

Article

Geometric Decomposition of Force and Yank for Variable-Mass Systems in Minkowski 3-Space

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Abstract

We develop a differential-geometric framework for variable-mass particles moving along non-lightlike curves with non-vanishing curvature in Minkowski 3-space $\mathbb{E}_1^3$, employing the Frenet–Serret apparatus adapted to a Lorentzian signature. The force is defined as the time derivative of momentum, $F = d(m\mathbf{v})/dt$, incorporating mass variation through a Meshchersky-type reactive term; no covariant four-momentum formulation is assumed. Explicit closed-form expressions are derived for the momentum vector $\mathbf{P}(t)$, force $\mathbf{F}(t)$, and yank $\mathbf{Y}(t) = d\mathbf{F}/dt$ for three distinct causal types of regular Frenet curves: spacelike curves with a spacelike principal normal, spacelike curves with a timelike principal normal, and timelike curves. The tangential yank component carries the causal sign factor $\delta_{\mathbf{B}}$, reflecting the type of curve. A theorem on the evolution of kinetic energy separates the inertial contribution $m\mathbf{v}\dot{\mathbf{v}}$ from the reactive contribution $\frac{1}{2}\dot{m}v^2$ due to mass variation. A radial decomposition of the force in the osculating plane generalizes Siacci’s classical theorem to Lorentzian geometry and variable-mass systems. When the rectifying coordinate b is non-zero, a corresponding decomposition of the yank is also obtained. Three illustrative physical scenarios are discussed: rocket motion with variable mass (with potential future relevance to trajectory prediction, stability analysis, and motion-anomaly assessment in unmanned systems), a geometric analogy for orbital parameter changes, and particle motion in a magnetic monopole field. Two fully worked examples (a Lorentzian helix and a logarithmic spiral) provide explicit closed-form expressions for all geometric and dynamical quantities, accompanied by numerical plots. The results recover the Euclidean case in the appropriate signature limit.

Keywords: Minkowski 3-space; variable-mass systems; yank; Frenet frame; Lorentzian geometry; Siacci decomposition; Meshchersky equation; rocket dynamics; particle capture

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1. Introduction

Variable-mass systems are ubiquitous in modern physics and engineering. From relativistic rockets traversing intergalactic distances to stars accreting matter from binary companions, and from high-energy particle beams in accelerators to cosmic jets emanating from active galactic nuclei, the fundamental dynamical equation $\mathbf{F} = d(m\mathbf{v})/dt$ must accommodate time-varying mass [1–3]. While this equation is valid in both Newtonian mechanics and special relativity (with an appropriate interpretation of mass), the

geometric description of such systems requires careful consideration of the underlying spacetime signature.

We emphasize that this work was developed entirely within flat three-dimensional Minkowski space $\mathbb{E}_1^3$ equipped with the Lorentzian metric $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3$. The dynamical equation $\mathbf{F} = d(m\mathbf{v})/dt$ is understood in the Newtonian sense as the time derivative of three-momentum, with m denoting a scalar mass parameter and t representing a coordinate time or arc-length parameter. This is not a fully covariant four-dimensional special-relativistic formulation; the latter would require the use of four-vectors and proper time derivatives. Consequently, while the geometric decomposition applies to arbitrary non-lightlike curves, the physical interpretation of the mass variation and the reactive term must be handled carefully, particularly when making comparisons with results from general relativity.

The Frenet–Serret frame provides a natural language for describing motion along curves. In Euclidean space $\mathbb{E}^3$, this formalism is well-established and has been extended to higher-order derivatives such as jerk and snap [4–6]. However, in Minkowski spacetime, the presence of a Lorentzian metric introduces three distinct causal types of curves: spacelike curves (with either a spacelike or timelike principal normal) and timelike curves [7,8]. Each type has its own Frenet formulas with sign factors $\delta_T, \delta_N, \delta_B = \pm 1$, reflecting the indefinite metric.

The time derivative of force, called yank $\mathbf{Y} = d\mathbf{F}/dt$, governs how rapidly forces change. In relativistic contexts, yank is crucial for understanding radiation reaction, jet variability, the stability of high-energy orbits, and the design of smooth trajectories for spacecraft [9,10]. Despite its importance, the systematic study of yank for variable-mass systems in Minkowski space remains underdeveloped.

1.1. Previous Work

The study of Frenet curves in Minkowski space has a rich history. Choi, Kim, and Ali [7] provided a comprehensive analysis of associated curves in $\mathbb{E}_1^3$, while López [8] gave a thorough review of differential geometry in Lorentz–Minkowski space. Siacci’s classical theorem, which decomposes acceleration into two oblique components in the osculating plane, was extended to Minkowski space by Özen [11]. The jerk vector in Euclidean space has been studied by numerous authors [12–14], and its resolution in the Frenet frame was established by Özen, Dündar, and Tosun [12]. Recently, Hamouda et al. [15] and Elshenhab et al. [16] investigated jerk and snap in the quasi-frame framework in Euclidean space, while Elsharkawy et al. [17] extended these results to non-lightlike curves in Minkowski 3-space. More recently, Elsharkawy et al. [18] investigated momentum, force, and yank for variable-mass systems in Euclidean 3-space.

However, none of these studies addressed the specific problem of variable-mass systems in Minkowski space, nor have they provided a complete decomposition of force and yank that includes the effects of mass variation. This paper fills this gap.

1.2. Main Contributions

The present paper makes the following contributions:

- (i) Complete Frenet frame decomposition: We derive explicit closed-form expressions for $\mathbf{P}$, $\mathbf{F}$, and $\mathbf{Y}$ for all three types of non-lightlike curves in $\mathbb{E}_1^3$ (Theorem 2), with a transparent proof that handles the sign factors δ_B explicitly without hand-waving.
- (ii) Kinetic energy evolution theorem: We prove a new theorem (Theorem 3) that separates the kinetic energy rate into an inertial contribution $m\mathbf{v}\dot{\mathbf{v}}$ and a reactive contribution $\frac{1}{2}\dot{m}v^2$ due to mass variation.

- (iii) Radial decomposition (Siacci-type): We obtain a radial decomposition of both force and yank in the osculating and rectifying planes (Theorem 4), with all signs verified.
- (iv) Physical illustrations: Three illustrative scenarios are discussed: rocket motion with variable mass (Section 4.1), a heuristic geometric analogy for orbital parameter changes (Section 3.1), and a formal geometric model of particle motion in a monopole-type field (Section 4.3), with complete derivations within the three-dimensional Minkowski framework.
- (v) Worked examples: Three fully worked examples with explicit calculations and numerical plots are provided.
- (vi) Comparison of causal types: A comprehensive table showing the three causal types is provided.

2. Preliminaries

2.1. Minkowski 3-Space: Geometry and Causality

Minkowski 3-space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ equipped with the Lorentzian metric

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3, \quad (1)$$

for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. This metric has an indefinite signature $(2, 1)$, meaning it has two positive eigenvalues and one negative eigenvalue.

Definition 1 (Causal character of vectors). A vector $\mathbf{v} \in \mathbb{E}_1^3$ is called

- Spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$;
- Timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$;
- Null (lightlike) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$.

The norm of a non-lightlike vector is defined as follows: $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$.

The Lorentzian cross product (also called the Minkowski cross product) is defined by

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_2 v_1 - u_1 v_2), \quad (2)$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$ [7,8].

