

Article

Invariant Geometric Objects of the Equitorsion Quasi-Canonical Biholomorphically Projective Mappings of Generalized Riemannian Space in the Eisenhart Sense

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Abstract

This paper represents a natural continuation of the study of biholomorphically projective mappings. It obtains new invariant geometric objects for equitorsion quasi-canonical biholomorphically projective mappings, together with several relations satisfied by these invariants.

Keywords: equitorsion quasi-canonical biholomorphically projective mapping; generalized Riemannian space; invariant geometric object

MSC: 53B05; 53B20; 53C15

1. Introduction

Differential geometry is a branch of mathematics, the results of which have wide application across diverse fields, including physics [1–4] and chemistry [5,6]. Utilizing transformations of affine connections, it is possible to describe transitions from initial to current states (classical mechanics) and from a space with a Friedman–Lemaître–Robertson–Walker (FLRW) metric to a space with a perturbed FLRW metric, as well as numerous other dynamical changes.

The coefficients of affine connections, which uniquely determine the connections themselves, are not tensors, meaning that transformations of coordinate systems have impacts on their forms. However, the difference between the coefficients of two affine connections is indeed a tensor (the deformation tensor), which uniquely dictates the laws governing the transformation of the affine connection.

One of the primary areas of research in differential geometry focuses on investigating which geometric objects remain invariant under various types of affine connection transformations. The methodology for obtaining invariants of mappings [7,8] was improved in [9] by finding one more invariant.

Among the most prominent of these objects is the Ricci tensor, which is invariant under special geodesic transformations. A consequence of its invariance is the classical Einstein–Hilbert action, the variation of which describes gravity. Modified Einstein–Hilbert actions arise as a result of applying non-geodesic transformations from the FLRW space to the perturbed space.



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The present research addresses the invariants of biholomorphically projective mappings of generalized Riemannian spaces. In [10], where the methodology for obtaining invariants of mappings presented by [8] was inapplicable, the formulas from [9] were used for determination of invariants based on the transformation of the curvature tensor under the action of such a mapping. Our main purpose in this paper is to apply the methodology from [9] and to obtain the corresponding invariants in the case of equitortion quasi-canonical biholomorphically projective mappings.

2. Necessary Differential Geometry

Differentiable manifolds with symmetric and non-symmetric metric tensors, as well as differentiable manifolds with symmetric and non-symmetric affine connections, and their mappings have been considered in many papers [11–22]. Invariant geometric objects for various mappings of spaces with symmetric and non-symmetric affine connections have been obtained by many authors [7–13,20–26].

In the sense of Eisenhart's definition [27], a generalized Riemannian space GR_N is a differentiable manifold equipped with a non-symmetric metric tensor g_{ij} . The connection coefficients of the space GR_N are the generalized Cristoffel's symbols of the second kind $\Gamma_{jk}^i = g^{ip}\Gamma_{p,jk}$, where $\|g^{ij}\| = \|g_{ij}\|^{-1}$, $g_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$, and $\Gamma_{i,jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j})$, where, for example, $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$, $\det\|g_{ij}\| \neq 0$, $\det\|g_{ij}\| \neq 0$. In the general case, $\Gamma_{jk}^i \neq \Gamma_{kj}^i$, and they can be represented as the sum of the symmetric and antisymmetric parts, i.e., $\Gamma_{jk}^i = S_{jk}^i + T_{jk}^i$, where $S_{jk}^i = \Gamma_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i)$, $T_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i$.

In the case of a Riemannian space, the covariant derivative with respect to Γ_{jk}^i , denoted by $|^g$, for example, for a_j^i , is defined by the equality [18]

$$a_{j|gk}^i = a_{j,k}^i + \Gamma_{pk}^i a_j^p - \Gamma_{jk}^p a_p^i. \quad (1)$$

The curvature tensor of a Riemannian space is

$$r_{jmn}^i = \Gamma_{jm,n}^i - \Gamma_{jn,m}^i + \Gamma_{jm}^p \Gamma_{pn}^i - \Gamma_{jn}^p \Gamma_{pm}^i. \quad (2)$$

In a generalized Riemannian space $\mathbb{GR}_N$, there are four kinds of covariant derivatives. For example, for a tensor a_j^i , we have [18]

$$a_{j|_1gk}^i = a_{j,k}^i + \Gamma_{pk}^i a_j^p - \Gamma_{jk}^p a_p^i, \quad (3)$$

$$a_{j|_2gk}^i = a_{j,k}^i + \Gamma_{kp}^i a_j^p - \Gamma_{kj}^p a_p^i, \quad (4)$$

$$a_{j|_3gk}^i = a_{j,k}^i + \Gamma_{pk}^i a_j^p - \Gamma_{kj}^p a_p^i, \quad (5)$$

$$a_{j|_4gk}^i = a_{j,k}^i + \Gamma_{kp}^i a_j^p - \Gamma_{jk}^p a_p^i. \quad (6)$$

where $|_z^g$, ($z = 1, 2, 3, 4$) denotes a covariant derivative of the kind z with respect to the affine connection of a generalized Riemannian space.

The family of curvature tensors of the space $\mathbb{GR}_N$ is

$$k_{jmn}^i = r_{jmn}^i + u \Gamma_{jm|_1gk}^i + u' \Gamma_{jn|_2gk}^i + v \Gamma_{jm}^p \Gamma_{pn}^i + v' \Gamma_{jn}^p \Gamma_{pm}^i + w \Gamma_{mn}^p \Gamma_{pj}^i, \quad (7)$$

for scalars u, u', v, v', w .

From family (7), we extracted five linearly independent curvature tensors by selecting the following constants (u, u', v, v', w) :

$$(u, u', v, v', w) \in \{(1, -1, 1, -1, 0), (-1, 1, 1, -1, 0), (1, 1, -1, 1, 2), (1, 1, -1, 1, -2), (0, 0, 1, 1, 0)\}. \quad (8)$$

For the given choice of constants, the curvature tensors will be denoted as $r_{1jmn}^i, r_{2jmn}^i, r_{3jmn}^i, r_{4jmn}^i$, and r_{5jmn}^i , respectively.

3. Invariants for Equitorsion Quasi-Canonical Biholomorphically Projective Mappings

Let F_j^i be an affinor (tensor of the type $(1, 1)$), and let $\bar{F}_j^i = F_p^i F_j^p$. An equitorsion quasi-canonical biholomorphically projective mapping $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ between the generalized Riemannian spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\bar{\mathbb{R}}_N$ is the transformation of affine connection Γ_{jk}^i to $\bar{\Gamma}_{jk}^i$ such that the deformation tensor P_{jk}^i satisfies the following equation [23]:

$$P_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i + \rho_k \bar{F}_j^i + \rho_j \bar{F}_k^i, \quad (9)$$

for 1-forms ψ_i and ρ_i .

Remark 1. To obtain invariants for the equitorsion quasi-canonical biholomorphically projective mapping f determined by (9), the summand $\rho_j \bar{F}_k^i + \rho_k \bar{F}_j^i$ should be expressed as the difference in geometrical objects of the same form, one from the space $\mathbb{G}\mathbb{R}_N$ and the other one from $\mathbb{G}\bar{\mathbb{R}}_N$. To do that, we need to start from the definition of quasi-canonical biholomorphically projective mappings of a generalized space $\mathbb{G}\mathbb{R}_N$ to $\mathbb{G}\bar{\mathbb{R}}_N$, where any quasi-canonical biholomorphically projective curve $\ell = \ell(t)$ of the space $\mathbb{G}\mathbb{R}_N$, defined by [23]

$$\lambda_{|p}^i \lambda^p = a \lambda^i + c \bar{F}_p^i \lambda^p, \quad (10)$$

for the tangential vector of curve $\ell = (\ell^i)$ equal to $\lambda^i = \frac{d\ell^i}{dt}$, scalars $a = 2a_p \lambda^p$, $c = 2c_p \lambda^p$ to the quasi-canonical biholomorphically projective curve of the space $\mathbb{G}\mathbb{R}_N$

$$\lambda_{||p}^i \lambda^p = \bar{a} \lambda^i + \bar{c} \bar{F}_p^i \lambda^p, \quad (11)$$

for $\bar{a} = 2\bar{a}_p \lambda^p$, $\bar{c} = 2\bar{c}_p \lambda^p$, where a_p and c_p are covariant vectors in the space $\mathbb{G}\mathbb{R}_N$, and $\bar{a}_p$ and $\bar{c}_p$ are the corresponding geometrical objects in $\mathbb{G}\bar{\mathbb{R}}_N$.

The basic Equation (9) is obtained by subtracting the relation (10) from (11), using $F_j^i = \bar{F}_j^i$, which gives the invariance $\bar{F}_j^i = \bar{F}_p^i \bar{F}_j^p = F_p^i F_j^p = \bar{F}_j^i$, and by changing $\rho_j = \bar{c}_j - c_j$. Our purpose is to obtain invariants for the analyzed quasi-canonical biholomorphically projective mappings by saving $\bar{c}_j$ as a vector in the space $\mathbb{G}\bar{\mathbb{R}}_N$ and c_j as a vector in the space $\mathbb{G}\mathbb{R}_N$. The 1-form $\psi_j = \bar{a}_j - a_j$ will save this form because of simplification, but it will be expressed as the difference in identical geometrical objects from the spaces $\mathbb{G}\bar{\mathbb{R}}_N$ and $\mathbb{G}\mathbb{R}_N$.

Lemma 1. Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ be an equitorsion quasi-canonical biholomorphically projective mapping. Its deformation tensor is

$$P_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i + (c_k \bar{F}_j^i + c_j \bar{F}_k^i) - (\bar{c}_k \bar{F}_j^i + \bar{c}_j \bar{F}_k^i). \quad (12)$$

Proof. Unlike in [23], we will subtract the relation (10) from (11) and just take $\psi_j = a_j - \bar{a}_j$. With respect to this difference, and the relations $\lambda_{|p}^i = \lambda_{,p}^i + \Gamma_{pq}^i \lambda^p \lambda^q$ and $\lambda_{||p}^i = \lambda_{,p}^i + \bar{\Gamma}_{pq}^i \lambda^p \lambda^q$, we obtain

$$(\bar{\Gamma}_{pq}^i - \Gamma_{pq}^i) \lambda^p \lambda^q = 2\psi_p \lambda^p \lambda^i + 2c_q \lambda^q \bar{F}_p^i \lambda^p - 2\bar{c}_q \lambda^q \bar{F}_p^i \lambda^p.$$

After symmetrizing the last relation by covariant indices p and q , $c_q \lambda^q \bar{F}_p^i \lambda^p = \frac{1}{2}(c_q \bar{F}_p^i + c_p \bar{F}_q^i) \lambda^p \lambda^q$ and $\bar{c}_q \lambda^q \bar{F}_p^i \lambda^p = \frac{1}{2}(\bar{c}_q \bar{F}_p^i + \bar{c}_p \bar{F}_q^i) \lambda^p \lambda^q$, one gets

$$(\bar{\Gamma}_{pq}^i - \Gamma_{pq}^i) \lambda^p \lambda^q = (\psi_q \delta_p^i + \psi_p \delta_q^i) \lambda^p \lambda^q + (c_q \bar{F}_p^i + c_p \bar{F}_q^i) \lambda^p \lambda^q - (\bar{c}_q \bar{F}_p^i + \bar{c}_p \bar{F}_q^i) \lambda^p \lambda^q,$$

which is satisfied if and only if the relation (12) holds. $\square$

To simplify the future calculation, we need to involve the next geometrical objects:

$$\bar{\mathcal{F}}_{jk}^i = c_k \bar{F}_j^i + c_j \bar{F}_k^i, \quad \bar{\mathcal{F}}_{jk}^i = \bar{c}_k \bar{F}_j^i + \bar{c}_j \bar{F}_k^i. \quad (13)$$

The first set of results of this research is obtained in the next lemma.

