

Article

Local-Time Sensitivity and Burst Instability for Threshold Functionals of One-Dimensional Diffusions

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Abstract

Let $X = (X_t)_{0 \leq t \leq T}$ be a real-valued continuous process. For a threshold a , the sub-threshold time set $E_T(a) = \{t \in [0, T] : X_t \leq a\}$ encodes several different threshold observables. The most elementary one is the cumulative occupation time $A_T(a) = \int_0^T \mathbf{1}_{\{X_t \leq a\}} dt$. For a regular one-dimensional diffusion, the classical occupation density formula gives $A_T(a) = \int_{-\infty}^a \frac{L_T^y(X)}{\sigma^2(y)} dy$, and hence $\frac{\partial A_T}{\partial a}(a) = \frac{L_T^a(X)}{\sigma^2(a)}$. Thus additive threshold occupation admits a local-time sensitivity calculus. In the terminology of barrier contracts, this additive clock is the cumulative, non-resetting Parisian clock, also called the Parasian clock. The purpose of this paper is to contrast this additive/Parasian regime with the behavior of resetting Parisian burst functionals. The connected components of $E_T(a)$ represent sub-threshold episodes. We study in particular the longest burst $M_T(a) = \sup\{|I| : I \text{ is a connected component of } E_T(a)\}$. While A_T is locally controlled by local time, M_T is governed by the connectivity of the sub-threshold time set. We prove that M_T is monotone, that its supremum is attained, and that the weak-sublevel version is right-continuous with left limits, while the strict-sublevel version is its left-continuous regularization. The jump at a level is the increase in the maximal connected-component length produced by adjoining the level set. This gives a deterministic càdlàg/càglàd calculus for longest-burst profiles. For regular one-dimensional diffusions, this yields a sharp structural contrast. At deterministic levels which are almost surely not local-extreme values, the weak and strict longest bursts agree almost surely. Whenever the path has a unique interior maximum, the level-indexed longest-burst profile has a positive jump at the maximum level and is therefore not absolutely continuous. Brownian motion satisfies this criterion almost surely. We further identify the deterministic mechanism behind this instability: small threshold increases may fill short temporal bridges and merge large sub-threshold components. Finally, we show that the longest burst is exactly a one-sided continuous Parisian functional. This yields an exact Laplace-transform representation of its Brownian law through the Chesney–Jeanblanc–Piqué–Yor Parisian transform, and an excursion-measure formulation in which local time enters only as the Itô excursion intensity. We also discuss smoothed burst statistics, moving thresholds, and diffusion examples. The paper is intended as a threshold-sensitivity comparison: local time controls cumulative Parasian occupation, whereas resetting Parisian burst observables are controlled by component mergers and excursion structure.



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1. Introduction

Threshold functionals appear throughout the theory of stochastic processes. Given a real-valued process $X = (X_t)_{t \geq 0}$, one may ask when it first crosses a level, how long it remains below a boundary, how often it returns to a prescribed band, or what is the longest interval during which it stays in a sub-threshold regime.

Fix a finite horizon $T > 0$. For a threshold $a \in \mathbb{R}$, define the sub-threshold time set

$$E_T(a) = \{t \in [0, T] : X_t \leq a\}. \quad (1)$$

This set contains more information than its Lebesgue measure. The cumulative occupation time

$$A_T(a) = \lambda(E_T(a)) = \int_0^T \mathbf{1}_{\{X_t \leq a\}} dt \quad (2)$$

records how much time the path spends below a , but it does not describe how this time is organized. The path may spend one long interval below the threshold, or it may return below the threshold many times for short durations. To capture this temporal organization, one may study the connected components of $E_T(a)$, which we call sub-threshold bursts. The longest such burst is

$$M_T(a) = \sup\{|I| : I \text{ is a connected component of } E_T(a)\}. \quad (3)$$

More generally, if $(\ell_i(a))_{i \geq 1}$ denotes the collection of lengths of the sub-threshold components, arranged in nonincreasing order, then

$$A_T(a) = \sum_i \ell_i(a) \quad (4)$$

whenever the level set $\{t : X_t = a\}$ has zero Lebesgue measure, whereas

$$M_T(a) = \ell_1(a). \quad (5)$$

Thus cumulative occupation is additive, while longest-burst duration is extremal and connectivity-sensitive.

This additive/extremal distinction is also the probabilistic counterpart of a distinction from the theory of Parisian-type barrier contracts. A continuous Parisian barrier monitors uninterrupted excursions beyond a barrier: the clock is reset whenever the underlying crosses back to the safe side. In the present threshold notation, the corresponding clock is the longest-burst or longest-excursion functional $M_T(a)$. By contrast, a cumulative Parisian barrier, often called a Parasian barrier, uses a non-resetting clock and records total time spent beyond the barrier. In the present notation, the corresponding clock is precisely the additive occupation functional $A_T(a)$. The Parasian trigger is of the form $A_T(a) > D$, while the continuous Parisian trigger is of the form $M_T(a) > D$. The central theme of the paper is that these two clocks have radically different threshold sensitivities.

The additive clock belongs to the classical theory of occupation times and local times. If X is a regular one-dimensional diffusion satisfying

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad (6)$$

then the occupation density formula yields

$$A_T(a) = \int_{-\infty}^a \frac{L_T^y(X)}{\sigma^2(y)} dy, \quad (7)$$

where $L_T^y(X)$ is the semimartingale local time of X at level y . Consequently,

$$\frac{\partial A_T}{\partial a}(a) = \frac{L_T^a(X)}{\sigma^2(a)} \quad (8)$$

at every level where the local-time density is continuous. In the Brownian case, this reduces to

$$\frac{\partial A_T}{\partial a}(a) = L_T^a(B). \quad (9)$$

This formula is a direct form of the occupation density formula for continuous semimartingales and diffusions; see, for example, Revuz–Yor [1], Karatzas–Shreve [2], and the collection of Brownian formulae in Borodin–Salminen [3]. Its role in the present paper is not to provide a new local-time identity. Rather, it serves as a benchmark: cumulative Parisian occupation has a local differential structure.

The non-additive clock belongs to a different lineage, governed by excursion lengths rather than by occupation density. The duration of the longest excursion was studied directly by Knight [4], and ranked excursion lengths, together with penalizations by long excursions, were developed by Pitman–Yor [5] and Roynette–Vallois–Yor [6]. The same excursion-age structure underlies the Parisian stopping times introduced into option pricing by Chesney, Jeanblanc–Picqué, and Yor [7], who obtained the law of the first excursion to reach a prescribed length through Brownian excursions, the Brownian meander, and the Azéma martingale. Schröder [8] later recast this transform for valuation at an intermediate date. Related excursion-length and Parisian stopping-time formulae have been developed by Gauthier [9] for height- and length-related stopping times, Dassios–Wu [10] for perturbed and jump-diffusion models, Dassios–Lim [11,12] for one-sided, two-sided and double-barrier Parisian windows, Dassios–Zhang [13] for the joint law with the hitting time, and Zhang–Li [14] and Liu–Yang–Zhang [15] for general Markov and time-inhomogeneous Markov models.

The cumulative, non-resetting side has its own valuation literature. Occupation-time derivatives and cumulative Parisian, or Parasian, options were developed by Hugonnier [16] and Moraux [17]. The explicit comparison between Parisian and Parasian structures is emphasized in Zhu–Chen [18] and Ai–Zhu [19]. These works primarily fix a barrier and a window and compute or approximate the law of a stopping time for pricing purposes. The present paper is not concerned with valuation. Instead, it fixes the finite horizon and studies the entire threshold-indexed profile of the two clocks: the Parasian profile $a \mapsto A_T(a)$ and the continuous Parisian profile $a \mapsto M_T(a)$.

The distinction isolated here is structural. Additive threshold occupation is local-time sensitive. Resetting burst duration is connectivity-sensitive. A small increase in the threshold may add very little occupation time while connecting two large sub-threshold intervals through a short temporal bridge; the occupation increment is then negligible, yet the longest burst may increase by a macroscopic amount.

This gives the central contrast:

- cumulative Parasian occupation is additive and local-time sensitive;
- continuous Parisian burst duration is extremal, resetting, and merger-sensitive.

The specific contributions of the paper are the following.

1. **Foundational threshold-clock decomposition.** We formulate the deterministic decomposition of weak and strict sublevel time sets for continuous paths and place it side by side with the additive occupation formula. This provides a common notation for the cumulative Parasian clock $A_T(a)$ and the resetting Parisian clock $M_T(a)$.

2. **One-sided regularity of the longest burst.** We prove that the weak longest-burst functional is nondecreasing, that its defining supremum is attained, and that it is right-continuous with left limits. Its strict-sublevel counterpart is the left-continuous regularization of the same monotone object. The jump at a level equals the increase in maximal component length produced by adjoining the level set.
3. **Diffusion-level contrast.** For regular one-dimensional diffusions, the additive/Parasian occupation profile is locally absolutely continuous with local-time density. By contrast, the continuous Parisian longest-burst profile has merger-driven regularity. At deterministic levels which are almost surely not local-extreme values, weak and strict longest bursts agree almost surely. Whenever the path has a unique interior maximum, the threshold-indexed profile has a positive jump at the maximum level and is therefore not absolutely continuous. Brownian motion satisfies this criterion almost surely.
4. **Distributional identity via Parisian times.** We show that the longest burst is exactly a one-sided continuous Parisian functional. This yields the Brownian Laplace-transform law through the Chesney–Jeanblanc–Picqué–Yor Parisian transform [7]. For Brownian motion started below the barrier, we give an explicit decomposition using the hitting time of the barrier, rather than appealing to the different intermediate-valuation problem treated by Schröder [8]. We also give an excursion-measure formulation in which local time enters only as the Itô excursion intensity.
5. **Extensions and model consequences.** We discuss smoothed burst statistics, moving thresholds, and standard diffusion examples. In the moving-threshold case, additive occupation is governed by local time on curves, in the sense of Peskir [20], while burst duration remains merger-sensitive.

To delineate the novelty precisely, we separate what is used from what is new. The excursion-theoretic results on ranked and longest excursions (Knight [4], Pitman–Yor [5], Roynette–Vallois–Yor [6]) and the Parisian stopping-time transforms (Chesney–Jeanblanc–Picqué–Yor [7] and its extensions [8,10–15]) are invoked as tools and are not re-derived here. What is new is the treatment of the entire threshold-indexed profile $a \mapsto M_T(a)$ as a deterministic pathwise object: its one-sided (càdlàg/càglàd) regularity together with the exact jump formula, the bridge-merger mechanism that generates the jumps and the consequent failure of absolute continuity, and the pathwise identification of the longest burst with a one-sided Parisian functional. This last point turns the classical Parisian transform into a statement about the profile M_T itself, rather than about a fixed pricing problem. In short, the excursion and Parisian machinery supplies the marginal Brownian law, whereas the threshold-profile regularity theory, the merger instability, and the additive-versus-extremal (Parasian-versus-Parisian) sensitivity dichotomy are the contribution of the paper.