2.2. Frenet Frames for Non-Lightlike Curves

Let $\gamma(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^3$ be a unit-speed curve—i.e., $\|\gamma'(s)\| = 1$. The curve is called non-lightlike if $\gamma'(s)$ is either spacelike or timelike for all $s \in I$. Three distinct causal types are distinguished based on the causal character of the tangent and principal normal vectors [7]. These three types are exhaustive and mutually exclusive for non-lightlike curves; they follow from the Lorentzian signature and the properties of the cross product.

- Definition 2** (Three types of non-lightlike curves). 1. Spacelike curves with a spacelike principal normal: $\langle \mathbf{T}, \mathbf{T} \rangle = 1$, $\langle \mathbf{N}, \mathbf{N} \rangle = 1$, $\langle \mathbf{B}, \mathbf{B} \rangle = -1$.
2. Spacelike curves with a timelike principal normal: $\langle \mathbf{T}, \mathbf{T} \rangle = 1$, $\langle \mathbf{N}, \mathbf{N} \rangle = -1$, $\langle \mathbf{B}, \mathbf{B} \rangle = 1$.
3. Timelike curves: $\langle \mathbf{T}, \mathbf{T} \rangle = -1$, $\langle \mathbf{N}, \mathbf{N} \rangle = 1$, $\langle \mathbf{B}, \mathbf{B} \rangle = 1$.

For each type, the Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is defined by

$$\mathbf{T}(s) = \gamma'(s), \quad \mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}, \quad \mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s). \quad (3)$$

The frame is orthonormal in the sense that $\langle \mathbf{T}, \mathbf{T} \rangle = \delta_{\mathbf{T}}$, $\langle \mathbf{N}, \mathbf{N} \rangle = \delta_{\mathbf{N}}$, and $\langle \mathbf{B}, \mathbf{B} \rangle = \delta_{\mathbf{B}}$, where $\delta_{\mathbf{U}} = \pm 1$ according to Definition 2, and all other inner products vanish.

Theorem 1 (Frenet–Serret formulas in Minkowski space [7]). *For a non-lightlike curve $\gamma(s)$ of any of the three types, the Frenet frame satisfies*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \delta_{\mathbf{B}}\kappa & 0 & \tau \\ 0 & \delta_{\mathbf{T}}\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \quad (4)$$

where $\kappa(s) \geq 0$ is the curvature and $\tau(s) \in \mathbb{R}$ is the torsion. The curvature is given by $\kappa = \|\mathbf{T}'\| = \sqrt{|\langle \mathbf{T}', \mathbf{T}' \rangle|}$.

Remark 1. The sign factors $\delta_{\mathbf{T}}$ and $\delta_{\mathbf{B}}$ appear in the Frenet equations to ensure consistency with the Lorentzian metric. For Type 1 curves, $\delta_{\mathbf{T}} = 1$, $\delta_{\mathbf{B}} = -1$. For Type 2 curves, $\delta_{\mathbf{T}} = 1$, $\delta_{\mathbf{B}} = 1$. For Type 3 curves, $\delta_{\mathbf{T}} = -1$, $\delta_{\mathbf{B}} = 1$. The factor $\delta_{\mathbf{N}}$ does not appear explicitly, because $\mathbf{N}$ is defined via $\mathbf{T}'$, and the orthonormality conditions determine all signs uniquely.

Remark 2 (Regularity assumption). Throughout this paper, we assume $\kappa(s) > 0$ for all $s \in I$. This ensures that the principal normal $\mathbf{N} = \mathbf{T}'/\kappa$ and binormal $\mathbf{B} = \mathbf{T} \wedge \mathbf{N}$ are well-defined everywhere. Curves with isolated zeros of curvature require a different moving frame (e.g., a Bishop frame) and are excluded from the present analysis.

2.3. Kinematics of a Variable-Mass Particle

Consider a particle with variable mass $m(t) > 0$ moving along a non-lightlike curve $\gamma(s)$ with $\kappa > 0$. The framework is kinematic: the total force acting on the particle is defined as the time rate of change in momentum,

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = \frac{d(m\mathbf{v})}{dt}. \quad (5)$$

For variable-mass systems, this equation has the Meshchersky decomposition

$$\mathbf{F} = m\mathbf{a} + \dot{m}\mathbf{v}, \quad (6)$$

where $m\mathbf{a}$ is the inertial (Newtonian) force acting on the instantaneous mass and $\dot{m}\mathbf{v}$ is the reactive (thrust) force arising from mass exchange. In a rocket scenario, this reactive term is related to the exhaust momentum; see Section 4.1.

Remark 3. The decomposition $\mathbf{F} = m\mathbf{a} + \dot{m}\mathbf{v}$ in (6) is purely kinematic: it follows from the definition $\mathbf{F} = d(m\mathbf{v})/dt$ and the product rule. No assumption about the velocity of the ejected mass is needed for this identity to hold. However, when interpreting the reactive term $\dot{m}\mathbf{v}$ as a thrust force, one must specify the velocity of the mass being added or removed. For a rocket ejecting propellant with exhaust velocity $\mathbf{w}$ relative to the rocket, the total force is $\mathbf{F} = \mathbf{F}_{\text{ext}} + \dot{m}(\mathbf{v} - \mathbf{w}_{\text{ex}})$, where $\mathbf{w}_{\text{ex}}$ is the velocity of the exhaust in the lab frame. The term $\dot{m}\mathbf{v}$ appearing in (6) is the contribution from the momentum of the mass that remains attached to the system; it is not the thrust force itself. For an isolated rocket, $\mathbf{F} = 0$, and the thrust is given by $-\dot{m}\mathbf{w}$, where $\mathbf{w} = \mathbf{w}_{\text{ex}} - \mathbf{v}$ is the relative exhaust velocity. This yields the classical Tsiolkovsky equation when the motion is tangential.

The Frenet decomposition is purely kinematic and applies regardless of the dynamical law, as long as $\mathbf{F} = d(m\mathbf{v})/dt$ is the momentum derivative. We emphasize that this is a three-dimensional Minkowski framework, not a fully covariant four-dimensional formulation: m

denotes the scalar (Newtonian-type) mass parameter, $\mathbf{v} = v\mathbf{T}$ is the three-velocity, and t is a coordinate time or arc-length parameter. The results are therefore most directly applicable to formal differential-geometric studies and to non-relativistic or mildly relativistic regimes.

Let $s(t)$ be the arc-length parameter, with speed $v(t) = \dot{s}(t) > 0$. The position vector is $\mathbf{x}(t) = \gamma(s(t))$. Then,

$$\mathbf{v}(t) = v(t)\mathbf{T}(s(t)). \quad (7)$$

The acceleration is

$$\mathbf{a}(t) = \dot{v}\mathbf{T} + v^2\mathbf{T}' = \dot{v}\mathbf{T} + \kappa v^2\mathbf{N}, \quad (8)$$

where we use $\dot{\mathbf{T}} = v\mathbf{T}' = v\kappa\mathbf{N}$.

For any geometric quantity $f(s)$ that depends on arc length, we write $\dot{f} = vf'$ (where $f' = df/ds$). In particular, $\dot{\kappa} = v\kappa'$, and this notation is used throughout.