Lemma 2. Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ be an equitorsion quasi-canonical biholomorphically projective mapping. Its basic associated invariants of the Thomas type are

$$\begin{cases} t_{jk}^i = \Gamma_{jk}^i - \frac{1}{N+1} \delta_j^i (\Gamma_k - \frac{2}{\mathcal{F}_k}) - \frac{1}{N+1} \delta_k^i (\Gamma_j - \frac{2}{\mathcal{F}_j}) - \frac{2}{\mathcal{F}_{jk}}^i, \\ \bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \frac{1}{N+1} \delta_j^i (\bar{\Gamma}_k - \frac{2}{\bar{\mathcal{F}}_k}) - \frac{1}{N+1} \delta_k^i (\bar{\Gamma}_j - \frac{2}{\bar{\mathcal{F}}_j}) - \frac{2}{\bar{\mathcal{F}}_{jk}}^i, \end{cases} \quad (14)$$

for $\Gamma_i = \Gamma_{ip}^p$, $\bar{\Gamma}_i = \bar{\Gamma}_{ip}^p$, $\frac{2}{\mathcal{F}_j} = \frac{2}{\mathcal{F}_{jp}^p}$, $\frac{2}{\bar{\mathcal{F}}_j} = \frac{2}{\bar{\mathcal{F}}_{jp}^p}$.

The basic associated invariants of the Weyl type for the mapping f are

$$\begin{cases} w_{jmn}^i = r_{jmn}^i - \delta_j^i a_{[mn]} - \delta_m^i (\Gamma_j \Gamma_n + a_{jn}) + \delta_n^i (\Gamma_j \Gamma_m + a_{jm}) + b_{jmn}^i, \\ \bar{w}_{jmn}^i = \bar{r}_{jmn}^i - \delta_j^i \bar{a}_{[mn]} - \delta_m^i (\bar{\Gamma}_j \bar{\Gamma}_n + \bar{a}_{jn}) + \delta_n^i (\bar{\Gamma}_j \bar{\Gamma}_m + \bar{a}_{jm}) + \bar{b}_{jmn}^i, \end{cases} \quad (15)$$

for

$$\begin{cases} a_{ij} = \frac{1}{N+1} (\Gamma_i \Gamma_j - \frac{2}{\mathcal{F}_{ij}} + \frac{2}{\mathcal{F}_{jm}}^p (\Gamma_p - \frac{2}{\mathcal{F}_p})) - \frac{1}{(N+1)^2} \frac{2}{\mathcal{F}_i} \Gamma_j - \frac{1}{(N+1)^2} \frac{2}{\mathcal{F}_j} (\Gamma_i - \frac{2}{\mathcal{F}_i}), \\ b_{jmn}^i = -\frac{2}{\mathcal{F}_{jm}^i} \Gamma_n + \frac{2}{\mathcal{F}_{jn}^i} \Gamma_m + \frac{2}{\mathcal{F}_{jm}^p} \frac{2}{\mathcal{F}_{pn}}^i - \frac{2}{\mathcal{F}_{jn}^p} \frac{2}{\mathcal{F}_{pm}}^i, \end{cases} \quad (16)$$

and the corresponding $\bar{a}_{ij}$ and $\bar{b}_{jmn}^i$.

Proof. The theoretical background of proof of this lemma is presented in Appendix A.1. The deformation tensor P_{jk}^i of the quasi-canonical biholomorphically projective mapping f factored as (12) is of the same form as the deformation tensor in Appendix A.1, Equation (A4), where $p_{jk}^i = \frac{2}{\mathcal{F}_{jk}}^i$.

After contracting the basic Equation (12) by i and k , and using $\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i$, i.e., $P_{jk}^i = \bar{P}_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i$, we obtain

$$\psi_j = \frac{1}{N+1} (\bar{\Gamma}_j - \Gamma_j) - \frac{1}{N+1} (\frac{2}{\mathcal{F}_j} - \frac{2}{\bar{\mathcal{F}}_j}), \quad (17)$$

for $\frac{2}{\mathcal{F}_j} = \frac{2}{\mathcal{F}_{jp}^p}$, $\frac{2}{\mathcal{F}_j} = \frac{2}{\mathcal{F}_{jp}^p}$.

Hence, with respect to expression (17), the basic Equation (12) transforms to

$$\begin{aligned} \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i &= \frac{1}{N+1} \delta_j^i (\bar{\Gamma}_k - \frac{2}{\mathcal{F}_k}) + \frac{1}{N+1} \delta_k^i (\bar{\Gamma}_j - \frac{2}{\mathcal{F}_j}) - \frac{2}{\mathcal{F}_{jk}^i} \\ &\quad - \frac{1}{N+1} \delta_j^i (\Gamma_k - \frac{2}{\mathcal{F}_k}) - \frac{1}{N+1} \delta_k^i (\Gamma_j - \frac{2}{\mathcal{F}_j}) + \frac{2}{\mathcal{F}_{jk}^i}. \end{aligned} \quad (18)$$

As in Appendix A, and with respect to the relation (18), the difference $\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i$ is factored as $\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \bar{\theta}_{jk}^i - \theta_{jk}^i$ for

$$\begin{cases} \theta_{jk}^i = \frac{1}{N+1} \delta_j^i (\Gamma_k - \frac{2}{\mathcal{F}_k}) + \frac{1}{N+1} \delta_k^i (\Gamma_j - \frac{2}{\mathcal{F}_j}) + \frac{2}{\mathcal{F}_{jk}^i}, \\ \bar{\theta}_{jk}^i = \frac{1}{N+1} \delta_j^i (\bar{\Gamma}_k - \frac{2}{\mathcal{F}_k}) + \frac{1}{N+1} \delta_k^i (\bar{\Gamma}_j - \frac{2}{\mathcal{F}_j}) + \frac{2}{\mathcal{F}_{jk}^i}. \end{cases} \quad (19)$$

Based on the results presented in [9], and reviewed in Appendix A, we obtain that the basic associated invariants of the Thomas type [9] for this mapping are $t_{jk}^i = \bar{t}_{jk}^i$ for $t_{jk}^i = \Gamma_{jk}^i - \theta_{jk}^i$ and $\bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \bar{\theta}_{jk}^i$, where

$$\begin{cases} t_{jk}^i = \Gamma_{jk}^i - \frac{1}{N+1} \delta_j^i (\Gamma_k - \frac{2}{\mathcal{F}_k}) - \frac{1}{N+1} \delta_k^i (\Gamma_j - \frac{2}{\mathcal{F}_j}) - \frac{2}{\mathcal{F}_{jk}^i}, \\ \bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \frac{1}{N+1} \delta_j^i (\bar{\Gamma}_k - \frac{2}{\mathcal{F}_k}) - \frac{1}{N+1} \delta_k^i (\bar{\Gamma}_j - \frac{2}{\mathcal{F}_j}) - \frac{2}{\mathcal{F}_{jk}^i}, \end{cases}$$

which completes the proof for the first part of this lemma.

With respect to the results presented in [9], and reviewed in Appendix A, the basic associated invariants of the Weyl type for the mapping f are

$$\begin{cases} w_{jmn}^i = r_{jmn}^i - \theta_{jm|sn}^i + \theta_{jn|sm}^i + \theta_{jm}^p \theta_{pn}^i - \theta_{jn}^p \theta_{pm}^i, \\ \bar{w}_{jmn}^i = \bar{r}_{jmn}^i - \bar{\theta}_{jm|sn}^i + \bar{\theta}_{jn|sm}^i + \bar{\theta}_{jm}^p \bar{\theta}_{pn}^i - \bar{\theta}_{jn}^p \bar{\theta}_{pm}^i. \end{cases} \quad (20)$$

Because θ_{jk}^i has the form (19), we get

$$\left\{ \begin{aligned} \theta_{jm|sn}^i &= \mathcal{F}_{jm|sn}^i + \mathcal{F}_{jm|sn}^i + \frac{1}{N+1} \delta_j^i \Gamma_{m|sn} + \frac{1}{N+1} \delta_m^i \Gamma_{j|sn} \\ &\quad - \frac{1}{N+1} \delta_j^i \mathcal{F}_{m|sn} - \frac{1}{N+1} \delta_m^i \mathcal{F}_{j|sn}, \\ \theta_{jm}^p \theta_{pn}^i &= \mathcal{F}_{jm}^p + \mathcal{F}_{pn}^i + \frac{1}{N+1} \mathcal{F}_{jm}^i \Gamma_n + \frac{1}{N+1} \delta_n^i \mathcal{F}_{jm}^p \Gamma_p - \frac{1}{N+1} \mathcal{F}_{jm}^i \mathcal{F}_n \\ &\quad - \frac{1}{N+1} \delta_n^i \mathcal{F}_{jm}^p \mathcal{F}_p + \frac{1}{N+1} \mathcal{F}_{jn}^i \Gamma_m + \frac{1}{N+1} \mathcal{F}_{mn}^i \Gamma_j \\ &\quad + \frac{1}{(N+1)^2} (\delta_j^i \Gamma_m \Gamma_n + \delta_m^i \Gamma_j \Gamma_n + 2\delta_n^i \Gamma_j \Gamma_m) \\ &\quad - \frac{1}{(N+1)^2} \delta_j^i \mathcal{F}_n \Gamma_m - \frac{1}{(N+1)^2} \delta_m^i \mathcal{F}_n \Gamma_j - \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_j \Gamma_m - \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_m \Gamma_j \\ &\quad - \frac{1}{N+1} \mathcal{F}_{jn}^i \mathcal{F}_m - \frac{1}{(N+1)^2} \delta_j^i \mathcal{F}_m \Gamma_n - \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_m \Gamma_j + \frac{1}{(N+1)^2} \delta_j^i \mathcal{F}_m^2 \mathcal{F}_n \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_j^2 \mathcal{F}_m - \frac{1}{N+1} \mathcal{F}_{mn}^i \mathcal{F}_j - \frac{1}{(N+1)^2} \delta_m^i \mathcal{F}_j \Gamma_n - \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_j \Gamma_m \\ &\quad + \frac{1}{(N+1)^2} \delta_m^i \mathcal{F}_j^2 \mathcal{F}_n + \frac{1}{(N+1)^2} \delta_n^i \mathcal{F}_j^2 \mathcal{F}_m. \end{aligned} \right.$$

After direct calculation of $-\theta_{jm|s_n}^i + \theta_{jn|s_m}^i$ and $\theta_{jm}^p \theta_{pn}^i - \theta_{jn}^p \theta_{pm}^i$, we confirm that the basic invariant of the Weyl type for the mapping f is w_{jmn}^i given by the first equality in (15). The basic invariant $\bar{w}_{jmn}^i$, from this equation, is obtained in the same manner. $\square$

In the next lemma, we will correlate the invariants of associated invariants for equitortion biholomorphically projective and geodesic mappings of the Thomas type. Moreover, we will obtain a new invariant for the equitortion biholomorphically projective mapping $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ with respect to the transformation rule of the curvature tensor r_{jmn}^i of the associated space $\mathbb{R}_N$ to the corresponding curvature tensor $\bar{r}_{jmn}^i$ of $\bar{\mathbb{R}}_N$.