The paper is organized as follows. Section 2 introduces the common threshold-clock framework: sublevel time sets, weak and strict burst decompositions, the cumulative Parisian occupation clock, and the local-time sensitivity formula. Section 3 proves the deterministic one-sided regularity theorem for longest bursts and isolates the bridge-merger mechanism, including a schematic illustration. Section 4 identifies the longest burst with a one-sided continuous Parisian functional and derives the Brownian Laplace-transform and excursion-intensity forms. Section 5 develops diffusion consequences and examples, including Brownian motion, Ornstein–Uhlenbeck processes, reflected Brownian motion, and sticky behavior. Section 6 treats two extensions, smoothed burst statistics and moving thresholds. Section 7 concludes with open directions.

2. Threshold Clocks: Additive Occupation and Burst Decomposition

This section introduces the common deterministic notation used throughout the paper and places the two threshold clocks side by side. The first clock is the cumulative, non-resetting clock

$$A_T(a), \quad (10)$$

which is the probabilistic analogue of a Parasiian clock. The second clock is the resetting, connected-component clock

$$M_T(a), \quad (11)$$

which is the finite-horizon version of a continuous Parisian clock.

Let $x \in C([0, T])$. For $a \in \mathbb{R}$, define the weak sublevel time set

$$E_T^{\leq, x}(a) = \{t \in [0, T] : x(t) \leq a\}, \quad (12)$$

and the strict sublevel time set

$$E_T^{<, x}(a) = \{t \in [0, T] : x(t) < a\}. \quad (13)$$

Since x is continuous, $E_T^{\leq, x}(a)$ is compact and $E_T^{<, x}(a)$ is relatively open in $[0, T]$.

The strict sublevel set $E_T^{<, x}(a)$ decomposes into a countable disjoint union of open intervals, with the convention that intervals may touch the endpoints 0 and T . We write

$$E_T^{<, x}(a) = \bigcup_{j \in J(a)} I_j^{<}(a), \quad (14)$$

where the intervals $I_j^{<}(a)$ are the connected components of $E_T^{<, x}(a)$. These intervals are the strict sub-threshold bursts of x below a .

For an interval I , write $|I|$ for its length. The strict occupation time is

$$A_T^{<, x}(a) = \lambda(E_T^{<, x}(a)). \quad (15)$$

The weak occupation time is

$$A_T^{\leq, x}(a) = \lambda(E_T^{\leq, x}(a)). \quad (16)$$

The difference between the weak and strict profiles is exactly the amount of time spent at level a :

$$A_T^{\leq, x}(a) - A_T^{<, x}(a) = \lambda\{t \in [0, T] : x(t) = a\}. \quad (17)$$

Thus, whenever

$$\lambda\{t : x(t) = a\} = 0, \quad (18)$$

the strict and weak occupation times coincide. This is the deterministic expression of the fact that a non-sticky path spends no positive amount of calendar time exactly at one level.

For burst duration, it is often useful to keep both versions. Let $\mathcal{C}_T^{\leq, x}(a)$ be the family of connected components of $E_T^{\leq, x}(a)$, and let $\mathcal{C}_T^{<, x}(a)$ be the family of connected components of $E_T^{<, x}(a)$. Components of the weak sublevel set are compact intervals or singletons; components of the strict sublevel set are relatively open intervals.

Define the weak longest burst by

$$M_T^{\leq, x}(a) = \sup\{|I| : I \in \mathcal{C}_T^{\leq, x}(a)\}, \quad (19)$$

with the convention $M_T^{\leq, x}(a) = 0$ if $E_T^{\leq, x}(a) = \emptyset$. Define the strict longest burst by

$$M_T^{\leq, x}(a) = \sup\{|I| : I \in \mathcal{C}_T^{\leq, x}(a)\}, \quad (20)$$

with the convention $M_T^{\leq, x}(a) = 0$ if $E_T^{\leq, x}(a) = \emptyset$.

For $\eta > 0$, define the number of macroscopic weak bursts longer than η by

$$N_T^{\leq, x}(a; \eta) = \#\{I \in \mathcal{C}_T^{\leq, x}(a) : |I| > \eta\}. \quad (21)$$

Because disjoint intervals of length greater than η have total length at most T ,

$$N_T^{\leq, x}(a; \eta) \leq \frac{T}{\eta}. \quad (22)$$

The contrast between $A_T^{\leq, x}$ and $M_T^{\leq, x}$ is the deterministic version of the Parasian/Parisian distinction. The functional $A_T^{\leq, x}(a)$ is cumulative: it adds the lengths of all sub-threshold components. The functional $M_T^{\leq, x}(a)$ is resetting and extremal: it keeps only the longest single sub-threshold component.

Proposition 1 (Sublevel burst decomposition). *Let $x \in C([0, T])$. Then, for every $a \in \mathbb{R}$:*

1. $E_T^{\leq, x}(a)$ is a countable disjoint union of relatively open intervals, and

$$A_T^{\leq, x}(a) = \sum_{I \in \mathcal{C}_T^{\leq, x}(a)} |I|. \quad (23)$$

2. $E_T^{\leq, x}(a)$ is compact, and its connected components are compact intervals or singletons.
3. The weak and strict longest bursts $M_T^{\leq, x}(a)$ and $M_T^{\leq, x}(a)$ are well defined and belong to $[0, T]$.
4. For every $\eta > 0$,

$$N_T^{\leq, x}(a; \eta) \leq \frac{T}{\eta}. \quad (24)$$

Proof. The set $E_T^{\leq, x}(a)$ is open in the relative topology of $[0, T]$, hence a countable union of disjoint open intervals; its Lebesgue measure is the sum of their lengths. The set $E_T^{\leq, x}(a)$ is closed in $[0, T]$, hence compact. Connected subsets of the real line are intervals, so its connected components are compact intervals or points. The longest component length is therefore well defined and bounded by T . Finally, any collection of disjoint intervals of length greater than η inside $[0, T]$ has at most T/η members.

We now recall the local-time sensitivity of the cumulative Parasian clock. The point is to isolate the differentiable benchmark against which resetting Parisian burst instability will be compared.

Let X be a one-dimensional diffusion satisfying

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad X_0 = x, \quad (25)$$

where σ is continuous and locally bounded away from zero.

We record the standing hypotheses used for the diffusion-level statements of the paper—the static occupation sensitivity below, the diffusion-level contrast of Section 3, and the diffusion examples of Section 5.

H1. X is a regular one-dimensional diffusion with continuous paths.

H2. σ is continuous and locally bounded away from zero.

H3. The local-time field $y \mapsto L_T^y(X)$ is continuous.

Hypotheses (H1)–(H3) are precisely those under which the occupation-density formula and its differential consequence below hold almost surely, and they are satisfied by the non-degenerate regular diffusions treated in Section 5.

Let $L_T^a(X)$ denote the semimartingale local time of X at level a . Define

$$A_T(a) = \int_0^T \mathbf{1}_{\{X_t \leq a\}} dt. \quad (26)$$

The occupation density formula states that, for suitable nonnegative Borel functions f ,

$$\int_0^T f(X_t) dt = \int_{\mathbb{R}} f(y) \frac{L_T^y(X)}{\sigma^2(y)} dy. \quad (27)$$

Taking $f = \mathbf{1}_{(-\infty, a]}$ recovers the cumulative occupation profile (7).

Thus the local-time field is the threshold density of cumulative occupation. In the terminology of the Introduction, it is the sensitivity density of the Parisian clock. $\square$

Proposition 2 (Static occupation sensitivity). *Assume that $y \mapsto L_T^y(X)$ is continuous and that σ is continuous and locally bounded away from zero. Then, almost surely, the map $a \mapsto A_T(a)$ is locally absolutely continuous and*

$$A'_T(a) = \frac{L_T^a(X)}{\sigma^2(a)}. \quad (28)$$

Moreover, for every sufficiently small ε ,

$$A_T(a + \varepsilon) - A_T(a) = \int_a^{a+\varepsilon} \frac{L_T^y(X)}{\sigma^2(y)} dy, \quad (29)$$

and consequently

$$A_T(a + \varepsilon) - A_T(a) = \varepsilon \frac{L_T^a(X)}{\sigma^2(a)} + R_T(a, \varepsilon), \quad (30)$$

where

$$|R_T(a, \varepsilon)| \leq |\varepsilon| \sup_{|y-a| \leq |\varepsilon|} \left| \frac{L_T^y(X)}{\sigma^2(y)} - \frac{L_T^a(X)}{\sigma^2(a)} \right|. \quad (31)$$

Proof. The representation (7) follows directly from the occupation density formula. Under the stated continuity assumptions, the fundamental theorem of calculus gives (28).

The finite-difference identity follows by subtracting the values at $a + \varepsilon$ and a . Subtracting the first-order term yields

$$R_T(a, \varepsilon) = \int_a^{a+\varepsilon} \left(\frac{L_T^y(X)}{\sigma^2(y)} - \frac{L_T^a(X)}{\sigma^2(a)} \right) dy, \quad (32)$$

and the displayed bound is immediate.

Proposition 2 gives the differentiable benchmark: the infinitesimal occupation gained by raising a is local time at a . The rest of the paper shows that this principle breaks down for the resetting Parisian clock M_T . $\square$

3. One-Sided Regularity and the Bridge-Merger Mechanism

Let $x \in C([0, T])$. Recall the closed and open sublevel sets (12) and (13) and the weak and strict longest-burst functionals (19) and (20).

We first record that “longest” is literal.

Lemma 1 (The supremum is attained). *For every $x \in C([0, T])$ and every $a \in \mathbb{R}$, if $M_T^{\leq, x}(a) > 0$, then there is a connected component of $E_T^{\leq, x}(a)$ of length exactly $M_T^{\leq, x}(a)$. The same holds for $M_T^{<, x}(a)$ whenever $M_T^{<, x}(a) > 0$.*

Proof. We prove the weak-sublevel statement first; the strict-sublevel statement is treated at the end.

Assume $M_T^{\leq, x}(a) > 0$. Take components $I_n = [\alpha_n, \beta_n]$ of $E_T^{\leq, x}(a)$ with

$$|I_n| \rightarrow M_T^{\leq, x}(a). \quad (33)$$

Since $[0, T]^2$ is compact, pass to a subsequence such that

$$\alpha_n \rightarrow \alpha, \quad \beta_n \rightarrow \beta. \quad (34)$$

Then

$$\beta - \alpha = M_T^{\leq, x}(a). \quad (35)$$

For any $t \in (\alpha, \beta)$, one has $t \in I_n$ for all sufficiently large n . Hence $x(t) \leq a$. By continuity, $x(t) \leq a$ for every $t \in [\alpha, \beta]$. Therefore $[\alpha, \beta] \subset E_T^{\leq, x}(a)$. This interval is contained in a connected component of $E_T^{\leq, x}(a)$, whose length is at least $M_T^{\leq, x}(a)$. By definition of the supremum, the length is exactly $M_T^{\leq, x}(a)$.