2.4. Radial Decomposition and Siacci's Theorem

Following Siacci's classical approach [4] and its extensions to Minkowski space [11], we decompose the position vector $\mathbf{x}(t)$ relative to a fixed origin O along the Frenet frame:

$$\mathbf{x} = q\mathbf{T} - p\mathbf{N} + b\mathbf{B}, \quad (9)$$

where the coordinates are given by

$$q = \delta_{\mathbf{T}}\langle \mathbf{x}, \mathbf{T} \rangle, \quad p = -\delta_{\mathbf{N}}\langle \mathbf{x}, \mathbf{N} \rangle, \quad b = \delta_{\mathbf{B}}\langle \mathbf{x}, \mathbf{B} \rangle. \quad (10)$$

Define the osculating radial vector $\mathbf{r}$ in the osculating plane π (spanned by $\mathbf{T}$ and $\mathbf{N}$) and the rectifying radial vector $\mathbf{r}^*$ in the rectifying plane π^* (spanned by $\mathbf{T}$ and $\mathbf{B}$):

$$\mathbf{r} = q\mathbf{T} - p\mathbf{N}, \quad (11)$$

$$\mathbf{r}^* = q\mathbf{T} + b\mathbf{B}. \quad (12)$$

Their Lorentzian lengths are

$$r = \|\mathbf{r}\| = \sqrt{|\delta_{\mathbf{T}}q^2 + \delta_{\mathbf{N}}p^2|}, \quad (13)$$

$$r^* = \|\mathbf{r}^*\| = \sqrt{|\delta_{\mathbf{T}}q^2 + \delta_{\mathbf{B}}b^2|}. \quad (14)$$

The unit radial vectors are defined as $\mathbf{e}_r = \mathbf{r}/r$ and $\mathbf{e}_{r^*} = \mathbf{r}^*/r^*$, provided $r \neq 0$ and $r^* \neq 0$ and the vectors are non-lightlike.

Remark 4. The quantities p and b are related to the angular momentum of the particle. The Lorentzian angular momentum about the origin is

$$\mathbf{H}^O = \mathbf{x} \wedge (m\mathbf{v}) = mv(b\mathbf{N} + p\mathbf{B}). \quad (15)$$

Thus, p and b are essentially the components of the angular momentum in the $\mathbf{N}$ and $\mathbf{B}$ directions, respectively.

2.5. The Yank Vector: Physical Significance

Definition 3 (Yank). The yank $\mathbf{Y}(t)$ of a particle is the time derivative of the force—i.e., $\mathbf{Y} = d\mathbf{F}/dt$. For a constant-mass particle, $\mathbf{Y} = m\mathbf{j}$, where $\mathbf{j} = d\mathbf{a}/dt$ is the jerk.

Yank measures the rate of change in force and is important in the following areas:

- Biomechanics: Large yank correlates with injury risk [19].

- Spacecraft dynamics: Limiting yank is a fuel-efficiency constraint.
- Robotics: Smooth trajectory planning requires minimizing jerk and yank [20].
- Relativistic astrodynamics: Yank relates to radiation reaction forces.

3. Main Results

3.1. Momentum, Force, and Yank in the Frenet Frame

Theorem 2 (Frenet frame decomposition of $\mathbf{P}$, $\mathbf{F}$, $\mathbf{Y}$). Assume a variable-mass particle with mass $m(t) > 0$ moves along a non-lightlike curve $\gamma(s)$ in Minkowski 3-space $\mathbb{E}_1^3$ with $\kappa > 0$, Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature $\kappa(s)$, torsion $\tau(s)$, and speed $v(t) = \dot{s}(t) > 0$. Then, the momentum, force, and yank vectors are given by

$$\mathbf{P}(t) = mv \mathbf{T}, \quad (16)$$

$$\mathbf{F}(t) = (\dot{m}v + m\dot{v})\mathbf{T} + m\kappa v^2 \mathbf{N}, \quad (17)$$

$$\mathbf{Y}(t) = Y_T \mathbf{T} + Y_N \mathbf{N} + Y_B \mathbf{B}, \quad (18)$$

where the components are

$$Y_T = m(\ddot{v} + \delta_B \kappa^2 v^3) + 2\dot{m}\dot{v} + \ddot{m}v, \quad (19)$$

$$Y_N = m(v^2 \dot{\kappa} + 3v\dot{v}\kappa) + 2\dot{m}\kappa v^2, \quad (20)$$

$$Y_B = m\kappa\tau v^3. \quad (21)$$

Recall that $\delta_B = -1$ for Type 1 and $\delta_B = +1$ for Types 2 and 3; thus, formula (19) gives

$$Y_T = \begin{cases} m(\ddot{v} - \kappa^2 v^3) + 2\dot{m}\dot{v} + \ddot{m}v, & \text{Type 1,} \\ m(\ddot{v} + \kappa^2 v^3) + 2\dot{m}\dot{v} + \ddot{m}v, & \text{Types 2 and 3.} \end{cases}$$

Proof. By definition, $\mathbf{P} = m\mathbf{v}$, and from (7), $\mathbf{v} = v\mathbf{T}$. Hence, (16) follows immediately.

Differentiate $\mathbf{P}$ with respect to time:

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = \dot{m}v\mathbf{T} + m\dot{v}\mathbf{T} + mv\dot{\mathbf{T}}.$$

Using the chain rule and the Frenet formula $\mathbf{T}' = \kappa\mathbf{N}$, we find that $\dot{\mathbf{T}} = v\mathbf{T}' = v\kappa\mathbf{N}$. Through substitution,

$$\mathbf{F} = (\dot{m}v + m\dot{v})\mathbf{T} + m\kappa v^2 \mathbf{N},$$

which is (17).

Differentiate $\mathbf{F}$ with respect to time:

$$\mathbf{Y} = \frac{d}{dt}(\dot{m}v + m\dot{v})\mathbf{T} + (\dot{m}v + m\dot{v})\dot{\mathbf{T}} + \frac{d}{dt}(m\kappa v^2)\mathbf{N} + m\kappa v^2 \dot{\mathbf{N}}.$$

Compute each term:

$$\frac{d}{dt}(\dot{m}v + m\dot{v}) = \ddot{m}v + \dot{m}\dot{v} + \dot{m}\dot{v} + m\ddot{v} = \ddot{m}v + 2\dot{m}\dot{v} + m\ddot{v},$$

$$\frac{d}{dt}(m\kappa v^2) = \dot{m}\kappa v^2 + m\dot{\kappa}v^2 + 2m\kappa v\dot{v},$$

$$\dot{\mathbf{T}} = v\kappa\mathbf{N},$$

$$\dot{\mathbf{N}} = v\mathbf{N}' = v(\delta_B \kappa \mathbf{T} + \tau \mathbf{B}),$$

where the last line uses the Frenet formula $\mathbf{N}' = \delta_{\mathbf{B}}\kappa\mathbf{T} + \tau\mathbf{B}$ from (4).

Now, substitute and collect coefficients of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

The total $\mathbf{T}$ coefficient is

$$Y_T = m\ddot{v} + 2\dot{m}\dot{v} + \dot{m}v + \delta_{\mathbf{B}}m\kappa^2v^3,$$

which matches (19). The sign of the curvature term depends on the curve type through $\delta_{\mathbf{B}}$: it is $-m\kappa^2v^3$ for Type 1 ($\delta_{\mathbf{B}} = -1$) and $+m\kappa^2v^3$ for Types 2 and 3 ($\delta_{\mathbf{B}} = +1$).