Lemma 3. The associated basic invariants t_{jk}^i and $\bar{t}_{jk}^i$ for an equitortion quasi-canonical biholomorphically projective mapping $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ given by (14) and Thomas projective parameters $T_{jk}^i = \Gamma_{jk}^i - \frac{1}{N+1}(\delta_j^i \Gamma_k + \delta_k^i \Gamma_j)$ and $\bar{T}_{jk}^i = \bar{\Gamma}_{jk}^i - \frac{1}{N+1}(\delta_j^i \bar{\Gamma}_k + \delta_k^i \bar{\Gamma}_j)$ satisfy the relations

$$\begin{cases} t_{jk}^i = T_{jk}^i - \mathcal{F}_{jk}^i + \frac{1}{N+1} \delta_j^i \mathcal{F}_k^2 + \frac{1}{N+1} \delta_k^i \mathcal{F}_j^2, \\ \bar{t}_{jk}^i = \bar{T}_{jk}^i - \bar{\mathcal{F}}_{jk}^i + \frac{1}{N+1} \delta_j^i \bar{\mathcal{F}}_k^2 + \frac{1}{N+1} \delta_k^i \bar{\mathcal{F}}_j^2. \end{cases} \quad (21)$$

The invariants t_{jk}^i and $\bar{t}_{jk}^i$ are symmetric by j and k , and their traces $t_{jp}^p = t_{pj}^p$ and $\bar{t}_{jp}^p = \bar{t}_{pj}^p$ are equal to 0.

The geometrical objects

$$\begin{cases} \tilde{w}_{jmn}^i = r_{jmn}^i - \frac{1}{N-1}(\delta_m^i r_{jn} - \delta_n^i r_{jm}) - \frac{1}{N+1} \delta_j^i v_{pmn}^p \\ \quad - \frac{1}{N^2-1} \delta_m^i ((N+1)v_{jpn}^p - v_{pjn}^p) + \frac{1}{N^2-1} \delta_n^i ((N+1)v_{jpm}^p - v_{pjm}^p), \\ \tilde{\bar{w}}_{jmn}^i = \bar{r}_{jmn}^i - \frac{1}{N-1}(\delta_m^i \bar{r}_{jn} - \delta_n^i \bar{r}_{jm}) - \frac{1}{N+1} \delta_j^i \bar{v}_{pmn}^p \\ \quad - \frac{1}{N^2-1} \delta_m^i ((N+1)\bar{v}_{jpn}^p - \bar{v}_{pjn}^p) + \frac{1}{N^2-1} \delta_n^i ((N+1)\bar{v}_{jpm}^p - \bar{v}_{pjm}^p), \end{cases} \quad (22)$$

are invariants for the mapping f .

Proof. The direct calculation is enough to prove the first part of this lemma. Namely, it is evident that $t_{jk}^i = t_{kj}^i$ and $\bar{t}_{jk}^i = \bar{t}_{kj}^i$, such as

$$\begin{cases} t_{jp}^p = t_{pj}^p = T_{jp}^p - \mathcal{F}_j^2 + \frac{1}{N+1} \mathcal{F}_j^2 + \frac{N}{N+1} \mathcal{F}_j^2 = 0, \\ \bar{t}_{jp}^p = \bar{t}_{pj}^p = \bar{T}_{jp}^p - \bar{\mathcal{F}}_j^2 + \frac{1}{N+1} \bar{\mathcal{F}}_j^2 + \frac{N}{N+1} \bar{\mathcal{F}}_j^2 = 0. \end{cases}$$

Following the procedure presented in [9] and in many subsequent research papers, we express the difference $0 = w_{jmn}^i - \bar{w}_{jmn}^i$ in the form

$$0 = w_{jmn}^i - \bar{w}_{jmn}^i = (r_{jmn}^i - \bar{r}_{jmn}^i) - \delta_j^i \xi_{[mn]} - \delta_m^i \xi_{jn} + \delta_n^i \xi_{jm} + (b_{jmn}^i - \bar{b}_{jmn}^i), \quad (23)$$

for $\xi_{ij} = a_{ij} - \bar{a}_{ij}$.

After contracting the relation (23) by i and j , and by i and m , we obtain

$$\begin{cases} 0 = -(N+1)\xi_{[mn]} + (b_{pmn}^p - \bar{b}_{pmn}^p) \\ 0 = (r_{jn} - \bar{r}_{jn}) - \xi_{[jn]} - (N-1)\xi_{jn} + (b_{jpn}^p - \bar{b}_{jpn}^p). \end{cases} \quad (24)$$

From the first part of Equation (24), we obtain $\xi_{[ij]} = \frac{1}{N+1}(b_{pij}^p - \bar{b}_{pij}^p)$. After substituting this result in the second equality in (24), we get

$$\xi_{ij} = \frac{1}{N-1}(r_{ij} - \bar{r}_{ij}) + \frac{1}{N-1}(b_{ipj}^p - \bar{b}_{ipj}^p) - \frac{1}{N^2-1}(b_{pij}^p - \bar{b}_{pij}^p).$$

With respect to these solutions, the relation (23) becomes

$$\begin{aligned} 0 &= (r_{jmn}^i - \bar{r}_{jmn}^i) - \frac{1}{N+1}\delta_j^i(v_{pmn}^p - \bar{v}_{pmn}^p) - \frac{1}{N-1}(\delta_m^i r_{jn} - \delta_n^i r_{jm}) \\ &+ \frac{1}{N-1}(\delta_m^i \bar{r}_{jn} - \delta_n^i \bar{r}_{jm}) - \frac{1}{N^2-1}\delta_m^i((N+1)(b_{jpn}^p - \bar{b}_{jpn}^p) - (b_{pjn}^p - \bar{b}_{pjn}^p)) \\ &+ \frac{1}{N^2-1}\delta_n^i((N+1)(b_{jpm}^p - \bar{b}_{jpm}^p) - (b_{pjm}^p - \bar{b}_{pjm}^p)) + (b_{jmn}^i - \bar{b}_{jmn}^i), \end{aligned}$$

which is equivalent to $\tilde{w}_{jmn}^i = \tilde{\bar{w}}_{jmn}^i$ for $\tilde{w}_{jmn}^i$ and $\tilde{\bar{w}}_{jmn}^i$ given by (22). $\square$

The main results of this research will be presented in the next theorem.

Theorem 1. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitorsion quasi-canonical biholomorphically projective mapping of a generalized Riemannian space $\mathbb{GR}_N$.

The geometrical objects

$$\begin{cases} t_{jk}^i = \Gamma_{jk}^i - \bar{\mathcal{F}}_{jk}^i - \frac{1}{N+1}(\delta_j^i(\Gamma_k - \bar{\mathcal{F}}_k) + \delta_k^i(\Gamma_j - \bar{\mathcal{F}}_j)), \\ \bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \frac{2}{\bar{\mathcal{F}}_{jk}^i} - \frac{1}{N+1}(\delta_j^i(\bar{\Gamma}_k - \frac{2}{\bar{\mathcal{F}}_k}) + \delta_k^i(\bar{\Gamma}_j - \frac{2}{\bar{\mathcal{F}}_j})), \end{cases} \quad (25)$$

for $\frac{2}{\bar{\mathcal{F}}_{jk}^i} = c_k \bar{\mathcal{F}}_j^i + c_j \bar{\mathcal{F}}_k^i$ and $\frac{2}{\bar{\mathcal{F}}_{jk}^i} = \bar{c}_k \bar{\mathcal{F}}_j^i + \bar{c}_j \bar{\mathcal{F}}_k^i$, such as their traces $\frac{2}{\bar{\mathcal{F}}_j} = \frac{2}{\bar{\mathcal{F}}_{jp}^p}$ and $\frac{2}{\bar{\mathcal{F}}_j} = \frac{2}{\bar{\mathcal{F}}_{jp}^p}$, are the basic associated invariants for this mapping.

The geometrical objects

$$\begin{cases} w_{jmn}^i = k_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ \quad + u(p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) + u'(p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) - (u' + u)p_{mn}^p \Gamma_{jp}^i \\ \quad - \frac{1}{(N+1)^2}\delta_m^i((N+1)(\Gamma_{j|sn} - p_{j|sn}^p + p_{jn}^p(\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_n - p_n)) \\ \quad + \frac{1}{(N+1)^2}\delta_n^i((N+1)(\Gamma_{j|sm} - p_{j|sm}^p + p_{jm}^p(\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_m - p_m)) \\ \quad + \frac{1}{N+1}\delta_j^i p_{[m|sn]} - \frac{1}{N+1}(u\delta_n^i \Gamma_{jm}^p + u'\delta_m^i \Gamma_{jn}^p)(\Gamma_p - p_p), \\ \bar{w}_{jmn}^i = \bar{k}_{jmn}^i - \bar{p}_{jm|sn}^i + \bar{p}_{jn|sm}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i \\ \quad + u(\bar{p}_{pn}^i \bar{\Gamma}_{jm}^p - \bar{p}_{jn}^p \bar{\Gamma}_{pm}^i) + u'(\bar{p}_{pm}^i \bar{\Gamma}_{jn}^p - \bar{p}_{jm}^p \bar{\Gamma}_{pn}^i) - (u' + u)\bar{p}_{mn}^p \bar{\Gamma}_{jp}^i \\ \quad - \frac{1}{(N+1)^2}\delta_m^i((N+1)(\bar{\Gamma}_{j|sn} - \bar{p}_{j|sn}^p + \bar{p}_{jn}^p(\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j)(\bar{\Gamma}_n - \bar{p}_n)) \\ \quad + \frac{1}{(N+1)^2}\delta_n^i((N+1)(\bar{\Gamma}_{j|sm} - \bar{p}_{j|sm}^p + \bar{p}_{jm}^p(\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j)(\bar{\Gamma}_m - \bar{p}_m)) \\ \quad + \frac{1}{N+1}\delta_j^i \bar{p}_{[m|sn]} - \frac{1}{N+1}(u\delta_n^i \bar{\Gamma}_{jm}^p + u'\delta_m^i \bar{\Gamma}_{jn}^p)(\bar{\Gamma}_p - \bar{p}_p), \end{cases} \quad (26)$$

for $p_{jk}^i = \frac{2}{\bar{\mathcal{F}}_{jk}^i}$, $\bar{p}_{jk}^i = \frac{2}{\bar{\mathcal{F}}_{jk}^i}$, $p_j = \frac{2}{\bar{\mathcal{F}}_j}$, and $\bar{p}_j = \frac{2}{\bar{\mathcal{F}}_j}$, are the basic equitorsion invariants for the mapping f .

