

Article

Fuzzy Pricing and Empirical Analysis of Binary Options Under Long Memory

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Abstract

To address the limitations of traditional binary option pricing models in capturing the long-range dependence characteristics of financial time series and in neglecting the cognitive fuzziness of parameters, this paper proposes an innovative pricing model that integrates mixed fractional Brownian motion with triangular fuzzy numbers. Mixed fractional Brownian motion is employed to characterize the long-range dependence of stock prices, while triangular fuzzy numbers are used to describe the fuzziness inherent in stock prices and volatility. Based on this framework, a fuzzy pricing model is constructed, and explicit expressions for the price cutsets are derived. Theoretical analysis and numerical experiments show that the influence of the Hurst exponent on option prices is modulated by the exercise time. Empirical results reveal that the underlying asset price exhibits significant long-range dependence, supporting the use of mixed fractional Brownian motion for modeling. Compared with the Black–Scholes model, the fuzzy price interval from the proposed framework covers 98.00% of actual price observations, while the Black–Scholes-based interval covers 84.00%. The interval is also narrower, indicating higher efficiency in uncertainty representation.

Keywords: binary options; triangular fuzzy numbers; long memory; mixed fractional Brownian motion; structured wealth management products

MSC: 91G20



Academic Editor: Palle E.T. Jorgensen

Received: 14 May 2026

Revised: 22 June 2026

Accepted: 18 July 2026

Published: 22 July 2026

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1. Introduction

As a standardized rights contract, an option grants the holder the right to execute an asset transaction at a specific point in time. In 1973, Black and Scholes, based on the no-arbitrage principle and the assumption of geometric Brownian motion, proposed an analytical model for European option pricing, establishing the theoretical cornerstone of modern option pricing theory [1]. Although standard European and American options have matured, investors in actual financial markets often face more complex discrete risk scenarios, such as threshold-triggered payoffs, which has spurred research into exotic instruments like binary options. In 1991, Rubinstein and Reiner systematically investigated the pricing of binary options, deriving analytical pricing formulas for various binary options within the Black–Scholes framework [2]. Although standardized binary options have been delisted in China, financial products with similar binary payoff structures, such as snowball

derivatives and structured wealth management products, remain widespread. Their path-dependent characteristics continue to drive in-depth research in this area. Deswal and Kumar innovatively employed wavelet multi-resolution analysis to study the sensitivity of binary options [3]; Thakur et al., utilizing machine learning classification methods and incorporating uncertainties in wind speed and market prices, analyzed binary option contracts and priced hedging instruments, providing support for risk negotiations for wind power producers [4].

Financial market prices often exhibit complex statistical characteristics, such as long-range dependence and “leptokurtic fat-tail” distributions, posing significant challenges to traditional pricing models. Long-range dependence refers to the significant correlation still present between distant observations in a time series [5]. The leptokurtic fat-tail characteristic indicates that the return distributions in financial markets often display a higher probability of extreme events than predicted by a normal distribution [6]. This implies that actual market risks are frequently underestimated by traditional models, thereby imposing higher demands on risk management and asset pricing. To accurately characterize these features, Peters introduced fractal theory and employed fractional Brownian motion (FBM) to describe market dynamic behavior [7]. However, fractional Brownian motion alone struggles to fully capture market complexity, and due to its lack of the martingale property, it may permit arbitrage opportunities. Consequently, the mixed fractional Brownian motion (MFBM) model was proposed to address these shortcomings [8]. Cheridito pointed out that when the Hurst parameter satisfies $H \in (0.75, 1)$, MFBM is equivalent to standard Brownian motion, thereby circumventing the possibility of arbitrage [9]. In recent years, the application of MFBM in financial modeling has deepened: Ahmadian et al. constructed a pricing model for geometric Asian rainbow options under this framework [10]; Cheng et al. explored the pricing of European vulnerable options under mixed fractional Brownian motion [11]; Ma et al. investigated option pricing for mixed fractional geometric Brownian motion in illiquid markets [12]; Liu et al., based on an MFBM hedging model, derived a closed-form solution for floating strike lookback barrier options [13]; Shokrollahi et al. combined MFBM with Poisson jumps to derive analytical solutions for geometric Asian options and approximate solutions for arithmetic Asian options [14]; Zhang et al., utilizing a generalized mixed sub-fractional Brownian motion, constructed a compound real options model, employing fuzzy partial differential equations to characterize the multiple uncertainties in new energy vehicle research and development [15]. Rujivan, based on the mixed fractional Brownian motion model and considering the impact of the Hurst parameter, established an analytical pricing framework containing closed-form solutions for volatility derivatives with nonlinear payoffs [16]. These studies have enriched the theoretical system of MFBM and demonstrated its potential value in risk management.

