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Weighted Higher-Order Beesack–Opial Inequalities on Time Scales

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Abstract

This work constructs a family of weighted higher-order Opial-type estimates on an arbitrary time scale $\mathbb{T}$, with the weight structure generated by a non-decreasing auxiliary function ω and its delta derivative ω^Δ . For n -times delta differentiable functions whose delta derivatives of orders $0, 1, \dots, n-1$ vanish at the left endpoint, the mixed functional $\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t$ is bounded by expressions involving only the highest-order delta derivative u^{Δ^n} . Three forms are obtained: a Hölder-type product estimate, a Young-type two-term estimate with a free balancing parameter $\theta > 0$, and an optimized single-integral estimate with the explicit constant $K = (r-1)^{1-1/r}/r$, where $r = (p+q)/q$. The proofs rely on the Taylor representation on time scales, Hölder's inequality, Jensen's inequality, and Fubini's theorem. The resulting bounds are governed by computable kernels that encode the interaction between the Taylor monomials, the auxiliary weight, and the external weight. Specializations to $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and the quantum lattice $q_0^{\mathbb{N}_0}$ recover classical, discrete, weighted, and quantum Opial inequalities with their standard constants and also yield several higher-order weighted estimates. As an application, a uniqueness criterion for higher-order dynamic initial value problems is established.

Keywords: Opial inequality; Beesack–Opial inequality; time scales; higher-order delta derivative; quantum calculus

MSC: 26D10; 26D15; 26E70; 34N05; 39A13



Academic Editor: Simeon Reich

Received: 1 June 2026

Revised: 7 July 2026

Accepted: 17 July 2026

Published: 19 July 2026

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1. Introduction

Opial-type inequalities constitute a fundamental class of integral inequalities whose importance stems from their ability to estimate mixed expressions involving a function and its derivative by quantities depending mainly, or entirely, on the corresponding derivative. This feature makes them useful in uniqueness theory, stability analysis, oscillation and disconjugacy, Lyapunov-type estimates, boundary value problems, and the qualitative theory of differential, difference, and dynamic equations; see, for example [1–4].

For clarity, we mention that some standard notions from time-scale calculus, such as the forward jump operator, delta derivative, delta integral, rd-continuity, and Taylor monomials on time scales, are used in the introductory discussion. Their precise definitions and basic properties are recalled in Section 2.

The classical starting point is the inequality of Opial [5]. If $u \in C^1[0, L]$, $u(0) = u(L) = 0$, and $u(t) > 0$ for $0 < t < L$, then

$$\int_0^L |u(t)u'(t)| dt \leq \frac{L}{4} \int_0^L |u'(t)|^2 dt. \quad (1)$$

A useful one-sided version, valid for absolutely continuous functions satisfying only $u(0) = 0$, is

$$\int_0^L |u(t)u'(t)| dt \leq \frac{L}{2} \int_0^L |u'(t)|^2 dt. \quad (2)$$

The one-sided boundary condition makes (2) especially suitable for initial value problems, while (1) remains the prototype for many two-endpoint Opial estimates. These inequalities have been developed within the broader theory of analytic inequalities; see, for instance, [1,3,6].

Power-type extensions appeared soon after. Hua [7] proved that if u is absolutely continuous on $[0, L]$, $u(0) = 0$, and $p \geq 0$, then

$$\int_0^L |u(t)|^p |u'(t)| dt \leq \frac{L^p}{p+1} \int_0^L |u'(t)|^{p+1} dt. \quad (3)$$

Yang [8] obtained the two-parameter form: for $p \geq 0$, $q \geq 1$, and $u(0) = 0$,

$$\int_0^L |u(t)|^p |u'(t)|^q dt \leq \frac{q}{p+q} L^p \int_0^L |u'(t)|^{p+q} dt, \quad (4)$$

whereas the two-sided condition $u(0) = u(L) = 0$ gives

$$\int_0^L |u(t)|^p |u'(t)|^q dt \leq \frac{q}{p+q} \left(\frac{L}{2}\right)^p \int_0^L |u'(t)|^{p+q} dt. \quad (5)$$

Taking $p = q = 1$ in (4) recovers (2), and the corresponding two-sided choice recovers (1).

Weighted variants introduced a second structural feature into the theory. If ω is a positive non-increasing weight on $[a, b]$, $u(a) = 0$, $p \geq 0$, and $q \geq 1$, then one has the estimate

$$\int_a^b \omega(t) |u(t)|^p |u'(t)|^q dt \leq \frac{q}{p+q} (b-a)^p \int_a^b \omega(t) |u'(t)|^{p+q} dt. \quad (6)$$

Another weighted form, involving a positive continuous function ρ with $\int_a^\tau dt/\rho(t) < \infty$, is

$$\int_a^\tau |u(t)| |u'(t)| dt \leq \frac{1}{2} \left(\int_a^\tau \frac{dt}{\rho(t)} \right) \left(\int_a^\tau \rho(t) |u'(t)|^2 dt \right). \quad (7)$$

Thus the early continuous theory already shows how endpoint information, powers of u and u' , and weights may interact in a single integral estimate.

A particularly influential recent contribution is due to Chesneau [9], who established a Beesack–Opial-type inequality involving two auxiliary functions v and ω and their interaction through ω' . More precisely, if $\omega(a) = 0$, $\omega'(t) > 0$, and

$$\Phi(x) = \int_x^b v(t) dt,$$

then, for $\gamma > 1$, $\gamma' = \gamma/(\gamma - 1)$, and $u(a) = 0$, one of Chesneau's main results asserts that

$$\int_a^b v(t) |u(t) u'(t)| dt \leq \left(\int_a^b v(t) \omega(t) |u'(t)|^\gamma dt \right)^{1/\gamma} \left(\int_a^b \Phi(t) \frac{|u'(t)|^{\gamma'}}{[\omega'(t)]^{\gamma-1}} dt \right)^{1/\gamma'}. \quad (8)$$

This result captures several earlier continuous inequalities of Beesack–Das, Olech, Yang, Cheung, and related authors as consequences, while introducing a flexible weight mechanism depending simultaneously on an auxiliary function and its derivative. The structure of (8) is particularly close to the present paper because the weight is not only inserted in the integrand but is also generated through an auxiliary function.

A further direction is obtained by replacing the first derivative by a higher-order derivative. Willett [10] showed that if $u \in C^n[a, b]$ and $u^{(i)}(a) = 0$ for $i = 0, 1, \dots, n - 1$, then

$$\int_a^b |u(t) u^{(n)}(t)| dt \leq \frac{(b-a)^n}{2} \int_a^b |u^{(n)}(t)|^2 dt. \quad (9)$$

A refinement due to Das [11], recorded in [12], states that under the same lower-order vanishing conditions, with $u^{(n-1)}$ absolutely continuous and $u^{(n)} \in L^2[a, b]$,

$$\int_a^b |u(t) u^{(n)}(t)| dt \leq \frac{1}{2n!} \left(\frac{n}{2n-1} \right)^{1/2} (b-a)^n \int_a^b |u^{(n)}(t)|^2 dt. \quad (10)$$

Such estimates are not formal repetitions of the first-order case: the proof depends on Taylor-type representations, the lower-order boundary conditions, and the way the n th derivative controls the function. A systematic treatment was given by Agarwal and Pang [1], and further developments include the works of Li [13], Pachpatte [14], Zhao and Cheung [15], and others; see also [12,16].

The continuous and discrete branches are naturally unified by time-scale calculus. A nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ is a time scale; the delta derivative and delta integral reduce to the ordinary derivative and the Riemann integral when $\mathbb{T} = \mathbb{R}$, and to the forward difference and summation when $\mathbb{T} = \mathbb{Z}$. This framework, introduced by Hilger [17] and developed by Bohner and Peterson [18], allows one theorem to produce continuous, discrete, nonuniform discrete, and q -dynamic consequences.

Bohner and Kaymakçalan [19] initiated the study of dynamic Opial inequalities. If $u : [0, L]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable and $u(0) = 0$, then

$$\int_0^L |u(t) + u^\sigma(t)| |u^\Delta(t)| \Delta t \leq L \int_0^L |u^\Delta(t)|^2 \Delta t. \quad (11)$$

They also proved weighted dynamic companions involving positive rd-continuous weights, in which the mixed term containing u , u^σ , and u^Δ is controlled by a product of two weighted integral factors. Bohner and Duman [20] later considered diamond- α versions, and Bohner, Mahmoud, and Saker [21] unified discrete, continuous, delta, nabla, and diamond- α Opial inequalities within one setting.

