FTS-CHEN-unique	9.87	0.05
motel (Takings)	IT2-KM-Cap	11.46	0.08	motion	IT2-KM-Cap	2.68	0.14
nail	FTS-CHEN-freq	14.21	0.09	oilprice	IT2-NT-Cap	14.04	25.32
petrol (Chemicals)	IT2-NT-CLASSIC	4.32	1.25	petrol (Coal)	FTS-CHEN-freq	9.31	0.13
petrol (Petrol)	IT2-KM-Cap	4.49	0.18	petrol (Vehicles)	IT2-KM-CLASSIC	16.09	0.09
pigs	FTS-SONG	7.77	0.13	plastics	IT2-KM-Cap	9.73	0.34
pollution	FTS-CHEN-unique	32.85	0.05	schizo	IT2-KM-odot	37.10	0.06
shampoo	IT2-NT-Cap	29.62	1.19	sheep	IT2-KM-Cap	7.33	0.05
strikes	IT2-NT-odot	14.40	0.59	telephone	IT2-NT-CLASSIC	9.78	0.02
ukdeaths	IT2-KM-odot	11.88	0.05	usdeaths	IT2-NT-Cap	6.41	15.36
uselec	IT2-KM-Cap	6.85	0.09	ustreas	IT2-KM-Cap	1.11	0.04
wagesuk	FTS-CHEN-freq	3.40	0.27	wheat	IT2-NT-Cap	15.10	98.92
wnoise	IT2-KM-odot	206.29	0.02	writing	IT2-KM-Cap	21.42	0.05

The aggregate pattern favors the proposed Cap-based configurations. As shown in Table 8, Cap obtains the largest number of wins according to MAPE, being the best-performing operator in 29 out of the 64 analyzed series. The odot-based models and Chen-type FTS models obtain 13 wins each, while the classical interval type-2 configuration obtains 8 wins and Song–Chissom obtains 1 win. These results suggest that the algebraic design of the type-2 operators has a direct impact on forecasting accuracy, although no single operator dominates uniformly across all time series.

Table 8. Number of winning methods per operator (criterion: MAPE).

Operator	Chen	Song	Classic T2	odot	Cap
Number of wins	13	1	8	13	29

The results clearly show that the operator Cap dominates in terms of MAPE, achieving the best performance in 29 of the analyzed series. It is followed by the odot operator and Chen-type models, both being the best choice in 13 series each, while the classical interval type-2 configuration attains the minimum MAPE in 8 series. Finally, Song–Chissom achieves the best MAPE in one series.

4.9. Sensitivity Analysis

The construction of the IT2FSs depends on two relevant parameters: the number of fuzzy partitions k and the shrinkage coefficient u used to define the lower membership functions. To evaluate the robustness of the proposed framework, we performed an additional sensitivity analysis.

First, we fixed $k = 8$, as in the main experiment, and evaluated $u \in \{0.1, 0.3, 0.5\}$. These values represent low, moderate and strong contraction of the lower membership functions. Second, we fixed $u = 0.25$ and evaluated $k \in \{7, 8, 9, 10, 11, 12\}$, which is the range considered in the preliminary tuning stage. All remaining components of the forecasting pipeline were kept unchanged.

Table 9 shows that the behavior of the proposed operators depends on the size of the footprint of uncertainty. When $u = 0.1$, the lower membership functions remain close to the upper membership functions, and the IT2 model behaves closer to a type-1 configuration; in this case, Chen-type models and odot-based models are highly competitive. However, for moderate and stronger contractions, $u = 0.3$ and $u = 0.5$, the Cap-based configurations clearly dominate, obtaining 34 and 36 wins, respectively. This indicates that Cap benefits from a richer interval type-2 representation of uncertainty.

Table 9. Sensitivity analysis with respect to the IT2 shrinkage parameter u . Number of wins according to MAPE.

u	Chen	Song	Classic T2	odot	Cap
0.1	19	5	7	19	14
0.3	11	2	5	12	34
0.5	10	4	4	10	36

Table 10 confirms that the conclusions are not tied to a single value of k . The Cap-based models obtain the largest number of wins for $k = 7, 8, 10, 11$ and 12 , and tie with Chen-type models for $k = 9$. The value $k = 8$, used in the main experiment, provides the largest number of wins for Cap and offers a favorable compromise between accuracy and stability. Smaller values of k may lead to overly coarse linguistic partitions, whereas larger values increase the granularity of the rule base and may reduce stability in shorter series.

