

## Article

# A Practical Stabilization Controller Design for Positive Linear Time-Varying Systems with Multiple Time Delays

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## Abstract

In this brief paper, we investigate the practical stabilization problem of multiple time-delayed positive linear time-varying systems. Firstly, a sufficient condition for the practical stability of the system is derived using the maximal separable Lyapunov–Krasovskii functional. Then, a state feedback controller is designed for the considered positive linear time-varying system with multiple time delays. Building on this method, the study further addresses the practical stabilization problem for positive linear time-varying systems with multiple state time delays and input time delays. The positivity of the linear time-varying system with input time delays is verified by designing a state feedback controller, and the practical stability is proven using the maximal separable Lyapunov–Krasovskii functional method. Finally, numerical examples are presented to validate the feasibility of the proposed method.

**Keywords:** practical stabilization; linear time-varying; positive system; time-delays; Lyapunov–Krasovskii functional

**MSC:** 35A01; 65L10; 65L12; 65L70

## 1. Introduction

Positive systems are dynamical systems whose state variables remain non-negative whenever the initial conditions and inputs are non-negative. They arise naturally in fields such as biology, economics, and chemistry, where quantities like population sizes, concentrations, and prices cannot be negative [1–3]. The study of positive linear time-varying (LTV) systems with time delays introduces additional complexity, arising from both the time-varying nature of the system matrices and the presence of delays, which can affect the stability and performance of the system [4,5]. But early works established that even with delays, positivity can be preserved if these structural conditions hold [3–9].

Practical stability refers to the property that the system state remains within predefined bounds over a time interval, rather than asymptotically converging to zero [10–12]. Traditional control theory primarily focuses on Lyapunov asymptotic stability (i.e., the convergence of states to zero) [13]. However, in practical engineering applications, due to environmental disturbances, measurement errors, or physical limitations, requiring the system states to strictly converge to zero is often too stringent and may even be unachievable. Practical stability concerns whether the system states can converge and remain within a predeter-



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mined bounded region, or whether they exceed a certain safety threshold [6,10–12,14–26]. This concept is crucial in applications where transient behavior is critical, such as in chemical processes, power systems, or networked control systems [2,12,25,27]. Recent advances in practical stability analysis have focused on deriving computable conditions that ensure the state remains within bounds over a time horizon, despite time delays and time-varying parameters [28,29].

The main research method for practical stability of the general linear system is the Lyapunov-based approach, which is the most fundamental approach. The key technique is not to require strict negative definiteness of the derivative function of the Lyapunov function globally, but to prove that the derivative of the Lyapunov function is less than 0 when the state exceeds a certain small region. For linear systems with time delays, most techniques are based on Lyapunov–Krasovskii functionals (LKFs in brief), coefficient conditions and delay-independent/dependent conditions. These conditions ensure that the state ultimately enters and stays within a bounded set (attraction domain). However, for positive systems, the strategy for checking the practical stability is to construct linear copositive Lyapunov functions in the form of  $V(x) = \omega^T x$  with  $\omega > 0$  ( $x$  is the state of the positive LTV system), which are adapted to positivity and lead to tractable inequalities. For LTV systems, conditions often also involve bounds on state matrices  $A(t)$ . Regarding the issue of system time delays, the maximal separable LKF is currently one of the more advanced approaches [26], which was firstly proposed by [30]. By incorporating a maximization term that accounts for both the state and its delays, this method provides a closer estimation of the decay rate and state boundaries. It is particularly effective in handling time-varying and distributed delays. Delay-independent criteria are simpler to implement but tend to be more conservative, whereas delay-dependent criteria exploit knowledge of the time-varying delay  $\tau(t)$  to achieve less conservative, tighter bounds [5,15,28,29,31,32]. The delay partitioning method is also used in [33] to deal with the time-varying delays for stability analysis of linear systems.

Positive LTV systems with time delays are used to model phenomena such as virus spread, gene regulatory networks, and traffic flows. Practical stability ensures that key quantities, like the infected population, remain below critical thresholds within a finite time window; the stabilization of the positive linear system with time delays has been studied in many papers [7,15,16,34,35]. A linear system with input time delays has also been investigated because it is introduced in human–cyber–physical systems [26,36,37]. Motivated by all these works, and building on the analysis and observations regarding the practical stability of positive LTV systems with multiple time delays, this paper explores the practical stabilization problem for these systems. By designing an appropriate state feedback controller, the positive LTV system with multiple state time delays is transformed into its corresponding closed-loop time-delayed system. Initially, the positivity of this resulting closed-loop system is established. Subsequently, using a maximal separable Lyapunov–Krasovskii functional approach, sufficient conditions for practical stability are derived. This study concentrates on two primary types of positive LTV systems: one that involves multiple state time delays, and another that encompasses both input time delays and multiple state time delays. The primary contribution of this paper is summarized as follows:

- (1) For the first time, a practical stability control design framework applicable to multiple time-delayed positive linear time-varying systems is proposed, which simultaneously addresses the practical stability problems of multiple state delays and input delays.
- (2) It provides sufficient conditions for system positivity and construction methods for controller parameters, and provides clear directions for future research directions.

The structure of this paper is organized as follows: Section 2 presents the preliminary knowledge of this paper and the practical stabilization controller design of positive LTV systems with multiple time delays. Section 3 focuses on the practical stabilization of positive LTV systems with multiple time delays and input time delays, and Section 4 concludes the paper.

**Notations:** The following symbols are used in this paper. Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space, and  $\mathbb{R}_+$  denotes the positive real number set.  $\mathbb{R}^{n \times m}$  represents the set of  $n \times m$  real matrices.  $x(t) \in \mathbb{R}^n$  is the state vector of the system;  $A(t) = [a_{ij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$ ,  $A_d(t) = [a_{dij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  and  $A_{dk}(t) = [a_{dkij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  are function state matrices and delay matrices of the system with respect to time  $t$ ; and  $A_{dk}^{(i)}$  means the  $i$ -th row of the delay matrix  $A_{dk}$ . The  $n$ -dimensional vector function  $\varphi(t)$  is the initial state, with  $\tau_k$  representing the state time delays. The transpose of the matrix  $A$  is written as  $A^\top$ . For a matrix  $A$ ,  $A \geq 0$  indicates that all the entries in matrix  $A$  are non-negative.  $I(p)$  denotes the integer set  $I(p) = \{1, 2, \dots, p\}$  and  $p$  can take any value when needed.

## 2. Practical Stabilization of Positive LTV Systems with Multiple Time Delays

### 2.1. Preliminary Knowledge

In this section, we mainly consider the following LTV system with multiple time delays:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{k=1}^p A_{dk}(t)x(t - \tau_k), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0], \end{cases} \quad (\text{i})$$

and the LTV system with control input

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t) + \sum_{k=1}^p A_{dk}(t)x(t - \tau_k), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0], \end{cases} \quad (\text{ii})$$

where  $x(t) \in \mathbb{R}^n$  is the state vector of the system;  $A(t) = [a_{ij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  and  $A_d(t) = [a_{dij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  are function matrices with respect to time  $t$ ;  $\tau_k$  is the state time delays satisfying  $0 \leq \tau_k \leq \tau_{\max}$ , where  $k \in I(p)$ ;  $u(t) \in \mathbb{R}_+$  is the control input; and  $\varphi(t) \in \mathbb{R}^n$  is the initial state of the aforementioned system.