Higher-order dynamic Opial inequalities were developed by Saker [22] and by Saker, Agarwal, and O'Regan [23] using Hölder's inequality, Keller's chain rule, and Taylor monomials on time scales. A representative form is

$$\int_a^\tau \omega(t) |u(t) + u^\sigma(t)|^p |u^{\Delta^n}(t)|^q \Delta t \leq K(a, \tau, p, q) \int_a^\tau \rho(t) |u^{\Delta^n}(t)|^{p+q} \Delta t, \quad (12)$$

where the coefficient $K(a, \tau, p, q)$ depends on the weights and on Taylor monomials on $\mathbb{T}$. Bohner, Osman, and Saker [24] proved general higher-order dynamic Opial inequalities and applied them to lower bounds for distances between zeros of solutions and their derivatives

for a fourth-order dynamic equation. Osman, Saker, and Anderson [25] obtained two-function weighted higher-order dynamic inequalities, in which products of derivatives of two different functions are bounded by weighted integral norms of the corresponding higher-order derivatives.

Weighted time-scale results provide the closest background to the present paper. Srivastava, Tseng, Tseng, and Lo [26] proved that if $\omega \in C_{rd}([a, m]_{\mathbb{T}}, \mathbb{R}_+)$ is non-increasing, $p \geq 0$, $q \geq 1$, and $u(a) = 0$, then

$$\int_a^m \omega(t) |u(t)|^p |u^\Delta(t)|^q \Delta t \leq \frac{q}{p+q} (m-a)^p \int_a^m \omega(t) |u^\Delta(t)|^{p+q} \Delta t, \quad (13)$$

and if $u(b) = 0$ on the complementary interval, then

$$\int_m^b \omega(t) |u^\sigma(t)|^p |u^\Delta(t)|^q \Delta t \leq \frac{q}{p+q} (b-m)^p \int_m^b \omega(t) |u^\Delta(t)|^{p+q} \Delta t. \quad (14)$$

The higher-order weighted delta version was treated by El-Deeb, El-Sennary, and Agarwal [27], who derived estimates involving $|u^{\Delta^n}(t)|$ under endpoint assumptions on lower-order delta derivatives. More recent developments include weighted higher-order inequalities with applications to dynamic equations [28], generalizations involving monotone functions [29], and the weighted forms of Al-Shehri, El-Sheikh, Zakarya, Ali, Rezk, and Khamis [30], where nonzero endpoint values lead to an additional boundary term, for example,

$$\begin{aligned} \int_a^m \omega(t) |u(t)|^p |u^\Delta(t)|^q \Delta t &\leq 2^{p-1} \frac{q}{p+q} (m-a)^p \int_a^m \omega(t) |u^\Delta(t)|^{p+q} \Delta t \\ &\quad + 2^{p-1} |u(a)|^p \int_a^m \omega(t) |u^\Delta(t)|^q \Delta t. \end{aligned} \quad (15)$$

Motivation, Novelty, and Contributions

The developments reviewed above have progressed along essentially separate lines. The continuous auxiliary-function mechanism of Chesneau [9] generates weights through a pair involving ω and ω' , but it is formulated on $\mathbb{R}$ in the first order and for the product $|uu'|$. Higher-order dynamic inequalities on time scales treat derivatives of arbitrary order, but their weights are usually inserted into the integrand rather than generated by an auxiliary function. Weighted first-order dynamic estimates, on the other hand, do not directly include higher-order delta derivatives. This leads to the following question:

Can one establish a higher-order Opial inequality on an arbitrary time scale in which the weight is generated by an auxiliary function and its delta derivative, thereby combining the continuous Beesack–Opial mechanism, higher-order derivative estimates, and time-scale calculus in a single framework?

We answer this question affirmatively. The main contributions of the paper are as follows.

- (i) We prove weighted higher-order Opial-type inequalities on arbitrary time scales for general exponents $p \geq 1$ and $q > 0$, where the weight structure is generated by ω and ω^Δ .
- (ii) We introduce two explicit kernels, Ω_n and G , which record the interaction between the Taylor monomial, the auxiliary function, and the external weight.
- (iii) We obtain three complementary estimates: a Hölder-type product form, a Young-type form with a free parameter $\theta > 0$, and an optimized single-integral form with an explicit constant depending only on the ratio p/q .

- (iv) We show that the same framework recovers known continuous, discrete, and quantum Opial inequalities and yields a uniqueness criterion for higher-order dynamic initial value problems.

The paper is organized as follows. Section 2 recalls the notation and time-scale tools used in the proofs. Section 3 establishes the three principal inequalities and compares them with representative earlier results. Section 4 presents continuous, discrete, weighted, and quantum consequences, together with an application to uniqueness. Section 5 summarizes the results and outlines possible directions for further work.

2. Time Scales: Preliminaries and Notation

In this section we fix notation and collect the standard time-scale notation and analytical tools that are used throughout the paper. We refer the reader to the monograph of Bohner and Peterson [18] and the survey of Agarwal, Bohner, and Peterson [31] for full details and proofs.

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

with the conventions $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Remark 1. The conventions $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$ are the standard endpoint conventions used in time-scale calculus for the jump operators. They are not intended as general order-theoretic conventions for arbitrary subsets of $\mathbb{R}$. Their purpose is only to make the forward and backward jump operators well defined at the endpoints of a bounded time scale. For example, if $t = \sup \mathbb{T}$ and $\sup \mathbb{T} \in \mathbb{T}$, then the set $\{s \in \mathbb{T} : s > t\}$ is empty, and the convention gives $\sigma(t) = t$. Similarly, if $t = \inf \mathbb{T}$ and $\inf \mathbb{T} \in \mathbb{T}$, then $\{s \in \mathbb{T} : s < t\}$ is empty, and the convention gives $\rho(t) = t$.

The *graininess function* is $\mu(t) = \sigma(t) - t \geq 0$. A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ (equivalently $\mu(t) > 0$), *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, and *left-dense* if $\rho(t) = t$. We write $u^\sigma(t) = u(\sigma(t))$ for any function $u : \mathbb{T} \rightarrow \mathbb{R}$.

The *derived time scale* is $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has a left-scattered maximum m ; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. A function $u : \mathbb{T} \rightarrow \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists $u^\Delta(t) \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a neighborhood U of t with

$$|u^\sigma(t) - u(s) - u^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U \cap \mathbb{T}.$$

Higher-order delta derivatives are defined inductively: $u^{\Delta^n} = (u^{\Delta^{n-1}})^\Delta$, with $u^{\Delta^0} = u$. A key identity: if u is delta differentiable at t , then

$$u^\sigma(t) = u(t) + \mu(t)u^\Delta(t). \quad (16)$$

A function $u : \mathbb{T} \rightarrow \mathbb{R}$ is *rd-continuous* if it is continuous at every right-dense point and has a finite left limit at every left-dense point. We write $u \in C_{rd}(\mathbb{T}, \mathbb{R})$, and $C_{rd}(\mathbb{T}, \mathbb{R}_+)$ for its non-negative members. Every rd-continuous function possesses a delta antiderivative, so the following definition is consistent.

If $F : \mathbb{T} \rightarrow \mathbb{R}$ satisfies $F^\Delta = u$ on $\mathbb{T}^\kappa$, the *Cauchy delta integral* of u over $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ is

$$\int_a^b u(t) \Delta t = F(b) - F(a).$$

The Taylor monomials and Taylor formula used below are recalled next. The *generalized monomials* $h_k(\cdot, s)$ on $\mathbb{T}$ are defined recursively by

$$h_0(t, s) = 1, \quad h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau, \quad k \geq 0.$$

These reduce to $(t - s)^k/k!$ when $\mathbb{T} = \mathbb{R}$ and to the binomial coefficient $\binom{t-s}{k}$ when $\mathbb{T} = \mathbb{Z}$. Under the vanishing conditions $u^{\Delta^k}(a) = 0$ for $k = 0, 1, \dots, n-1$, the Taylor formula on time scales gives

$$u(t) = \int_a^t h_{n-1}(t, \sigma(\tau)) u^{\Delta^n}(\tau) \Delta\tau, \quad t \in [a, b]_{\mathbb{T}}, \quad (17)$$

and in particular

$$|u(t)| \leq \int_a^t |h_{n-1}(t, \sigma(\tau))| |u^{\Delta^n}(\tau)| \Delta\tau. \quad (18)$$

The analytical tools used in the proofs are collected in the following cited lemmas. The three lemmas below are the principal tools used throughout the paper.