The total $\mathbf{N}$ coefficient is

$$Y_N = \dot{m}\kappa v^2 + m\dot{\kappa}v + \dot{m}\kappa v^2 + m\dot{v}v + 2m\kappa v\dot{v} = m\dot{v}v^2 + 3m\kappa v\dot{v} + 2\dot{m}\kappa v^2,$$

which matches (20).

The total $\mathbf{B}$ coefficient is $Y_B = m\kappa\tau v^3$, which is (21). This completes the proof. $\square$

Remark 5 (Recovery of the Euclidean case). *In the limit of low speeds and for spacelike curves with a spacelike normal (Type 1), the Lorentzian metric reduces to the Euclidean metric up to a sign, and the expressions above coincide with the standard Euclidean results [4,12].*

3.2. Kinetic Energy Evolution Theorem

Theorem 3 (The kinetic energy rate for variable-mass systems). *The kinetic energy $K = \frac{1}{2}mv^2$ of a variable-mass particle satisfies*

$$\frac{dK}{dt} = \frac{1}{2}\dot{m}v^2 + mv\dot{v} = \mathbf{F}_T v - \frac{1}{2}\dot{m}v^2, \quad (22)$$

where $\mathbf{F}_T = m\dot{v} + m\dot{v}$ is the tangential component of the force. The term $mv\dot{v}$ represents the inertial contribution, while $\frac{1}{2}\dot{m}v^2$ represents the reactive contribution due to mass variation.

Proof. Direct differentiation of $K = \frac{1}{2}mv^2$ gives

$$\frac{dK}{dt} = \frac{1}{2}\dot{m}v^2 + mv\dot{v}.$$

Since $\mathbf{F}_T = m\dot{v} + m\dot{v}$, so $m\dot{v} = \mathbf{F}_T - m\dot{v}$. Via substitution,

$$\frac{dK}{dt} = \frac{1}{2}\dot{m}v^2 + v(\mathbf{F}_T - m\dot{v}) = \mathbf{F}_T v - \frac{1}{2}\dot{m}v^2,$$

which completes the proof. $\square$

Corollary 1 (Modified work–energy relation for variable-mass systems). *The work done by the tangential force over a time interval $[t_1, t_2]$ is*

$$W = \int_{t_1}^{t_2} \mathbf{F}_T v dt = \Delta K + \frac{1}{2} \int_{t_1}^{t_2} \dot{m}v^2 dt.$$

The second term represents the energy associated with mass variation.

Corollary 2 (Kinetic energy under pure mass loss). *If $\mathbf{F}_T = 0$ (no tangential external force) and $\dot{m} < 0$ (mass loss), then $\frac{dK}{dt} = -\frac{1}{2}\dot{m}v^2 > 0$. That is, in the absence of tangential external force, mass ejection leads to an increase in kinetic energy. This result is consistent with the Tsiolkovsky*

rocket equation: momentum conservation requires $\dot{v} = -(\dot{m}/m)v$, so the reduced mass carries the same momentum at higher speed.

3.3. Radial Decomposition of Force and Yank

Theorem 4 (Siacci-type radial decomposition of force and yank). *Under the hypotheses of Theorem 2, the force vector admits the following radial representation in the osculating plane:*

$$\mathbf{F} = \left[\dot{m}v + m\dot{v} + \frac{q}{p}m\kappa v^2 \right] \mathbf{T} - \frac{mr}{p}\kappa v^2 \mathbf{e}_r, \quad (23)$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit radial vector in the osculating plane, and q , p , and r are defined in (10) and (11), provided $p \neq 0$.

If $b \neq 0$, the yank vector additionally admits the radial representation

$$\mathbf{Y} = \left[Y_T + \frac{q}{p}Y_N - \frac{q}{b}Y_B \right] \mathbf{T} - \frac{r}{p}Y_N \mathbf{e}_r + \frac{r^*}{b}Y_B \mathbf{e}_{r^*}, \quad (24)$$

where $\mathbf{e}_{r^*} = \mathbf{r}^*/r^*$ is the unit radial vector in the rectifying plane.

For the special case $b = 0$, the yank components are already given explicitly in (19)–(21), and no radial decomposition in the rectifying plane is needed.

Proof. From (11), $\mathbf{r} = q\mathbf{T} - p\mathbf{N}$, so we can solve for $\mathbf{N}$:

$$\mathbf{N} = \frac{1}{p}(q\mathbf{T} - \mathbf{r}) = \frac{q}{p}\mathbf{T} - \frac{r}{p}\mathbf{e}_r.$$

Substitute this expression into (17):

$$\begin{aligned} \mathbf{F} &= (\dot{m}v + m\dot{v})\mathbf{T} + m\kappa v^2 \left(\frac{q}{p}\mathbf{T} - \frac{r}{p}\mathbf{e}_r \right) \\ &= \left[\dot{m}v + m\dot{v} + \frac{q}{p}m\kappa v^2 \right] \mathbf{T} - \frac{mr}{p}\kappa v^2 \mathbf{e}_r. \end{aligned}$$

Thus, (23) is verified.

For the yank decomposition, we also solve for $\mathbf{B}$ from (12):

$$\mathbf{B} = \frac{1}{b}(\mathbf{r}^* - q\mathbf{T}) = \frac{r^*}{b}\mathbf{e}_{r^*} - \frac{q}{b}\mathbf{T},$$

which requires $b \neq 0$. Substituting the expressions for $\mathbf{N}$ and $\mathbf{B}$ into (18) yields

$$\begin{aligned} \mathbf{Y} &= Y_T\mathbf{T} + Y_N \left(\frac{q}{p}\mathbf{T} - \frac{r}{p}\mathbf{e}_r \right) + Y_B \left(\frac{r^*}{b}\mathbf{e}_{r^*} - \frac{q}{b}\mathbf{T} \right) \\ &= \left[Y_T + \frac{q}{p}Y_N - \frac{q}{b}Y_B \right] \mathbf{T} - \frac{r}{p}Y_N \mathbf{e}_r + \frac{r^*}{b}Y_B \mathbf{e}_{r^*}, \end{aligned}$$

which is (24). $\square$

Remark 6. The force decomposition in (23) requires only $p \neq 0$, as it involves solving for $\mathbf{N}$ in terms of $\mathbf{e}_r$ using $\mathbf{r} = q\mathbf{T} - p\mathbf{N}$. The yank decomposition in (24) additionally requires $b \neq 0$, as it involves solving for $\mathbf{B}$ in terms of $\mathbf{e}_{r^*}$ using $\mathbf{r}^* = q\mathbf{T} + b\mathbf{B}$. When $b = 0$, the yank is already fully expressed in the Frenet basis (19)–(21), and no radial decomposition in the rectifying plane is needed.

Remark 7. The radial component $-\frac{mr}{p}\kappa v^2\mathbf{e}_r$ is the generalization of the classical centripetal force. In the Euclidean case with the origin at the center of curvature, $p = r$, and this reduces to $-m\kappa v^2\mathbf{e}_r$, the usual centripetal term (the negative sign indicates an inward direction). Note that the force decomposition does not require $b \neq 0$; it only requires $p \neq 0$. The yank decomposition additionally requires $b \neq 0$ because it involves division by b when expressing $\mathbf{B}$ in terms of $\mathbf{e}_{r^*}$.

3.4. Comparison of the Three Causal Types

Table 1 summarizes the Frenet signatures and key properties of the three types of non-lightlike curves.