Similar trends are observed when using RMSE and MAE as evaluation metrics, confirming that the dominance of the Cap-based configurations is not specific to MAPE.

Overall, the sensitivity analysis supports the robustness of the proposed approach. In particular, the Cap-based configurations remain consistently competitive across different choices of u and k , which reinforces the conclusion that the proposed type-2 operators provide useful alternatives for fuzzy time series forecasting.

Table 10. Sensitivity analysis with respect to the number of fuzzy partitions k . Number of wins according to MAPE.

k	Chen	Song	Classic T2	odot	Cap
7	20	1	6	11	26
8	13	1	8	13	29
9	22	1	9	10	22
10	15	3	10	16	20
11	16	2	6	15	25
12	11	5	7	15	26

5. Conclusions

In this paper, we provide new operators acting as t-norms and t-conorms in different subsets of type-2 fuzzy sets. In particular, we introduce the novel operations $\odot$, $\oplus$, $\mathfrak{M}$ and $\mathfrak{U}$.

We analyze the posets of M where these operations satisfy the required axioms for them to be t-norms (operators $\mathfrak{M}$ and $\odot$) or t-conorms (operators $\mathfrak{U}$ and $\oplus$). More precisely, the proposed operators are studied in M , C , N , L , K and K_c^F with respect to the two most commonly used partial orders, namely, $\sqsubseteq$ and $\preceq$. It should be noted that this is the first time that t-norms and t-conorms in N , K and K_c^F are determined for both orders. The following list contains the main results obtained in this work:

1. The operator $\odot$ is a t-norm in M , C , N , L , K and K_c^F with respect to the order $\sqsubseteq$.
2. The operator $\oplus$ is t-conorm in M , C , N , L , K and K_c^F with respect to the order $\preceq$.
3. The operator $\mathfrak{M}$ is t-norm in N , L , K and K_c^F with respect to both partial order $\sqsubseteq$ and $\preceq$.
4. The operator $\mathfrak{U}$ is t-conorm in N , L , K and K_c^F with respect to both partial orders $\sqsubseteq$ and $\preceq$.

T-norms and t-conorms are fundamental tools to compute logical operators in fuzzy logic. In particular, they represent the conjunction and disjunction, respectively. This is the reason why we expect the proposed novel operators to show their utility in type-2 fuzzy logic systems, providing new alternative ways of reasoning with T2FSs. In future work, we plan to study other interesting structures in type-2 fuzzy sets, such as similarity measures or fuzzy implications, and to analyze their connection with proposed negations, t-norms and t-conorms.

Beyond the algebraic analysis, a second contribution of the paper is the integration of the proposed operators into a fuzzy time series forecasting framework. We have presented odot and Cap in IT2 Mamdani-type models, combined with both Nie–Tan and Karnik–Mendel type-reduction, and we have carried out an extensive empirical study on a large collection of real time series from the fma package. All models share the same fuzzy partition, rule base and experimental protocol, and differ only in the underlying type-2 operators and reduction schemes.

The forecasting results, primarily evaluated in terms of MAPE, reveal a clear and consistent pattern. Models based on the $\mathfrak{M}$ operator (Cap models) achieve the best performance in the largest number of series (29 out of the analysed datasets, around 45% of the cases), followed by the $\odot$ operator (odot models) and Chen-type fuzzy time series models, which attain the minimum MAPE in 13 series each. The classical interval type-2 extension based on minimum and maximum operators also remains a competitive baseline, achieving the best results in eight series.

These findings indicate that the algebraic design of type-2 operators has a tangible impact on predictive accuracy in time series applications, and that operators such as $\mathfrak{M}$ and $\mathfrak{U}$ can provide a favorable balance between robustness and flexibility.

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Abbreviations

The following abbreviations are used in this manuscript:

MAPE	Mean absolute percentage error
RSME	Root mean square deviation
FSs	Fuzzy sets
T2FSs	Type-2 fuzzy sets
IT2FSs	Interval type-2 fuzzy sets
FTS	Fuzzy time series
UMF	Upper membership function

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