Lemma 1 (Fubini's theorem on time scales [18] (Theorem 5.9)). *Let X and Y be time scales and let $u : X \times Y \rightarrow \mathbb{R}$ be Δ -integrable. Setting $\varphi(y) = \int_X u(x, y) \Delta x$ and $\psi(x) = \int_Y u(x, y) \Delta y$, we have $\varphi \in L^1_{\Delta}(Y)$, $\psi \in L^1_{\Delta}(X)$, and*

$$\int_X \Delta x \int_Y u(x, y) \Delta y = \int_Y \Delta y \int_X u(x, y) \Delta x. \quad (19)$$

Lemma 2 (Hölder's inequality on time scales [18] (Theorem 6.13)). *Let $u, v \in C_{\text{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$, $\gamma > 1$, and $1/\gamma + 1/\nu = 1$. Then*

$$\int_a^b |u(t) v(t)| \Delta t \leq \left[\int_a^b |u(t)|^{\gamma} \Delta t \right]^{1/\gamma} \left[\int_a^b |v(t)|^{\nu} \Delta t \right]^{1/\nu}. \quad (20)$$

Lemma 3 (Jensen's inequality on time scales [31]). *Let $a, b \in \mathbb{T}$ with $a < b$, and let $v \in C_{\text{rd}}([a, b]_{\mathbb{T}}, (c, d))$. If $u \in C([c, d], \mathbb{R})$ is convex, then*

$$u\left(\frac{1}{b-a} \int_a^b v(s) \Delta s\right) \leq \frac{1}{b-a} \int_a^b (u \circ v)(s) \Delta s. \quad (21)$$

Remark 2. *The preceding lemmas play specific and complementary roles in the proofs of the main inequalities. Hölder's inequality is used first to split the mixed integrand*

$$v(t)|u(t)|^p |u^{\Delta^n}(t)|^q$$

into two conjugate factors, one containing the weighted highest-order derivative and the other containing the power of u . It is then used a second time in combination with the Taylor representation on time scales to estimate $|u(t)|$ in terms of u^{Δ^n} and the auxiliary weight ω^{Δ} . Jensen's inequality is used only through its standard power-integral consequence

$$\left(\int_a^t F(x) \Delta x \right)^p \leq (t-a)^{p-1} \int_a^t F(x)^p \Delta x, \quad p \geq 1,$$

which converts a power of an integral into an integral of a power. Finally, Fubini's theorem is used to interchange the order of integration over the triangular region $a \leq x \leq t \leq b$. This interchange is what produces the auxiliary kernel G appearing in the statement of the main inequalities. Thus, the three lemmas are not only technical preliminaries; they determine the structure of the kernels and the final weighted estimates.

Finally, we fix the notation used in the main results. Throughout the remainder of the paper, $p \geq 1$ and $q > 0$ are fixed constants. We define the conjugate exponents

$$r = \frac{p+q}{q}, \quad s = \frac{p+q}{p}, \quad \frac{1}{r} + \frac{1}{s} = 1. \quad (22)$$

Note that $qr = p+q$, $ps = p+q$, $s/r = q/p$, and $p/(p+q) = 1/s$.

For $a, b \in \mathbb{T}$ with $a < b$, a weight $v \in C_{\text{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}_+)$, and a Δ -differentiable non-decreasing function $\omega : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ with $\omega(a) = 0$, we define the Taylor-kernel weight

$$\Omega_n(t) = \int_a^t |h_{n-1}(t, \sigma(x))|^r \omega^\Delta(x) \Delta x, \quad t \in [a, b]_{\mathbb{T}}, \quad (23)$$

and the auxiliary kernel

$$G(x) = \int_x^b v(t) \omega(t)^{-q/p} \Omega_n(t)^q (t-a)^{p-1} \Delta t, \quad x \in [a, b]_{\mathbb{T}}. \quad (24)$$

For $n = 1$, $h_0 \equiv 1$ and hence $\Omega_1(t) = \omega(t)$; in this case the kernel G reduces to the first-order form determined by the auxiliary function ω .

3. Main Results

In this section we establish three weighted higher-order Opial-type inequalities on an arbitrary time scale. Throughout, the following *standing hypotheses* are in force:

- $\mathbb{T}$ is a time scale and $a, b \in \mathbb{T}$ with $a < b$.
- $v \in C_{\text{rd}}([a, b]_{\mathbb{T}}, \mathbb{R}_+)$.
- $\omega : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ is Δ -differentiable and non-decreasing, with $\omega(a) = 0$.
- $u \in C_{\text{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$ is n -times delta differentiable and satisfies

$$u^{\Delta^k}(a) = 0, \quad k = 0, 1, \dots, n-1.$$

- $p \geq 1$ and $q > 0$, and r, s, Ω_n , and G are as defined in (22)–(24).

Remark 3. The lower-order vanishing conditions

$$u^{\Delta^k}(a) = 0, \quad k = 0, 1, \dots, n-1,$$

are used in an essential way in the proof. They allow the Taylor formula on time scales to represent $u(t)$ only in terms of its highest-order delta derivative:

$$u(t) = \int_a^t h_{n-1}(t, \sigma(\tau)) u^{\Delta^n}(\tau) \Delta \tau.$$

Consequently, the mixed term involving u can be estimated directly by integrals involving u^{Δ^n} . If the initial values $u(a), u^\Delta(a), \dots, u^{\Delta^{n-1}}(a)$ are not all zero, then the Taylor representation contains additional polynomial-type terms depending on these initial data. In that case, corresponding Opial-type estimates are expected to contain extra boundary contributions. Thus, the present results are most directly suited to zero-initial-data problems, while extensions to nonzero initial data form a natural direction for future work.

The first theorem is the foundational result, expressing the left-hand side as a geometric mean of two factors controlled by integrals of $|u^{\Delta^n}|^{p+q}$.

Theorem 1 (Hölder form). *Under the standing hypotheses,*

$$\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t \leq \left[\int_a^b v(t) \omega(t) |u^{\Delta^n}(t)|^{p+q} \Delta t \right]^{1/r} \times \left[\int_a^b G(t) \frac{|u^{\Delta^n}(t)|^{p+q}}{[\omega^{\Delta}(t)]^q} \Delta t \right]^{1/s}. \quad (25)$$

Proof. Write the integrand as a product and apply Lemma 2 with exponents r and s :

$$\begin{aligned} & \int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t \\ &= \int_a^b [v(t)^{1/r} \omega(t)^{1/r} |u^{\Delta^n}(t)|^q] [v(t)^{1/s} \omega(t)^{-1/r} |u(t)|^p] \Delta t \\ &\leq \underbrace{\left[\int_a^b v(t) \omega(t) |u^{\Delta^n}(t)|^{qr} \Delta t \right]^{1/r}}_{=: A^{1/r}} \underbrace{\left[\int_a^b v(t) \omega(t)^{-s/r} |u(t)|^{ps} \Delta t \right]^{1/s}}_{=: B^{1/s}}. \end{aligned} \quad (26)$$

The identities $qr = p + q$, $ps = p + q$, and $s/r = q/p$ (from (22)) simplify

$$A = \int_a^b v(t) \omega(t) |u^{\Delta^n}(t)|^{p+q} \Delta t, \quad B = \int_a^b v(t) \omega(t)^{-q/p} |u(t)|^{p+q} \Delta t. \quad (27)$$

Using the Taylor representation (18) together with the vanishing conditions, we first obtain

$$|u(t)| \leq \int_a^t |h_{n-1}(t, \sigma(x))| |u^{\Delta^n}(x)| \Delta x.$$

Applying Lemma 2 to the last integral with the factors

$$|h_{n-1}(t, \sigma(x))| [\omega^{\Delta}(x)]^{1/r} \quad \text{and} \quad \frac{|u^{\Delta^n}(x)|}{[\omega^{\Delta}(x)]^{1/r}},$$

we get

$$|u(t)| \leq \Omega_n(t)^{1/r} \left(\int_a^t \frac{|u^{\Delta^n}(x)|^s}{[\omega^{\Delta}(x)]^{s/r}} \Delta x \right)^{1/s}. \quad (28)$$

Since $s/r = q/p$, raising (28) to the power $p + q$ gives

$$|u(t)|^{p+q} \leq \Omega_n(t)^q \left(\int_a^t \frac{|u^{\Delta^n}(x)|^s}{[\omega^{\Delta}(x)]^q} \Delta x \right)^p.$$

By Lemma 3, in its elementary power-integral form,

$$\left(\int_a^t F(x) \Delta x \right)^p \leq (t-a)^{p-1} \int_a^t F(x)^p \Delta x, \quad p \geq 1,$$

with

$$F(x) = \frac{|u^{\Delta^n}(x)|^s}{[\omega^{\Delta}(x)]^{q/p}},$$

we obtain

$$|u(t)|^{p+q} \leq \Omega_n(t)^q (t-a)^{p-1} \int_a^t \frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q} \Delta x. \quad (29)$$

Substituting (29) into (27), we obtain

$$B \leq \int_a^b v(t) \omega(t)^{-q/p} \Omega_n(t)^q (t-a)^{p-1} \left(\int_a^t \frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q} \Delta x \right) \Delta t.$$

Applying Fubini's theorem on time scales, Lemma 1, to the nonnegative integrand

$$v(t)\omega(t)^{-q/p}\Omega_n(t)^q(t-a)^{p-1}\frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q},$$

we may interchange the order of integration in the triangular region

$$a \leq x \leq t \leq b.$$

Thus,

$$\begin{aligned} B &\leq \int_a^b v(t)\omega(t)^{-q/p}\Omega_n(t)^q(t-a)^{p-1} \left(\int_a^t \frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q} \Delta x \right) \Delta t \\ &= \int_a^b \left(\int_x^b v(t)\omega(t)^{-q/p}\Omega_n(t)^q(t-a)^{p-1} \Delta t \right) \frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q} \Delta x. \end{aligned}$$