**Definition 1** (Practically stable [1]). Consider system (i) with initial condition  $x(t) = \varphi(t), t \in [-\tau_{\max}, 0]$ . Given a vector  $v \in \mathbb{R}_+^n, v > 0$ , and a pair of positive numbers  $(\lambda, \rho)$  with  $0 < \lambda < \rho$ , system (i) is said to be practically stable with respect to  $(\lambda, \rho)$  if  $\varphi(t) \leq \lambda v, \forall t \in [-\tau_{\max}, 0]$ ; then,  $x(t) \leq \rho v, \forall t \geq 0$ , where the inequalities are understood component-wise.

**Definition 2** (Metzler matrix [36]). For a matrix  $M \in \mathbb{R}^{n \times n}$ , if all its off-diagonal elements are non-negative, i.e.,  $m_{ij} \geq 0, i \neq j, i, j \in I(n)$ , then the matrix  $M$  is called a Metzler matrix.

**Definition 3** (Positive LTV System [38]). Consider LTV system (i) with initial condition  $x(t) = \varphi(t), t \in [-\tau_{\max}, 0]$ ; if the system state  $x(t) \in \mathbb{R}_+^n, x(t) \geq 0$  for any  $t \geq 0$ , which is component-wise, then system (i) is said to be positive.

**Lemma 1** (Positive system [39]). System (i) is a positive system if and only if matrix  $A(t)$  is a Metzler matrix and matrices  $A_{dk}(t) \geq 0$  for any  $k \in I(p), p \in \mathbb{R}_+^n$  and  $t \geq 0$ .

**Lemma 2** (Gronwall–Bellman Inequality [13]). For a continuous function  $x(t) \geq 0, f(t) \geq 0$ , if there exists a monotonic non-decreasing continuous positive function  $n(t)$  such that the inequality

ity  $x(t) \leq n(t) + \int_a^t f(s)x(s)ds, a \leq t \leq b$  holds, then it follows that  $x(t) \leq n(t)e^{\int_a^t f(s)ds}, a \leq t \leq b$ .

## 2.2. Practical Stabilization for Positive LTV Systems with Multiple Time Delays

Considering the positivity and practical stability of positive LTV system (i), we have the following Theorem.

**Theorem 1.** Consider a positive pair  $(\lambda, \rho)$  and a vector  $v \in R_n^+$ , where  $0 < \lambda < \rho$ , such that the initial state function  $\phi(t)$  satisfies the inequality  $\phi(t) < \lambda v$  for  $t \in [-\tau_{max}, 0]$ . For linear time-varying system (i) with multiple time delays, suppose that  $A(t)$  is a Metzler matrix, and  $A_{dk} \geq 0, k \in I(p)$ . If there exists a function  $\gamma(t) \in C[0, +\infty]$  such that the inequalities

$$a_{ii}(t) + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{j=1, j \neq i}^n (a_{ij}(t) + \sum_{k=1}^p a_{dkij}(t + \tau_k)) \frac{v_i}{v_j} \leq \gamma(t), \quad (1)$$

hold for all  $t \geq 0$ , and

$$\int_0^t \gamma(s)ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{max} H_{max})}, \quad (2)$$

where  $a_{ij}(t), a_{dkij}(t)$  are the entries of matrices  $A(t), A_{dk}(t)$ , respectively;  $v_i, v_j$  are the  $i$ th and  $j$ th components of vector  $v, i, j \in I(p)$ ; and

$$H_{max} = \max_{i \in I(n), s_1 \in (0, \tau_{max})} \left\{ \frac{\sum_{k=1}^p A_{dk}^{(i)}(s_1)v}{v_i} \right\}.$$

then multiple time-delayed LTV system (i) is positive and practically stable with respect to the positive pair  $(\lambda, \rho)$ .

**Proof.** Since  $A_0(t)$  is a Metzler matrix and  $A_{dk}(t) \geq 0, k \in I(p)$ , from Lemma 1, system (i) is positive. Next, we prove that multiple time-delayed LTV system (i) is practically stable. Consider the maximal separable Lyapunov function  $V(t, x(t))$  defined as follows:

$$V(t, x(t)) = \max_{i \in I(n)} \left\{ \frac{x_i(t) + \sum_{k=1}^p \int_{t-\tau_k}^t A_{dk}^{(i)}(s + \tau_k)x(s)ds}{v_i} \right\}. \quad (3)$$

Because  $A_{dk} \geq 0$  and system (i) is positive, we have

$$\sum_{k=1}^p \int_{t-\tau_k}^t A_{dk}^{(i)}(s + \tau_k)x(s)ds \geq 0.$$

Then, from (3), we obtain

$$\max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\} \leq V(t, x(t)). \quad (4)$$

When  $t = 0$ , the initial condition of the separable Lyapunov function  $V(t, x_t)$  is

$$V(0, x_0) = \max_{i \in I(n)} \left\{ \frac{x_i(0) + \sum_{k=1}^p \int_{-\tau_k}^0 A_{dk}^{(i)}(s + \tau_k)x(s)ds}{v_i} \right\}. \quad (5)$$

Since the initial state of system (i) satisfies  $x(t) = \phi(t) \leq \lambda v, t \in [-\tau_{max}, 0]$ , then we have

$$\frac{x_i(0)}{v_i} \leq \lambda, x(s) \leq \lambda v,$$

where  $s \in [-\tau_{\max}, 0], i \in I(p)$ . Then (5) can be rewritten as

$$V(0, x_0) \leq \max_{i \in I(n)} \left\{ \lambda + \lambda \frac{\sum_{k=1}^p \int_0^{\tau_k} A_{dk}^{(i)}(s_1) v ds_1}{v_i} \right\}. \quad (6)$$

Let  $s_1 = s + \tau_k$ . Then  $s_1 \in [0, \tau_{\max}]$ , and we have

$$V(0, x_0) \leq \max_{i \in I(n)} \left\{ \lambda + \lambda \frac{\sum_{k=1}^p \int_0^{\tau_k} A_{dk}^{(i)}(s_1) v ds_1}{v_i} \right\}. \quad (7)$$