For the strict-sublevel statement, let $m = M_T^{<, x}(a) > 0$. Choose strict components whose lengths tend to m . For all sufficiently large indices, these lengths are larger than $m/2$. Since disjoint intervals of length greater than $m/2$ inside $[0, T]$ are finite in number, the supremum over those components is a maximum. Hence one strict component has length $m = M_T^{<, x}(a)$. $\square$

Theorem 1 (One-sided regularity and jump structure). *Let $x \in C([0, T])$. Then:*

1. Both $M_T^{\leq, x}$ and $M_T^{<, x}$ are nondecreasing.
2. For every $a \in \mathbb{R}$,

$$M_T^{\leq, x}(a+) = M_T^{\leq, x}(a), \quad (36)$$

and

$$M_T^{\leq, x}(a-) = M_T^{<, x}(a). \quad (37)$$

Consequently $M_T^{\leq, x}$ is right-continuous with left limits, while $M_T^{<, x}$ is left-continuous with right limits. The two functions are the càdlàg and càglàd regularizations of the same monotone object.

3. The jump of the weak longest-burst profile at a is

$$\Delta M_T^{\leq, x}(a) = M_T^{\leq, x}(a) - M_T^{\leq, x}(a-) = M_T^{\leq, x}(a) - M_T^{<, x}(a). \quad (38)$$

It is the increase in maximal connected-component length produced by adjoining the level set

$$\{t : x(t) = a\} \quad (39)$$

to the strict sublevel set

$$\{t : x(t) < a\}. \quad (40)$$

In particular, $M_T^{\leq, x}$ is continuous at a if and only if adjoining the level set creates no weak sublevel component whose length is larger than the previous strict longest burst.

4. The set of discontinuities of $M_T^{\leq, x}$ is at most countable.

5. If, moreover, $x \in C^1([0, T])$ has finitely many critical points, all strict and with pairwise distinct critical values, then $M_T^{\leq, x}$ is continuous away from the local-maximum values. At an interior local-maximum value, a two-sided component-merger event occurs. This event produces a positive jump of $M_T^{\leq, x}$ if and only if the welded component exceeds the previous maximal component length. Endpoint extrema produce only one-sided endpoint changes and are not part of the two-sided merger mechanism. Between consecutive critical values, $M_T^{\leq, x}$ varies continuously.

Proof. Monotonicity is immediate: if $a_1 < a_2$, then

$$E_T^{\leq, x}(a_1) \subseteq E_T^{\leq, x}(a_2), \quad E_T^{<, x}(a_1) \subseteq E_T^{<, x}(a_2). \quad (41)$$

Every component of the smaller sublevel set lies in a component of the larger one, so the maximal component length cannot decrease.

We use the pointwise set identities

$$E_T^{\leq, x}(a) = \bigcap_{\varepsilon > 0} E_T^{\leq, x}(a + \varepsilon), \quad (42)$$

and

$$E_T^{<, x}(a) = \bigcup_{\varepsilon > 0} E_T^{<, x}(a - \varepsilon). \quad (43)$$

For right-continuity, suppose by contradiction that

$$M_T^{\leq, x}(a+) > M_T^{\leq, x}(a). \quad (44)$$

Then there exist $\varepsilon_n \downarrow 0$ and components $I_n = [\alpha_n, \beta_n]$ of $E_T^{\leq, x}(a + \varepsilon_n)$ such that

$$|I_n| \rightarrow \ell > M_T^{\leq, x}(a). \quad (45)$$

Passing to a subsequence,

$$\alpha_n \rightarrow \alpha, \quad \beta_n \rightarrow \beta. \quad (46)$$

For any $t \in (\alpha, \beta)$, one has $t \in I_n$ for all sufficiently large n , hence

$$x(t) \leq a + \varepsilon_n. \quad (47)$$

Letting $n \rightarrow \infty$, we obtain $x(t) \leq a$. By continuity, $[\alpha, \beta] \subset E_T^{\leq, x}(a)$. Thus $E_T^{\leq, x}(a)$ contains a connected subset of length ℓ , contradicting $\ell > M_T^{\leq, x}(a)$. Therefore

$$M_T^{\leq, x}(a+) = M_T^{\leq, x}(a). \quad (48)$$

Next we prove the left-limit identity. Since

$$E_T^{\leq, x}(a - \varepsilon) \subset E_T^{<, x}(a), \quad (49)$$

we have

$$M_T^{\leq, x}(a-) \leq M_T^{<, x}(a). \quad (50)$$

Here $x(t) \leq a - \varepsilon$ implies $x(t) < a$, which gives the displayed inclusion $E_T^{\leq, x}(a - \varepsilon) \subset E_T^{<, x}(a)$ for every $\varepsilon > 0$. By monotonicity of $\varepsilon \mapsto M_T^{\leq, x}(a - \varepsilon)$ the left limit equals the supremum, $M_T^{\leq, x}(a-) = \sup_{\varepsilon > 0} M_T^{\leq, x}(a - \varepsilon)$; and since any connected component of $E_T^{\leq, x}(a - \varepsilon)$ is an interval contained in $E_T^{<, x}(a)$, hence in one of its components, its length is at most $M_T^{<, x}(a)$. Passing to the supremum over ε yields the stated bound.

Conversely, let $I = (\alpha, \beta)$ be a strict component of $E_T^{<,x}(a)$. For every compact subinterval $[\alpha', \beta'] \subset I$, continuity gives

$$\sup_{t \in [\alpha', \beta']} x(t) < a. \quad (51)$$

The supremum of the continuous function x over the compact interval $[\alpha', \beta'] \subset I$ is attained and, since $x < a$ throughout I , is strictly below a . Setting $\varepsilon := a - \sup_{t \in [\alpha', \beta']} x(t) > 0$ gives $x(t) \leq a - \varepsilon$ for all $t \in [\alpha', \beta']$, which is exactly the inclusion recorded next.

Hence there exists $\varepsilon > 0$ such that

$$[\alpha', \beta'] \subset E_T^{<,x}(a - \varepsilon). \quad (52)$$

Therefore

$$M_T^{<,x}(a - \varepsilon) \geq \beta' - \alpha'. \quad (53)$$

For the ε just chosen, monotonicity gives $M_T^{<,x}(a - \varepsilon) \geq M_T^{<,x}(a - \varepsilon) \geq \beta' - \alpha'$. As $[\alpha', \beta']$ increases to I we have $\beta' - \alpha' \rightarrow |I|$, so $M_T^{<,x}(a - \varepsilon) \geq |I|$ for every strict component I ; taking the supremum over such components gives the reverse inequality.

Letting $[\alpha', \beta']$ exhaust I , and then taking the supremum over strict components I , yields

$$M_T^{<,x}(a - \varepsilon) \geq M_T^{<,x}(a). \quad (54)$$

Thus

$$M_T^{<,x}(a - \varepsilon) = M_T^{<,x}(a). \quad (55)$$

The càdlàg and càglàd conclusions follow from monotonicity and these one-sided identities. The jump formula follows immediately.

A nondecreasing real function has at most countably many discontinuities, proving the fourth assertion.

For the finite-critical-points statement, suppose a is not a critical value of x . Then every solution of

$$x(t) = a \quad (56)$$

is transversal, meaning $x'(t) \neq 0$. By the implicit function theorem, the crossing times vary continuously under small perturbations of the level. Hence the number and ordering of sublevel components remain locally unchanged, and their endpoint locations vary continuously. Therefore $M_T^{<,x}$ is continuous at a . At an interior strict local minimum, a new component is born with zero length, so no positive jump of the maximum can occur at the instant of birth. At an interior strict local maximum, two adjacent components merge. Such a merger produces a positive jump precisely when the newly welded component has length larger than the previous strict longest component. Endpoint extrema may change one endpoint of a component, but they do not create the two-sided welding mechanism described above. This proves the final assertion. $\square$

Corollary 1 (Diffusion-level contrast). *Let X be a regular one-dimensional diffusion with continuous paths and nondegenerate diffusion coefficient σ . Fix $T > 0$.*

1. *The additive occupation profile $a \mapsto A_T(a)$ is almost surely locally absolutely continuous, with*

$$A'_T(a) = \frac{L_T^a(X)}{\sigma^2(a)} \quad (57)$$

at every level at which the local-time density is continuous.

2. At any deterministic level a such that

$$\mathbb{P}(a \text{ is a local-maximum value of } X \text{ on } [0, T]) = 0, \quad (58)$$

one has

$$M_T^{\leq}(a) = M_T^{\leq}(a) \quad (59)$$

almost surely, and the weak longest-burst profile is almost surely continuous at a .

3. If a continuous path x attains its maximum on $[0, T]$ at a unique point $\theta \in (0, T)$, then $a \mapsto M_T^{\leq, x}(a)$ has a positive jump at the maximum level

$$m = \max_{0 \leq t \leq T} x(t). \quad (60)$$

In fact,

$$M_T^{\leq, x}(m) = T, \quad (61)$$

whereas

$$M_T^{\leq, x}(m-) = M_T^{\leq, x}(m) = \max\{\theta, T - \theta\}. \quad (62)$$

Thus

$$\Delta M_T^{\leq, x}(m) = \min\{\theta, T - \theta\} > 0. \quad (63)$$

Consequently $a \mapsto M_T^{\leq, x}(a)$ is not absolutely continuous.

4. In particular, for standard Brownian motion on $[0, T]$, $a \mapsto M_T^{\leq}(a)$ is almost surely not absolutely continuous.

Proof. The first assertion is Proposition 2.

For the second assertion, Theorem 1 gives

$$M_T^{\leq}(a) - M_T^{\leq}(a-) = M_T^{\leq}(a) - M_T^{\leq}(a). \quad (64)$$

A positive jump can occur only if adjoining the level set at a welds strict sublevel components into a longer weak component. Such a weld requires a to be a local-maximum value of the path. If a is almost surely not a local-maximum value, then the jump vanishes almost surely, and

$$M_T^{\leq}(a) = M_T^{\leq}(a). \quad (65)$$

For the third assertion, suppose x has a unique interior maximizer $\theta \in (0, T)$. At the maximum level m , the weak sublevel set is all of $[0, T]$, hence

$$M_T^{\leq, x}(m) = T. \quad (66)$$

The strict sublevel set is

$$E_T^{\leq, x}(m) = [0, T] \setminus \{\theta\}, \quad (67)$$

which has two connected components of lengths θ and $T - \theta$. Therefore

$$M_T^{\leq, x}(m) = \max\{\theta, T - \theta\}. \quad (68)$$

The jump at m is

$$T - \max\{\theta, T - \theta\} = \min\{\theta, T - \theta\} > 0. \quad (69)$$

Any monotone function with a positive jump is not absolutely continuous.