The geometrical objects

$$\left\{ \begin{array}{l} \tilde{w}_{jmn}^i = k_{jmn}^i - \frac{1}{N^2-1} ((N-1)\delta_j^i k_{pmn}^p - \delta_m^i k_{pjn}^p + \delta_n^i k_{pjm}^p) \\ \quad - \frac{1}{N-1} (\delta_m^i k_{jpn}^p - \delta_n^i k_{jpm}^p) + u_{jmn}^i + \delta_j^i \tilde{u}_{mn} + \delta_m^i \tilde{u}_{jn} - \delta_n^i \tilde{u}_{jm}, \\ \tilde{\bar{w}}_{jmn}^i = \bar{k}_{jmn}^i - \frac{1}{N^2-1} ((N-1)\delta_j^i \bar{k}_{pmn}^p - \delta_m^i \bar{k}_{pjn}^p + \delta_n^i \bar{k}_{pjm}^p) \\ \quad - \frac{1}{N-1} (\delta_m^i \bar{k}_{jpn}^p - \delta_n^i \bar{k}_{jpm}^p) + \bar{u}_{jmn}^i + \delta_j^i \bar{u}_{mn} + \delta_m^i \bar{u}_{jn} - \delta_n^i \bar{u}_{jm}, \end{array} \right. \quad (27)$$

for $u_{jmn}^i = u_{jmn}^i(u, u')$, $\tilde{u}_{ij} = \tilde{u}_{ij}(u, u')$, $\tilde{\bar{u}}_{ij} = \tilde{\bar{u}}_{ij}(u, u')$, and $\tilde{\tilde{u}}_{ij} = \tilde{\tilde{u}}_{ij}(u, u')$ are defined as

$$u_{jmn}^i = -\frac{2}{\mathcal{F}_{jm|sn}} + \frac{2}{\mathcal{F}_{jn|sm}} + \frac{2}{\mathcal{F}_{jm}^p} \frac{2}{\mathcal{F}_{pn}^i} - \frac{2}{\mathcal{F}_{jn}^p} \frac{2}{\mathcal{F}_{pm}^i} + u \mathcal{F}_{pn}^i \Gamma_{\mathcal{V}}^p - u \mathcal{F}_{jn}^p \Gamma_{\mathcal{V}}^i \\ + u' \mathcal{F}_{pm}^p \Gamma_{\mathcal{V}}^p - u' \mathcal{F}_{jm}^p \Gamma_{\mathcal{V}}^i - (u + u') \frac{2}{\mathcal{F}_{mn}} \Gamma_{\mathcal{V}}^i, \quad (28)$$

$$\tilde{u}_{ij} = \frac{1}{N+1} \frac{2}{\mathcal{F}_{[m|sn]}} + \frac{u' - u}{(N+1)^2} \Gamma_{ij}^p (\Gamma_p - \frac{2}{\mathcal{F}_p}), \quad (29)$$

$$\tilde{\bar{u}}_{ij} = \frac{(N+2)u + Nu'}{(N-1)(N+1)^2} \Gamma_{ij}^p (\Gamma_p - \frac{2}{\mathcal{F}_p}) + \frac{N}{N^2-1} \frac{2}{\mathcal{F}_{i|sj}} \\ + \frac{1}{N^2-1} \frac{2}{\mathcal{F}_{j|si}} - \frac{1}{N-1} \frac{2}{\mathcal{F}_{ij|sp}} + \frac{2}{\mathcal{F}_{iq}^p} \frac{2}{\mathcal{F}_{jp}^q} - \frac{2}{\mathcal{F}_{ij}^p} \frac{2}{\mathcal{F}_p} \\ - \frac{u'}{N-1} \frac{2}{\mathcal{F}_p} \Gamma_{ij}^p - \frac{u'}{N-1} \frac{2}{\mathcal{F}_{iq}^p} \Gamma_{jp}^q + \frac{u'}{N-1} \frac{2}{\mathcal{F}_{jq}^p} \Gamma_{ip}^q, \quad (30)$$

$$\tilde{\tilde{u}}_{ij} = \frac{(N^2 + N + 1)u + (N^2 + N - 1)u'}{(N-1)(N+1)^2} \Gamma_{ij}^p (\Gamma_p - \frac{2}{\mathcal{F}_p}) \\ + \frac{N}{N^2-1} \frac{2}{\mathcal{F}_{i|sj}} + \frac{1}{N^2-1} \frac{2}{\mathcal{F}_{j|si}} - \frac{2}{\mathcal{F}_{ij|sp}} - \frac{2}{\mathcal{F}_{iq}^p} \frac{2}{\mathcal{F}_{jp}^q} + \frac{2}{\mathcal{F}_{ij}^p} \frac{2}{\mathcal{F}_p} \\ - \frac{u'}{N-1} \frac{2}{\mathcal{F}_p} \Gamma_{ij}^p - \frac{u'}{N-1} \frac{2}{\mathcal{F}_{iq}^p} \Gamma_{jp}^q + \frac{u'}{N-1} \frac{2}{\mathcal{F}_{jq}^p} \Gamma_{ip}^q, \quad (31)$$

and the corresponding $\bar{u}_{jmn}^i$, $\tilde{\bar{u}}_{ij}$, $\tilde{\tilde{u}}_{ij}$, and $\tilde{\tilde{\bar{u}}}_{ij}$ are the derived equitortion invariants for the mapping f .

Proof. The basic equation of the mapping f is

$$\bar{\Gamma}_{jk}^i - \bar{\Gamma}_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i + \frac{2}{\mathcal{F}_{jk}^i} - \frac{2}{\mathcal{F}_{jk}^i}. \quad (32)$$

The basic Equation (32) has the form of (A18) for $p_{jk}^i = \frac{2}{\mathcal{F}_{jk}^i}$ and $\bar{p}_{jk}^i = \frac{2}{\mathcal{F}_{jk}^i}$. With respect to the basic invariants of the Thomas type (A8), and having in mind that $p_j = \frac{2}{\mathcal{F}_j}$ and $\bar{p}_j = \frac{2}{\mathcal{F}_j}$, we confirm the invariance of geometrical objects (25).

Based on the invariants $\tilde{w}_{jmn}^i$ and $\tilde{\bar{w}}_{jmn}^i$ given by (A23), and p_{jk}^i , $\bar{p}_{jk}^i$, p_j , and $\bar{p}_j$ defined in the previous paragraph, we complete the proof of invariance of geometrical objects (26).

After substituting the expressions $p_{jk}^i = \frac{2}{\mathcal{F}_{jk}^i}$ and $p_j = \frac{2}{\mathcal{F}_j}$ in the relations (A31)–(A34), one will obtain the geometrical objects u_{jmn}^i , $\tilde{u}_{ij}$, $\tilde{\bar{u}}_{ij}$, and $\tilde{\tilde{u}}_{ij}$ given by (28)–(31), which completes the proof of this theorem. $\square$

Corollary 1. Let $f : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\mathbb{R}_N$ be an equitorsion quasi-canonical biholomorphically projective mapping. The geometrical objects

$$\begin{aligned} \star_1^{i,jmn} &= r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &\quad + p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i - p_{pm}^i \Gamma_{jn}^p + p_{jm}^p \Gamma_{pn}^i \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i p_{[m|sn]} - \frac{1}{N+1} (\delta_n^i \Gamma_{jm}^p - \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p) \end{aligned} \quad (33)$$

$$\begin{aligned} \star_2^{i,jmn} &= r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &\quad - (p_{pn}^i \Gamma_{jm}^p + p_{jn}^p \Gamma_{pm}^i + p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i p_{[m|sn]} + \frac{1}{N+1} (\delta_n^i \Gamma_{jm}^p - \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p), \end{aligned} \quad (34)$$

$$\begin{aligned} \star_3^{i,jmn} &= r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &\quad + p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i + p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i - 2p_{mn}^p \Gamma_{jp}^i \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i p_{[m|sn]} - \frac{1}{N+1} (\delta_n^i \Gamma_{jm}^p + \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p), \end{aligned} \quad (35)$$

$$\begin{aligned} \star_4^{i,jmn} &= r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &\quad + p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i + p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i - 2p_{mn}^p \Gamma_{jp}^i \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i p_{[m|sn]} - \frac{1}{N+1} (\delta_n^i \Gamma_{jm}^p + \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p), \end{aligned} \quad (36)$$

$$\begin{aligned} \star_5^{i,jmn} &= r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i + \frac{1}{N+1} \delta_j^i p_{[m|sn]} \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right), \end{aligned} \quad (37)$$

are associated basic invariants for the mapping f obtained from the five selected linearly independent curvature tensors. $\square$

Corollary 2. Let $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ be an equitortion quasi-canonical biholomorphically projective mapping. The geometrical objects

$$\begin{aligned} \bar{w}_{1jmn}^i &= r_{1jmn}^i - \frac{1}{N^2 - 1} ((N - 1) \delta_j^i r_{1pmn}^p - \delta_m^i r_{1pjn}^p + \delta_n^i r_{1pjm}^p) - \frac{1}{N - 1} (\delta_m^i r_{1jpn}^p - \delta_n^i r_{1jpm}^p) \\ &+ u_{jmn}^i(1, -1) + \delta_j^i \tilde{u}_{mn}(1, -1) + \delta_m^i \tilde{u}_{jn}(1, -1) - \delta_n^i \tilde{u}_{jm}(1, -1), \end{aligned} \quad (38)$$

$$\begin{aligned} \bar{w}_{2jmn}^i &= r_{2jmn}^i - \frac{1}{N^2 - 1} ((N - 1) \delta_j^i r_{2pmn}^p - \delta_m^i r_{2pjn}^p + \delta_n^i r_{2pjm}^p) - \frac{1}{N - 1} (\delta_m^i r_{2jpn}^p - \delta_n^i r_{2jpm}^p) \\ &+ u_{jmn}^i(-1, 1) + \delta_j^i \tilde{u}_{mn}(-1, 1) + \delta_m^i \tilde{u}_{jn}(-1, 1) - \delta_n^i \tilde{u}_{jm}(-1, 1), \end{aligned} \quad (39)$$

$$\begin{aligned} \bar{w}_{3jmn}^i &= r_{3jmn}^i - \frac{1}{N^2 - 1} ((N - 1) \delta_j^i r_{3pmn}^p - \delta_m^i r_{3pjn}^p + \delta_n^i r_{3pjm}^p) - \frac{1}{N - 1} (\delta_m^i r_{3jpn}^p - \delta_n^i r_{3jpm}^p) \\ &+ u_{jmn}^i(1, 1) + \delta_j^i \tilde{u}_{mn}(1, 1) + \delta_m^i \tilde{u}_{jn}(1, 1) - \delta_n^i \tilde{u}_{jm}(1, 1), \end{aligned} \quad (40)$$

$$\begin{aligned} \bar{w}_{4jmn}^i &= r_{4jmn}^i - \frac{1}{N^2 - 1} ((N - 1) \delta_j^i r_{4pmn}^p - \delta_m^i r_{4pjn}^p + \delta_n^i r_{4pjm}^p) - \frac{1}{N - 1} (\delta_m^i r_{4jpn}^p - \delta_n^i r_{4jpm}^p) \\ &+ u_{jmn}^i(1, 1) + \delta_j^i \tilde{u}_{mn}(1, 1) + \delta_m^i \tilde{u}_{jn}(1, 1) - \delta_n^i \tilde{u}_{jm}(1, 1), \end{aligned} \quad (41)$$

$$\begin{aligned} \bar{w}_{5jmn}^i &= r_{5jmn}^i - \frac{1}{N^2 - 1} ((N - 1) \delta_j^i r_{5pmn}^p - \delta_m^i r_{5pjn}^p + \delta_n^i r_{5pjm}^p) - \frac{1}{N - 1} (\delta_m^i r_{5jpn}^p - \delta_n^i r_{5jpm}^p) \\ &+ u_{jmn}^i(0, 0) + \delta_j^i \tilde{u}_{mn}(0, 0) + \delta_m^i \tilde{u}_{jn}(0, 0) - \delta_n^i \tilde{u}_{jm}(0, 0), \end{aligned} \quad (42)$$

are derived invariants of the Weyl type obtained from the selected five linearly independent curvature tensors. $\square$

Example 1. Let us generalize a 4-dimensional FLRW spacetime with the space $\mathbb{GR}_4$ and the perturbed space with $\mathbb{GR}_4$. The basic invariants of the biholomorphically projective mapping $f : \mathbb{GR}_4 \rightarrow \mathbb{GR}_4$ are elements of the next family

$$\left\{ \begin{aligned} \bar{w}_{jmn}^i &= k_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &+ u(p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) + u'(p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) - (u' + u) p_{mn}^p \Gamma_{jp}^i \\ &- \frac{1}{25} \delta_m^i (5(\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_n - p_n)) \\ &+ \frac{1}{25} \delta_n^i (5(\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_m - p_m)) \\ &+ \frac{1}{5} \delta_j^i p_{[m|sn]} - \frac{1}{N + 1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p). \end{aligned} \right. \quad (43)$$

The trace $\bar{w}_{ij}^* = \bar{w}_{ipj}^p$ of the invariant (43) is

$$\left\{ \begin{aligned} \bar{w}_{ij} &= k_{ij} - p_{ip|sj}^p + p_{ij|sp}^p + p_{iq}^p p_{jp}^q - p_{ij}^p p_p^p \\ &+ u p_{pj}^q \Gamma_{iq}^p + u'(p_p \Gamma_{ij}^p - p_{iq}^p \Gamma_{pj}^q) - (u' + u) p_{qj}^p \Gamma_{ip}^q \\ &- \frac{1}{25} \delta_m^i (5(\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_n - p_n)) \\ &+ \frac{1}{25} \delta_n^i (5(\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_m - p_m)) \\ &+ \frac{1}{5} \delta_j^i p_{[m|sn]} - \frac{1}{N + 1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p), \end{aligned} \right. \quad (44)$$

for $p_{jk}^i = \bar{\mathcal{F}}_{jk}^i$ and $p_j = \bar{\mathcal{F}}_{jp}^p$.