Using the definition (24) of G , this becomes

$$B \leq \int_a^b G(x) \frac{|u^{\Delta^n}(x)|^{p+q}}{[\omega^{\Delta}(x)]^q} \Delta x. \quad (30)$$

Combining (26)–(30) yields (25). $\square$

Remark 4. In the proof of Theorem 1, several auxiliary factors are inserted and later cancel algebraically in special cases. This is intentional. The factorization

$$v(t)|u(t)|^p|u^{\Delta^n}(t)|^q = \left(v(t)^{1/r}\omega(t)^{1/r}|u^{\Delta^n}(t)|^q \right) \left(v(t)^{1/s}\omega(t)^{-1/r}|u(t)|^p \right)$$

is chosen so that Hölder's inequality produces the weighted derivative term

$$\int_a^b v(t)\omega(t)|u^{\Delta^n}(t)|^{p+q} \Delta t$$

and the complementary term involving $v(t)\omega(t)^{-q/p}|u(t)|^{p+q}$. Similarly, the factors involving ω^{Δ} in the estimate of the Taylor representation are inserted to produce the kernel

$$\Omega_n(t) = \int_a^t |h_{n-1}(t, \sigma(x))|^r \omega^{\Delta}(x) \Delta x.$$

Thus, these factors are not added for simplification; they determine the weighted structure of the inequality and lead naturally to the kernels Ω_n and G .

In particular examples, such as $\omega(t) = t - a$, some factors reduce to 1 or cancel inside the expression for G . Performing such cancellations at the beginning would give simpler formulas in those special cases, but it would hide the mechanism that works on a general time scale and for a general auxiliary function. Therefore, we keep the factors in the proof until the general kernels have been identified. Whenever an endpoint value such as $\omega(a) = 0$ produces a formal quotient at the lower limit, the corresponding expression is understood through the integral identity obtained before specialization, or by the natural limiting value when this is needed in an explicit example.

The following two corollaries record the direct specializations of Theorem 1 to the most important time scales.

Corollary 1 (Continuous case, $\mathbb{T} = \mathbb{R}$). Let $\mathbb{T} = \mathbb{R}$, $u \in C^n([a, b], \mathbb{R})$ with $u^{(k)}(a) = 0$ for $k = 0, \dots, n - 1$, and let $\omega \in C^1([a, b], \mathbb{R})$ be non-decreasing with $\omega(a) = 0$. With r, s, G as in (22)–(24) (with Δt replaced by dt), Theorem 1 gives

$$\int_a^b v(t) |u(t)|^p |u^{(n)}(t)|^q dt \leq \left(\int_a^b v(t) \omega(t) |u^{(n)}(t)|^{p+q} dt \right)^{q/(p+q)} \times \left(\int_a^b G(t) |u^{(n)}(t)|^{p+q} [\omega'(t)]^{-q} dt \right)^{p/(p+q)}. \quad (31)$$

This gives a higher-order continuous consequence in the classical line of [1].

Corollary 2 (Discrete case, $\mathbb{T} = \mathbb{Z}$). Let $\mathbb{T} = \mathbb{Z}$, $a < b$ integers, $u : [a, b]_{\mathbb{Z}} \rightarrow \mathbb{R}$ with $\Delta^k u(a) = 0$ for $k = 0, \dots, n-1$, and $\omega : [a, b]_{\mathbb{Z}} \rightarrow \mathbb{R}$ non-decreasing with $\omega(a) = 0$. For $j \in \{a, a+1, \dots, b-1\}$, the discrete kernel is

$$\Omega_n(j) = \sum_{k=a}^{j-1} |h_{n-1}(j, \sigma(k))|^r \Delta \omega(k).$$

Since $\sigma(k) = k+1$ on $\mathbb{Z}$, and

$$h_{n-1}(j, k+1) = \binom{j-k-1}{n-1},$$

we have the computable form

$$\Omega_n(j) = \sum_{k=a}^{j-1} \binom{j-k-1}{n-1}^r \Delta \omega(k).$$

Here the binomial coefficient is understood to be zero whenever $j-k-1 < n-1$. Here G is given by (24); explicitly,

$$G(x) = \sum_{j=x}^{b-1} v(j) \omega(j)^{-q/p} \Omega_n(j)^q (j-a)^{p-1}.$$

Theorem 1 gives

$$\sum_{k=a}^{b-1} v(k) |u(k)|^p |\Delta^n u(k)|^q \leq \left(\sum_{k=a}^{b-1} v(k) \omega(k) |\Delta^n u(k)|^{p+q} \right)^{q/(p+q)} \times \left(\sum_{k=a}^{b-1} G(k) |\Delta^n u(k)|^{p+q} [\Delta \omega(k)]^{-q} \right)^{p/(p+q)}. \quad (32)$$

Setting $n = 1$, $p = q = 1$, $v \equiv 1$, $\omega(k) = k - a$, and applying the AM–GM inequality to (32) yields the classical discrete Opial inequality [32,33]:

$$\sum_{k=a}^{b-1} |u(k) \Delta u(k)| \leq \frac{b-a}{2} \sum_{k=a}^{b-1} |\Delta u(k)|^2.$$

Corollary 3 (q_0 -time scale, $\mathbb{T} = q_0^{\mathbb{N}_0}$). Let $\mathbb{T} = q_0^{\mathbb{N}_0} = \{q_0^k : k \in \mathbb{N}_0\}$ with $q_0 > 1$, $a = q_0^{k_0}$, $b = q_0^{k_1}$, $k_0 < k_1$. The Δ -derivative is the Jackson q_0 -derivative $D_{q_0} u(t) = [u(q_0 t) - u(t)] / [(q_0 - 1)t]$, the Δ -integral is $\int_a^b u \Delta t = \sum_{k=k_0}^{k_1-1} u(q_0^k) (q_0 - 1) q_0^k$, and $\mu(t) = (q_0 - 1)t$. Under the analogue of the standing hypotheses on $q_0^{\mathbb{N}_0}$, Theorem 1 yields

$$\begin{aligned} & \sum_{k=k_0}^{k_1-1} v(q_0^k) |u(q_0^k)|^p |D_{q_0}^n u(q_0^k)|^q (q_0 - 1) q_0^k \\ & \leq \left(\sum_{k=k_0}^{k_1-1} v(q_0^k) \omega(q_0^k) |D_{q_0}^n u(q_0^k)|^{p+q} (q_0 - 1) q_0^k \right)^{q/(p+q)} \\ & \quad \times \left(\sum_{k=k_0}^{k_1-1} G(q_0^k) |D_{q_0}^n u(q_0^k)|^{p+q} [D_{q_0} \omega(q_0^k)]^{-q} (q_0 - 1) q_0^k \right)^{p/(p+q)}, \end{aligned} \quad (33)$$

where $G(q_0^{k_0})$ is evaluated using the q_0 -integral. This extends the q_0 -Opial inequality of Mirkovic, Trickovic, and Stankovic [34] to higher-order and to general weights.

The second main theorem introduces a free parameter $\theta > 0$ via Young's inequality, giving greater flexibility in applications.

Theorem 2 (Young's form). *Under the standing hypotheses, for any $\theta > 0$,*

$$\begin{aligned} \int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t & \leq \frac{\theta^r}{r} \int_a^b v(t) \omega(t) |u^{\Delta^n}(t)|^{p+q} \Delta t \\ & \quad + \frac{1}{s\theta^s} \int_a^b G(t) \frac{|u^{\Delta^n}(t)|^{p+q}}{[\omega^{\Delta}(t)]^q} \Delta t. \end{aligned} \quad (34)$$

Proof. From the proof of Theorem 1, we have

$$\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t \leq A^{1/r} B^{1/s},$$

where A is the first integral in (27) and B is the second integral in (27). Moreover, B is bounded by the right-hand side of (30).

We use the following parameterized form of Young's inequality: if $r, s > 1$, $1/r + 1/s = 1$, $X, Y \geq 0$, and $\theta > 0$, then

$$X^{1/r} Y^{1/s} \leq \frac{\theta^r}{r} X + \frac{1}{s\theta^s} Y.$$

Indeed, this follows from the classical Young inequality $ab \leq a^r/r + b^s/s$ by taking $a = \theta X^{1/r}$ and $b = Y^{1/s}/\theta$. In the present situation, $A \geq 0$ and $B \geq 0$. If $A = 0$ or $B = 0$, the desired estimate is immediate. Otherwise, applying the above inequality with $X = A$ and $Y = B$ gives

$$A^{1/r} B^{1/s} \leq \frac{\theta^r}{r} A + \frac{1}{s\theta^s} B.$$

Using the upper bound for B from (30) completes the proof. $\square$

Corollary 4 (Continuous case, $\mathbb{T} = \mathbb{R}$). *Let $\mathbb{T} = \mathbb{R}$ with the same assumptions as in Corollary 1. For any $\theta > 0$, Theorem 2 specializes to*

$$\begin{aligned} \int_a^b v(t) |u(t)|^p |u^{(n)}(t)|^q dt & \leq \frac{\theta^r}{r} \int_a^b v(t) \omega(t) |u^{(n)}(t)|^{p+q} dt \\ & \quad + \frac{1}{s\theta^s} \int_a^b G(t) \frac{|u^{(n)}(t)|^{p+q}}{|\omega'(t)|^q} dt. \end{aligned} \quad (35)$$

This gives a parametrized higher-order consequence in the spirit of [22,27].