For Brownian motion, the maximum on a compact interval is almost surely attained at a unique time; see, for instance, the standard treatment of Brownian extrema in Karatzas–Shreve ([2] Section 2.9). The arcsine law for the time of the maximum gives no mass to the endpoints 0 and T . Therefore the Brownian maximum is attained at a unique interior time almost surely. The preceding deterministic criterion applies, and $a \mapsto M_T^{\leq}(a)$ is almost surely not absolutely continuous. $\square$

Theorem 2 (Bridge-merger jump). *Let $x \in C([0, T])$ and $a \in \mathbb{R}$. Suppose that a two-sided merger occurs at level a : the weak sublevel set $E_T^{\leq, x}(a)$ has a connected component J whose intersection with the strict sublevel set $E_T^{<, x}(a)$ has exactly two connected components I_1 and I_2 , with lengths $\ell_1 = |I_1| \geq \ell_2 = |I_2| > 0$; writing $B = J \cap \{t : x(t) = a\}$ for the bridge, $J = I_1 \cup I_2 \cup B$ is a disjoint decomposition. Assume moreover that J realizes the weak longest burst and I_1 the strict longest burst, i.e. $|J| = M_T^{\leq, x}(a)$ and $\ell_1 = M_T^{<, x}(a)$. Then*

$$\Delta M_T^{\leq, x}(a) = M_T^{\leq, x}(a) - M_T^{<, x}(a) = |J| - \ell_1 = \ell_2 + |B|. \quad (70)$$

In particular, if the level set $\{t : x(t) = a\}$ has zero Lebesgue measure, then $|B| = 0$ and the jump equals the length of the shorter merged component,

$$\Delta M_T^{\leq, x}(a) = \ell_2 = \min\{\ell_1, \ell_2\}. \quad (71)$$

Proof. By the jump formula of Theorem 1, $\Delta M_T^{\leq, x}(a) = M_T^{\leq, x}(a) - M_T^{<, x}(a)$. By hypothesis $M_T^{\leq, x}(a) = |J|$ and $M_T^{<, x}(a) = \ell_1$. Each point of J satisfies either $x(t) < a$, and then lies in I_1 or I_2 , or $x(t) = a$, and then lies in B ; hence $J = I_1 \cup I_2 \cup B$ is a disjoint union, and additivity of Lebesgue measure gives $|J| = \ell_1 + \ell_2 + |B|$. Therefore $\Delta M_T^{\leq, x}(a) = |J| - \ell_1 = \ell_2 + |B|$. If $\lambda\{t : x(t) = a\} = 0$ then $|B| \leq \lambda\{t : x(t) = a\} = 0$, so $|B| = 0$ and the jump reduces to $\ell_2 = \min\{\ell_1, \ell_2\}$. $\square$

Thus a two-sided merger raises the longest burst by exactly the shorter of the two flanking components (plus the bridge length, in the sticky case), a quantity unrelated to the occupation gained across the merger. The explicit family of Remark 1 makes the resulting sensitivity ratio unbounded.

Remark 1 (The explicit bridge example). *This is the explicit family behind Theorem 2: it realizes a two-sided merger in which the occupation increment can be made arbitrarily small while the burst jump stays macroscopic, so that the sensitivity ratio is unbounded. The preceding corollary gives a robust pathwise mechanism for non-absolute-continuity. A simpler deterministic bridge construction illustrates the same phenomenon at a prescribed level.*

Fix $T > 0$, $\delta > 0$, and a scale $m > 0$ with $2m < T$. Choose $s < \delta$. Take $a = 0$ and consider three time intervals

$$I_1 = [0, m], \quad J = (m, m + s), \quad I_2 = [m + s, 2m + s]. \quad (72)$$

Choose a continuous path x such that $x < 0$ on most of $I_1 \cup I_2$, such that $0 < x < \varepsilon$ on the bridge J , such that $x > \varepsilon$ outside $I_1 \cup J \cup I_2$, and such that all smoothing transitions have total length less than $\delta - s$.

At threshold 0, the two long pieces are separate sub-threshold components. At threshold ε , the bridge J is filled and the two pieces merge. Thus

$$A_T^{\leq, x}(\varepsilon) - A_T^{\leq, x}(0) < \delta, \quad (73)$$

while

$$M_T^{\leq, x}(\varepsilon) - M_T^{\leq, x}(0) > m \quad (74)$$

after choosing the smoothing intervals sufficiently short. In particular, there is no universal constant C such that

$$M_T^{\leq, x}(a + \varepsilon) - M_T^{\leq, x}(a) \leq C \left(A_T^{\leq, x}(a + \varepsilon) - A_T^{\leq, x}(a) \right) \quad (75)$$

for all continuous paths, all a , and all $\varepsilon > 0$.

The construction is illustrated in Figure 1. At the lower threshold, the two long sub-threshold blocks are separated by a short bridge lying just above the threshold. After the threshold is raised, the bridge is filled and the two components are welded into one component. The newly captured occupation is only of order s , while the longest-burst increment is of order $m + s$.

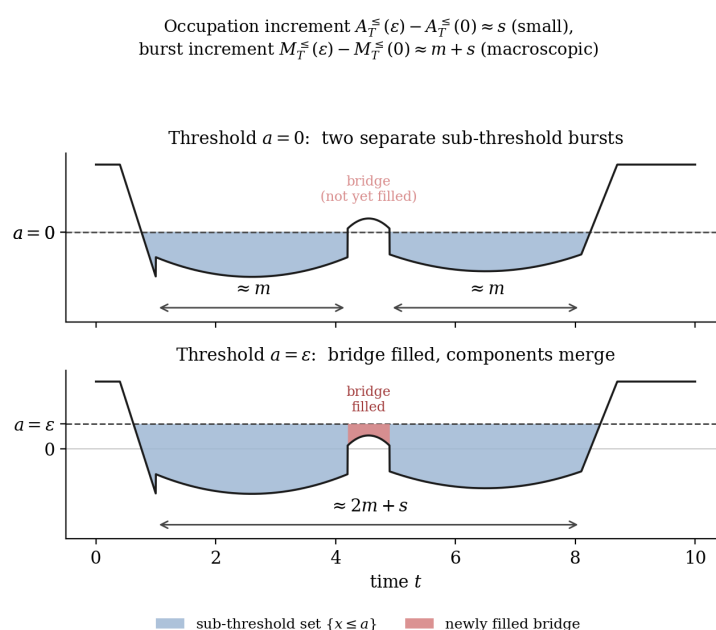


Figure 1. Bridge-merger schematic. At threshold $a = 0$, the sub-threshold set consists of two separate bursts, each of length approximately m . Raising the threshold to $a = \varepsilon$ fills the short bridge of length s , welding the two components into a single burst of length approximately $2m + s$. The occupation increment is only approximately s , while the longest-burst increment is approximately $m + s$; their ratio $(m + s)/s$ is unbounded as $s \downarrow 0$.

Remark 2 (Mechanisms of singularity). Corollary 1 obtained non-absolute-continuity from one sufficient condition, a unique interior global maximum. Theorem 2 identifies the general source of the jumps, and several distinct mechanisms produce them. (a) Non-global local maxima. A two-sided merger at an interior local-maximum value a that is not the global maximum still produces a positive jump $\ell_2 + |B|$ as soon as the welded component J becomes the longest burst; the global maximum is not required. (b) Level sets of positive measure. At a sticky level (Section 5.5) the bridge has $|B| > 0$, contributing the extra term $|B|$ to the jump; there the additive profile A_T itself jumps, so both clocks are discontinuous. (c) Accumulation of merger levels. A path may possess infinitely many interior local maxima, so that the jump set of $M_T^{\leq, x}$ is countable but possibly dense; the profile is then a monotone function with a nontrivial jump part, and a single positive jump already precludes absolute continuity. The unique-maximizer argument of Corollary 1 is therefore only the simplest instance of the merger mechanism.

Example 1 (Weak and strict longest bursts differ). Take $T = 2$ and the symmetric tent $x(t) = -|t - 1|$, $t \in [0, 2]$, so that $x(0) = x(2) = -1$ and the unique maximum $m = 0$ is attained at $t = 1$. For $a \in (-1, 0)$, $E_T^{\leq, x}(a) = [0, 1 + a] \cup [1 - a, 2]$ has two components each of length $1 + a$; since $\{t : x(t) = a\}$ has zero length, the strict set agrees up to endpoints, and

$$M_T^{\leq, x}(a) = M_T^{<, x}(a) = 1 + a, \quad a \in (-1, 0). \quad (76)$$

At the maximum level $a = 0$, however, $E_T^{\leq, x}(0) = [0, 2]$ is connected, so $M_T^{\leq, x}(0) = 2$, whereas $E_T^{<, x}(0) = [0, 2] \setminus \{1\}$ has two components of length 1, giving $M_T^{<, x}(0) = 1$. Hence

$$M_T^{\leq, x}(0) - M_T^{<, x}(0) = 1 = \frac{T}{2}. \quad (77)$$

This is a two-sided merger in the sense of Theorem 2, with $\ell_1 = \ell_2 = 1$, $|B| = 0$, and jump $\min\{\ell_1, \ell_2\} = 1$. The gap is macroscopic—half the horizon—so the two formulations genuinely differ: the càdlàg profile $M_T^{\leq, x}$ records the merged burst at the maximum level, while its càglàd regularization $M_T^{<, x} = M_T^{\leq, x}(\cdot -)$ retains the longer arm just below it. Both are therefore needed.

4. The Longest Burst as a Continuous Parisian Functional

Throughout this section we fix one normalization of local time and carry it consistently. We use the semimartingale (Tanaka) local time L^a , normalized so that the occupation density formula of Section 2 holds in the form $A_T^a(a) = \frac{L_T^a}{\sigma^2(a)}$, and so that the inverse local time $\tau_\ell = \inf\{t \geq 0 : L_t > \ell\}$ is the stable subordinator with $E[e^{-q\tau_\ell}] = e^{-\ell\sqrt{2q}}$. The one-sided Parisian transform $\Pi_D(q)$ of Chesney, Jeanblanc-Picqué, and Yor [7] is expressed below in this same normalization, so that the excursion-intensity computation of Proposition 3 and the transform of Theorem 3 refer to the identical local-time clock. With this convention the negative-excursion Itô measure $n_-(\zeta > D)$ used in Proposition 3 is the one induced by the inverse local time above; any other normalization rescales $n_-(\zeta > D)$ and the transform by the same factor, leaving every displayed identity invariant.

The preceding section described the longest-burst profile as a deterministic connected-component functional. We now identify the same object with the continuous Parisian, or resetting, clock. This is the exact counterpart of the cumulative Parisian clock $A_T(a)$ studied in Section 2. The Parisian event $\{A_T(a) > D\}$ depends on total time below the threshold; the continuous Parisian event $\{M_T(a) > D\}$ depends on one uninterrupted below-threshold episode.

The longest-burst functional admits an exact probabilistic description: it is a one-sided Parisian functional. This section records the pathwise identity and then specializes to Brownian motion.

Fix $a \in \mathbb{R}$ and $D > 0$. For a continuous path x , define the strict below- a age at time t by

$$\alpha_t^a(x) = \begin{cases} t - \sup\{s \leq t : x(s) = a\}, & x(t) < a \text{ and } \{s \leq t : x(s) = a\} \neq \emptyset, \\ t, & x(u) < a \text{ for all } 0 \leq u \leq t, \\ 0, & x(t) \geq a. \end{cases} \quad (78)$$

Thus $\alpha_t^a(x)$ is the time already spent in the current strict below- a excursion. Define the one-sided Parisian time with window D by

$$\tau_{a,D}^{Par,-}(x) = \inf\{t \geq 0 : \alpha_t^a(x) \geq D\}. \quad (79)$$

This is the first time at which the current below-threshold episode has lasted at least D .

Lemma 2 (Pathwise Parisian identity). For every continuous path x ,

$$\{M_T^{<,x}(a) > D\} = \{\tau_{a,D}^{Par,-}(x) < T\}. \quad (80)$$

If, in addition,

$$M_T^{<,x}(a) = M_T^{<,x}(a), \quad (81)$$

then the same identity holds with $M_T^{<,x}(a)$ in place of $M_T^{<,x}(a)$.