From (4), we obtain

$$V(0, x_0) \leq \lambda(1 + \tau_{\max} H_{\max}), \quad (8)$$

Let the subscript  $m$  denote the component corresponding to the maximal of the upper right Dini derivative of the separable Lyapunov functional  $V(t, x(t))$ . Then we have

$$\begin{aligned} D^+ V(t, x_t) &= \frac{\dot{x}_m(t) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k) x(t) - \sum_{k=1}^p A_{dk}^{(m)}(t) x(t - \tau_k)}{v_m} \\ &= \frac{A_m(t) x(t) + \sum_{k=1}^p A_{dk}^{(m)}(t) x(t - \tau_k) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k) x(t) - \sum_{k=1}^p A_{dk}^{(m)}(t) x(t - \tau_k)}{v_m} \\ &= \frac{A_m(t) x(t) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k) x(t)}{v_m} \\ &= (a_{mm}(t) + \sum_{k=1}^p a_{dkmm}(t + \tau_k)) \frac{x_m(t) x(t)}{v_m} + \sum_{j=1, j \neq m}^n (a_{mj}(t) + \sum_{k=1}^p a_{dkmj}(t + \tau_k)) \frac{v_j}{v_m} \frac{x_j(t)}{v_j} \\ &\leq (a_{mm}(t) + \sum_{k=1}^p a_{dkmm}(t + \tau_k)) \frac{x_m(t)}{v_m} + \sum_{k=1}^p (a_{mj}(t) + \sum_{i=1}^p a_{dkmj}(t + \tau_k)) \frac{v_j}{v_m} \frac{x_m(t)}{v_m} \\ &= (a_{mm}(t) + \sum_{k=1}^p a_{dkmm}(t + \tau_k) + \sum_{k=1}^p (a_{mj}(t) + \sum_{i=1}^p a_{dkmj}(t + \tau_k)) \frac{v_j}{v_m}) \frac{x_m(t)}{v_m}. \end{aligned}$$

From (1), we obtain

$$D^+ V(t, x_t) \leq \gamma(t) \frac{x_m(t)}{v_m},$$

and because  $\frac{x_m(t)}{v_m} \in \max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\}$ , from (4), we get

$$D^+ V(t, x_t) \leq \gamma(t) V(t, x_t). \quad (9)$$

By Lemma 2, we have

$$V(t, x_t) \leq V(0, x_0) e^{\int_0^t \gamma(s) ds}. \quad (10)$$

Using (2), (10) can be rewritten as

$$V(t, x_t) \leq \lambda(1 + \tau_{\max} H_{\max}) e^{\ln \frac{\rho}{\lambda(1 + \tau_{\max} H_{\max})}} = \rho, \quad t \geq 0.$$

Substituting (8) into the above inequality yields

$$V(t, x_t) \leq \lambda(1 + \tau_{\max} H_{\max}) e^{\ln \frac{\rho}{\lambda(1 + \tau_{\max} H_{\max})}} = \rho, \quad t \geq 0. \quad (11)$$

From inequality (4), it follows that

$$\max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\} \leq V(t, x_t) \leq \rho. \quad (12)$$

Based on inequality (12), it is easy to see that  $x_i(t) \leq \rho v_i, i \in I(p), t > 0$ . Therefore,  $x(t) \leq \rho v$  for all  $t \geq 0$ . According to Definition 1 and Definition 3, multiple time-delayed LTV system (i) is positive and practically stable with respect to the positive pair  $(\lambda, \rho)$ .  $\square$

For aforementioned system (ii), we design a state feedback control law  $u(t) = H(t)x(t)$ , where  $(H(t))^T \in \mathbb{R}^n$ ; thus, we can obtain the following closed loop system:

$$\begin{cases} \dot{x}(t) = (A(t) + B(t)H(t))x(t) + \sum_{k=1}^p A_{dk}(t)x(t - \tau_k), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0]. \end{cases} \quad (13)$$

**Theorem 2.** Consider a positive pair  $(\lambda, \rho)$  and a vector  $v \in \mathbb{R}_+^n$ , where  $0 < \lambda < \rho$ . The initial state function  $\varphi(t)$  satisfies the inequality  $\varphi(t) \leq \lambda v$ , where  $t \in [-\tau_{\max}, 0]$ , for the considered system (13), satisfying  $A_{dk}(t) \geq 0, k \in I(p)$ . If there exists a function  $\gamma(t) \in C([0, \infty), \mathbb{R})$ , a constant  $\xi > 0$ , and a vector  $z(t) > 0, \sigma_1(t) > 0, \sigma_2(t) > 0$ , and the following inequalities hold:

$$A(t)(\sigma_1^T(t)\sigma_2(t)) + B(t)z^T(t) + \xi I \geq 0, \quad (14)$$

$$\left( a_{ii}(t) + \frac{b_i(t)z_i(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \frac{b_i(t)z_j(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \right) \leq \gamma(t), \quad (15)$$

$$\int_0^t \gamma(s)ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{\max}H_{\max})}, \quad (16)$$

which hold for any  $t \geq 0$ , where  $a_{ij}(t)$  and  $a_{dkij}(t)$  are the  $ij$ -th matrix elements of  $A(t)$  and  $A_{dk}(t)$ , respectively, and  $b_i(t)$  and  $z_i(t)$  are the  $i$ -th element of the vectors  $B(t)$  and  $z(t)$ , respectively, and

$$H_{\max} = \max_{\substack{i \in I(n), \\ s_1 \in [0, \tau_{\max}]}} \left\{ \frac{\sum_{k=1}^p A_{dk}^{(i)}(s_1)v}{v_i} \right\}, \quad (17)$$

then with state feedback control

$$u(t) = \frac{z^T(t)}{\sigma_1^T(t)\sigma_2(t)}x(t), \quad (18)$$

the closed-loop system (13) is positive and practically stable with respect to positive pairs  $(\lambda, \rho)$ .

**Proof.** First, we prove that the system (13) is a positive system. Given that  $\sigma_1(t) > 0, \sigma_2(t) > 0, \xi > 0$ , it is easy to obtain

$$\sigma_1^T(t)\sigma_2(t) > 0, \quad \frac{\xi}{\sigma_1^T(t)\sigma_2(t)} > 0,$$

According to (14), we can obtain

$$A(t) + B(t) \frac{z^T(t)}{\sigma_1^T(t)\sigma_2(t)} + \frac{\xi}{\sigma_1^T(t)\sigma_2(t)} I \geq 0. \quad (19)$$

Combining the state feedback control input (18), we can obtain

$$A(t) + B(t)H(t) + \frac{\xi}{\sigma_1^T(t)\sigma_2(t)}I \geq 0. \quad (20)$$

It is easy to obtain that  $(A(t) + B(t)H(t))$  is a Metzler matrix. Since  $A_{dk}(t) \geq 0, k \in I(p)$ , according to Lemma 1, the closed-loop system (13) is a positive system. Next, we prove the practical stability of the system (13). Consider the maximal separable Lyapunov functional  $V(t, x_t)$  as follows:

$$V(t, x_t) = \max_{i \in I(n)} \left\{ \frac{x_i(t) + \sum_{k=1}^p \int_{t-\tau_k}^t A_{dk}^{(i)}(s + \tau_k)x(s)ds}{v_i} \right\}, \quad (21)$$

Since  $A_{dk}(t) \geq 0, k \in I(p)$ , and system (13) is a positive system, it follows that

$$\sum_{k=1}^p \int_{t-\tau_k}^t A_{dk}^{(i)}(s + \tau_k)x(s)ds \geq 0.$$

According to (21), we obtain

$$\max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\} \leq V(t, x_t). \quad (22)$$