Corollary 5 (Discrete case, $\mathbb{T} = \mathbb{Z}$). *Let $\mathbb{T} = \mathbb{Z}$ with the same assumptions as in Corollary 2. For any $\theta > 0$, Theorem 2 gives*

$$\sum_{k=a}^{b-1} v(k) |u(k)|^p |\Delta^n u(k)|^q \leq \frac{\theta^r}{r} \sum_{k=a}^{b-1} v(k) \omega(k) |\Delta^n u(k)|^{p+q} + \frac{1}{s\theta^s} \sum_{k=a}^{b-1} G(k) \frac{|\Delta^n u(k)|^{p+q}}{|\Delta \omega(k)|^q}. \quad (36)$$

This gives a discrete higher-order consequence related to [1,32].

The third theorem derives a compact single-integral bound from Theorem 2 by choosing θ optimally.

Theorem 3 (Single-integral form). *Under the standing hypotheses,*

$$\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t \leq K \int_a^b \left[v(t) \omega(t) |u^{\Delta^n}(t)|^{p+q} + G(t) \frac{|u^{\Delta^n}(t)|^{p+q}}{[\omega^{\Delta}(t)]^q} \right] \Delta t, \quad (37)$$

where

$$K = \frac{(r-1)^{1-1/r}}{r}, \quad r = \frac{p+q}{q}. \quad (38)$$

Proof. Starting from Theorem 2, we have, for every $\theta > 0$,

$$\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t \leq \frac{\theta^r}{r} \int_a^b v(t) \omega(t) |u^{\Delta^n}(t)|^{p+q} \Delta t + \frac{1}{s\theta^s} \int_a^b G(t) \frac{|u^{\Delta^n}(t)|^{p+q}}{[\omega^{\Delta}(t)]^q} \Delta t.$$

To obtain a single coefficient multiplying the sum of the two integrals, we choose θ so that the two coefficients in the preceding estimate are equal. Thus we impose

$$\frac{\theta^r}{r} = \frac{1}{s\theta^s}.$$

Multiplying both sides by $rs\theta^s$ gives

$$s\theta^{r+s} = r,$$

or equivalently

$$\theta^{r+s} = \frac{r}{s}. \quad (39)$$

Using

$$r = \frac{p+q}{q}, \quad s = \frac{p+q}{p},$$

we compute

$$r+s = \frac{p+q}{q} + \frac{p+q}{p} = \frac{(p+q)p + (p+q)q}{pq} = \frac{(p+q)^2}{pq},$$

and

$$\frac{r}{s} = \frac{(p+q)/q}{(p+q)/p} = \frac{p}{q}.$$

Therefore, from (39),

$$\theta = \left(\frac{r}{s}\right)^{1/(r+s)} = \left(\frac{p}{q}\right)^{pq/(p+q)^2}.$$

With this choice of θ , the common coefficient is

$$K = \frac{\theta^r}{r}.$$

Since

$$\theta^r = \left(\frac{p}{q}\right)^{rpq/(p+q)^2} = \left(\frac{p}{q}\right)^{p/(p+q)},$$

we obtain

$$K = \frac{1}{r} \left(\frac{p}{q}\right)^{p/(p+q)}.$$

Finally, because

$$\frac{p}{q} = r - 1, \quad \frac{p}{p+q} = 1 - \frac{1}{r},$$

we get

$$K = \frac{(r-1)^{1-1/r}}{r}.$$

Substituting this balanced value into Theorem 2 gives (37). $\square$

Remark 5. The constant

$$K = \frac{(r-1)^{1-1/r}}{r}, \quad r = \frac{p+q}{q},$$

depends only on the ratio of the exponents p and q . In the canonical case $p = q = 1$, we have $r = 2$, and therefore

$$K = \left. \frac{(r-1)^{1-1/r}}{r} \right|_{r=2} = \frac{1^{1/2}}{2} = \frac{1}{2}.$$

This is the value that gives the classical one-sided Opial constant in the first-order continuous and discrete specializations. The study of best constants in related integral inequalities is classical; see, for example [35]. For example, if $p = 2$ and $q = 1$, then $r = 3$, and

$$K = \frac{(3-1)^{1-1/3}}{3} = \frac{2^{2/3}}{3}.$$

If $p = 1$ and $q = 2$, then $r = 3/2$, and

$$K = \frac{(1/2)^{1-2/3}}{3/2} = \frac{(1/2)^{1/3}}{3/2} = \frac{2^{2/3}}{3}.$$

Thus the two choices $(p, q) = (2, 1)$ and $(p, q) = (1, 2)$ give the same numerical value of K . This coincidence is specific to these parameter values; the formula for K is not symmetric in the conjugate exponents.

Comparison with Existing Results

We now indicate how Theorems 1–3 relate to the principal results discussed in the introduction.

- (a) **Relation to Chesneau.** If $\mathbb{T} = \mathbb{R}$, $n = 1$, and $p = q = 1$, then $r = s = 2$, $\Omega_1 = \omega$, and the factor $\omega(t)^{-q/p} \Omega_1(t)^q (t-a)^{p-1}$ reduces to 1. Hence $G(x) = \int_x^b v(t) dt$, and Theorem 1 gives exactly the case $\gamma = 2$ of Chesneau's estimate [9]. The present

result extends this overlap by allowing arbitrary order, arbitrary time scales, and the exponents (p, q) in the mixed term.

- (b) **Relation to Srivastava et al.** The weighted first-order time-scale estimates of Srivastava, Tseng, Tseng, and Lo [26] involve an inserted non-increasing weight and a global length factor. In contrast, Theorem 3 uses a generated non-decreasing auxiliary weight and replaces the global length factor by the pointwise kernel $v(t)\omega(t) + G(t)[\omega^\Delta(t)]^{-q}$. In the basic case $n = 1$, $p = q = 1$, $v \equiv 1$, and $\omega(t) = t - a$, both approaches recover the same classical constant.
- (c) **Relation to Willett and Das.** For the unweighted continuous higher-order functional $\int_a^b |u(t)u^{(n)}(t)| dt$, the sharp constant of Das [11] is not improved here. The present contribution is structural: it attaches an auxiliary-generated weight mechanism to the higher-order setting and keeps the kernels explicit on arbitrary time scales.
- (d) **Relation to higher-order dynamic inequalities.** In the higher-order dynamic inequalities of Saker [22], Saker, Agarwal, and O'Regan [23], and El-Deeb, El-Sennary, and Agarwal [27], the weights are inserted into the estimates, and the constants are fixed by the chosen weights and Taylor monomials. Theorem 2 gives instead a parametrized family with a free balancing parameter θ , while Theorem 3 yields an optimized coefficient depending only on p/q .
- (e) **Relation to nonzero-endpoint and two-function estimates.** The estimates of Al-Shehri et al. [30] allow nonzero endpoint values in the first order and therefore include additional boundary terms. Our results treat arbitrary order under zero initial data, so the two settings are complementary. The two-function inequalities of Osman, Saker, and Anderson [25] address products of derivatives of two different functions and are not directly comparable with the present single-function auxiliary-generated framework.

Table 1 summarizes these structural differences.

Table 1. Structural comparison with representative Opial-type results.

Result	Domain	Order	Weight Mechanism	Free Parameter
Opial [5]	$\mathbb{R}$	1	none	—
Yang [8]	$\mathbb{R}$	1	none	—
Chesneau [9]	$\mathbb{R}$	1	generated by (ω, ω')	$\gamma > 1$
Willett [10], Das [11]	$\mathbb{R}$	n	none	—
Bohner–Kaymakçalan [19]	$\mathbb{T}$	1	inserted	—
Srivastava et al. [26]	$\mathbb{T}$	1	inserted, non-increasing	—
Saker et al. [22,23]	$\mathbb{T}$	n	inserted	—
El-Deeb et al. [27]	$\mathbb{T}$	n	inserted	—
Al-Shehri et al. [30]	$\mathbb{T}$	1	inserted, boundary terms	—
This paper	$\mathbb{T}$	n	generated by (ω, ω^Δ)	$\theta > 0$

4. Applications and Illustrative Examples

The general theorems become concrete once the time scale, the exponents, the external weight, and the auxiliary function are chosen. This section presents several such specializations with two aims: to verify that the framework reproduces the known classical, discrete, weighted, and quantum inequalities with their standard constants, and to display estimates whose kernels can be computed explicitly.