Proof. The event $M_T^{<,x}(a) > D$ means that the strict sublevel set $\{t : x(t) < a\}$ contains a connected component of length strictly larger than D . Equivalently, there exists a time $t < T$ at which the path has completed a full below- a window of length D since its most recent visit to a , or since time 0 if the path started below a and has not yet returned to a . This is exactly

$$\tau_{a,D}^{Par,-}(x) < T. \quad (82)$$

The final assertion follows immediately from the equality $M_T^{<,x}(a) = M_T^{<,x}(a)$. In particular, for regular diffusions at deterministic levels satisfying the no-local-maximum condition of Corollary 1, strict and weak burst durations coincide almost surely.

The preceding identity is the bridge between threshold-burst profiles and Parisian stopping times. $\square$

Theorem 3 (Brownian longest-burst law through the Parisian transform). *Let B be standard Brownian motion started at x , and fix a barrier $a \in \mathbb{R}$, a window $D > 0$, and a Laplace parameter $q > 0$. Let*

$$H_a = \inf\{t \geq 0 : B_t = a\}. \quad (83)$$

Let

$$\tau_D^{Par,-} \quad (84)$$

denote the one-sided below-zero Parisian time for Brownian motion started at the barrier 0, with window D . Define the Chesney–Jeanblanc–Picqué–Yor [7] one-sided Parisian transform by

$$\Pi_D(q) = \mathbb{E}_0 \left[e^{-q\tau_D^{Par,-}} \right], \quad (85)$$

with the standard normalization used in the Parisian-excursion literature [7].

The tail is written with the strict inequality $M_T^{<,B}(a) > D$, which is the exact pathwise counterpart of $\tau_{a,D}^{Par,-} < T$ in Lemma 2. Replacing strict by weak inequalities is harmless at continuity points of the corresponding tail distribution.

If $x > a$, then

$$\mathbb{E}_x \left[e^{-q\tau_{a,D}^{Par,-}} \right] = e^{-(x-a)\sqrt{2q}} \Pi_D(q). \quad (86)$$

Equivalently,

$$\int_0^\infty e^{-qT} \mathbb{P}_x \left(M_T^{<,B}(a) > D \right) dT = \frac{1}{q} e^{-(x-a)\sqrt{2q}} \Pi_D(q). \quad (87)$$

If $x = a$, the hitting factor is absent and

$$\mathbb{E}_a \left[e^{-q\tau_{a,D}^{Par,-}} \right] = \Pi_D(q), \quad (88)$$

so

$$\int_0^\infty e^{-qT} \mathbb{P}_a(M_T^{<,B}(a) > D) dT = \frac{1}{q} \Pi_D(q). \quad (89)$$

If $x < a$, the process starts inside a below- a excursion whose age at time 0 is zero. In this case

$$\tau_{a,D}^{Par,-} = D \quad \text{on } \{H_a > D\}, \quad (90)$$

whereas, on $\{H_a \leq D\}$,

$$\tau_{a,D}^{Par,-} = H_a + \tau_D^{Par,-} \circ \theta_{H_a}, \quad (91)$$

where the second term is independent of $\mathcal{F}_{H_a}$ and has the at-the-barrier Parisian law. Consequently,

$$\mathbb{E}_x[e^{-q\tau_{a,D}^{Par,-}}] = e^{-qD} \mathbb{P}_x(H_a > D) + \Pi_D(q) \mathbb{E}_x[e^{-qH_a}; H_a \leq D]. \quad (92)$$

Equivalently,

$$\int_0^\infty e^{-qT} \mathbb{P}_x(M_T^{<,B}(a) > D) dT = \frac{1}{q} (e^{-qD} \mathbb{P}_x(H_a > D) + \Pi_D(q) \mathbb{E}_x[e^{-qH_a}; H_a \leq D]). \quad (93)$$

The two quantities involving H_a are the standard one-sided Brownian hitting-time terms.

We stress which parts of this statement are new. The transform $\Pi_D(q)$ is not derived here: it is the at-the-barrier one-sided Parisian Laplace transform obtained by Chesney, Jeanblanc-Picqué and Yor [7] through Brownian excursions, the Brownian meander and the Azéma martingale, and it is used as a known input. The contribution of the theorem is the reduction of the longest-burst tail $\mathbb{P}_x(M_T^{<,B}(a) > D)$ to $\Pi_D(q)$ via the pathwise identity of Lemma 2, together with the explicit hitting-time decomposition in the case $x < a$, where calendar time begins inside a below-barrier excursion of zero age. This last case lies outside the at-the-barrier framework and is distinct from the intermediate-valuation problem of Schröder [8].

Proof. By Lemma 2,

$$\{M_T^{<,B}(a) > D\} = \{\tau_{a,D}^{Par,-} < T\}. \quad (94)$$

For any nonnegative stopping time τ ,

$$\int_0^\infty e^{-qT} \mathbf{1}_{\{\tau < T\}} dT = \frac{e^{-q\tau}}{q}. \quad (95)$$

Taking expectations gives

$$\int_0^\infty e^{-qT} \mathbb{P}(\tau < T) dT = \frac{1}{q} \mathbb{E}[e^{-q\tau}]. \quad (96)$$

Assume first that $x > a$. The Brownian path must hit a before it can begin a below- a excursion. By the strong Markov property at H_a ,

$$\tau_{a,D}^{Par,-} = H_a + \tau_D^{Par,-} \circ \theta_{H_a}, \quad (97)$$

where the second term is independent of $\mathcal{F}_{H_a}$ and has the one-sided Parisian law for Brownian motion started at the barrier. Hence

$$\mathbb{E}_x[e^{-q\tau_{a,D}^{Par,-}}] = \mathbb{E}_x[e^{-qH_a}] \mathbb{E}_0[e^{-q\tau_D^{Par,-}}]. \quad (98)$$

The Brownian hitting-time transform is

$$\mathbb{E}_x[e^{-qH_a}] = e^{-(x-a)\sqrt{2q}}, \quad (99)$$

and the second factor is $\Pi_D(q)$. This proves the case $x > a$. The case $x = a$ follows by setting $H_a = 0$.

Assume now that $x < a$. The process starts below the barrier. If $H_a > D$, then Brownian motion stays below a throughout $[0, D]$, so the Parisian time is exactly D . If $H_a \leq D$, the initial below- a episode ends before the strict Parisian event has occurred. At H_a the process is at the barrier, and the strong Markov property gives a fresh independent at-the-barrier Parisian time. Therefore

$$\tau_{a,D}^{Par,-} = D \quad \text{on } \{H_a > D\}, \quad (100)$$

and

$$\tau_{a,D}^{Par,-} = H_a + \tau_D^{Par,-} \circ \theta_{H_a} \quad \text{on } \{H_a \leq D\}. \quad (101)$$

Taking Laplace transforms gives the announced decomposition. $\square$

Remark 3 (Normalization, starting inside an excursion, and drift). *The transform $\Pi_D(q)$ in Theorem 3 is deliberately expressed in the notation of the Parisian-excursion literature. Chesney, Jeanblanc-Picqué, and Yor compute the Brownian one-sided Parisian Laplace transform using Brownian excursions, the Brownian meander, and the Azéma martingale [7]. Schröder's note [8] concerns a related but distinct issue: valuation during the lifetime of the contract, when the observation time is intermediate and the currently observed excursion age may already be positive. This is different from the $x < a$ case of Theorem 3, where calendar time starts at 0 and the initial below-barrier age is 0. The latter case is handled by the explicit hitting-time decomposition above.*

For Brownian motion with drift, the structural identity of Lemma 2 remains unchanged, because it is pathwise. Its Brownian law reduces to the driftless one by a Cameron–Martin change of measure: writing $Y_t = x + \mu t + B_t$ for Brownian motion with drift μ under $\mathbb{P}$, the process Y is a standard Brownian motion W started at x under the tilted measure $\mathbb{P}$ with $d\mathbb{P}/d\mathbb{P}|_{\mathcal{F}_T} = \exp(-\mu B_T - \frac{1}{2}\mu^2 T)$, so that for every bounded path functional G on $[0, T]$,

$$\mathbb{E}[G(Y)] = \mathbb{E}_x \left[G(W) e^{\mu(W_T - x) - \frac{1}{2}\mu^2 T} \right]. \quad (102)$$

Choosing $G = \mathbf{1}_{\{M_T^{< \cdot}(a) > D\}}$ and applying Lemma 2 expresses the drifted longest-burst law as an exponentially tilted driftless Parisian functional,

$$\mathbb{P}(M_T^{< \cdot}(a) > D) = \mathbb{E}_x \left[e^{\mu(W_T - x) - \frac{1}{2}\mu^2 T}; \tau_{a,D}^{Par,-}(W) < T \right]. \quad (103)$$

The drift therefore enters only through the exponential weight, and the reduction itself needs no new excursion theory. The explicit transform is modified by the drifted excursion and meander terms; see the Parisian option literature [7,10–13].

Proposition 3 (Local time as intensity; excursion tail as global object). *Throughout this proposition, L^0 denotes the right-sided Tanaka local time at zero,*

$$L_t^0 = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0 < B_s < \varepsilon\}} ds, \quad (104)$$

so that the negative-excursion Itô measure satisfies

$$n_-(\zeta > D) = \frac{1}{\sqrt{2\pi D}}. \quad (105)$$

Let B be standard Brownian motion at level 0, and let n_- denote the Itô excursion measure of negative Brownian excursions under this normalization. If ζ denotes the excursion lifetime and R_ℓ is the longest negative excursion completed before local time ℓ , then

$$\mathbb{P}(R_\ell \leq D) = \exp\{-\ell n_-(\zeta > D)\} = \exp\left\{-\frac{\ell}{\sqrt{2\pi D}}\right\}. \quad (106)$$

Equivalently,

$$\mathbb{P}(R_\ell > D) = 1 - \exp\left\{-\frac{\ell}{\sqrt{2\pi D}}\right\}. \quad (107)$$

Proof. In the local-time clock, Brownian excursions away from zero form a Poisson point process with characteristic measure n . Negative excursions form one side of this excursion process, with intensity measure n_- . The number of negative excursions with lifetime greater than D completed before local time ℓ is therefore Poisson with parameter

$$\ell n_-(\zeta > D). \quad (108)$$

Hence the probability that no such excursion has occurred is

$$\exp\{-\ell n_-(\zeta > D)\}. \quad (109)$$

Under the right-sided Tanaka normalization specified above,

$$n_-(\zeta > D) = \frac{1}{\sqrt{2\pi D}}. \quad (110)$$

This proves the formula.

The value $n_-(\zeta > D) = 1/\sqrt{2\pi D}$ is tied to the right-sided Tanaka normalization of L^0 fixed above. Under any other admissible normalization $\tilde{L} = cL^0$ with $c > 0$, the local-time horizon rescales as $\tilde{\ell} = c\ell$ and the excursion intensity as $\tilde{n}_- = c^{-1}n_-$, so the product $\ell n_-(\zeta > D) = \tilde{\ell} \tilde{n}_-(\zeta > D)$ that governs the law of R_ℓ is invariant. The distribution in the proposition is therefore independent of the chosen normalization, provided the horizon and the intensity are read in the same clock; this is the excursion-level counterpart of the invariance noted for $\Pi_D(q)$ at the start of Section 4.