When  $t = 0$ , the initial condition of the maximum separable Lyapunov functional  $V(t, x_t)$  is

$$V(0, x_0) = \max_{i \in I(n)} \left\{ \frac{x_i(0) + \sum_{k=1}^p \int_{-\tau_k}^0 A_{dk}^{(i)}(s + \tau_k)x(s)ds}{v_i} \right\}. \quad (23)$$

Since the initial state of the system (13) satisfies  $x(t) = \varphi(t) \leq \lambda v, t \in [-\tau_{\max}, 0]$ , we have

$$\frac{x_i(0)}{v_i} \leq \lambda, \quad x(s) \leq \lambda v,$$

where  $s \in [-\tau_{\max}, 0], i \in I(n)$ , and (23) can be rewritten as

$$V(0, x_0) \leq \max_{i \in I(n)} \left\{ \lambda + \lambda \frac{\sum_{k=1}^p \int_{-\tau_k}^0 A_{dk}^{(i)}(s + \tau_k)v ds}{v_i} \right\}. \quad (24)$$

Let  $s_1 = s + \tau_k$ , where  $s_1 \in [0, \tau_{\max}]$ .

$$V(0, x_0) \leq \max_{i \in I(n)} \left\{ \lambda + \lambda \frac{\sum_{k=1}^p \int_0^{\tau_k} A_{dk}^{(i)}(s_1)v ds_1}{v_i} \right\}. \quad (25)$$

From (17), and by combining the geometric meaning of definite integral, we can obtain

$$V(0, x_0) \leq \lambda(1 + \tau_{\max}H_{\max}). \quad (26)$$

Let the subscript  $m$  denote the largest component of the upper right Dini derivative of the maximal separable Lyapunov function  $V(t, x_t)$ ; then,

$$\begin{aligned} D^+V(t, x_t) &= \frac{\dot{x}_m(t) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k)x(t) - \sum_{k=1}^p A_{dk}^{(m)}(t)x(t - \tau_k)}{v_m} \\ &= \frac{(A_m(t) + B_m(t)H_m(t))x(t) + \sum_{k=1}^p A_{dk}^{(m)}(t)x(t - \tau_k) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k)x(t)}{v_m} \\ &\quad - \frac{\sum_{k=1}^p A_{dk}^{(m)}(t)x(t - \tau_k)}{v_m} \\ &= \frac{(A_m(t) + B_m(t)H_m(t))x(t) + \sum_{k=1}^p A_{dk}^{(m)}(t + \tau_k)x(t)}{v_m}. \end{aligned}$$

Because  $H(t) = \frac{z^T(t)}{\sigma_1^T(t)\sigma_2(t)}$ , substituting into the above equality, we can obtain

$$\begin{aligned} D^+V(t, x_t) &= \left( a_{mm}(t) + \frac{b_m(t)z_m(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmm}(t + \tau_k) \right) \frac{x_m(t)}{v_m} \\ &\quad + \sum_{\substack{j=1, \\ j \neq m}}^n \left( a_{mj}(t) + \frac{b_m(t)z_j(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmj}(t + \tau_k) \right) \frac{v_j}{v_m} \frac{x_j(t)}{v_j} \\ &\leq \left( a_{mm}(t) + \frac{b_m(t)z_m(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmm}(t + \tau_k) \right) \frac{x_m(t)}{v_m} \\ &\quad + \sum_{\substack{j=1, \\ j \neq m}}^n \left( a_{mj}(t) + \frac{b_m(t)z_j(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmj}(t + \tau_k) \right) \frac{v_j}{v_m} \frac{x_m(t)}{v_m} \\ &= \left( a_{mm}(t) + \frac{b_m(t)z_m(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmm}(t + \tau_k) \right) \frac{x_m(t)}{v_m} \\ &\quad + \sum_{\substack{j=1, \\ j \neq m}}^n \left( a_{mj}(t) + \frac{b_m(t)z_j(t)}{\omega_1^T(t)\omega_2(t)} + \sum_{k=1}^p a_{dkmj}(t + \tau_k) \right) \frac{v_j}{v_m} \frac{x_m(t)}{v_m}. \end{aligned}$$

According to (15), we obtain

$$D^+V(t, x_t) \leq \gamma(t) \frac{x_m(t)}{v_m}.$$

From the fact that  $\frac{x_m(t)}{v_m} \in \max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\}$ , according to Equation (22), we can obtain

$$D^+V(t, x_t) \leq \gamma(t)V(t, x_t). \quad (27)$$

According to Lemma 2, we obtain

$$V(t, x_t) \leq V(0, x_0)e^{\int_0^t \gamma(s)ds}. \quad (28)$$

Based on Equation (16), Equation (28) can be rewritten as

$$V(t, x_t) \leq V(0, x_0)e^{\ln \frac{\rho}{\lambda(1+\tau_{\max}H_{\max})}}.$$

By substituting the above inequality with Equation (26), we can obtain

$$V(t, x_t) \leq \lambda(1 + \tau_{\max}H_{\max})e^{\ln \frac{\rho}{\lambda(1+\tau_{\max}H_{\max})}} = \rho, \quad t \geq 0, \quad (29)$$

and from inequality (22), we have

$$\max_{i \in I(n)} \left\{ \frac{x_i(t)}{v_i} \right\} \leq V(t, x_t) \leq \rho. \quad (30)$$

Based on the aforementioned inequality, it is easy to obtain  $x_i(t) \leq \rho v_i, i \in I(n), t \geq 0$ . Therefore, we have  $x(t) \leq \rho v, \forall t \geq 0$ . According to Definition 1 and Definition 3, the closed-loop system (13) is a positive system and is practically stable with respect to the positive pair  $(\lambda, \rho)$ .  $\square$

Through the aforementioned proof, it can be concluded that the closed-loop system (13) is positive and practically stable. Therefore, multiple time-delayed LTV control system (ii) is a positive system and practically stable with respect to the positive pair  $(\lambda, \rho)$ .

### 2.3. Numerical Example

In this section, the correctness of Theorem 1 and Theorem 2 for the LTV system with multiple time delays is verified through numerical examples, followed by simulation results of the considered system for further verification.

**Example 1.** In this example, the correctness of Theorem 1 for the LTV system with multiple time delays is verified. Consider system (i) with the parameters

$$A(t) = \begin{bmatrix} -0.7e^{-3\sin t} & 0.7e^{-3\sin t} \\ 2.5e^{-0.5\sin t} & -3.5e^{-0.5\sin t} \end{bmatrix}, \quad A_{d1}(t) = \begin{bmatrix} 2e^{-3t} & 0 \\ 0 & 2e^{-4t} \end{bmatrix},$$

$$A_{d2}(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}, \quad \tau_1 = 0.01, \tau_2 = 0.02,$$

and take the initial state as  $\varphi(0.3, 0.3)^T$ .