When applying the estimates, the individual ingredients play distinct roles. The external weight v may model damping, density, or another nonuniform distribution; the auxiliary function ω determines how endpoint information enters the bound through ω^Δ ; the exponents p and q tune the relative strength of $|u|$ and $|u^\Delta|$; and the order n selects

the Taylor kernel through which lower-order behavior is controlled by u^{Δ^n} . The examples below should therefore be read as templates rather than as isolated formulas.

We begin by checking that the single-integral theorem returns the one-sided Opial inequality with its classical constant.

Corollary 6 (Classical Opial inequality [5]). *Let $\mathbb{T} = \mathbb{R}$, $n = 1$, $p = q = 1$, $v \equiv 1$, and $\omega(t) = t - a$. If $u \in C^1([a, b], \mathbb{R})$ and $u(a) = 0$, then*

$$\int_a^b |u(t)u'(t)| \, dt \leq \frac{b-a}{2} \int_a^b |u'(t)|^2 \, dt. \quad (40)$$

Proof. For these choices we have $\omega'(t) = 1$, $\omega(a) = 0$, $r = s = 2$, $K = 1/2$, and

$$G(x) = \int_x^b (t-a)^0 \, dt = b-x.$$

Applying Theorem 3, we obtain

$$\int_a^b |u(t)u'(t)| \, dt \leq \frac{1}{2} \int_a^b [(t-a) + (b-t)] |u'(t)|^2 \, dt.$$

Since $(t-a) + (b-t) = b-a$, this gives (40). $\square$

Corollary 7 (Parametrized refinement). *Under the hypotheses of Corollary 6, for every $\theta > 0$,*

$$\int_a^b |u(t)u'(t)| \, dt \leq \frac{\theta^2}{2} \int_a^b (t-a) |u'(t)|^2 \, dt + \frac{1}{2\theta^2} \int_a^b (b-t) |u'(t)|^2 \, dt. \quad (41)$$

In particular, taking $\theta = 1$ gives the classical one-sided Opial inequality (40).

Proof. The result follows directly from Theorem 2 with

$$\mathbb{T} = \mathbb{R}, \quad n = 1, \quad p = q = 1, \quad v \equiv 1, \quad \omega(t) = t - a.$$

For these choices $r = s = 2$, $G(t) = b - t$, and $\omega'(t) = 1$. Substitution in Theorem 2 gives (41). Setting $\theta = 1$ recovers Corollary 6. This yields a parametrized refinement of the classical inequality; see also [1]. $\square$

Remark 6. Corollaries 6 and 7 illustrate the normalizing role of the general theory. When $p = q = 1$, $v \equiv 1$, and $\omega(t) = t - a$, the abstract kernels reduce to the elementary endpoint weights $t - a$ and $b - t$. Consequently, the single-integral estimate gives the classical one-sided Opial constant $(b-a)/2$. This shows that the general weighted framework preserves the sharp scale of the classical estimate in the simplest continuous case.

The parameter θ in Corollary 7 describes how the estimate distributes the derivative energy between the left endpoint weight $t - a$ and the right endpoint weight $b - t$. The choice $\theta = 1$ gives a symmetric balance and recovers the standard Opial bound. Values $\theta > 1$ increase the contribution of the term weighted by $t - a$, whereas $0 < \theta < 1$ increase the contribution of the term weighted by $b - t$. Thus, the Young-type form is useful when additional information about $|u'|$ is concentrated closer to one endpoint of the interval.

Corollary 8 (Discrete Opial inequality). *Let $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = N$, where $N \in \mathbb{N}$, and let $n = 1$, $p = q = 1$, $v \equiv 1$, and $\omega(k) = k$. If $u : \{0, 1, \dots, N\} \rightarrow \mathbb{R}$ satisfies $u(0) = 0$, then*

$$\sum_{k=0}^{N-1} |u(k)\Delta u(k)| \leq \frac{N}{2} \sum_{k=0}^{N-1} |\Delta u(k)|^2. \quad (42)$$

Proof. Here $\Delta\omega(k) = 1$, $r = s = 2$, $K = 1/2$, and, after cancellation in the definition of G ,

$$G(x) = \sum_{k=x}^{N-1} 1 = N - x.$$

Applying Theorem 3, we get

$$\sum_{k=0}^{N-1} |u(k)\Delta u(k)| \leq \frac{1}{2} \sum_{k=0}^{N-1} [k + (N - k)] |\Delta u(k)|^2.$$

Since $k + (N - k) = N$, this gives (42). This is the discrete Opial inequality of Lasota [32] and Wong [33]. $\square$

We next give higher-order continuous consequences obtained by choosing particular auxiliary functions and weights.

Corollary 9 (Power weight, order n). *Let $\mathbb{T} = \mathbb{R}$, $a = 0$, $b > 0$, $n \geq 1$, $p = 2$, $q = 1$, $v \equiv 1$, and $\omega(t) = t^2$. If $u \in C^n([0, b], \mathbb{R})$ satisfies*

$$u(0) = u'(0) = \dots = u^{(n-1)}(0) = 0,$$

then

$$\int_0^b |u(t)|^2 |u^{(n)}(t)| \, dt \leq \frac{2^{2/3}}{3} \int_0^b \left[t^2 |u^{(n)}(t)|^3 + \frac{G(t)}{2t} |u^{(n)}(t)|^3 \right] dt, \quad (43)$$

where

$$G(x) = \frac{2(b^{3n} - x^{3n})}{3n((n-1)!)^3(3n-2)(3n-1)}. \quad (44)$$

Proof. For these choices,

$$r = 3, \quad s = \frac{3}{2}, \quad K = \frac{2^{2/3}}{3}, \quad \omega'(t) = 2t.$$

Since $\mathbb{T} = \mathbb{R}$,

$$h_{n-1}(t, x) = \frac{(t-x)^{n-1}}{(n-1)!}.$$

Hence

$$\Omega_n(t) = \int_0^t \left(\frac{(t-x)^{n-1}}{(n-1)!} \right)^3 2x \, dx = \frac{2t^{3n-1}}{((n-1)!)^3(3n-2)(3n-1)}.$$

Since $p = 2$, $q = 1$, and $\omega(t) = t^2$, the factor $\omega(t)^{-q/p}(t-a)^{p-1}$ becomes

$$(t^2)^{-1/2}t = 1.$$

Thus

$$G(x) = \int_x^b \Omega_n(t) \, dt = \frac{2(b^{3n} - x^{3n})}{3n((n-1)!)^3(3n-2)(3n-1)}.$$

Substitution in Theorem 3 gives (43). $\square$

Corollary 10 (Trigonometric weight). *Let $\mathbb{T} = \mathbb{R}$, $a = 0$, $b = \pi/2$, $n = 1$, $p = 1$, $q = 2$, $v(t) = \cos t$, and $\omega(t) = \sin t$. If $u \in C^1([0, \pi/2], \mathbb{R})$ satisfies $u(0) = 0$, then*

$$\int_0^{\pi/2} \cos t |u(t)| |u'(t)|^2 \, dt \leq \frac{2^{2/3}}{3} \int_0^{\pi/2} \left[\sin t |u'(t)|^3 + \frac{1 - \sin t}{\cos^2 t} |u'(t)|^3 \right] dt. \quad (45)$$

Proof. We have

$$\omega(0) = 0, \quad \omega'(t) = \cos t > 0, \quad 0 < t < \frac{\pi}{2}.$$

Moreover,

$$r = \frac{p+q}{q} = \frac{3}{2}, \quad s = \frac{p+q}{p} = 3, \quad K = \frac{2^{2/3}}{3}.$$

Since $n = 1$, $h_0 \equiv 1$, and hence

$$\Omega_1(t) = \int_0^t \omega'(x) \, dx = \omega(t) = \sin t.$$

Therefore,

$$\omega(t)^{-q/p} \Omega_1(t)^q = (\sin t)^{-2} (\sin t)^2 = 1.$$

Also, since $p = 1$,

$$(t-a)^{p-1} = t^0 = 1.$$

Thus

$$G(x) = \int_x^{\pi/2} v(t) \, dt = \int_x^{\pi/2} \cos t \, dt = 1 - \sin x.$$

Finally,

$$[\omega'(t)]^q = (\cos t)^2.$$

Substitution in Theorem 3 gives (45). $\square$

Corollary 11 (Exponential weight, second order). *Let $\mathbb{T} = \mathbb{R}$, $a = 0$, $b > 0$, $n = 2$, $p = q = 1$, $v(t) = e^{-t}$, and $\omega(t) = 1 - e^{-t}$. If $u \in C^2([0, b], \mathbb{R})$ satisfies $u(0) = u'(0) = 0$, then*

$$\int_0^b e^{-t} |u(t)u''(t)| \, dt \leq \frac{1}{2} \int_0^b [e^{-t}(1 - e^{-t}) + e^t G(t)] |u''(t)|^2 \, dt, \quad (46)$$

where

$$G(x) = \int_x^b e^{-t}(1 - e^{-t})^{-1} \Omega_2(t) \, dt, \quad \Omega_2(t) = \int_0^t (t-x)^2 e^{-x} \, dx. \quad (47)$$

Proof. We have

$$\omega'(t) = e^{-t}, \quad r = s = 2, \quad K = \frac{1}{2}.$$

Since $n = 2$,

$$\Omega_2(t) = \int_0^t (t-x)^2 e^{-x} \, dx.$$

Moreover,

$$G(x) = \int_x^b e^{-t}(1 - e^{-t})^{-1} \Omega_2(t) \, dt.$$

Substituting these quantities into Theorem 3 gives (46). This is a higher-order weighted consequence in the spirit of the dynamic inequalities in [22,27]. $\square$

Remark 7. Corollaries 9–11 show how the auxiliary function ω , the external weight v , and the exponents p and q influence the resulting estimate. In the power-weight example, the choice $\omega(t) = t^2$ gives a higher-order inequality in which the kernel G depends explicitly on the order n . This makes the estimate sensitive to the growth of the Taylor monomial and shows how the constant and the kernel change as the order of the derivative increases.