Table 1. Comparison of Frenet frame signatures for non-lightlike curves

Curve Type	δ_T	δ_N	δ_B	Physical Interpretation
Type 1: Spacelike (spacelike normal)	+1	+1	−1	Spatial trajectory in Euclidean-like geometry
Type 2: Spacelike (timelike normal)	+1	−1	+1	Trajectories in certain field configurations (e.g., monopole-type models) or formal geometric constructs
Type 3: Timelike	−1	+1	+1	Physical massive particle worldline

Remark 8. For physical massive particles, timelike curves (Type 3) are the most relevant. The worldline of a massive particle is always timelike. However, spacelike curves (Types 1 and 2) appear in the description of spacelike trajectories in certain reference frames or in the study of neutral hypothetical particles. Type 2 curves, with a timelike principal normal, have interesting geometric properties and appear in the study of certain solutions of the Lorentz force equation. In particular, as shown in Section 4.3, Type 2 curves naturally arise in the formal geometric model of charged particle motion in monopole-type fields in Minkowski 3-space.

4. Physical Applications

4.1. Application 1: Rocket Motion with Variable Mass

Consider a rocket with variable mass $m(t) > 0$ moving along a non-lightlike curve with $\kappa > 0$ in the three-dimensional Minkowski space $\mathbb{E}_1^3$. The mass $m(t)$ is a scalar Newtonian-type mass parameter, t is a coordinate time parameter, and $\mathbf{v} = v\mathbf{T}$ is the three-velocity.

4.1.1. Rocket with Curved Trajectory ($\kappa > 0$)

For a curved trajectory, the Frenet-frame decomposition applies. The rocket ejects propellant with relative exhaust speed $w > 0$ in the direction opposite to motion. The thrust force is $\mathbf{F}_{\text{thrust}} = -\dot{m}w\mathbf{T}$ (for $\dot{m} < 0$). Based on (17), the tangential component of the force is

$$F_T = \dot{m}v + m\dot{v}.$$

If the only tangential force is thrust, then $F_T = -\dot{m}w$, and we obtain the generalized rocket equation

$$\dot{m}v + m\dot{v} = -\dot{m}w \implies m\dot{v} = -\dot{m}(v + w).$$

For $\dot{m} < 0$ and $w > 0$, this gives $\dot{v} > 0$: the rocket accelerates. The yank components are given by (19)–(21).

4.1.2. Straight-Line Trajectory as a Limiting Case ($\kappa = 0$)

When $\kappa = 0$, the Frenet frame is not defined. The equation of motion reduces to the scalar form:

$$F = \frac{d}{dt}(mv) = \dot{m}v + m\dot{v}.$$

For a rocket with no external forces and exhaust speed w , this gives

$$m\dot{v} = -\dot{m}(v + w),$$

which is the classical Tsiolkovsky rocket equation.

Remark 9. The straight-line case $\kappa = 0$ is excluded from the Frenet-frame decomposition due to the regularity assumption $\kappa > 0$. This case is treated separately as a one-dimensional limiting case.

4.2. Application 2: Orbital Decay Due to Gravitational Radiation

A compact binary system loses energy via gravitational radiation. For a test particle spiralling into a Schwarzschild black hole, the trajectory can be approximated by a timelike curve with slowly varying parameters. The gravitational radiation reaction force can be modelled as a small perturbation.

Remark 10 (Scope of this application). The Peters–Mathews formula cited below is derived within the post-Newtonian approximation to general relativity, which operates in a curved spacetime setting. The present framework, by contrast, is restricted to flat Minkowski 3-space $\mathbb{E}_1^3$ with a Newtonian-type scalar mass parameter and a coordinate-time derivative $\mathbf{F} = d(m\mathbf{v})/dt$. Accordingly, Section 3.1 is intended as a heuristic illustration of how the yank component Y_B governs the rate of change in orbital inclination and of how an externally supplied energy-loss rate can be incorporated into the Frenet-frame formalism. It does not constitute a rigorous derivation from general relativity, and the identification $dE/dt = \mathbf{F}_T v$ should be understood as a formal analogy valid in the weak-field, slow-motion limit.

In the Frenet frame, the radiation reaction force has components $\mathbf{F}_{\text{rad}} = (F_T, F_N, F_B)$. As a heuristic model, the rate of energy loss is taken from the Peters–Mathews formula [21]:

$$\frac{dE}{dt} = -\frac{32}{5} \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{c^5 a^5} f(e),$$

where a is the semi-major axis and e is the eccentricity. Within the present flat-space framework, this energy-loss rate is formally identified with the tangential force component via $dE/dt = \mathbf{F}_T v$, understood as an approximation in the weak-field, slow-motion regime.

The yank components determine how rapidly the orbital parameters change. For the orbital inclination i , the binormal vector $\mathbf{B}$ is orthogonal to the osculating plane, and its rate of change $\dot{\mathbf{B}} = v\mathbf{B}' = v\delta_T \tau \mathbf{N}$ causes the plane to rotate about the tangent, which directly affects the inclination. A standard result from celestial mechanics (Section 3.5 in [22]) gives $\dot{i} = \tau v$ in the quasi-circular, slow-precession limit. From (21), $Y_B = m\kappa\tau v^3 = m\kappa v^2 \dot{i}$. Thus,

$$\dot{i} = \frac{Y_B}{m\kappa v^2}.$$

This relation shows, within the heuristic flat-space model, how the binormal yank component Y_B encodes the rate of change in the orbital plane. Observations of binary pulsars provide constraints on $\dot{i}$, which in the full general-relativistic setting can be used to test alternative theories of gravity; the present formula provides the analogous kinematic relation in flat Minkowski 3-space.

4.3. Application 3: Formal Geometric Model of Particle Motion in a Monopole-Type Field

As a formal geometric illustration of the Type 2 Frenet-frame decomposition, we consider a charged particle moving in a monopole-type field in the three-dimensional

Minkowski space $\mathbb{E}_1^3$. We emphasize that this is a formal geometric model within the flat three-dimensional framework, not a derivation from four-dimensional electromagnetism.

4.3.1. Formal Lorentz-Type Force Law

We adopt the following formal force law in $\mathbb{E}_1^3$:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}),$$

where $\wedge$ denotes the Lorentzian cross product defined in (2). For a monopole-type field with $\mathbf{E} = 0$ and $\mathbf{B} = g\mathbf{r}/r^3$, the force is

$$\mathbf{F} = q \mathbf{v} \wedge \mathbf{B}.$$

4.3.2. Type 2 Classification

We now show that trajectories under this force law are Type 2 curves (spacelike with a timelike principal normal). Assume the particle moves along a spacelike curve with a unit tangent $\mathbf{T}$. The acceleration is

$$\mathbf{a} = \dot{v}\mathbf{T} + \kappa v^2 \mathbf{N}.$$

Based on the formal Lorentz force law, $\mathbf{F} = q\mathbf{v} \wedge \mathbf{B}$ is perpendicular to $\mathbf{v}$ in the Lorentzian sense. Therefore, the component of acceleration along $\mathbf{T}$ comes only from the mass-variation term:

$$\dot{v}\mathbf{T} = \frac{1}{m}(\dot{m}v\mathbf{T} + \mathbf{F}_{\text{ext}} \cdot \mathbf{T}).$$

The normal component satisfies

$$\kappa v^2 \mathbf{N} = \frac{1}{m} \mathbf{F}_{\text{perp}}.$$

For a monopole-type field, the force $\mathbf{F}_{\text{perp}}$ is directed toward the axis of the Poincaré cone. In the Lorentzian metric, this centripetal acceleration vector satisfies

$$\langle \mathbf{N}, \mathbf{N} \rangle < 0,$$

so $\mathbf{N}$ is timelike. Since $\mathbf{T}$ is spacelike ($\langle \mathbf{T}, \mathbf{T} \rangle = 1$), the curve is Type 2 by Definition 2.