It is easy to find that  $A(t)$  is a Metzler matrix,  $A_{dk}(t) \geq 0, k \in I(2)$ . From Lemma 1, system (i) with the above parameters is a positive system. Next, the practical stability of system (i) will be validated based on Theorem 1. Given a pair of positive numbers  $(\lambda, \rho) = (0.5, 2)$  and a positive vector  $v(t) = (1, 1)^T$ , there exists a function  $\gamma(t) = 3e^{-2t}$  such that the following holds:

Case 1. When  $i = 1$ , according to (1), we have

$$\begin{aligned} & a_{ii}(t) + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=1}^p a_{dkij}(t + \tau_k) \right) \frac{v_2}{v_1} \\ &= a_{11}(t) + \sum_{k=1}^2 a_{dk11}(t + \tau_k) + (a_{12}(t) + \sum_{k=1}^2 a_{dk12}(t + \tau_k)) \\ &= 2e^{-3(t+0.01)} + e^{-3(t+0.02)}. \end{aligned}$$

Case 2. When  $i = 2$ , from (1), we have

$$\begin{aligned} & a_{ii}(t) + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=1}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \\ &= a_{22}(t) + \sum_{k=1}^2 a_{dk22}(t + \tau_k) + (a_{21}(t) + \sum_{k=1}^2 a_{dk21}(t + \tau_k)) \frac{v_1}{v_2} \\ &= -1.5e^{-0.5\sin t} + 2e^{-4(t+0.01)} + e^{-4(t+0.02)}. \end{aligned}$$

In summary, it can be concluded that

$$\max_{i=1,2} \left\{ a_{ii}(t) + \sum_{k=1}^2 a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^2 \left( a_{ij}(t) + \frac{b_i(t)z_j(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^2 a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \leq \gamma(t) = 3e^{-2t} \right\},$$

and because

$$H_{\max} = \max_{\substack{i \in I(n), \\ s_1 \in [0, \tau_{\max}]}} \left\{ \frac{\sum_{k=1}^p A_{dkii}(s_1)v}{v_i} \right\} = 3,$$

then we have

$$\int_0^t \gamma(s)ds = \int_0^t 3e^{-2s}ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{\max}H_{\max})} = \ln \frac{2}{0.53} \approx 1.328.$$

Therefore, conditions (1) and (2) in Theorem 1 are satisfied, and thus system (i) is positive and is practically stable with respect to the positive pair (0.5, 2).

The simulation results of the state trajectories of system (i) with the given initial state over time are shown in Figure 1. From Figure 1, it can be observed that the state trajectories of the LTV system with multiple time delays of (i) remain within the positive quadrant when  $t \geq 0$ , and we have  $x(t) \leq \rho v = (2, 2)^T$ .

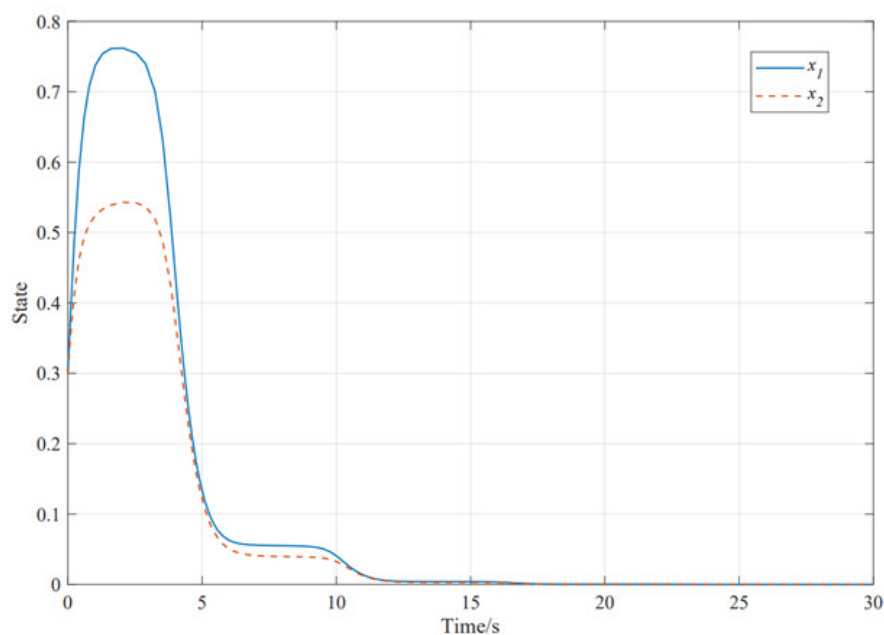


Figure 1. State trajectory of system (i) satisfying given parameters.

**Example 2.** Consider positive LTV control system (ii) with multiple state time delays with the following parameters:

$$A(t) = \begin{bmatrix} -1.5 & 0.5e^{-3t} \\ 0.2e^{-t} & 0 \end{bmatrix}, \quad A_{d1}(t) = \begin{bmatrix} 0.5e^{-2t} & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{d2}(t) = \begin{bmatrix} 0.7e^{-2t} & 0 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

The state time delays of multiple time-delayed LTV system (ii) are  $\tau_1 = 0.02$  and  $\tau_2 = 0.03$ , and the initial state of the system is  $\varphi(0.5, 0.5)^T$ . The state feedback control is taken as  $u(t) = H(t)x(t)$ .

Given a positive pair  $(\lambda, \rho) = (0.5, 2)$  and a positive vector  $v = (1, 1)^T$ , then there exists a positive vector  $\sigma_1(t) = (1, 1)^T$ ,  $\sigma_2(t) = (e^{-2t}, e^{-2t})^T$  and  $z(t) = (0.8e^{-2t}, 0.8e^{-4t})^T$ , where the function  $\gamma(t) = e^{-t}$  and the constants  $\xi = 4$ , where

$$H(t) = \frac{z^T(t)}{\sigma_1^T(t)\sigma_2(t)} = (0.4, 0.4e^{-2t}),$$

which make the following equation valid:

$$A(t) + B(t)H(t) + \frac{\xi}{\sigma_1^T(t)\sigma_2(t)}I = \begin{bmatrix} 0.4e^{-t} - 1.5 & 0.9e^{-3t} \\ 0.6e^{-t} & 0.4e^{-3t} \end{bmatrix} + \begin{bmatrix} 2e^{2t} & 0 \\ 0 & 2e^{2t} \end{bmatrix} \geq 0.$$

From the above formulas,  $(A(t) + B(t)H(t))$  is a Metzler matrix. Since  $A_{dk}(t) \geq 0$ ,  $k \in I(2)$ , it is easy to prove the positivity of the considered system through Lemma 1. Next, we will prove that the considered system is practically stable.