The trigonometric example illustrates a situation in which the auxiliary function and its derivative naturally adapt to an interval with a finite endpoint singular behavior. Since $\omega(t) = \sin t$ and $\omega'(t) = \cos t$, the final estimate reflects the interaction between the vanishing of $\sin t$ at the left end-

point and the decay of $\cos t$ near $\pi/2$. The exponential example, by contrast, uses $\omega(t) = 1 - e^{-t}$ and $v(t) = e^{-t}$, producing an estimate suitable for quantities with exponential damping.

The exponents also affect the estimation performance. The constant

$$K = \frac{(r-1)^{1-1/r}}{r}, \quad r = \frac{p+q}{q},$$

depends only on the ratio p/q . Hence changing p and q changes the balance between the power of $|u|$ and the power of the highest-order derivative, while the auxiliary kernels determine how the estimate is distributed across the interval. These examples demonstrate that the main theorems are not merely formal generalizations; they provide adjustable estimates whose strength depends on the choice of ω , v , p , q , and the order n .

We also record consequences on the q_0 -time scale, where the delta derivative becomes the Jackson derivative.

Corollary 12 (Refined q_0 -Opial inequality, first order). *Let $\mathbb{T} = q_0^{\mathbb{N}_0}$, $q_0 > 1$, $n = 1$, $p = q = 1$, $v \equiv 1$, and $\omega(t) = t - a$. If $u(a) = 0$, then*

$$\int_a^b |u(t) D_{q_0} u(t)| d_{q_0} t \leq \frac{b-a}{2} \int_a^b |D_{q_0} u(t)|^2 d_{q_0} t. \quad (48)$$

Proof. For this choice,

$$D_{q_0} \omega(t) = 1, \quad r = s = 2, \quad K = \frac{1}{2},$$

and

$$G(x) = b - x$$

in the q_0 -integral sense. Theorem 3 yields

$$\begin{aligned} \int_a^b |u(t) D_{q_0} u(t)| d_{q_0} t &\leq \frac{1}{2} \int_a^b [(t-a) + (b-t)] |D_{q_0} u(t)|^2 d_{q_0} t \\ &= \frac{b-a}{2} \int_a^b |D_{q_0} u(t)|^2 d_{q_0} t. \end{aligned}$$

This proves (48). It is the corresponding q_0 -Opial inequality and extends [34] to general $[a, b]$. $\square$

Corollary 13 (q_0 -Opial inequality, second order). *Let $\mathbb{T} = q_0^{\mathbb{N}_0}$, $q_0 > 1$, $a = 1$, $b = q_0^N$, $n = 2$, $p = 2$, $q = 1$, $v \equiv 1$, and $\omega(t) = t - 1$. If $u(1) = D_{q_0} u(1) = 0$, then*

$$\int_1^{q_0^N} |u(t)|^2 |D_{q_0}^2 u(t)| d_{q_0} t \leq \frac{2^{2/3}}{3} \int_1^{q_0^N} [(t-1) + G(t)] |D_{q_0}^2 u(t)|^3 d_{q_0} t, \quad (49)$$

where, for $m = 0, 1, \dots, N-1$,

$$G(q_0^m) = \sum_{j=m}^{N-1} (q_0^j - 1)^{1/2} \Omega_2(q_0^j) (q_0 - 1) q_0^j \quad (50)$$

and

$$\Omega_2(q_0^j) = \sum_{i=0}^{j-1} \left| h_1(q_0^i, q_0^{i+1}) \right|^3 (q_0 - 1) q_0^i. \quad (51)$$

Proof. Here

$$p + q = 3, \quad r = 3, \quad s = \frac{3}{2}, \quad K = \frac{2^{2/3}}{3},$$

and

$$D_{q_0} \omega(t) = 1.$$

For $t = q_0^j$, the kernel is

$$\Omega_2(q_0^j) = \sum_{i=0}^{j-1} \left| h_1(q_0^i, q_0^{i+1}) \right|^3 (q_0 - 1) q_0^i.$$

Consequently,

$$G(q_0^m) = \sum_{j=m}^{N-1} (q_0^j - 1)^{1/2} \Omega_2(q_0^j) (q_0 - 1) q_0^j.$$

Substitution in Theorem 3 gives (49). This is a second-order quantum consequence of the general time-scale result. $\square$

Remark 8. Corollaries 12 and 13 demonstrate that the same weighted higher-order structure remains valid in the quantum setting. On the time scale $\mathbb{T} = q_0^{\mathbb{N}_0}$, the delta derivative becomes the Jackson q_0 -derivative and the delta integral becomes a weighted sum over the geometric mesh. Therefore, the resulting inequalities are adapted to multiplicative discretizations rather than to uniform additive meshes.

The first-order q_0 -Opial inequality shows that the classical one-sided constant is preserved in the quantum setting when the natural auxiliary function $\omega(t) = t - a$ is used. The second-order example shows how the kernels Ω_2 and G encode the geometry of the q_0 -mesh through the factors $(q_0 - 1)q_0^j$ and the Taylor monomial h_1 . Thus, variation in the parameter q_0 changes not only the mesh spacing but also the effective weight distribution in the estimate. This illustrates the usefulness of the time-scale formulation for comparing continuous, discrete, and quantum models within one framework.

The examples above show how the general inequalities produce concrete continuous, discrete, and quantum consequences once the time scale, the exponents, and the auxiliary function are specified. Some of the displayed estimates recover known inequalities exactly, while others should be read as new consequences of the present framework in the same line as the cited sources.

Beyond these illustrative specializations, we present an application to uniqueness problems for higher-order dynamic initial value problems. Opial-type inequalities have long been useful tools in the qualitative theory of differential and dynamic equations; see, for example, Willett [36] and Pachpatte [4,37]. In the present setting, the single-integral form of our inequality allows each lower-order delta derivative of the difference of two possible solutions to be controlled by the highest-order delta derivative. Combined with a Lipschitz condition on the right-hand side of the dynamic equation, this yields a contraction-type criterion for uniqueness on an arbitrary time scale.

Theorem 4 (Uniqueness for higher-order dynamic initial value problems). *Let $\mathbb{T}$ be a time scale, let $a, b \in \mathbb{T}$ with $a < b$, and let $n \geq 1$. Consider the initial value problem*

$$\begin{cases} y^{\Delta^n}(t) = F(t, y(t), y^{\Delta}(t), \dots, y^{\Delta^{n-1}}(t)), \\ y(a) = y^{\Delta}(a) = \dots = y^{\Delta^{n-1}}(a) = 0. \end{cases} \quad (52)$$

Assume that

$$F : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \rightarrow \mathbb{R}$$

is rd-continuous in t and satisfies the Lipschitz condition

$$|F(t, \xi_0, \xi_1, \dots, \xi_{n-1}) - F(t, \eta_0, \eta_1, \dots, \eta_{n-1})| \leq L \sum_{j=0}^{n-1} |\xi_j - \eta_j| \quad (53)$$

for all $t \in [a, b]_{\mathbb{T}}$ and all $\xi_j, \eta_j \in \mathbb{R}$.

For $m = 1, 2, \dots, n$, define

$$\Omega_m(t) := \int_a^t |h_{m-1}(t, \sigma(\tau))|^2 \Delta \tau$$

and

$$G_m(x) := \int_x^b \frac{\Omega_m(t)}{t-a} \Delta t,$$

where the integrand at $t = a$ is understood by its natural limiting value whenever needed. Set

$$H_m := \sup_{t \in [a, b]_{\mathbb{T}}} ((t-a) + G_m(t)).$$

If

$$\frac{L}{2} \sum_{m=1}^n H_m < 1, \quad (54)$$

then problem (52) has at most one solution.