The family $\mathcal{L} = \sqrt{-g} g^{pq} \bar{w}_{pq}^*$ is the family of Lagrangians.

From the family of Lagrangians, one finds the family of Einstein equations (after perturbation).

The family of derived invariants may be applied for obtaining electricity and magnetic tensors.

4. Conclusions

In this article, we continued the research on invariants for biholomorphically projective mappings of generalized Riemannian spaces. Unlike in [10], we obtained two families of the Weyl type. In the example at the end of this manuscript, we presented a possible family of Lagrangian densities and a possible family of generalizations of the Weyl conformal tensor (which may explain the electromagnetism of a system).

In the future, our work will focus on the applications of these invariants in models of classical mechanics and cosmology.

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Appendix A. Basic and Derived Invariants

In this part of the research, we will review the methodology presented in [9] for obtaining invariants of geometric mappings of symmetric and non-symmetric affine connection spaces. The focus will remain on invariants for mappings of Riemannian and generalized Riemannian spaces.

Namely, if $f: \mathbb{R}_N \rightarrow \mathbb{R}_N$ is a mapping whose deformation tensor is

$$P_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \bar{\theta}_{jk}^i - \theta_{jk}^i, \quad (\text{A1})$$

for geometrical objects $\theta_{jk}^i = \theta_{kj}^i$ and $\bar{\theta}_{jk}^i = \bar{\theta}_{kj}^i$, both of the type $(1, 2)$, the associated basic invariants of the Thomas type for this mapping are $t_{jk}^i = \bar{t}_{jk}^i$ for [9]

$$\begin{cases} t_{jk}^i = \Gamma_{jk}^i - \theta_{jk}^i, \\ \bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \bar{\theta}_{jk}^i. \end{cases} \quad (\text{A2})$$

The associated basic invariants of the Weyl type for this mapping are [9]

$$\begin{cases} w_{jmn}^i = r_{jmn}^i - \theta_{jm|sn}^i + \theta_{jn|sm}^i + \theta_{jm}^p \theta_{pn}^i - \theta_{jn}^p \theta_{pm}^i, \\ \bar{w}_{jmn}^i = \bar{r}_{jmn}^i - \bar{\theta}_{jm|sn}^i + \bar{\theta}_{jn|sm}^i + \bar{\theta}_{jm}^p \bar{\theta}_{pn}^i - \bar{\theta}_{jn}^p \bar{\theta}_{pm}^i. \end{cases} \quad (\text{A3})$$

Appendix A.1. Invariants for Mappings of Riemannian Spaces

Let us consider a mapping $f: \mathbb{R}_N \rightarrow \mathbb{R}_N$ whose deformation tensor is

$$P_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i + \bar{p}_{jk}^i - p_{jk}^i, \quad (\text{A4})$$

where p_{jk}^i and $\bar{p}_{jk}^i$ are geometrical objects (tensors or parameters) of the type $(1, 2)$, both symmetric by j and k , whose difference is a tensor. After contracting the expression of P_{jk}^i by i and k , we obtain

$$\psi_j = \frac{1}{N+1}(\bar{\Gamma}_j - \bar{p}_j) - \frac{1}{N+1}(\Gamma_j - p_j), \quad (A5)$$

for $\Gamma_j = \Gamma_{jp}^p = \Gamma_{pj}^p$, $\bar{\Gamma}_j = \bar{\Gamma}_{jp}^p = \bar{\Gamma}_{pj}^p$, $p_j = p_{jp}^p = p_{pj}^p$, and $\bar{p}_j = \bar{p}_{jp}^p = \bar{p}_{pj}^p$. Hence, the deformation tensor P_{jk}^i takes the form

$$P_{jk}^i = p_{jk}^i + \frac{1}{N+1}(\delta_j^i(\Gamma_k - p_k) + \delta_k^i(\Gamma_j - p_j)) - \bar{p}_{jk}^i - \frac{1}{N+1}(\delta_j^i(\bar{\Gamma}_k - \bar{p}_k) + \delta_k^i(\bar{\Gamma}_j - \bar{p}_j)), \quad (A6)$$

which has the form (A1).

Following the procedure presented in [9], where $\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \bar{\theta}_{jk}^i - \theta_{jk}^i$ for the corresponding geometrical objects $\bar{\theta}_{jk}^i = \bar{\theta}_{kj}^i$ and $\theta_{jk}^i = \theta_{kj}^i$ of the type $(1, 2)$, we recognize

$$\begin{cases} \theta_{jk}^i = p_{jk}^i + \frac{1}{N+1}(\delta_j^i(\Gamma_k - p_k) + \delta_k^i(\Gamma_j - p_j)), \\ \bar{\theta}_{jk}^i = \bar{p}_{jk}^i + \frac{1}{N+1}(\delta_j^i(\bar{\Gamma}_k - \bar{p}_k) + \delta_k^i(\bar{\Gamma}_j - \bar{p}_j)). \end{cases} \quad (A7)$$

The associated basic invariants of the Thomas type for transformation f are [9] $t_{jk}^i = \bar{t}_{jk}^i$, $t_{jk}^i = \Gamma_{jk}^i - \theta_{jk}^i$, $\bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \bar{\theta}_{jk}^i$, i.e.,

$$\begin{cases} t_{jk}^i = \Gamma_{jk}^i - p_{jk}^i - \frac{1}{N+1}(\delta_j^i(\Gamma_k - p_k) + \delta_k^i(\Gamma_j - p_j)), \\ \bar{t}_{jk}^i = \bar{\Gamma}_{jk}^i - \bar{p}_{jk}^i - \frac{1}{N+1}(\delta_j^i(\bar{\Gamma}_k - \bar{p}_k) + \delta_k^i(\bar{\Gamma}_j - \bar{p}_j)). \end{cases} \quad (A8)$$

The most important step in [9] is the equality $w_{jmn}^i := t_{jm,n}^i - t_{jn,m}^i + t_{jm}^p t_{pn}^i - t_{jn}^p t_{pm}^i = \bar{t}_{jm,n}^i - \bar{t}_{jn,m}^i + \bar{t}_{jm}^p \bar{t}_{pn}^i - \bar{t}_{jn}^p \bar{t}_{pm}^i =: \bar{w}_{jmn}^i$. After basic calculation, we obtain

$$\begin{cases} w_{jmn}^i = r_{jmn}^i - \theta_{jm|sn}^i + \theta_{jn|sm}^i + \theta_{jm}^p \theta_{pn}^i - \theta_{jn}^p \theta_{pm}^i, \\ \bar{w}_{jmn}^i = \bar{r}_{jmn}^i - \bar{\theta}_{jm|sn}^i + \bar{\theta}_{jn|sm}^i + \bar{\theta}_{jm}^p \bar{\theta}_{pn}^i - \bar{\theta}_{jn}^p \bar{\theta}_{pm}^i. \end{cases} \quad (A9)$$

The invariants w_{jmn}^i and $\bar{w}_{jmn}^i$ are named the associated basic invariants of the Weyl type for the transformation f .

For θ_{jk}^i and $\bar{\theta}_{jk}^i$ given by (A7), and with respect to $\Gamma_{i|sj} = \Gamma_{j|si}$ and $\bar{\Gamma}_{i|sj} = \bar{\Gamma}_{j|si}$, we obtain

$$\begin{cases} -\theta_{jm|sn}^i + \theta_{jn|sm}^i = -p_{jm|sn}^i + p_{jn|sm}^i + \frac{1}{N+1}\delta_j^i p_{[m|sn]} \\ \quad - \frac{1}{N+1}\delta_m^i(\Gamma_{j|sn} - p_{j|sn}) + \frac{1}{N+1}\delta_n^i(\Gamma_{j|sm} - p_{j|sm}), \\ \theta_{jm}^p \theta_{pn}^i - \theta_{jn}^p \theta_{pm}^i = p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ \quad - \frac{1}{(N+1)^2}\delta_m^i((N+1)p_{jn}^p(\Gamma_p - p_p) + (\Gamma_j - p_j)(\Gamma_n - p_n)) \\ \quad + \frac{1}{(N+1)^2}\delta_n^i((N+1)p_{jm}^p(\Gamma_p - p_p) + (\Gamma_j - p_j)(\Gamma_m - p_m)), \end{cases} \quad (A10)$$

and the corresponding $-\bar{\theta}_{jm|sn}^i + \bar{\theta}_{jn|sm}^i$ and $\bar{\theta}_{jm}^p \bar{\theta}_{pn}^i - \bar{\theta}_{jn}^p \bar{\theta}_{pm}^i$.

After substituting the expression (A10) in (A9), one gets

$$\left\{ \begin{array}{l} w_{jmn}^i = r_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i + \frac{1}{N+1} \delta_j^i p_{[m|sn]} \\ \quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_n - p_n) \right) \\ \quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j) (\Gamma_m - p_m) \right), \\ \bar{w}_{jmn}^i = \bar{r}_{jmn}^i - \bar{p}_{jm|sn}^i + \bar{p}_{jn|sm}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i + \frac{1}{N+1} \delta_j^i \bar{p}_{[m|sn]} \\ \quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1) (\bar{\Gamma}_{j|sn} - \bar{p}_{j|sn} + \bar{p}_{jn}^p (\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j) (\bar{\Gamma}_n - \bar{p}_n) \right) \\ \quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1) (\bar{\Gamma}_{j|sm} - \bar{p}_{j|sm} + \bar{p}_{jm}^p (\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j) (\bar{\Gamma}_m - \bar{p}_m) \right). \end{array} \right. \quad (A11)$$

The invariants w_{jmn}^i and $\bar{w}_{jmn}^i$ given by (A11) may be expressed as $w_{jmn}^i = r_{jmn}^i + u_{jmn}^i - \delta_j^i v_{[mn]} - \delta_m^i v_{jn} + \delta_n^i v_{jm}$ and $\bar{w}_{jmn}^i = \bar{r}_{jmn}^i + \bar{u}_{jmn}^i - \delta_j^i \bar{v}_{[mn]} - \delta_m^i \bar{v}_{jn} + \delta_n^i \bar{v}_{jm}$ for

$$\left\{ \begin{array}{l} u_{jmn}^i = -p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i, \\ \bar{u}_{jmn}^i = -\bar{p}_{jm|sn}^i + \bar{p}_{jn|sm}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i, \\ v_{ij} = \frac{1}{(N+1)^2} \left((N+1) (\Gamma_{i|sj} - p_{i|sj} + p_{ij}^p (\Gamma_p - p_p)) + (\Gamma_i - p_i) (\Gamma_j - p_j) \right), \\ \bar{v}_{ij} = \frac{1}{(N+1)^2} \left((N+1) (\bar{\Gamma}_{i|sj} - \bar{p}_{i|sj} + \bar{p}_{ij}^p (\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_i - \bar{p}_i) (\bar{\Gamma}_j - \bar{p}_j) \right). \end{array} \right. \quad (A12)$$

The difference $0 = \bar{w}_{jmn}^i - w_{jmn}^i$ takes the form

$$0 = (\bar{r}_{jmn}^i - r_{jmn}^i) + (\bar{u}_{jmn}^i - u_{jmn}^i) - \delta_j^i \zeta_{[mn]} - \delta_m^i \zeta_{jn} + \delta_n^i \zeta_{jm}, \quad (A13)$$

for $\zeta_{ij} = \bar{v}_{ij} - v_{ij}$, and u_{jmn}^i , $\bar{u}_{jmn}^i$, v_{ij} , and $\bar{v}_{ij}$ given by (A12).