Proposition 3 should be compared with the additive, Parisian benchmark (28). The same local time enters the additive functional as a pointwise differential density, whereas for the extremal burst functional it appears as the clock intensity of a Poisson process of excursions. The relevant quantity is no longer a local density at level a , but the global excursion-lifetime tail

$$n_-(\zeta > D). \quad (111)$$

Passing from a local-time horizon ℓ to a deterministic time horizon T requires accounting for the inverse local-time process and the terminal excursion straddling T . This is precisely the meander correction encoded in the Chesney–Jeanblanc–Picqué–Yor transform and related Parisian-transform formulae [7,8]. $\square$

5. Diffusion Consequences and Examples

Let X be a regular one-dimensional diffusion. For fixed T , the path $t \mapsto X_t$ is continuous almost surely, so all pathwise burst results apply almost surely.

The point of this section is not to derive complete model-specific laws for $M_T(a)$. Even for classical diffusions, longest-burst laws are generally Parisian or excursion-length laws, rather than local formulas. Instead, the goal is to show how the additive/extremal dichotomy manifests itself in standard diffusion examples.

The additive profile $a \mapsto A_T(a)$ is controlled by local time. For a regular diffusion satisfying

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad (112)$$

the occupation-density formula gives the cumulative occupation profile (7), and hence the local-time sensitivity (28) at levels where the local-time density is continuous.

The continuous Parisian burst profile $a \mapsto M_T(a)$, by contrast, is controlled by the component structure of the random sublevel time set

$$E_T(a) = \{t \in [0, T] : X_t \leq a\}. \quad (113)$$

A threshold increase may add only a small amount of occupation time, but if the newly captured time lies between two pre-existing sub-threshold components, it may weld them into a much longer burst. Thus A_T has local-time sensitivity, while M_T has merger sensitivity.

5.1. Non-Sticky Regular Diffusions

For nondegenerate regular diffusions, the level set

$$\{t \in [0, T] : X_t = a\} \quad (114)$$

has zero Lebesgue measure almost surely for each fixed a . Consequently,

$$A_T^<(a) = A_T^{\leq}(a) \quad (115)$$

almost surely at fixed a , and A_T is locally absolutely continuous with derivative (28).

At the same time, by Theorem 1, the weak longest-burst profile $M_T^{\leq}$ is monotone and càdlàg, while the strict profile $M_T^<$ is its càglàd regularization. Its jumps are caused by component mergers at local-maximum levels.

This distinction is not merely qualitative. The additive Parisian functional can be recovered by integrating a local density over levels. The longest-burst functional cannot. A short interval of newly captured time may weld two large existing components. Therefore the size of the burst increment depends not only on the amount of newly captured occupation, but also on the location of that occupation within the time axis.

5.2. Brownian Motion

For Brownian motion

$$X_t = x + B_t, \quad (116)$$

the occupation derivative becomes

$$A_T'(a) = L_T^a(X). \quad (117)$$

The expectation-level derivative is

$$\frac{d}{da} \mathbb{E}[A_T(a)] = \int_0^T \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(a-x)^2}{2t}\right) dt. \quad (118)$$

Thus the expected local-time profile is the time-integrated heat kernel.

The longest-burst profile has no analogous local formula. It is a monotone càdlàg level-indexed functional, and by Corollary 1 it is almost surely not absolutely continuous, because Brownian motion attains its maximum at a unique interior time almost surely.

The jump at the maximum level can be described explicitly. Let

$$\Theta_T = \arg \max_{0 \leq t \leq T} B_t, \quad (119)$$

which is almost surely unique. At the maximum level

$$S_T = \sup_{0 \leq t \leq T} B_t, \quad (120)$$

one has

$$M_T^{\leq}(S_T) = T, \quad (121)$$

whereas

$$M_T^{\leq}(S_T) = \max\{\Theta_T, T - \Theta_T\}. \quad (122)$$

Hence the jump size at the global maximum is

$$J_T = \Delta M_T^{\leq}(S_T) = \min\{\Theta_T, T - \Theta_T\}. \quad (123)$$

Since Θ_T/T has the arcsine distribution on $[0,1]$, a result going back to Lévy [21] (see also Karatzas-Shreve ([2] Section 2.8) for a modern derivation through the reflection principle), one obtains, for $0 \leq r \leq T/2$,

$$\mathbb{P}(J_T \leq r) = \frac{4}{\pi} \arcsin \sqrt{\frac{r}{T}}. \quad (124)$$

Equivalently,

$$\mathbb{P}(J_T > r) = 1 - \frac{4}{\pi} \arcsin \sqrt{\frac{r}{T}}. \quad (125)$$

This elementary formula is a useful diagnostic. The additive profile A_T is locally described by $L_T^a(X)$, but the burst profile has a macroscopic jump whose size is governed by the location of the global maximum.

The law of the event

$$\{M_T^{\leq,B}(a) > D\} \quad (126)$$

is the one-sided Parisian tail described in Theorem 3. Thus the Brownian example displays both sides of the theory: a local-time density for additive occupation and a Parisian excursion law for the extremal burst.

5.3. Ornstein–Uhlenbeck Process

Proposition 4 (Ornstein–Uhlenbeck occupation sensitivity). *Let X be the stationary Ornstein–Uhlenbeck process solving*

$$dX_t = -\theta X_t dt + \sigma dB_t, \quad \theta > 0, \quad (127)$$

with invariant density $p_\infty(a)$. Then, for every fixed level a , the expected first-order occupation increment satisfies

$$E[A_T(a + \varepsilon) - A_T(a)] \sim \varepsilon T \sqrt{\frac{\theta}{\pi \sigma^2}} e^{-\frac{\theta a^2}{\sigma^2}}, \quad \varepsilon \downarrow 0. \quad (128)$$

In particular, the first-order density of newly captured occupation is $T p_\infty(a)$, while the first-order increment is $\varepsilon T p_\infty(a)$. This density is maximal at the long-run mean and decays away from it with the stated Gaussian factor.

Proof. For the Ornstein–Uhlenbeck process defined in Proposition 4, one has

$$A'_T(a) = \frac{L_T^a(X)}{\sigma^2}. \quad (129)$$

The transition law is Gaussian; the mean, variance and invariant density recalled below are standard for the Ornstein-Uhlenbeck process and may be found, for example, in Karatzas-Shreve ([2] Section 5.6) or in the explicit tabulation of Borodin-Salminen [3]. If $X_0 = x$, then X_t has mean

$$xe^{-\theta t} \quad (130)$$

and variance

$$v_t = \frac{\sigma^2}{2\theta} (1 - e^{-2\theta t}). \quad (131)$$

Thus the transition density is

$$p_t(x, a) = \frac{1}{\sqrt{2\pi v_t}} \exp\left(-\frac{(a - xe^{-\theta t})^2}{2v_t}\right). \quad (132)$$

Taking expectations in the occupation-density formula yields

$$\mathbb{E}_x[L_T^a(X)] = \sigma^2 \int_0^T p_t(x, a) dt. \quad (133)$$

Consequently, for small $\varepsilon > 0$,

$$\mathbb{E}_x[A_T(a + \varepsilon) - A_T(a)] = \int_0^T \mathbb{P}_x(a < X_t \leq a + \varepsilon) dt, \quad (134)$$

and therefore

$$\mathbb{E}_x[A_T(a + \varepsilon) - A_T(a)] \sim \varepsilon \int_0^T p_t(x, a) dt = \frac{\varepsilon}{\sigma^2} \mathbb{E}_x[L_T^a(X)], \quad \varepsilon \downarrow 0. \quad (135)$$

In stationarity, the invariant density is

$$p_\infty(a) = \sqrt{\frac{\theta}{\pi\sigma^2}} \exp\left(-\frac{\theta a^2}{\sigma^2}\right). \quad (136)$$

If X_0 has this invariant distribution, then X_t has density p_∞ for every t , and hence

$$\mathbb{E}[A_T(a + \varepsilon) - A_T(a)] \sim \varepsilon T p_\infty(a), \quad \varepsilon \downarrow 0. \quad (137)$$

Equivalently,

$$\mathbb{E}[A_T(a + \varepsilon) - A_T(a)] \sim \varepsilon T \sqrt{\frac{\theta}{\pi\sigma^2}} \exp\left(-\frac{\theta a^2}{\sigma^2}\right). \quad (138)$$

The first-order expected amount of newly captured time is therefore maximal at the long-run mean 0, and decays away from 0 with Gaussian factor

$$\exp\left(-\frac{\theta a^2}{\sigma^2}\right). \quad (139)$$

This gives a quantitative version of the qualitative statement that levels near the long-run mean are revisited frequently. Under stationarity, the expected temporal material captured by raising the threshold from a to $a + \varepsilon$ has first-order density $T p_\infty(a)$. Hence the density of possible bridge-filling opportunities is largest near the mean-reversion level 0.

This statement should not be interpreted as a formula for the jump sizes of $M_T(a)$. The local-time density measures how much time is added when the threshold is raised. A

jump of $M_T(a)$, however, depends on where the newly added time lies in $[0, T]$. A very small amount of newly captured occupation may produce no change in the longest burst, or it may connect two long pre-existing components and produce a macroscopic jump.

Thus the mean-reverting drift makes the qualitative burst picture different from Brownian motion. Near the long-run mean, the sub-threshold time set tends to be highly fragmented, and threshold increases may progressively fill small bridges between adjacent visits. Far above or far below the mean, the expected amount of near-level occupation is smaller, and the burst structure is typically less fragmented.

Although an explicit Parisian transform for the Ornstein–Uhlenbeck process is not as elementary as in the Brownian case, the pathwise identity of Lemma 2 still holds. Therefore

$$\{M_T^{<,X}(a) > D\} \quad (140)$$

is the event that the Ornstein–Uhlenbeck process has a below- a excursion of duration strictly greater than D . This is the natural diffusion analogue of the Brownian Parisian event. $\square$

5.4. Reflected Brownian Motion

Let X be reflected Brownian motion on $[0, \infty)$:

$$dX_t = dB_t + dK_t, \quad (141)$$

where K_t is the reflection term. Thus K is nondecreasing, $X_t \geq 0$, and K increases only when $X_t = 0$. Equivalently,

$$\int_0^T \mathbf{1}_{\{X_t > 0\}} dK_t = 0. \quad (142)$$

For $a > 0$, the cumulative load below a is just the usual additive occupation functional

$$A_T(a) = \int_0^T \mathbf{1}_{\{X_t \leq a\}} dt. \quad (143)$$

Since the diffusion coefficient is 1 away from the boundary, the occupation-density formula gives

$$A'_T(a) = L_T^a(X), \quad a > 0. \quad (144)$$

Near $a = 0$, the behavior is influenced by the boundary local time generated by reflection. With the local-time normalization used in the occupation-density formula,

$$\frac{1}{a} A_T(a) = \frac{1}{a} \int_0^T \mathbf{1}_{\{0 \leq X_t \leq a\}} dt \rightarrow L_T^0(X), \quad a \downarrow 0. \quad (145)$$

In the Skorokhod construction of reflected Brownian motion [22], where the reflection term is the minimal nondecreasing process keeping the path nonnegative, this boundary local time satisfies

$$L_T^0(X) = 2K_T. \quad (146)$$

Therefore,

$$\frac{1}{a} A_T(a) \rightarrow 2K_T, \quad a \downarrow 0. \quad (147)$$

Thus the cumulative time spent in a vanishing boundary layer is asymptotically proportional to the boundary local time generated by reflection.