Proof. Let y_1 and y_2 be two solutions of (52), and set

$$U = y_1 - y_2.$$

Then

$$U(a) = U^{\Delta}(a) = \dots = U^{\Delta^{n-1}}(a) = 0.$$

Moreover,

$$U^{\Delta^n}(t) = F(t, y_1(t), y_1^{\Delta}(t), \dots, y_1^{\Delta^{n-1}}(t)) - F(t, y_2(t), y_2^{\Delta}(t), \dots, y_2^{\Delta^{n-1}}(t)).$$

By the Lipschitz condition (53), we have

$$|U^{\Delta^n}(t)| \leq L \sum_{i=0}^{n-1} |U^{\Delta^i}(t)|. \quad (55)$$

Multiplying (55) by $|U^{\Delta^n}(t)|$ and integrating over $[a, b]_{\mathbb{T}}$, we obtain

$$\int_a^b |U^{\Delta^n}(t)|^2 \Delta t \leq L \sum_{i=0}^{n-1} \int_a^b |U^{\Delta^i}(t)| |U^{\Delta^n}(t)| \Delta t. \quad (56)$$

Fix $i \in \{0, 1, \dots, n-1\}$, and put

$$Y_i = U^{\Delta^i}, \quad m = n - i.$$

Then

$$Y_i(a) = Y_i^{\Delta}(a) = \dots = Y_i^{\Delta^{m-1}}(a) = 0,$$

and

$$Y_i^{\Delta^m} = U^{\Delta^n}.$$

Applying Theorem 3 with

$$p = q = 1, \quad v \equiv 1, \quad \omega(t) = t - a,$$

to the function Y_i and with order $m = n - i$, we get

$$\int_a^b |U^{\Delta^i}(t)| |U^{\Delta^n}(t)| \Delta t \leq \frac{1}{2} \int_a^b ((t - a) + G_{n-i}(t)) |U^{\Delta^n}(t)|^2 \Delta t. \quad (57)$$

Therefore, by the definition of H_{n-i} ,

$$\int_a^b |U^{\Delta^i}(t)| |U^{\Delta^n}(t)| \Delta t \leq \frac{H_{n-i}}{2} \int_a^b |U^{\Delta^n}(t)|^2 \Delta t. \quad (58)$$

Substituting (58) into (56) gives

$$\int_a^b |U^{\Delta^n}(t)|^2 \Delta t \leq \frac{L}{2} \left(\sum_{i=0}^{n-1} H_{n-i} \right) \int_a^b |U^{\Delta^n}(t)|^2 \Delta t.$$

Equivalently,

$$\left(1 - \frac{L}{2} \sum_{m=1}^n H_m \right) \int_a^b |U^{\Delta^n}(t)|^2 \Delta t \leq 0.$$

By condition (54), it follows that

$$\int_a^b |U^{\Delta^n}(t)|^2 \Delta t = 0.$$

Hence

$$U^{\Delta^n} \equiv 0 \quad \text{on } [a, b]_{\mathbb{T}}.$$

Using the initial conditions successively, we obtain

$$U^{\Delta^{n-1}} \equiv 0, \quad U^{\Delta^{n-2}} \equiv 0, \quad \dots, \quad U^{\Delta} \equiv 0, \quad U \equiv 0.$$

Thus $y_1 = y_2$, and the solution of (52) is unique. $\square$

The following simple example illustrates how the preceding uniqueness criterion can be applied in a concrete second-order case.

Example 1. Let $\mathbb{T} = \mathbb{R}$, $a = 0$, $b = 1$, and consider the second-order initial value problem

$$\begin{cases} y''(t) = \lambda(t)(y(t) + y'(t)), & 0 \leq t \leq 1, \\ y(0) = y'(0) = 0, \end{cases}$$

where $\lambda \in C([0, 1], \mathbb{R})$ and

$$|\lambda(t)| \leq \Lambda, \quad 0 \leq t \leq 1.$$

Here

$$F(t, \xi_0, \xi_1) = \lambda(t)(\xi_0 + \xi_1)$$

satisfies

$$|F(t, \xi_0, \xi_1) - F(t, \eta_0, \eta_1)| \leq \Lambda(|\xi_0 - \eta_0| + |\xi_1 - \eta_1|).$$

Thus the Lipschitz constant in Theorem 4 is $L = \Lambda$.

For $\mathbb{T} = \mathbb{R}$, $a = 0$, and $b = 1$, the quantities H_1 and H_2 are computed from

$$\Omega_1(t) = \int_0^t 1 \, dx = t, \quad G_1(x) = \int_x^1 1 \, dt = 1 - x,$$

and

$$\Omega_2(t) = \int_0^t (t - x)^2 \, dx = \frac{t^3}{3}, \quad G_2(x) = \int_x^1 \frac{t^2}{3} \, dt = \frac{1 - x^3}{9}.$$

Therefore,

$$H_1 = \sup_{0 \leq t \leq 1} \{t + G_1(t)\} = 1,$$

and

$$H_2 = \sup_{0 \leq t \leq 1} \left\{ t + \frac{1 - t^3}{9} \right\} = 1.$$

The uniqueness condition becomes

$$\frac{\Lambda}{2} (H_1 + H_2) < 1,$$

that is,

$$\Lambda < 1.$$

Hence, if $\|\lambda\|_\infty < 1$, the above second-order initial value problem has at most one solution on $[0, 1]$. Since $y \equiv 0$ is a solution, it is the unique solution. This example shows explicitly how the weighted higher-order Opial estimate yields a concrete uniqueness criterion for a dynamic equation.

5. Conclusions

In this paper, we established a unified family of weighted higher-order Beesack–Opial-type inequalities on an arbitrary time scale $\mathbb{T}$. The main estimates control the mixed expression

$$\int_a^b v(t) |u(t)|^p |u^{\Delta^n}(t)|^q \Delta t$$

by quantities involving only the highest-order delta derivative u^{Δ^n} , under the lower-order vanishing conditions $u^{\Delta^k}(a) = 0$, $k = 0, 1, \dots, n - 1$. Three complementary forms were proved: the Hölder-type product estimate in Theorem 1, the Young-type estimate with a free parameter $\theta > 0$ in Theorem 2, and the optimized single-integral estimate in Theorem 3. The constant in the latter form is given explicitly by

$$K = \frac{(r - 1)^{1-1/r}}{r}, \quad r = \frac{p + q}{q},$$

and hence depends only on the ratio of the exponents p and q .

The main innovation of the paper is the combination of three features in one framework: higher-order delta derivatives, an auxiliary-function-generated weight structure involving ω and ω^Δ , and the time-scale setting. This combination extends the continuous Beesack–Opial mechanism to higher-order dynamic inequalities and simultaneously produces continuous, discrete, and q_0 -dynamic consequences. In particular, the special cases recover the classical one-sided Opial inequality, the discrete Opial inequalities of Lasota and Wong, and several weighted higher-order estimates related to the existing dynamic Opial literature. The examples also show how the choices of v , ω , p , q , and the order n influence the kernels and the final estimates.

The results have a direct qualitative application to higher-order dynamic initial value problems. By applying the single-integral form to successive lower-order delta derivatives,

we derived a uniqueness criterion under a Lipschitz condition on the nonlinear term. This illustrates how the proposed inequalities can be used as analytical tools in the study of dynamic equations on arbitrary time scales.

There are also natural limitations in the present work. The main estimates depend essentially on the assumptions

$$u^{\Delta^k}(a) = 0, \quad k = 0, 1, \dots, n-1,$$

which allow the Taylor representation to express u entirely through its highest-order delta derivative. These conditions are suitable for many initial value problems, but they restrict direct applicability to problems with nonzero initial data. In such situations, additional boundary terms are expected to appear, as occurs in related nonzero-endpoint weighted Opial-type inequalities.

Several directions remain open for future research. A first direction is to extend the present auxiliary-function-generated framework to nonzero initial conditions and to two-sided boundary conditions. A second direction is to develop corresponding nabla and diamond- α versions. A third direction is to investigate sharpness and best-constant questions for special choices of the time scale, the auxiliary function, and the weights. It would also be interesting to apply the resulting inequalities to oscillation, stability, disconjugacy, and boundary value problems for higher-order dynamic equations. These directions would further clarify the role of auxiliary weights in the qualitative theory of continuous, discrete, and quantum models.

Author Contributions: Conceptualization, R.R.M., S.H.S. and K.R.A.; methodology, R.R.M., S.H.S. and K.R.A.; validation, R.R.M., S.H.S., K.R.A. and D.R.A.; formal analysis, R.R.M., S.H.S. and K.R.A.; investigation, R.R.M., S.H.S. and K.R.A.; writing—original draft preparation, R.R.M., S.H.S. and K.R.A.; writing—review and editing, R.R.M., S.H.S., K.R.A. and D.R.A.; supervision, R.R.M., S.H.S. and K.R.A.; project administration, R.R.M., S.H.S., K.R.A. and D.R.A. All authors have read and agreed to the published version of the manuscript

Funding: D.R.A. is supported by the AMS–Simons Research Enhancement Grant for Primarily Undergraduate Institutions ASPUI-25-ANDGRB.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors thank the reviewers for their careful reading and constructive comments.

Conflicts of Interest: The authors declare no conflicts of interest.

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