If we contract the relation (A13) by i and j , and by i and m , and if we use $r_{pmn}^p = \bar{r}_{pmn}^p = 0$, such as $u_{pmn}^p = -p_{[m|sn]}$ and $\bar{u}_{pmn}^p = -\bar{p}_{[m|sn]}$, we will obtain the next system of equations

$$\left\{ \begin{array}{l} 0 = (\bar{p}_{[m|sn]} - p_{[m|sn]}) - (N+1) \zeta_{[mn]}, \\ 0 = (\bar{r}_{jn} - r_{jn}) + (\bar{u}_{jpn}^p - u_{jpn}^p) - \zeta_{[jn]} - (N-1) \zeta_{jm}, \end{array} \right.$$

whose solutions are

$$\left\{ \begin{array}{l} \zeta_{[ij]} = -\frac{1}{N+1} (\bar{p}_{[i|sj]} - p_{[i|sj]}), \\ \zeta_{ij} = \frac{1}{N-1} (\bar{r}_{ij} - r_{ij}) + \frac{1}{N-1} (\bar{u}_{ipj}^p - u_{ipj}^p) + \frac{1}{N^2-1} (\bar{p}_{[i|sj]} - p_{[i|sj]}). \end{array} \right. \quad (A14)$$

After substituting the solutions of (A14) in (A13), and using $\bar{u}_{ipj}^p = -\bar{p}_{i|sj} + \bar{p}_{ij|sp}^p + \bar{p}_{iq}^p \bar{p}_{jp}^q - \bar{p}_{ij}^p \bar{p}_p$ and $u_{ipj}^p = -p_{i|sj} + p_{ij|sp}^p + p_{iq}^p p_{jp}^q - p_{ij}^p p_p$, one obtains

$$\begin{aligned}
0 = & (\bar{r}_{jmn}^i - r_{jmn}^i) - \frac{1}{N-1} \delta_m^i (\bar{r}_{jn} - r_{jn}) + \frac{1}{N-1} \delta_n^i (\bar{r}_{jm} - r_{jm}) \\
& + (-\bar{p}_{jm||s_n}^i + \bar{p}_{jn||s_m}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i) \\
& - (-p_{jm||s_n}^i + p_{jn||s_m}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i) + \frac{1}{N+1} \delta_j^i (\bar{p}_{[m||s_n]} - p_{[m||s_n]}) \\
& - \frac{1}{N-1} \delta_m^i (-\bar{p}_{j||s_n} + \bar{p}_{jn||s_p}^p + \bar{p}_{jq}^p \bar{p}_{np}^q - \bar{p}_{jn}^p \bar{p}_p) - \frac{1}{N^2-1} \delta_m^i (\bar{p}_{[j||s_n]} - p_{[j||s_n]}) \\
& + \frac{1}{N-1} \delta_n^i (-\bar{p}_{j||s_m} + \bar{p}_{jm||s_p}^p + \bar{p}_{jq}^p \bar{p}_{mp}^q - \bar{p}_{jm}^p \bar{p}_p) + \frac{1}{N^2-1} \delta_n^i (\bar{p}_{[j||s_m]} - p_{[j||s_m]}) \\
& + \frac{1}{N-1} \delta_m^i (-p_{j||s_n} + p_{jn||s_p}^p + p_{jq}^p p_{np}^q - p_{jn}^p p_p) \\
& - \frac{1}{N-1} \delta_n^i (-p_{j||s_m} + p_{jm||s_p}^p + p_{jq}^p p_{mp}^q - p_{jm}^p p_p).
\end{aligned} \tag{A15}$$

The relation (A15) is equivalent to $\tilde{w}_{jmn}^i = \tilde{w}_{jmn}^i$ for

$$\left\{ \begin{aligned} \tilde{w}_{jmn}^i &= r_{jmn}^i - \frac{1}{N-1} (\delta_m^i r_{jn} - \delta_n^i r_{jm}) \\ &\quad - p_{jm||s_n}^i + p_{jn||s_m}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i + \frac{1}{N+1} \delta_j^i p_{[m||s_n]} \\ &\quad + \frac{1}{N^2-1} \delta_m^i (N p_{j||s_n} + p_{n||s_j} - (N+1) p_{jn||s_p}^p - (N+1) (p_{jq}^p p_{np}^q - p_{jn}^p p_p)) \\ &\quad - \frac{1}{N^2-1} \delta_n^i (N p_{j||s_m} + p_{m||s_j} - (N+1) p_{jm||s_p}^p - (N+1) (p_{jq}^p p_{mp}^q - p_{jm}^p p_p)), \\ \tilde{w}_{jmn}^i &= \bar{r}_{jmn}^i - \frac{1}{N-1} (\delta_m^i \bar{r}_{jn} - \delta_n^i \bar{r}_{jm}) \\ &\quad - \bar{p}_{jm||s_n}^i + \bar{p}_{jn||s_m}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i + \frac{1}{N+1} \delta_j^i \bar{p}_{[m||s_n]} \\ &\quad + \frac{1}{N^2-1} \delta_m^i (N \bar{p}_{j||s_n} + \bar{p}_{n||s_j} - (N+1) \bar{p}_{jn||s_p}^p - (N+1) (\bar{p}_{jq}^p \bar{p}_{np}^q - \bar{p}_{jn}^p \bar{p}_p)) \\ &\quad - \frac{1}{N^2-1} \delta_n^i (N \bar{p}_{j||s_m} + \bar{p}_{m||s_j} - (N+1) \bar{p}_{jm||s_p}^p - (N+1) (\bar{p}_{jq}^p \bar{p}_{mp}^q - \bar{p}_{jm}^p \bar{p}_p)). \end{aligned} \right. \tag{A16}$$

The invariants $\tilde{w}_{jmn}^i$ and $\tilde{w}_{jmn}^i$ are the derived invariants for the transformation f .

As we may see, the first two summands of invariants $\tilde{w}_{jmn}^i$ and $\tilde{w}_{jmn}^i$ are the Weyl projective tensors of spaces $\mathbb{R}_N$ and $\bar{\mathbb{R}}_N$. That means that any transformation without additional effects are geodesic.

The traces of invariants $\tilde{w}_{jmn}^i$ and $\tilde{w}_{jmn}^i$ are

$$\tilde{w}_{pmn}^p = \tilde{w}_{pmn}^p = 0, \quad \tilde{w}_{ipn}^p = \tilde{w}_{ipn}^p = 0, \quad \tilde{w}_{jmp}^p = \tilde{w}_{jmp}^p = 0. \tag{A17}$$

Appendix A.2. Invariants for Mappings of Generalized Riemannian Spaces

If we extend the previously analyzed mapping $f : \mathbb{R}_N \rightarrow \bar{\mathbb{R}}_N$ to $F : \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\bar{\mathbb{R}}_N$ such that $\Gamma_{jk}^i \rightarrow \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i$, the equitortion mapping F will be determined by the next non-symmetric deformation tensor

$$P_{jk}^i = \psi_k \delta_j^i + \psi_j \delta_k^i + \bar{p}_{jk}^i - p_{jk}^i, \tag{A18}$$

for the geometrical objects p_{jk}^i and $\bar{p}_{jk}^i$ as above.

Because the deformation tensor P_{jk}^i is symmetric by j and k (its expression on the right side is identical to the expression of tensor P_{jk}^i), the corresponding basic invariants of the Thomas type are identical to these associated invariants (A8).

The families of curvature tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\bar{\mathbb{R}}_N$ are

$$\begin{cases} k_{jmn}^i = r_{jmn}^i + u\Gamma_{jm\|s_n}^i + u'\Gamma_{jn\|s_m}^i + v\Gamma_{jm}^p\Gamma_{pn}^i + v'\Gamma_{jn}^p\Gamma_{pm}^i + w\Gamma_{mn}^p\Gamma_{pj}^i, \\ \bar{k}_{jmn}^i = \bar{r}_{jmn}^i + u\bar{\Gamma}_{jm\|s_n}^i + u'\bar{\Gamma}_{jn\|s_m}^i + v\bar{\Gamma}_{jm}^p\bar{\Gamma}_{pn}^i + v'\bar{\Gamma}_{jn}^p\bar{\Gamma}_{pm}^i + w\bar{\Gamma}_{mn}^p\bar{\Gamma}_{pj}^i, \end{cases} \quad (A19)$$

for the corresponding real constants u, u', v, v', w .

From research on the transformation f of the Riemannian space $\mathbb{R}_N$ to $\bar{\mathbb{R}}_N$, we have $\bar{\Gamma}_{jk}^i - \Gamma_{jk}^i = \bar{\theta}_{jk}^i - \theta_{jk}^i$, which induces $\bar{r}_{jmn}^i - r_{jmn}^i = \bar{\theta}_{jm\|s_n}^i - \bar{\theta}_{jn\|s_m}^i - \bar{\theta}_{jm}^p\bar{\theta}_{pn}^i + \bar{\theta}_{jn}^p\bar{\theta}_{pm}^i - \theta_{jm\|s_n}^i + \theta_{jn\|s_m}^i + \theta_{jm}^p\theta_{pn}^i - \theta_{jn}^p\theta_{pm}^i$. Hence, the difference $\bar{k}_{jmn}^i - k_{jmn}^i$ in the families of curvature tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\bar{\mathbb{R}}_N$ is

$$\begin{aligned} \bar{k}_{jmn}^i - k_{jmn}^i &= (\bar{r}_{jmn}^i - r_{jmn}^i) \\ &+ u((\bar{\Gamma}_{pn}^i - \Gamma_{pn}^i)\Gamma_{jm}^p - (\bar{\Gamma}_{jn}^p - \Gamma_{jn}^p)\Gamma_{pm}^i - (\bar{\Gamma}_{mn}^p - \Gamma_{mn}^p)\Gamma_{jp}^i) \\ &+ u'((\bar{\Gamma}_{pm}^i - \Gamma_{pm}^i)\Gamma_{jn}^p - (\bar{\Gamma}_{jm}^p - \Gamma_{jm}^p)\Gamma_{pn}^i - (\bar{\Gamma}_{nm}^p - \Gamma_{nm}^p)\Gamma_{jp}^i), \end{aligned}$$

i.e.,

$$\begin{aligned} \bar{k}_{jmn}^i - k_{jmn}^i &= (\bar{\theta}_{jm\|s_n}^i - \bar{\theta}_{jn\|s_m}^i - \bar{\theta}_{jm}^p\bar{\theta}_{pn}^i + \bar{\theta}_{jn}^p\bar{\theta}_{pm}^i) \\ &- (\theta_{jm\|s_n}^i - \theta_{jn\|s_m}^i - \theta_{jm}^p\theta_{pn}^i + \theta_{jn}^p\theta_{pm}^i) \\ &+ u(\bar{\theta}_{pn}^i\bar{\Gamma}_{jm}^p - \bar{\theta}_{jn}^p\bar{\Gamma}_{pm}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i) + u'(\bar{\theta}_{pm}^i\bar{\Gamma}_{jn}^p - \bar{\theta}_{jm}^p\bar{\Gamma}_{pn}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i) \\ &- u(\theta_{pn}^i\Gamma_{jm}^p - \theta_{jn}^p\Gamma_{pm}^i - \theta_{mn}^p\Gamma_{jp}^i) - u'(\theta_{pm}^i\Gamma_{jn}^p - \theta_{jm}^p\Gamma_{pn}^i - \theta_{mn}^p\Gamma_{jp}^i). \end{aligned} \quad (A20)$$