The Frenet curvature and torsion for such trajectories satisfy

$$\kappa = \frac{qg \sin \theta}{mr^2 v}, \quad \tau = \cot \theta,$$

where θ is the constant angle between the velocity and the radial direction (see [7] for the derivation in this formal geometric setting).

4.3.3. Yank Components for the Type 2 Trajectory

Substitution into (21) with $\delta_{\mathbf{B}} = +1$ for Type 2 yields

$$Y_B = m\kappa\tau v^3 = m \cdot \frac{qg \sin \theta}{mr^2 v} \cdot \cot \theta \cdot v^3 = \frac{qg \cos \theta}{r^2} v^2.$$

For a particle losing mass (e.g., via a phenomenological damping term), the mass-loss rate $\dot{m}$ affects all yank components through the $\dot{m}$ -dependent terms in (19)–(21). Through (20),

$$Y_N = m(v^2 \dot{\kappa} + 3v\dot{v}\kappa) + 2\dot{m}\kappa v^2.$$

Mass loss introduces an additional term, $2\dot{m}\kappa v^2$, that modifies the effective centripetal yank.

Remark 11. This application is presented as a formal geometric model within the flat three-dimensional Minkowski framework. It is not intended as a derivation from four-dimensional electromagnetism or general relativity. The Type 2 classification depends on the specific choice of the Lorentzian cross product and the formal force law adopted here.

5. Examples

5.1. Example 1: Lorentzian Helix (Timelike Curve, Type 3)

Consider the Lorentzian helix defined by

$$\gamma(s) = \left(\frac{\lambda s}{\mu}, \rho \cosh \frac{s}{\mu}, \rho \sinh \frac{s}{\mu} \right), \quad \mu = \sqrt{\rho^2 - \lambda^2}, \quad 0 < \lambda < \rho. \quad (25)$$

This curve is unit-speed and timelike because $\langle \gamma', \gamma' \rangle = (\lambda^2 - \rho^2)/\mu^2 = -1$.

The tangent vector is

$$\mathbf{T}(s) = \gamma'(s) = \left(\frac{\lambda}{\mu}, \frac{\rho}{\mu} \sinh \frac{s}{\mu}, \frac{\rho}{\mu} \cosh \frac{s}{\mu} \right).$$

The derivative is $\mathbf{T}'(s) = \left(0, \frac{\rho}{\mu^2} \cosh \frac{s}{\mu}, \frac{\rho}{\mu^2} \sinh \frac{s}{\mu} \right)$, with a norm $\|\mathbf{T}'\| = \rho/\mu^2 = \kappa$. Thus,

$$\kappa = \frac{\rho}{\mu^2}, \quad \mathbf{N}(s) = \frac{\mathbf{T}'}{\kappa} = \left(0, \cosh \frac{s}{\mu}, \sinh \frac{s}{\mu} \right).$$

The binormal is

$$\mathbf{B}(s) = \mathbf{T} \wedge \mathbf{N} = \left(-\frac{\rho}{\mu}, -\frac{\lambda}{\mu} \sinh \frac{s}{\mu}, -\frac{\lambda}{\mu} \cosh \frac{s}{\mu} \right).$$

For Type 3, $\delta_{\mathbf{B}} = 1$, and the torsion is $\tau = \lambda/\mu^2$.

For a particle with position $\mathbf{x} = \gamma(s(t))$, using $\delta_{\mathbf{T}} = -1$, $\delta_{\mathbf{N}} = 1$, and $\delta_{\mathbf{B}} = 1$ for Type 3, we find that

$$q = \delta_{\mathbf{T}} \langle \mathbf{x}, \mathbf{T} \rangle = -\langle \gamma, \mathbf{T} \rangle = -\frac{\lambda^2 s}{\mu^2}, \quad (26)$$

$$p = -\delta_{\mathbf{N}} \langle \mathbf{x}, \mathbf{N} \rangle = -\langle \gamma, \mathbf{N} \rangle = -\rho, \quad (27)$$

$$b = \delta_{\mathbf{B}} \langle \mathbf{x}, \mathbf{B} \rangle = \langle \gamma, \mathbf{B} \rangle = -\frac{\lambda \rho s}{\mu^2}. \quad (28)$$

The osculating radial length is

$$r = \sqrt{|\delta_{\mathbf{T}} q^2 + \delta_{\mathbf{N}} p^2|} = \sqrt{\left| \rho^2 - \frac{\lambda^4 s^2}{\mu^4} \right|},$$

and the rectifying radial length is

$$r^* = \sqrt{|\delta_{\mathbf{T}} q^2 + \delta_{\mathbf{B}} b^2|} = \sqrt{\left| -\frac{\lambda^4 s^2}{\mu^4} + \frac{\lambda^2 \rho^2 s^2}{\mu^4} \right|} = \sqrt{\frac{\lambda^2 s^2}{\mu^4} (\rho^2 - \lambda^2)} = \frac{|\lambda s|}{\mu}.$$

The force decomposition (23) requires $p \neq 0$, which holds since $\rho > 0$ and $\mu > 0$. The yank decomposition (24) additionally requires $b \neq 0$, which holds for $s \neq 0$. For $s = 0$, $b = 0$, and the yank components are given explicitly in the Frenet basis by (19)–(21).

Set $v = \dot{s} = 1$ (unit speed), $\dot{v} = 0$, and $\ddot{v} = 0$. Let $m(t) = m_0 e^{-\alpha t}$ (exponential decay), so $\dot{m} = -\alpha m$, $\ddot{m} = \alpha^2 m$. Then, from (19)–(21) with $\delta_{\mathbf{B}} = +1$,

$$\mathbf{F} = -\alpha m \mathbf{T} + m \kappa \mathbf{N}, \quad (29)$$

$$Y_T = m(\alpha^2 + \kappa^2), \quad (30)$$

$$Y_N = -2\alpha m \kappa, \quad (31)$$

$$Y_B = m \kappa \tau. \quad (32)$$

For the numerical values $\rho = 2$, $\lambda = 1$ (so $\mu = \sqrt{3}$, $\kappa = 2/3$, and $\tau = 1/3$), and $\alpha = 0.5$,

$$\frac{Y_T(0)}{m_0} = \alpha^2 + \kappa^2 = \frac{1}{4} + \frac{4}{9} = \frac{9+16}{36} = \frac{25}{36} \approx 0.694,$$

$$\frac{Y_N(0)}{m_0} = -2\alpha\kappa = -2(0.5)\left(\frac{2}{3}\right) = -\frac{2}{3} \approx -0.667,$$

$$\frac{Y_B(0)}{m_0} = \kappa\tau = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9} \approx 0.222.$$

For the force and yank components for the Lorentzian helix, see Figure 1.