Case 1. When  $i = 1$ , from (4), we have

$$\begin{aligned} a_i(t) + \frac{b_i(t)z_i(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \frac{b_i(t)z_j(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \\ = a_1(t) + 0.4e^{-t} + \sum_{k=1}^2 a_{dk11}(t + \tau_k) + a_{12}(t) + 0.4e^{-3t} + \sum_{k=1}^2 a_{dk12}(t + \tau_k) \\ = 0.4e^{-t} + 0.9e^{-3t} + 0.5e^{-2(t+0.02)} + 0.7e^{-2(t+0.03)} - 1.5. \end{aligned}$$

Case 2. When  $i = 2$ , according to (4), we have

$$\begin{aligned} a_{ii}(t) + \frac{b_i(t)z_i(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \frac{b_i(t)z_j(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \\ = a_{22}(t) + 0.4e^{-3t} + \sum_{k=1}^2 a_{dk22}(t + \tau_k) + a_{21}(t) + 0.4e^{-t} + \sum_{k=1}^2 a_{dk21}(t + \tau_k) \\ = 0.6e^{-t} + 0.4e^{-3t}. \end{aligned}$$

In summary, it can be concluded that

$$\begin{aligned} \max_{i=1,2} \left\{ a_{ii}(t) + \frac{b_i(t)z_i(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^2 a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^2 \left( a_{ij}(t) + \frac{b_i(t)z_j(t)}{\sigma_1^T(t)\sigma_2(t)} + \sum_{k=1}^2 a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \right\} \\ \leq \gamma(t) = e^{-t}. \end{aligned}$$

And because

$$H_{\max} = \max_{\substack{i \in I(n), \\ s_1 \in [0, \tau_{\max}]}} \left\{ \frac{\sum_{k=1}^p A_{dki}(s_1)v}{v_i} \right\} = 1.2,$$

then,

$$\int_0^t \gamma(s)ds = \int_0^t e^{-s}ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{\max}H_{\max})} = \ln \frac{2}{0.518} \approx 1.351.$$

Therefore, according to Theorem 2, multiple time-delayed LTV control system (ii) with given parameters is positive and practically stable with respect to the positive pair  $(0.5, 2)$ .

The result curve of the state  $(x_1, x_2)^T$  of system (ii) with initial state  $\varphi(0.5, 0.5)^T$  as a function of time  $t$  is shown in Figure 1. From Figure 2, it can be observed that the state trajectory of multiple time-delayed LTV positive system (ii) remains in the positive quadrant after  $t \geq 0$ , and satisfies  $x(t) \leq \rho v = (2, 2)^T$ .

Meanwhile, we present the state trajectory of the system (ii) with the same parameters when there is no control input  $u(t)$ , as shown in Figure 3. From Figure 3, it can be observed that when time  $t > 0$ , the state trajectory of the system (ii) with initial state  $\varphi(0.5, 0.5)^T$  cannot be maintained within a certain range without control input. Within the range, the system is not practically stable. Therefore, for the considered system, we can make the originally unstable system achieve practical stability by adding appropriate control inputs. Example 2 validates the feasibility of Theorem 2.

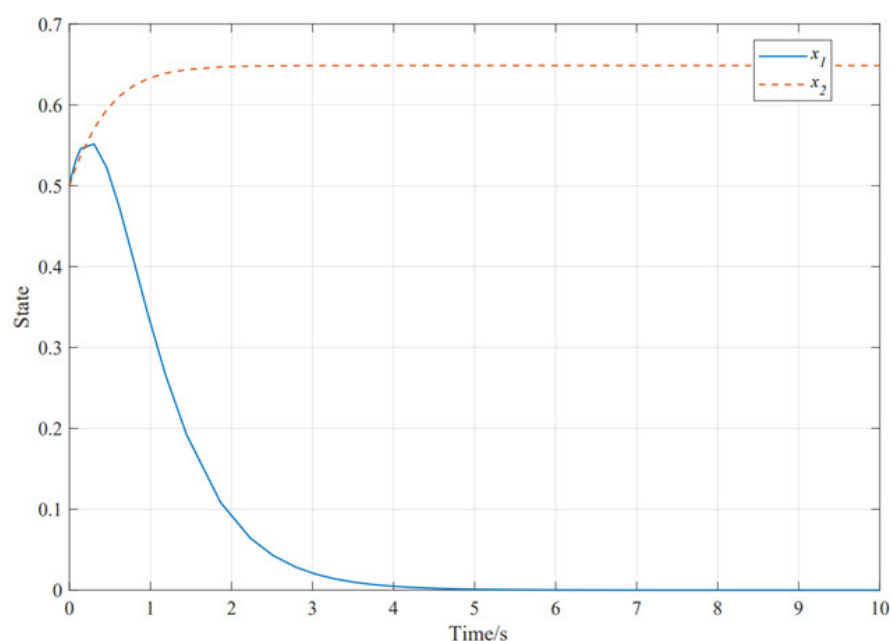


Figure 2. State trajectory of system (ii) satisfying given parameters.

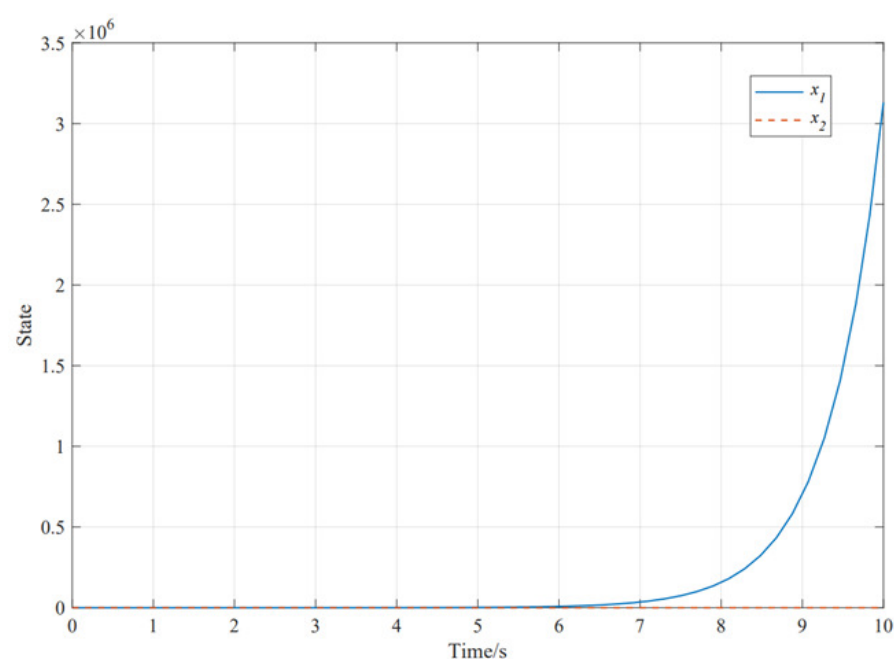


Figure 3. State trajectory of system (ii) without control inputs.

### 3. Practical Stabilization of Positive LTV Systems with Multiple State Time Delays and Input Time Delays

In this section, we consider the following LTV control system with multiple state time delays and input time delays:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{k=1}^p A_{dk}(t)x(t - \tau_k) + B(t)u(t - \tau_0), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0], \end{cases} \quad (\text{iii})$$

where  $x(t) \in \mathbb{R}^n$  is the state vector of the aforementioned system;  $A(t) = [a_{ij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  and  $A_{dk}(t) = [a_{dkij}(t)]^{n \times n} \in \mathbb{R}^{n \times n}$  are function matrices with respect to time  $t$ , where  $\tau_k$  denotes the state time delay and satisfies  $0 \leq \tau_k \leq \tau_{\max}$ ,  $k \in I(p)$ ,  $\tau_0$  is the input time delay, and  $u(t) \in \mathbb{R}_+$  is the control input;  $B(t) = [b_i(t)]^{n \times 1} \in \mathbb{R}^n$  is the input matrix; and  $\varphi(t) \in \mathbb{R}^n$  is the initial state of aforementioned system (iii).