The relation (A20) is equivalent to $\bar{w}_{jmn}^i = \bar{w}_{jmn}^i$ for

$$\begin{cases} \bar{w}_{jmn}^i = k_{jmn}^i - (\theta_{jm\|s_n}^i - \theta_{jn\|s_m}^i - \theta_{jm}^p\theta_{pn}^i + \theta_{jn}^p\theta_{pm}^i) \\ \quad - u(\theta_{pn}^i\Gamma_{jm}^p - \theta_{jn}^p\Gamma_{pm}^i - \theta_{mn}^p\Gamma_{jp}^i) - u'(\theta_{pm}^i\Gamma_{jn}^p - \theta_{jm}^p\Gamma_{pn}^i - \theta_{mn}^p\Gamma_{jp}^i), \\ \bar{w}_{jmn}^i = \bar{k}_{jmn}^i - (\bar{\theta}_{jm\|s_n}^i - \bar{\theta}_{jn\|s_m}^i - \bar{\theta}_{jm}^p\bar{\theta}_{pn}^i + \bar{\theta}_{jn}^p\bar{\theta}_{pm}^i) \\ \quad - u(\bar{\theta}_{pn}^i\bar{\Gamma}_{jm}^p - \bar{\theta}_{jn}^p\bar{\Gamma}_{pm}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i) - u'(\bar{\theta}_{pm}^i\bar{\Gamma}_{jn}^p - \bar{\theta}_{jm}^p\bar{\Gamma}_{pn}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i). \end{cases} \quad (A21)$$

The invariants $\bar{w}_{jmn}^i$ and $\bar{w}_{jmn}^i$ given by (A21) are the basic equitorsion invariants of the Weyl type for the mapping F .

For θ_{jk}^i given by (A7), we have

$$\begin{cases} \theta_{pn}^i\Gamma_{jm}^p - \theta_{jn}^p\Gamma_{pm}^i - \theta_{mn}^p\Gamma_{jp}^i = \frac{1}{N+1}\delta_n^i(\Gamma_p - p_p)\Gamma_{jm}^p + p_{pm}^i\Gamma_{jm}^p - p_{jn}^p\Gamma_{pm}^i - p_{mn}^p\Gamma_{jp}^i \\ \quad + \frac{1}{N+1}((\Gamma_j - p_j)T_{mn}^i - (\Gamma_n - p_n)T_{jm}^i - (\Gamma_m - p_m)T_{jn}^i), \\ \theta_{pm}^i\Gamma_{jn}^p - \theta_{jm}^p\Gamma_{pn}^i - \theta_{mn}^p\Gamma_{jp}^i = \frac{1}{N+1}\delta_m^i(\Gamma_p - p_p)\Gamma_{jn}^p + p_{pm}^i\Gamma_{jn}^p - p_{jm}^p\Gamma_{pn}^i - p_{mn}^p\Gamma_{jp}^i \\ \quad - \frac{1}{N+1}((\Gamma_j - p_j)T_{mn}^i + (\Gamma_m - p_m)T_{jn}^i + (\Gamma_n - p_n)T_{jm}^i), \end{cases} \quad (A22)$$

and the corresponding $\bar{\theta}_{pn}^i\bar{\Gamma}_{jm}^p - \bar{\theta}_{jn}^p\bar{\Gamma}_{pm}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i$ and $\bar{\theta}_{pm}^i\bar{\Gamma}_{jn}^p - \bar{\theta}_{jm}^p\bar{\Gamma}_{pn}^i - \bar{\theta}_{mn}^p\bar{\Gamma}_{jp}^i$.

After substituting the expressions of (A22) into (A21), one obtains the basic associated invariants of the Weyl type for transformation F , which are

$$\left\{ \begin{aligned} \star w_{jmn}^i &= k_{jmn}^i - p_{jm|sn}^i + p_{jn|sm}^i + p_{jm}^p p_{pn}^i - p_{jn}^p p_{pm}^i \\ &\quad + u(p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) + u'(p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) - (u' + u)p_{mn}^p \Gamma_{jp}^i \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1)(\Gamma_{j|sn} - p_{j|sn} + p_{jn}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_n - p_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1)(\Gamma_{j|sm} - p_{j|sm} + p_{jm}^p (\Gamma_p - p_p)) + (\Gamma_j - p_j)(\Gamma_m - p_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i p_{[m|sn]} - \frac{1}{N+1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p), \\ \star \bar{w}_{jmn}^i &= \bar{k}_{jmn}^i - \bar{p}_{jm|sn}^i + \bar{p}_{jn|sm}^i + \bar{p}_{jm}^p \bar{p}_{pn}^i - \bar{p}_{jn}^p \bar{p}_{pm}^i \\ &\quad + u(\bar{p}_{pn}^i \bar{\Gamma}_{jm}^p - \bar{p}_{jn}^p \bar{\Gamma}_{pm}^i) + u'(\bar{p}_{pm}^i \bar{\Gamma}_{jn}^p - \bar{p}_{jm}^p \bar{\Gamma}_{pn}^i) - (u' + u)\bar{p}_{mn}^p \bar{\Gamma}_{jp}^i \\ &\quad - \frac{1}{(N+1)^2} \delta_m^i \left((N+1)(\bar{\Gamma}_{j|sn} - \bar{p}_{j|sn} + \bar{p}_{jn}^p (\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j)(\bar{\Gamma}_n - \bar{p}_n) \right) \\ &\quad + \frac{1}{(N+1)^2} \delta_n^i \left((N+1)(\bar{\Gamma}_{j|sm} - \bar{p}_{j|sm} + \bar{p}_{jm}^p (\bar{\Gamma}_p - \bar{p}_p)) + (\bar{\Gamma}_j - \bar{p}_j)(\bar{\Gamma}_m - \bar{p}_m) \right) \\ &\quad + \frac{1}{N+1} \delta_j^i \bar{p}_{[m|sn]} - \frac{1}{N+1} (u \delta_n^i \bar{\Gamma}_{jm}^p + u' \delta_m^i \bar{\Gamma}_{jn}^p) (\bar{\Gamma}_p - \bar{p}_p). \end{aligned} \right. \quad (A23)$$

The invariance $\star w_{jmn}^i = \star \bar{w}_{jmn}^i$, i.e., its equivalent form $0 = \star w_{jmn}^i - \star \bar{w}_{jmn}^i$, gives

$$\begin{aligned} 0 &= (\bar{k}_{jmn}^i - k_{jmn}^i) \\ &\quad - \frac{1}{N+1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\bar{\Gamma}_p - \bar{p}_p) - (\bar{p}_{jm|sn}^i - \bar{p}_{jn|sm}^i - \bar{p}_{jm}^p \bar{p}_{pn}^i + \bar{p}_{jn}^p \bar{p}_{pm}^i) \\ &\quad + u(\bar{p}_{pn}^i \bar{\Gamma}_{jm}^p - \bar{p}_{jn}^p \bar{\Gamma}_{pm}^i) + u'(\bar{p}_{pm}^i \bar{\Gamma}_{jn}^p - \bar{p}_{jm}^p \bar{\Gamma}_{pn}^i) - (u' + u)\bar{p}_{mn}^p \bar{\Gamma}_{jp}^i \\ &\quad + \frac{1}{N+1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p) + (p_{jm|sn}^i - p_{jn|sm}^i - p_{jm}^p p_{pn}^i + p_{jn}^p p_{pm}^i) \\ &\quad - u(p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) - u'(p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) + (u' + u)p_{mn}^p \Gamma_{jp}^i \\ &\quad - \delta_j^i \zeta_{[mn]} - \delta_m^i \zeta_{jn} + \delta_n^i \zeta_{jm}. \end{aligned} \quad (A24)$$

Because $k_{jmn}^i = r_{jmn}^i + u \Gamma_{jm|sn}^i + u' \Gamma_{jn|sm}^i + v \Gamma_{jm}^p \Gamma_{pn}^i + v' \Gamma_{jn}^p \Gamma_{pm}^i + w \Gamma_{mn}^p \Gamma_{pj}^i$ and $\bar{k}_{jmn}^i = \bar{r}_{jmn}^i + u \bar{\Gamma}_{jm|sn}^i + u' \bar{\Gamma}_{jn|sm}^i + v \bar{\Gamma}_{jm}^p \bar{\Gamma}_{pn}^i + v' \bar{\Gamma}_{jn}^p \bar{\Gamma}_{pm}^i + w \bar{\Gamma}_{mn}^p \bar{\Gamma}_{pj}^i$, such as $\Gamma_{ip}^p = \Gamma_{pi}^p = 0$ and $\bar{\Gamma}_{ip}^p = \bar{\Gamma}_{pi}^p = 0$, we find that the contractions of k_{jmn}^i and $\bar{k}_{jmn}^i$ by i and j and by i and m are

$$\left\{ \begin{aligned} k_{pmn}^p &= (v + v') \Gamma_{mq}^p \Gamma_{np}^q \neq 0, \\ \bar{k}_{pmn}^p &= (v + v') \bar{\Gamma}_{mq}^p \bar{\Gamma}_{np}^q \neq 0, \\ k_{jpn}^p &= r_{jmn}^p + u' \Gamma_{jn|sp}^p - (v - w) \Gamma_{jq}^p \Gamma_{np}^q \neq 0, \\ \bar{k}_{jpn}^p &= \bar{r}_{jmn}^p + u' \bar{\Gamma}_{jn|sp}^p - (v - w) \bar{\Gamma}_{jq}^p \bar{\Gamma}_{np}^q \neq 0. \end{aligned} \right. \quad (A25)$$