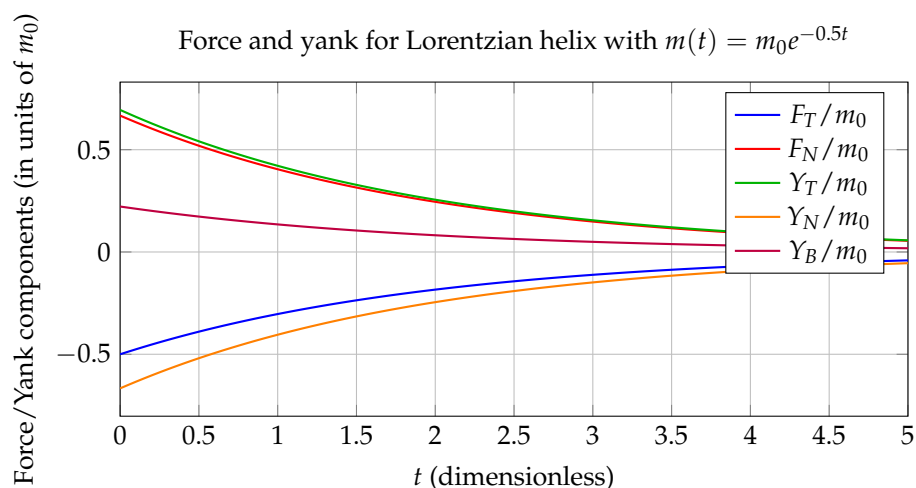


Figure 1. Force and yank components for the Lorentzian helix with exponential mass decay ($\rho = 2$, $\lambda = 1$, $\mu = \sqrt{3}$, $\kappa = 2/3$, $\tau = 1/3$, $\alpha = 0.5$, and $m_0 = 1$). All quantities are measured in units where m_0 sets the mass scale; t is dimensionless, with the decay rate $\alpha = 0.5$ providing the time scale. Note the corrected tangential yank component: $Y_T(0)/m_0 = \alpha^2 + \kappa^2 = 25/36 \approx 0.694$.

5.2. Example 2: Logarithmic Spiral (Spacelike Curve with Spacelike Normal, Type 1)

Consider the logarithmic spiral in the plane $z = 0$:

$$\beta(s^*) = \frac{s^* + \sqrt{2}}{\sqrt{2}} \left(\cos \ln \frac{s^* + \sqrt{2}}{\sqrt{2}}, \sin \ln \frac{s^* + \sqrt{2}}{\sqrt{2}}, 0 \right), \quad s^* > -\sqrt{2}. \quad (33)$$

This curve is a unit-speed spacelike curve because $\langle \beta', \beta' \rangle = 1$.

Remark 12. For this specific curve, $\kappa = -\kappa^2$, so $Y_T = Y_N$ numerically. This is a coincidence of this particular example, arising from the special relation $\kappa = -\kappa^2$ and $\tau = 0$, and should not be interpreted as a general feature of the formalism. Example 3 below demonstrates the general case with all three components being distinct.

The tangent vector is

$$\mathbf{T} = \frac{1}{\sqrt{2}}(\cos \theta - \sin \theta, \sin \theta + \cos \theta, 0), \quad \theta = \ln \frac{s^* + \sqrt{2}}{\sqrt{2}}.$$

The normal vector is

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\sin \theta - \cos \theta, \cos \theta - \sin \theta, 0).$$

The curvature is $\kappa = 1/(s^* + \sqrt{2})$, and the torsion is $\tau = 0$ (the curve is planar). The binormal is $\mathbf{B} = (0, 0, -1)$.

For a particle with position $\mathbf{x} = \beta(s^*(t))$, using $\delta_{\mathbf{T}} = 1$, $\delta_{\mathbf{N}} = 1$, and $\delta_{\mathbf{B}} = -1$ for Type 1, we find that

$$\begin{aligned} q &= \delta_{\mathbf{T}} \langle \mathbf{x}, \mathbf{T} \rangle = \langle \mathbf{x}, \mathbf{T} \rangle = \frac{s^* + \sqrt{2}}{2}, \\ p &= -\delta_{\mathbf{N}} \langle \mathbf{x}, \mathbf{N} \rangle = -\langle \mathbf{x}, \mathbf{N} \rangle = \frac{s^* + \sqrt{2}}{2}, \\ b &= \delta_{\mathbf{B}} \langle \mathbf{x}, \mathbf{B} \rangle = -\langle \mathbf{x}, \mathbf{B} \rangle = 0 \quad (\text{since } z = 0). \end{aligned}$$

Thus, $r = \sqrt{q^2 + p^2} = (s^* + \sqrt{2})/\sqrt{2}$, and $r^* = |q| = (s^* + \sqrt{2})/2$.

Assume $m = 1$ (constant mass) and $v = 1$ (unit speed), so $\dot{v} = 0$, $\dot{m} = 0$, and $\dot{m} = 0$. Then, from (17) and (18),

$$\begin{aligned} \mathbf{F} &= \kappa \mathbf{N}, \\ Y_T &= -\kappa^2, \\ Y_N &= \dot{\kappa} = -\frac{1}{(s^* + \sqrt{2})^2}, \\ Y_B &= 0. \end{aligned}$$

The radial decomposition (23) gives

$$\mathbf{F} = \left[\dot{m}v + m\dot{v} + \frac{q}{p}m\kappa v^2 \right] \mathbf{T} - \frac{mr}{p}\kappa v^2 \mathbf{e}_r.$$

With $\dot{m} = 0$, $\dot{v} = 0$, $m = 1$, $v = 1$, $q = p$, and $r = \sqrt{2}p$, this becomes

$$\mathbf{F} = \left[0 + 0 + \frac{p}{p}\kappa \right] \mathbf{T} - \frac{\sqrt{2}p}{p}\kappa \mathbf{e}_r = \kappa \mathbf{T} - \sqrt{2}\kappa \mathbf{e}_r.$$

which coincides with the decomposition above. The negative sign on the radial term correctly indicates an inward centripetal component.

Note that for this specific curve, $\dot{\kappa} = -\kappa^2$, so $Y_T = Y_N$ numerically. This is a coincidence of the example, not a general relation. For the force and yank components for the logarithmic spiral, see Figure 2.

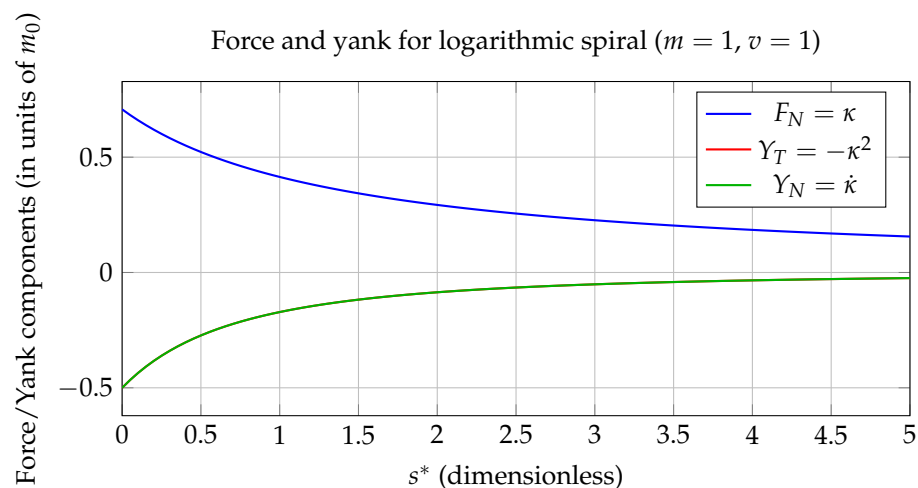


Figure 2. Force and yank components for the logarithmic spiral. Lengths are measured in units of the initial scale s_0^* ; force components are in units of m_0 (with $m = 1$ in this example). For this specific curve, $\dot{\kappa} = -\kappa^2$, so $Y_T = Y_N$ and the two curves coincide. This is a coincidence of the example, not a general relation.