#### 3.1. Practical Stabilization for Positive LTV Systems with Multiple Time Delays and Input Time Delays

For aforementioned system (iii), take the state feedback control law as  $u(t - \tau_0) = H(t - \tau_0)x(t - \tau_0)$  and  $(H(t - \tau_0))^T \in \mathbb{R}^n$ ; we can obtain the following closed loop system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{k=1}^p A_{dk}(t)x(t - \tau_k) + B(t)H(t - \tau_0)x(t - \tau_0), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0]. \end{cases} \quad (31)$$

**Theorem 3.** Given a positive pair  $(\lambda, \rho)$  and vector function  $v \in \mathbb{R}_+^n$ , where  $0 < \lambda < \rho$ , such that the initial state function  $\varphi(t)$  satisfies the inequality  $\varphi(t) \leq \lambda v$ , where  $t \in [-\tau_{\max}, 0]$ , for the considered system (31), satisfying  $A_{dk}(t) \geq 0$ ,  $k \in I(p)$ , if there exists a function  $\gamma(t) \in C([0, \infty), \mathbb{R})$ , a constant  $\xi > 0$  and a vector  $z(t) > 0$ ,  $\sigma_1(t) > 0$ ,  $\sigma_2(t) > 0$ , such that the following inequalities hold:

$$B(t) \frac{z^T(t - \tau_0)}{\sigma_1^T(t - \tau_0)\sigma_2(t - \tau_0)} \geq 0, \quad (32)$$

$$a_{ii}(t) + \sum_{k=0}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=0}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \leq \gamma(t), \quad (33)$$

$$\int_0^t \gamma(s) ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{\max} H_{\max})}, \quad (34)$$

which hold for any  $t \geq 0$ , where  $a_{ij}(t)$  and  $a_{dkij}(t)$  are the  $ij$ -th matrix elements of  $A(t)$  and  $A_{dk}(t)$ , respectively, and  $b_i(t)$  and  $z_i(t)$  are the  $i$ -th elements of vectors  $B(t)$  and  $z(t)$ , respectively, and

$$H_{\max} = \max_{\substack{i \in I(n), \\ s_1 \in [0, \tau_{\max}]}} \left\{ \frac{\sum_{k=1}^p A_{dk}^{(i)}(s_1)v}{v_i} \right\}, \quad (35)$$

then with state feedback control law

$$u(t - \tau_0) = \frac{z^T(t - \tau_0)}{\sigma_1^T(t - \tau_0)\sigma_2(t - \tau_0)} x(t - \tau_0), \quad (36)$$

the closed-loop system (31) is positive and practically stable with respect to positive pairs  $(\lambda, \rho)$ .

**Proof.** According to control law (36), we can obtain

$$H(t - \tau_0) = \frac{z^T(t - \tau_0)}{\sigma_1^T(t - \tau_0)\sigma_2(t - \tau_0)}.$$

Substituting the above expression into Equation (32), we obtain  $B(t)H(t - \tau_0) \geq 0$ . Let  $A_{d0}(t) = B(t)H(t - \tau_0)$ ; then, system (31) can be rewritten in the following form:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{k=0}^p A_{dk}(t)x(t - \tau_k), t \geq 0, \\ x(t) = \varphi(t), t \in [-\tau_{\max}, 0]. \end{cases}$$

Since  $A_{dk}(t) \geq 0$  for all  $k \in I(p)$  and  $A_{d0}(t) \geq 0$ , it follows that  $\sum_{k=0}^p A_{dk}(t) \geq 0$ .

Given that  $A(t)$  is a Metzler matrix, it can be easily proven that system (31) is a positive system according to Lemma 1. Then, the maximal separable Lyapunov function  $V(t, x_t)$  takes the following form:

$$V(t, x_t) = \max_{i \in I(n)} \left\{ \frac{x_i(t) + \sum_{k=0}^p \int_{t-\tau_k}^t A_{dki}(s + \tau_k)x(s)ds}{v_i} \right\}.$$

The subsequent proof is similar to the proof of Theorem 1, also involving the derivation of the maximal separable Lyapunov function. The upper right Dini derivative  $D^+V(t, x_t)$  of  $V(t, x_t)$  can be obtained by satisfying Equations (33) and (34), thus deriving that  $x(t) \leq \rho v$ , which holds for all  $\forall t \geq 0$ . According to Definition 2, we obtain that closed-loop system (31) is positive and practically stable with respect to the positive pair  $(\lambda, \rho)$  (the detailed proof is omitted).  $\square$

Then time-delayed LTV control system (iii) with multiple state time delays and input control time delays is positive, practical stable, and stabilized with respect to the positive pair  $(\lambda, \rho)$ .

### 3.2. Numerical Examples

In this section, for the multiple time-delayed LTV control system with input control time delays, numerical examples are provided to demonstrate the correctness of Theorem 3, followed by simulation results of the considered system for further verification.

**Example 3.** Consider positive LTV control system (31) with multiple time delays and input delays with the following parameters:

$$A(t) = \begin{bmatrix} -0.5e^{-3 \sin t} & 0.5e^{-3 \sin t} \\ 2e^{-0.5 \sin t} & -2e^{-0.5 \sin t} \end{bmatrix}, \quad A_{d1}(t) = \begin{bmatrix} 0.12e^{-0.01 \sin t} & 0 \\ 0 & 0.12e^{-0.02 \sin t} \end{bmatrix},$$

$$A_{d2}(t) = \begin{bmatrix} 0.1e^{-0.01 \sin t} & 0 \\ 0 & 0.1e^{-0.02 \sin t} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0.09e^{-0.1 \sin t} \\ 0.05e^{-0.1 \sin t} \end{bmatrix}.$$

Among them, the state time delay of multiple time-delayed LTV (31) with input time delay is  $\tau_1 = 0.02$  and  $\tau_2 = 0.03$ , and the input time delay is  $\tau_0 = 0.04$ . The initial state of the system is  $\varphi = (0.5, 0.5)^T$ . The state feedback control is taken as

$$u(t - \tau_0) = H(t - \tau_0)x(t - \tau_0).$$

Given the positive pair  $(\lambda, \rho) = (0.5, 3)$  and the positive vector  $v = (1, 1)^T$ , then there exists a positive vector  $\sigma_1(t) = (1, 1)^T$ ,  $\sigma_2(t) = (e^{0.02 \sin t}, e^{0.02 \sin t})^T$  and  $z(t) = (2e^{0.11 \sin t}, 2e^{0.10 \sin t})^T$ , where the functions  $\gamma(t) = e^{-0.005 \sin t}$ , where

$$H(t - \tau_0) = \frac{z^T(t - \tau_0)}{\sigma_1^T(t - \tau_0)\sigma_2(t - \tau_0)} = (e^{0.09 \sin(t - \tau_0)}, e^{0.08 \sin(t - \tau_0)}).$$