A closed expectation-level statement is available in the canonical case $X_t = |B_t|$ started from zero. In that case,

$$A_T(a) = \int_0^T \mathbf{1}_{\{|B_t| \leq a\}} dt, \quad (148)$$

and hence

$$\mathbb{E}[A_T(a)] = \int_0^T \mathbb{P}(|B_t| \leq a) dt. \quad (149)$$

Since $B_t \sim N(0, t)$, the symmetry of the centred Gaussian law about the origin gives $\mathbb{P}(|B_t| \leq a) = \mathbb{P}(-a \leq B_t \leq a) = 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1$, where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. So,

$$\mathbb{E}[A_T(a)] = \int_0^T \left[2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1 \right] dt. \quad (150)$$

As $a \downarrow 0$,

$$2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1 \sim \sqrt{\frac{2}{\pi}} \frac{a}{\sqrt{t}}. \quad (151)$$

Therefore,

$$\mathbb{E}[A_T(a)] \sim 2a\sqrt{\frac{2T}{\pi}}, \quad a \downarrow 0. \quad (152)$$

Equivalently,

$$\frac{1}{a} \mathbb{E}[A_T(a)] \rightarrow 2\sqrt{\frac{2T}{\pi}}. \quad (153)$$

This is consistent with

$$\frac{1}{a} A_T(a) \rightarrow 2K_T, \quad (154)$$

because, in this normalization,

$$\mathbb{E}[K_T] = \sqrt{\frac{2T}{\pi}}. \quad (155)$$

The continuous Parisian burst profile below a small threshold $a > 0$, however, does not admit a corresponding first-order local-time formula. It describes episodes during which the reflected process remains close to the boundary. Such episodes may merge when a threshold increase fills a brief upward excursion away from the boundary. Hence the near-boundary longest burst is a connectivity statistic, not a local-time density.

This is relevant in applications where one distinguishes cumulative time spent near a constraint from the longest uninterrupted time spent near that constraint. The first quantity is asymptotically governed by boundary local time. The second is governed by the organization of near-boundary visits into connected temporal components.

5.5. Sticky Behavior

If X is a sticky diffusion at level a_0 , then

$$\lambda\{t : X_t = a_0\} \quad (156)$$

may be positive with positive probability. More precisely, a sticky point arises when the scale function and speed measure of the diffusion assign a positive atom of the speed

measure to the single point a_0 . Equivalently, in the stochastic-differential-equation description of sticky Brownian motion of Engelbert-Peskir [23], the dynamics at the sticky point are governed by a stickiness parameter that forces the local time at a_0 to grow on a set of positive Lebesgue time, so that the occupation indicator $\mathbf{1}_{\{X_t=a_0\}}$ has strictly positive expected integral. In that case the additive occupation profile has a genuine jump at a_0 .

Indeed,

$$A_T^{\leq}(a_0) - A_T^{\leq}(a_0) = \lambda\{t : X_t = a_0\}. \quad (157)$$

Equivalently,

$$A_T^{\leq}(a_0) - \lim_{a \uparrow a_0} A_T^{\leq}(a) = \lambda\{t : X_t = a_0\}. \quad (158)$$

More generally, for every $\varepsilon > 0$,

$$A_T^{\leq}(a_0 + \varepsilon) - A_T^{\leq}(a_0 - \varepsilon) = \lambda\{t : X_t = a_0\} + \lambda\{t : a_0 - \varepsilon \leq X_t \leq a_0 + \varepsilon, X_t \neq a_0\}. \quad (159)$$

Letting $\varepsilon \downarrow 0$, the second term vanishes under the usual non-sticky behavior away from a_0 , while the first term remains. Thus the sticky level contributes an atom in the threshold variable.

In schematic Stieltjes form, the threshold measure associated with A_T contains a term of the form

$$\lambda\{t : X_t = a_0\} \delta_{a_0}(da). \quad (160)$$

Sticky behavior therefore marks a different regime. For non-sticky diffusions, additive occupation is locally absolutely continuous, while burst duration may be unstable because of component mergers. For sticky diffusions, even additive occupation may fail to be continuous at the sticky level.

This example clarifies the role of the non-sticky assumption. In the non-sticky case, the additive functional is regular and the burst functional is singular. In the sticky case, the additive functional itself acquires a discontinuity because the process spends positive Lebesgue time at a single level. Thus the additive/extremal contrast is sharpest in the non-sticky diffusion setting.

5.6. Numerical Validation

We record the numerical experiments that produce the figures and validate the analytic predictions of this section. Throughout, sample paths are generated on $[0, T]$ with $T = 1$ by an Euler-Maruyama scheme on a uniform grid of n steps of size $dt = T/n$; the occupation profile $a \mapsto A_T(a)$ is evaluated as dt times the number of grid points below a , computed from the same simulated ensemble across all levels (common random numbers), and the longest burst $M_T(a)$ as dt times the longest run of consecutive sub-threshold grid points.

Construction of Figure 2. Panel (a) uses a single Brownian path: the profiles $a \mapsto A_T(a)$ and $a \mapsto M_T(a)$ are read off that path over a grid of levels, giving the continuous occupation profile and the step-like longest-burst profile with its jump to T at the maximum level $a = S_T$. Panel (b) is a Monte-Carlo histogram of the jump $J_T = \min\{\Theta_T, T - \Theta_T\}$ over $N = 60,000$ independent paths, with $\Theta_T = dt \arg \max_k B_{t_k}$, overlaid with the arcsine density, the inset comparing the empirical and theoretical CDFs. An independent run of $N = 48,000$ paths reproduces the arcsine law $\mathbb{P}(J_T \leq r) = \frac{4}{\pi} \arcsin \sqrt{r/T}$ to within 4×10^{-3} uniformly over $r \in \{0.1, \dots, 0.5\}$.

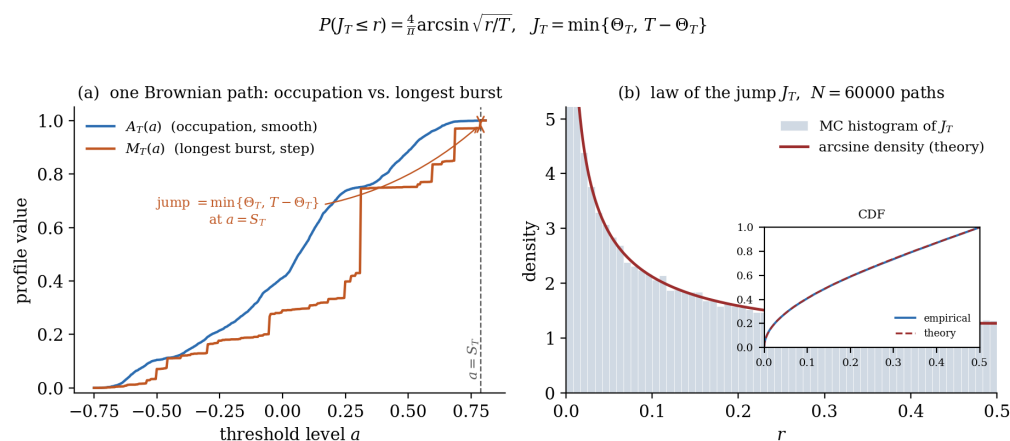


Figure 2. The Brownian longest-burst jump at the maximum level (Section 5.2). (a) For a single simulated path on $[0, T]$, the occupation profile $a \mapsto A_T(a)$ is continuous, whereas the longest-burst profile $a \mapsto M_T(a)$ is step-like, with a macroscopic jump at the maximum level $a = S_T$ where M_T jumps to T . (b) Monte-Carlo law of the jump $J_T = \min\{\Theta_T, T - \Theta_T\}$ over $N = 60,000$ paths, compared with the exact arcsine density; the inset shows the empirical and theoretical CDFs $\mathbb{P}(J_T \leq r) = \frac{4}{\pi} \arcsin \sqrt{r/T}$.

Validation of the diffusion predictions (Figure 3). Panel (a) tests the Ornstein–Uhlenbeck occupation sensitivity of Proposition 4: for the stationary process with $\theta = \sigma = 1$, the finite-difference sensitivity $\varepsilon^{-1} \mathbb{E}[A_T(a + \varepsilon) - A_T(a)]$ over 16,000 paths matches the predicted density $TP_\infty(a)$ across the level range, peaking at the long-run mean $a = 0$ (simulated 0.567 versus $TP_\infty(0) = 0.564$, and 0.187 versus 0.187 at $a = -1$). Panel (b) tests the reflected-Brownian near-boundary asymptotics of Section 5.4: for $X = |B|$ the ratio $a^{-1} \mathbb{E}[A_T(a)]$ over 40,000 paths rises toward the predicted limit $2\sqrt{2T/\pi} \approx 1.596$ as $a \downarrow 0$ (for instance 1.566 at $a = 0.02$ and 1.542 at $a = 0.05$). Together with the arcsine jump law of Figure 2, these experiments confirm the local-time prediction for the additive clock and the excursion/merger prediction for the burst clock.

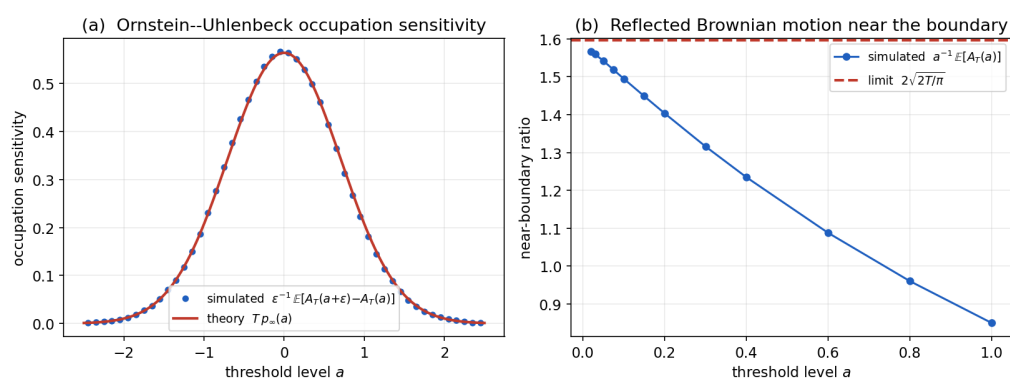


Figure 3. Numerical validation (Section 5.6). (a) Ornstein–Uhlenbeck occupation sensitivity: the simulated finite-difference sensitivity $\varepsilon^{-1} \mathbb{E}[A_T(a + \varepsilon) - A_T(a)]$ (points, 16,000 stationary paths, $\theta = \sigma = 1$) against the theoretical density $TP_\infty(a)$ of Proposition 4. (b) Reflected Brownian motion $X = |B|$: the near-boundary ratio $a^{-1} \mathbb{E}[A_T(a)]$ (points, 40,000 paths) converging to the predicted limit $2\sqrt{2T/\pi}$ (dashed) as $a \downarrow 0$, cf. Section 5.4.