5.3. Example 3: A Type 2 Spacelike Curve with All Yank Components being Distinct

To illustrate the full generality of the yank decomposition with all three components being non-trivially distinct, consider the curve in $\mathbb{R}_1^3$ defined by

$$\gamma(s) = \frac{1}{\sqrt{3}}(s, \sqrt{2} \sinh s, \sqrt{2} \cosh s), \quad s \in \mathbb{R}. \quad (34)$$

The tangent vector is

$$\gamma'(s) = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \cosh s, \frac{\sqrt{2}}{\sqrt{3}} \sinh s \right),$$

and

$$\langle \gamma', \gamma' \rangle = \frac{1}{3} + \frac{2}{3} \cosh^2 s - \frac{2}{3} \sinh^2 s = 1.$$

Thus, $v = 1$. The second derivative is

$$\gamma''(s) = \left(0, \frac{\sqrt{2}}{\sqrt{3}} \sinh s, \frac{\sqrt{2}}{\sqrt{3}} \cosh s \right),$$

with

$$\langle \gamma'', \gamma'' \rangle = \frac{2}{3} \sinh^2 s - \frac{2}{3} \cosh^2 s = -\frac{2}{3}.$$

Since $\langle \gamma', \gamma' \rangle = 1 > 0$ and $\langle \gamma'', \gamma'' \rangle < 0$, this is a Type 2 curve (spacelike with a timelike principal normal), for which $\delta_T = +1$, $\delta_N = -1$, and $\delta_B = +1$.

The Frenet frame is

$$\mathbf{T}(s) = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \cosh s, \frac{\sqrt{2}}{\sqrt{3}} \sinh s \right), \quad (35)$$

$$\mathbf{N}(s) = (0, \sinh s, \cosh s), \quad (36)$$

$$\mathbf{B}(s) = \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\cosh s}{\sqrt{3}}, -\frac{\sinh s}{\sqrt{3}} \right). \quad (37)$$

The curvature is

$$\kappa = \|\gamma''(s)\| = \sqrt{\frac{2}{3}},$$

and the torsion, obtained from $\mathbf{N}' = \kappa\mathbf{T} + \tau\mathbf{B}$, is

$$\tau = -\frac{1}{\sqrt{3}}.$$

Consider a particle with exponentially decaying mass

$$m(t) = m_0 e^{-\alpha t}, \quad \alpha = \frac{1}{2},$$

moving at unit speed ($v = 1, \dot{v} = \ddot{v} = 0$). Based on (19)–(21), with $\delta_{\mathbf{B}} = +1$,

$$Y_T = m(\kappa^2 + \alpha^2) = m\left(\frac{2}{3} + \frac{1}{4}\right) = \frac{11}{12}m, \quad (38)$$

$$Y_N = -2\alpha m\kappa = -\sqrt{\frac{2}{3}}m, \quad (39)$$

$$Y_B = m\kappa\tau = -\frac{\sqrt{2}}{3}m. \quad (40)$$

Thus,

$$(Y_T, Y_N, Y_B) = \left(\frac{11}{12}m, -\sqrt{\frac{2}{3}}m, -\frac{\sqrt{2}}{3}m\right) \approx (0.917m, -0.816m, -0.471m),$$

which are all non-zero and distinct. For the Yank components of the Type 2 spacelike curve $\gamma(s)$, see Figure 3.

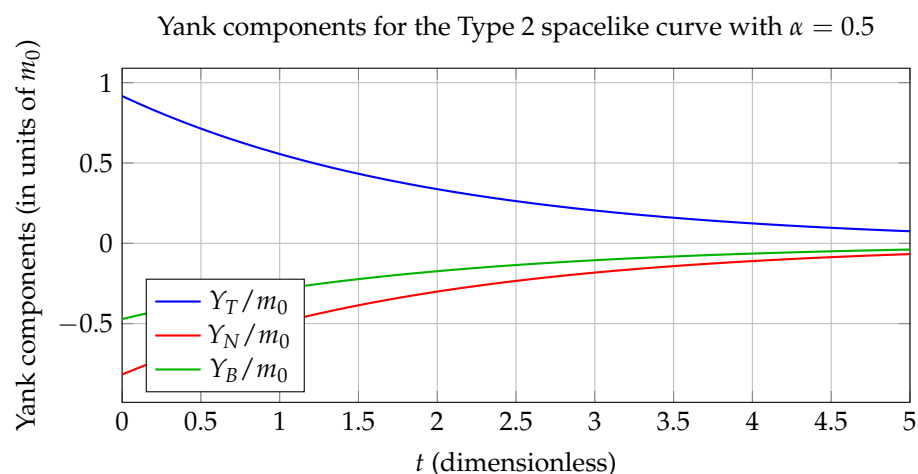


Figure 3. Yank components for the Type 2 spacelike curve $\gamma(s) = \frac{1}{\sqrt{3}}(s, \sqrt{2}\sinh s, \sqrt{2}\cosh s)$ with $m(t) = m_0 e^{-0.5t}$. Time t is dimensionless with $\alpha = 0.5$ providing the time scale; yank components are measured in units of m_0 . All three components are distinct and non-zero, demonstrating the full generality of the yank decomposition.

Remark 13. This example is particularly instructive because

- (i) It is a genuine Type 2 curve (spacelike with a timelike principal normal), complementing the Type 3 example (Lorentzian helix) and the planar Type 1 example (logarithmic spiral);
- (ii) The curvature and torsion are constant, so the geometric contributions to the yank components are clearly separated from the mass-variation contributions;

- (iii) All three yank components are non-zero and distinct, demonstrating the full generality of the yank decomposition.

6. Conclusions

We have presented a comprehensive geometric framework for variable-mass particles moving along non-lightlike curves in Minkowski 3-space, using the Frenet–Serret frame adapted to a Lorentzian signature. The main results are

1. Explicit closed-form expressions for momentum, force, and yank (Theorem 2) valid for all three causal types of curves, with a proof that transparently handles the sign factors;
2. A kinetic energy evolution theorem (Theorem 3) that separates inertial and reactive contributions;
3. A radial decomposition (Theorem 4) that generalizes Siacci’s classical theorem to Lorentzian geometry, with all signs verified;
4. Three illustrative physical scenarios (relativistic rocket motion, a heuristic geometric analogy for orbital parameter changes, and a formal geometric model of particle motion in a monopole-type field);
5. Two fully worked examples (a Lorentzian helix and a logarithmic spiral) with explicit calculations and numerical plots, plus a third example demonstrating the case where all yank components are distinct.

The framework developed here is restricted to flat three-dimensional Minkowski space $\mathbb{E}_1^3$ and does not constitute a fully covariant four-dimensional special-relativistic formulation. The physical applications should be understood as illustrative examples within this restricted setting.

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