Thus, the following equation is valid:

$$B(t) \frac{z^T(t - \tau_0)}{\sigma_1^T(t - \tau_0)\sigma_2(t - \tau_0)} = \begin{bmatrix} 0.09e^{0.09 \sin(t - \tau_0) - 0.1 \sin t} & 0.09e^{0.08 \sin(t - \tau_0) - 0.1 \sin t} \\ 0.05e^{0.09 \sin(t - \tau_0) - 0.1 \sin t} & 0.05e^{0.08 \sin(t - \tau_0) - 0.1 \sin t} \end{bmatrix} \geq 0.$$

Then, given that  $B(t)H(t - \tau_0) \geq 0$ , let  $A_{d0}(t) = B(t)H(t - \tau_0)$ ; then,  $\sum_{k=0}^p A_{dk}(t) \geq 0$ , and since  $A(t)$  is a Metzler matrix, by virtue of Lemma 1, it is easy to prove the positivity of the considered system. The following shows the positivity of the system:

Case 1. When  $i = 1$ , according to Equation (33), we have

$$\begin{aligned} a_{ii}(t) + \sum_{k=0}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=0}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \\ = a_{11}(t) + \sum_{k=0}^2 a_{dk11}(t + \tau_k) + a_{12}(t) + \sum_{k=0}^2 a_{dk12}(t + \tau_k) \\ = 0.12e^{-0.01 \sin(t+0.02)} + 0.1e^{-0.01 \sin(t+0.03)} + 0.09e^{0.09 \sin(t) - 0.1 \sin(t+0.04)} + 0.09e^{0.08 \sin(t) - 0.1 \sin(t+0.04)}. \end{aligned}$$

Case 2. When  $i = 2$ , according to Equation (33), we have

$$\begin{aligned} a_{ii}(t) + \sum_{k=0}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=0}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \\ = a_{22}(t) + \sum_{k=0}^2 a_{dk22}(t + \tau_k) + a_{21}(t) + \sum_{k=0}^2 a_{dk21}(t + \tau_k) \\ = 0.12e^{-0.02 \sin(t+0.02)} + 0.1e^{-0.02 \sin(t+0.03)} + 0.05e^{0.09 \sin(t) - 0.1 \sin(t+0.04)} + 0.05e^{0.08 \sin(t) - 0.1 \sin(t+0.04)}. \end{aligned}$$

In summary, it can be concluded that

$$\max_{i=1,2} \left\{ a_{ii}(t) + \sum_{k=0}^p a_{dkii}(t + \tau_k) + \sum_{\substack{j=1, \\ j \neq i}}^n \left( a_{ij}(t) + \sum_{k=0}^p a_{dkij}(t + \tau_k) \right) \frac{v_j}{v_i} \right\} \leq \gamma(t) = e^{-0.005 \sin t}.$$

And because

$$H_{\max} = \max_{\substack{i \in I(n), \\ s_1 \in [0, \tau_{\max}]}} \left\{ \frac{\sum_{k=1}^p A_{dk}^{(i)}(s_1)v}{v_i} \right\} = 0.31$$

then,

$$\int_0^t \gamma(s) ds = \int_0^t e^{-0.005 \sin(s)} ds \leq \ln \frac{\rho}{\lambda(1 + \tau_{\max} H_{\max})} = \ln \frac{3}{0.5062} \approx 1.779.$$

Therefore, according to Theorem 2, multiple time-delayed LTV control system (iii) with input time delays is positive and practically stable with respect to the positive pair  $(0.5, 3, (1, 1)^T)$ .

The simulation curve of the state  $(x_1, x_2)^T$  of system (iii) with initial state  $\varphi = (0.5, 0.5)^T$  as a function of time  $t$  is shown in Figure 4. From Figure 4, it can be observed that the state trajectory of considered system (iii) remains within the positive quadrant for  $t \geq 0$ , and satisfies  $x(t) \leq \rho v = (3, 3)^T$ . Example 3 validates the feasibility of Theorem 3.

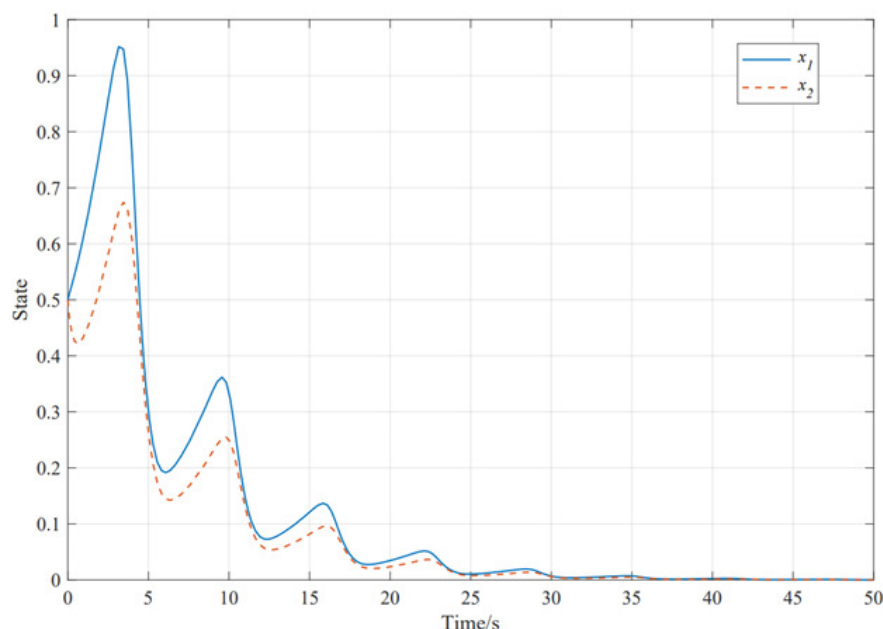


Figure 4. State trajectory of system (iii) satisfying the given parameters.

## 4. Conclusions

This paper investigates the practical stabilization problem of positive LTV systems subject to multiple time delays. For such a delayed system, an appropriate state feedback controller is designed, and the positivity of the closed loop system is verified via relevant lemmas and the maximal LKF method. Sufficient conditions for practical stabilization are then derived by applying the maximal separable Lyapunov function method to establish practical stability. Numerical examples and simulation results are provided to validate the theoretical findings. In addition, simulations of the uncontrolled system under the same parameters are presented, which demonstrate instability without control input. This comparison underscores the importance of introducing suitable control inputs to achieve practical stability in multiple time-delayed positive LTV systems. Similarly, for the case of an LTV system with both multiple state time delays and input delays, a delayed state feedback controller is designed to ensure system positivity and practical stability. The practical stabilization analysis employs the maximal separable LKF method, and the effectiveness of the proposed approach is confirmed through numerical simulations. Future research topics are interesting and challenging: (1) nonlinear positive systems: extending practical stability to nonlinear positive time-delayed systems (e.g., Lotka–Volterra models); (2) adaptive delays: systems where delay depends on state or parameters (common in networked control); (3) data-driven approaches: using machine learning to identify stability certificates for complex positive LTV systems [40]; (4) hybrid delays: mixing discrete, distributed, and time-varying delays.

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