6. Extensions: Smoothed Burst Statistics and Moving Thresholds

The main theory focuses on the cumulative Parisian clock $A_T(a)$ and the continuous Parisian longest-burst clock $M_T(a)$. This section records two extensions showing that the same contrast persists beyond the basic fixed-threshold longest-burst functional. The first

extension replaces the maximum component length by smoothed statistics of ranked burst lengths. The second replaces the fixed threshold by a deterministic moving boundary.

6.1. Smoothed Burst Statistics

The longest-burst functional is the most singular member of a broader family of burst statistics. Let

$$\ell_1(a) \geq \ell_2(a) \geq \dots \quad (161)$$

denote the lengths of the connected components of $E_T^{\leq x}(a)$, arranged in nonincreasing order and repeated according to multiplicity. Then

$$M_T^{\leq x}(a) = \ell_1(a), \quad (162)$$

and if level sets have zero Lebesgue measure,

$$A_T^{\leq x}(a) = \sum_{j \geq 1} \ell_j(a). \quad (163)$$

For a bounded function $\varphi : [0, T] \rightarrow \mathbb{R}$, define

$$\Psi_T^{x, \varphi}(a) = \sum_{j \geq 1} \varphi(\ell_j(a)). \quad (164)$$

To avoid contributions from infinitely many microscopic components, assume that φ is supported away from zero:

$$\text{supp } \varphi \subset [\eta, T] \quad (165)$$

for some $\eta > 0$. Then only components of length at least η contribute, and there are at most T/η of them.

The ranked family $(\ell_j(a))$ is the natural threshold-indexed analogue of ranked excursion lengths. In the Brownian case, ranked excursion lengths are classically connected with stable subordinators and Poisson–Dirichlet distributions [5].

Proposition 5 (Finiteness of smoothed burst statistics). *Let φ be bounded and supported in $[\eta, T]$, with $\eta > 0$. Then, for every continuous path x and every threshold a ,*

$$\Psi_T^{x, \varphi}(a) \quad (166)$$

is finite and satisfies

$$\left| \Psi_T^{x, \varphi}(a) \right| \leq \frac{T}{\eta} \|\varphi\|_{\infty}. \quad (167)$$

Proof. Only components of length at least η contribute. Since all such components are disjoint subintervals of $[0, T]$, there are at most T/η of them. The bound follows. $\square$

Proposition 6 (Stability away from critical levels). *Let $x \in C^1([0, T])$ have finitely many strict critical points with distinct critical values. Let a be a noncritical value of x . If φ is continuous and supported away from zero, then*

$$\Psi_T^{x, \varphi}(a + \varepsilon) \rightarrow \Psi_T^{x, \varphi}(a) \quad (168)$$

as $\varepsilon \rightarrow 0$.

Proof. Since a is not a critical value, the crossing times of x through level a move continuously under small perturbations of the level. The number and ordering of the relevant

macroscopic components remain unchanged for sufficiently small ε , and their lengths vary continuously. Since φ is continuous and only finitely many components contribute, the conclusion follows.

Smoothed burst statistics lie between cumulative occupation and the longest burst: they are less singular than M_T , but still depend on the component structure of $E_T(a)$. The mechanism remains nonlocal—burst statistics depend on how threshold perturbations reorganize connected components. $\square$

6.2. Moving Thresholds

Proposition 7 (Moving-boundary occupation sensitivity). *Let X be a continuous semimartingale with $d\langle X \rangle_t = q_t dt$, where $q_t > 0$, let b be a finite-variation boundary, set $Y = X - b$, and let h be bounded and continuous. Under the assumptions ensuring the validity of the local-time-on-curves change-of-variable formula of Peskir [20], the moving-boundary occupation functional is Gâteaux differentiable in the direction h , with*

$$DA_T(b)[h] = \int_0^T h(t) \frac{1}{q_t} dL_t^0(Y). \quad (169)$$

In the autonomous diffusion case, this specializes to

$$DA_T(b)[h] = \int_0^T h(t) \frac{1}{\sigma^2(X_t)} dL_t^0(X - b). \quad (170)$$

Thus moving-boundary occupation retains a local-time sensitivity calculus, now governed by the local time of $X - b$ on the boundary curve, exactly as in the static case of Proposition 2.

Proof. Let $b : [0, T] \rightarrow \mathbb{R}$ be a deterministic moving boundary. Define

$$E_T^X(b) = \{t \in [0, T] : X_t \leq b(t)\}, \quad (171)$$

and set

$$Y_t = X_t - b(t). \quad (172)$$

Then

$$E_T^X(b) = \{t : Y_t \leq 0\}. \quad (173)$$

Thus all deterministic burst results for fixed thresholds apply to moving thresholds after replacing X by Y .

The moving-boundary occupation functional is

$$A_T(b) = \int_0^T \mathbf{1}_{\{X_t \leq b(t)\}} dt = \int_0^T \mathbf{1}_{\{Y_t \leq 0\}} dt. \quad (174)$$

The moving-boundary longest burst is

$$M_T(b) = \sup\{|I| : I \text{ is a connected component of } E_T^X(b)\}. \quad (175)$$

If the boundary is perturbed in the direction h ,

$$b_\varepsilon(t) = b(t) + \varepsilon h(t), \quad (176)$$

then

$$A_T(b_\varepsilon) = \int_0^T \mathbf{1}_{\{Y_t \leq \varepsilon h(t)\}} dt. \quad (177)$$

For additive occupation, the sensitivity is governed by local time of Y at zero, equivalently local time of X on the curve b . Suppose X is a continuous semimartingale with strictly positive quadratic-variation density

$$d\langle X \rangle_t = q_t dt, \quad (178)$$

and b has finite variation. Then

$$d\langle Y \rangle_t = d\langle X \rangle_t = q_t dt. \quad (179)$$

Under the standard assumptions ensuring the validity of the local-time-on-curves change-of-variable formula [20].

Indeed, the quotient $\varepsilon^{-1}(A_T(b + \varepsilon h) - A_T(b))$ measures the first-order occupation of Y in the thin moving strip between 0 and $\varepsilon h(t)$. The local-time-on-curves formula identifies the limit of this strip occupation with the local time accumulated by Y along the zero curve, weighted by $h(t)q_t^{-1}$.

$$DA_T(b)[h] = \int_0^T h(t) \frac{1}{q_t} dL_t^0(Y). \quad (180)$$

In the autonomous diffusion case

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t, \quad (181)$$

one has

$$q_t = \sigma^2(X_t), \quad (182)$$

so

$$DA_T(b)[h] = \int_0^T h(t) \frac{1}{\sigma^2(X_t)} dL_t^0(X - b). \quad (183)$$

This is the moving-boundary analogue of the static identity

$$A'_T(a) = \frac{L_T^a(X)}{\sigma^2(a)}. \quad (184)$$

For burst functionals, the same local-time calculus does not apply. The moving-threshold longest burst $M_T(b)$ is the longest connected component of

$$\{t : Y_t \leq 0\}. \quad (185)$$

Perturbing b changes the zero-sublevel set of Y . A small perturbation may fill a short bridge between two large sub-threshold components, producing the same merger instability as in Section 3 (Theorem 2). Hence the moving-boundary case reinforces the central contrast:

- moving-boundary occupation is governed by curve local time
 - moving-boundary burst duration is governed by component mergers
-

7. Conclusions

This paper compared two classes of threshold functionals attached to the sub-threshold time set $E_T(a) = \{t \in [0, T] : X_t \leq a\}$: the additive, cumulative *Parasian* occupation time $A_T(a) = \lambda(E_T(a))$, which for regular one-dimensional diffusions obeys the local-time

sensitivity Formula (28), and the resetting *Parisian* longest-burst functional $M_T(a)$, the longest connected sub-threshold episode.

We proved that the weak longest-burst profile is monotone, that its supremum is attained, and that it is right-continuous with left limits; the strict-sublevel functional is the corresponding left-continuous regularization. The jump at a level is the increase in maximal component length produced by adjoining the level set. Theorem 2 makes this jump explicit: at a two-sided merger it equals the length of the shorter flanking component (plus the bridge, in the sticky case). This gives a deterministic one-sided regularity theory for burst profiles.

We then showed that this burst profile is not an additive local-time functional. At deterministic levels which are almost surely not local-extreme values, weak and strict longest bursts agree. But a path with a unique interior maximum has a positive jump in its longest-burst profile at the maximum level, and, more generally, so does any two-sided merger (Remark 2). Brownian motion satisfies this condition almost surely. Thus, while additive occupation is locally governed by local time, extreme burst duration is globally governed by connectivity.

Finally, we identified the longest burst with a one-sided Parisian functional. For Brownian motion this gives a Laplace-transform representation through the Chesney–Jeanblanc–Picqué–Yor Parisian transform, extended to the drifted case by a Cameron–Martin tilt; in the local-time clock the law is governed by the Itô excursion-lifetime tail $n_-(\zeta > D) = 1/\sqrt{2\pi D}$. The same local time that appears as a density for additive occupation therefore appears, for burst extremes, only as the clock intensity of a Poisson process of excursions. These analytic predictions were confirmed numerically in Section 5.6.

The extensions preserve the same distinction: additive occupation below a moving boundary is governed by local time on the boundary curve, while burst duration remains sensitive to bridge filling and component mergers.

These two clocks also have a direct reading for monitoring problems in finance and risk management. In barrier-contract terms, the Parisian trigger $\{A_T(a) > D\}$ fires on the *total* time spent beyond a level and has a local-time-smooth threshold sensitivity, whereas the continuous Parisian trigger $\{M_T(a) > D\}$ fires on a single *uninterrupted* spell beyond the level and has a discontinuous threshold sensitivity. The merger instability shows that these two monitors react very differently to a small change in the level: raising the barrier slightly can leave the cumulative exposure A_T essentially unchanged while abruptly increasing the longest continuous breach M_T by a macroscopic amount, as soon as a short bridge is filled. A duration-based risk limit on the longest uninterrupted time spent in a stressed region is therefore far more sensitive to the precise placement of the threshold than a cumulative-time limit, and may jump even when average exposure is stable. The same contrast underlies occupation-time versus continuous-Parisian barrier options, drawdown-duration constraints, and any covenant phrased through the longest continuous, rather than the total, time a process spends beyond a level.

Several directions remain open. The Brownian marginal law of $M_T(a)$ is described by the Parisian and excursion-measure forms of Section 4, but the joint law of the jump levels and jump sizes of $a \mapsto M_T(a)$ remains to be characterized. For general regular diffusions, explicit Parisian transforms are typically unavailable and the corresponding marginal laws remain model-dependent. Other natural problems include multidimensional and rough-path analogues, the behavior of burst profiles under discrete observation, and statistical estimation of bridge-merger levels from sampled trajectories. These questions lie beyond the local-time calculus of additive occupation and require a genuinely component-based theory of threshold episodes.

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