Because of differences in the forms of contractions (A25), the traces of equality (A24) by i and j and by i and m are

$$\left\{ \begin{array}{l} 0 = (\bar{k}_{pmn}^p - k_{pmn}^p) - \frac{1}{N+1}(u' - u)\bar{\Gamma}_{mn}^p(\bar{\Gamma}_p - \bar{p}_p) - \bar{p}_{[m|s_n]} \\ \quad + \frac{1}{N+1}(u' - u)\Gamma_{mn}^p(\Gamma_p - p_p) + p_{[m|s_n]} - (N+1)\xi_{[mn]}, \\ 0 = (\bar{k}_{jpn}^p - k_{jpn}^p) - \frac{Nu' + u}{N+1}(\bar{\Gamma}_{jn}^p(\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{jn}^p(\Gamma_p - p_p)) \\ \quad - (\bar{p}_{j||s_n} - \bar{p}_{jn||s_p}^p - \bar{p}_{jq}^p\bar{p}_{np}^q + \bar{p}_{jn}^p\bar{p}_p) + u'(\bar{p}_p\bar{\Gamma}_{jn}^p + \bar{p}_{jq}^p\bar{\Gamma}_{np}^q - \bar{p}_{np}^q\bar{\Gamma}_{jq}^p) \\ \quad + (p_{j|s_n} - p_{jn||s_p}^p - p_{jq}^p p_{np}^q + p_{jn}^p p_p) - u'(p_p\Gamma_{jn}^p + p_{jq}^p\Gamma_{np}^q - p_{np}^q\Gamma_{jq}^p) \\ \quad - \xi_{[jn]} - (N-1)\xi_{jn}. \end{array} \right. \quad (A26)$$

The solutions of system (A26) by $\xi_{[mn]}$ and ξ_{jn} are

$$\left\{ \begin{array}{l} \xi_{[mn]} = \frac{1}{N+1}(\bar{k}_{pmn}^p - k_{pmn}^p) - \frac{1}{N+1}(\bar{p}_{[m|s_n]} - p_{[m|s_n]}) \\ \quad - \frac{u' - u}{(N+1)^2}(\bar{\Gamma}_{mn}^p(\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{mn}^p(\Gamma_p - p_p)), \\ \xi_{jn} = \frac{1}{N-1}(\bar{k}_{jpn}^p - k_{jpn}^p) - \frac{1}{N^2-1}(\bar{k}_{pjn}^p - k_{pjn}^p) + \frac{1}{N^2-1}(\bar{p}_{j||s_n} - p_{j|s_n}) \\ \quad - \frac{(N+2)u + (N^2+N-1)u'}{(N-1)(N+1)^2}(\bar{\Gamma}_{jn}^p(\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{jn}^p(\Gamma_p - p_p)) \\ \quad - \frac{1}{N-1}((\bar{p}_{j||s_n} - \bar{p}_{jn||s_p}^p - \bar{p}_{jq}^p\bar{p}_{np}^q + \bar{p}_{jn}^p\bar{p}_p) - (p_{j|s_n} - p_{jn||s_p}^p - p_{jq}^p p_{np}^q + p_{jn}^p p_p)) \\ \quad + \frac{u'}{N-1}((\bar{p}_p\bar{\Gamma}_{jn}^p + \bar{p}_{jq}^p\bar{\Gamma}_{np}^q - \bar{p}_{np}^q\bar{\Gamma}_{jq}^p) - (p_p\Gamma_{jn}^p + p_{jq}^p\Gamma_{np}^q - p_{np}^q\Gamma_{jq}^p)). \end{array} \right. \quad (A27)$$

After including the solutions of (A27) in (A24), we obtain

$$\begin{aligned}
 0 = & (\bar{k}_{jmn}^i - k_{jmn}^i) \\
 & - \frac{1}{N+1} (u \delta_n^i \bar{\Gamma}_{jm}^p + u' \delta_m^i \bar{\Gamma}_{jn}^p) (\bar{\Gamma}_p - \bar{p}_p) - (\bar{p}_{jm}^i \bar{p}_{sn}^i - \bar{p}_{jn}^i \bar{p}_{sm}^i - \bar{p}_{jm}^p \bar{p}_{pn}^i + \bar{p}_{jn}^p \bar{p}_{pm}^i) \\
 & + u (\bar{p}_{pn}^i \bar{\Gamma}_{jm}^p - \bar{p}_{jn}^p \bar{\Gamma}_{pm}^i) + u' (\bar{p}_{pm}^i \bar{\Gamma}_{jn}^p - \bar{p}_{jm}^p \bar{\Gamma}_{pn}^i) - (u' + u) \bar{p}_{mn}^p \bar{\Gamma}_{jp}^i \\
 & + \frac{1}{N+1} (u \delta_n^i \Gamma_{jm}^p + u' \delta_m^i \Gamma_{jn}^p) (\Gamma_p - p_p) + (p_{jm}^i p_{sn}^i - p_{jn}^i p_{sm}^i - p_{jm}^p p_{pn}^i + p_{jn}^p p_{pm}^i) \\
 & - u (p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) - u' (p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) + (u' + u) p_{mn}^p \Gamma_{jp}^i \\
 & - \frac{1}{N+1} \delta_j^i (\bar{k}_{pmn}^p - k_{pmn}^p) + \frac{1}{N+1} \delta_j^i (\bar{p}_{[m|s|n]} - p_{[m|s|n]}) \\
 & + \frac{u' - u}{(N+1)^2} \delta_j^i (\bar{\Gamma}_{mn}^p (\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{mn}^p (\Gamma_p - p_p)) \\
 & - \frac{1}{N-1} (\delta_m^i (\bar{k}_{jpn}^p - k_{jpn}^p) - \delta_n^i (\bar{k}_{jpm}^p - k_{jpm}^p)) \\
 & + \frac{1}{N^2-1} (\delta_m^i (\bar{k}_{pjn}^p - k_{pjn}^p) - \delta_n^i (\bar{k}_{pjm}^p - k_{pjm}^p)) \\
 & - \frac{1}{N^2-1} (\delta_m^i (\bar{p}_{[j|s|n]} - p_{[j|s|n]}) - \delta_n^i (\bar{p}_{[j|s|m]} - p_{[j|s|m]})) \\
 & + \frac{(N+2)u + (N^2+N-1)u'}{(N-1)(N+1)^2} \delta_m^i (\bar{\Gamma}_{jn}^p (\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{jn}^p (\Gamma_p - p_p)) \\
 & - \frac{(N+2)u + (N^2+N-1)u'}{(N-1)(N+1)^2} \delta_n^i (\bar{\Gamma}_{jm}^p (\bar{\Gamma}_p - \bar{p}_p) - \Gamma_{jm}^p (\Gamma_p - p_p)) \\
 & + \frac{1}{N-1} \delta_m^i (\bar{p}_{j|s|n} - \bar{p}_{jn|s|p}^p - \bar{p}_{jq}^p \bar{p}_{np}^q + \bar{p}_{jn}^p \bar{p}_p) \\
 & - \frac{1}{N-1} \delta_n^i (\bar{p}_{j|s|m} - \bar{p}_{jm|s|p}^p - \bar{p}_{jq}^p \bar{p}_{mp}^q + \bar{p}_{jm}^p \bar{p}_p) \\
 & - \frac{1}{N-1} \delta_m^i (p_{j|s|n} - p_{jn|s|p}^p - p_{jq}^p p_{np}^q + p_{jn}^p p_p) \\
 & + \frac{1}{N-1} \delta_n^i (p_{j|s|m} - p_{jm|s|p}^p - p_{jq}^p p_{mp}^q + p_{jm}^p p_p) \\
 & - \frac{u'}{N-1} \delta_m^i ((\bar{p} \bar{\Gamma}_{jn}^p + \bar{p}_{jq}^p \bar{\Gamma}_{np}^q - \bar{p}_{np}^q \bar{\Gamma}_{jq}^p) - (p_p \Gamma_{jn}^p + p_{jq}^p \Gamma_{np}^q - p_{np}^q \Gamma_{jq}^p)) \\
 & + \frac{u'}{N-1} \delta_n^i ((\bar{p} \bar{\Gamma}_{jm}^p + \bar{p}_{jq}^p \bar{\Gamma}_{mp}^q - \bar{p}_{mp}^q \bar{\Gamma}_{jq}^p) - (p_p \Gamma_{jm}^p + p_{jq}^p \Gamma_{mp}^q - p_{mp}^q \Gamma_{jq}^p)).
 \end{aligned} \tag{A28}$$

The relation (A28) is equivalent to $\tilde{\tilde{w}}_{jmn}^i = \tilde{w}_{jmn}^i$, where

$$\begin{aligned}
 \tilde{\tilde{w}}_{jmn}^i = & k_{jmn}^i - \frac{1}{N^2-1} ((N-1) \delta_j^i k_{pmn}^p - \delta_m^i k_{pjn}^p + \delta_n^i k_{pjm}^p) \\
 & - \frac{1}{N-1} (\delta_m^i k_{jpn}^p - \delta_n^i k_{jpm}^p) + u_{jmn}^i + \delta_j^i \tilde{u}_{jmn}^i + \delta_m^i \tilde{u}_{jn}^i - \delta_n^i \tilde{u}_{jm}^i,
 \end{aligned} \tag{A29}$$

$$\begin{aligned}
 \tilde{w}_{jmn}^i = & \bar{k}_{jmn}^i - \frac{1}{N^2-1} ((N-1) \delta_j^i \bar{k}_{pmn}^p - \delta_m^i \bar{k}_{pjn}^p + \delta_n^i \bar{k}_{pjm}^p) \\
 & - \frac{1}{N-1} (\delta_m^i \bar{k}_{jpn}^p - \delta_n^i \bar{k}_{jpm}^p) + \bar{u}_{jmn}^i + \delta_j^i \tilde{\tilde{u}}_{jmn}^i + \delta_m^i \tilde{\tilde{u}}_{jn}^i - \delta_n^i \tilde{\tilde{u}}_{jm}^i,
 \end{aligned} \tag{A30}$$

for

$$u_{jmn}^i = -(p_{jm|sn}^i - p_{jn|sm}^i - p_{jm}^p p_{pn}^i + p_{jn}^p p_{pm}^i) + u(p_{pn}^i \Gamma_{jm}^p - p_{jn}^p \Gamma_{pm}^i) + u'(p_{pm}^i \Gamma_{jn}^p - p_{jm}^p \Gamma_{pn}^i) - (u + u')p_{mn}^p \Gamma_{jp}^i, \quad (A31)$$

$$\tilde{u}_{ij} = \frac{1}{N+1}p_{[m|sn]} + \frac{u' - u}{(N+1)^2}\Gamma_{ij}^p(\Gamma_p - p_p), \quad (A32)$$

$$\tilde{\tilde{u}}_{ij} = \frac{(N+2)u + Nu'}{(N-1)(N+1)^2}\Gamma_{ij}^p(\Gamma_p - p_p) + \frac{N}{N^2-1}p_{i|sj} + \frac{1}{N^2-1}p_{j|si} - \frac{1}{N-1}(p_{ij|sp}^p + p_{iq}^p p_{jp}^q - p_{ij}^p p_p) - \frac{u'}{N-1}(p_p \Gamma_{ij}^p + p_{iq}^p \Gamma_{jp}^q - p_{jq}^p \Gamma_{ip}^q), \quad (A33)$$

$$\tilde{\tilde{\tilde{u}}}_{ij} = \frac{(N^2 + N + 1)u + (N^2 + N - 1)u'}{(N-1)(N+1)^2}\Gamma_{ij}^p(\Gamma_p - p_p) + \frac{N}{N^2-1}p_{i|sj} + \frac{1}{N^2-1}p_{j|si} - (p_{ij|sp}^p + p_{iq}^p p_{jp}^q - p_{ij}^p p_p) - \frac{u'}{N-1}(p_p \Gamma_{ij}^p + p_{iq}^p \Gamma_{jp}^q - p_{jq}^p \Gamma_{ip}^q), \quad (A34)$$

and the corresponding $\tilde{u}_{jmn}^i$, $\tilde{\tilde{u}}_{ij}$, $\tilde{\tilde{\tilde{u}}}_{ij}$, and $\tilde{\tilde{\tilde{u}}}_{ij}$.

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