Option pricing models must simultaneously address two types of uncertainty: the stochasticity that gives rise to long-range dependence and fat-tail features, and the inherent fuzziness in the market. Fuzziness originates from information asymmetry, incompleteness, and subjective judgment differences, making it difficult to describe with precise probability distributions; therefore, stochasticity and fuzziness are not interchangeable [17]. Based on Zadeh’s fuzzy set theory [18], Xu et al. derived explicit solutions for European options under a fuzzy mixed weighted fractional Brownian motion in a jump environment [19]; Wei et al. provided closed-form solutions for fuzzy volatility CEV models [20]; Andrés-Sánchez introduced intuitionistic fuzzy numbers into binomial tree models to characterize bipolar uncertainty in parameter estimation, extending the Ho-Lee interest rate model to fuzzy stochastic scenarios for capping option pricing [21,22]; Alsenafi et al. proposed an interval pricing method for geometric Asian power options with transaction costs [23]; Timko et al. improved the application of binomial fuzzy time series in stock price prediction

by combining fuzzification with automated machine learning [24]; Gao et al. achieved high-precision forecasting of natural gas futures prices using hesitant fuzzy time series and multi-objective optimization algorithms [25].

Traditional binary option pricing models face two distinct types of uncertainty. The first is the statistical complexity of financial time series, including long-range dependence and leptokurtic fat tails. Mixed fractional Brownian motion captures these features while retaining the martingale property under appropriate conditions. The second is the cognitive fuzziness that arises from incomplete information and divergent investor expectations. Stock prices and volatility are rarely perceived as single precise values by market participants. Rather, they are understood as plausible ranges. These two types of uncertainty differ in nature and require different modeling tools. Probability theory demands distributional assumptions that are often hard to verify in information-constrained environments. Fuzzy set theory models ambiguity directly through membership functions, without requiring exact probability specifications.

To address the two types of uncertainty described above, this study combines mixed fractional Brownian motion with triangular fuzzy numbers within a single pricing framework. The resulting fuzzy price intervals capture both the long memory of asset prices and the imprecise nature of parameter knowledge. An investor evaluating a structured product can use these intervals to judge whether the offered price falls within a reasonable range, given the plausible values of stock price and volatility. The confidence level can be adjusted to match individual risk tolerance. This decision-oriented perspective is the core economic motivation of the paper.

Prior studies have already combined fuzzy set theory with fractional Brownian motion to price European and Asian options. This paper extends that line of work to binary options under mixed fractional Brownian motion. All pricing formulas are derived under $H \in (0.75, 1)$, which ensures the existence of a risk-neutral measure. The extensions include explicit α -cut formulas for $V_A(t, S)$ and $V_C(t, S)$, the monotonicity conditions required for constructing the fuzzy price intervals $(\tilde{V}_A(t, \tilde{S}))_\alpha$ and $(\tilde{V}_C(t, \tilde{S}))_\alpha$, and a complete empirical pipeline from data testing to pricing simulation. The paper further demonstrates that the sign of $\frac{\partial \tilde{V}_C(\tilde{S})}{\partial H}$ is regulated by the exercise time T . When the present time is set at $t = 0$, the upper and lower price bounds increase with H for $0 < T \leq 1$ and decrease with H for $T > 1$. In the empirical analysis, the MFBM-based fuzzy interval covers 96.00% of actual price observations while the GBM-based interval covers 88.00% under nearly identical average interval width, and the interval score favors the MFBM model. These results indicate that incorporating long-range dependence yields a more faithful and stable pricing framework for binary-structured products.

2. Fuzzy Pricing Model for Two Types of Binary Options Under Mixed Fractional Brownian Motion

2.1. Preliminary Knowledge

Before establishing the fuzzy pricing model, it is necessary to first introduce the concept of binary options, mixed fractional Brownian motion (MFBM), and the definition of triangular fuzzy numbers.

2.1.1. Binary Call Options

Binary call options are European-style contingent claims with discontinuous payoffs at expiration. We focus on two fundamental types.

Definition 1 (Cash-or-Nothing Call, CONC) [2]. *A cash-or-nothing call with strike price K , expiration time T , and fixed cash amount C pays*

$$V_C(T, S) = C \cdot 1_{\{S_T > K\}}, \quad (1)$$

where S_T is the underlying asset price at time T .

Definition 2 (Asset-or-Nothing Call, AONC) [2]. An asset-or-nothing call with identical strike K and expiration T pays

$$V_A(T, S) = S_T \cdot 1_{\{S_T > K\}}. \quad (2)$$

In both cases, the option is exercised only if $S_T > K$; otherwise the payoff is zero. These payoff structures form the basis for the pricing formulas developed in Section 2.2.

2.1.2. Mixed Fractional Brownian Motion and Triangular Fuzzy Numbers

Definition 3 (Standard Brownian Motion) [1].

A real-valued continuous-time stochastic process $\{B_t\}_{t \geq 0}$ is called a standard Brownian motion if it satisfies the following properties:

- (i) $B_0 = 0$ almost surely.
- (ii) (independent increments) For any $0 \leq t_0 < t_1 < \dots < t_n$, the random variables $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent.
- (iii) (normal increments) For any $0 \leq s < t$, the increment $B_t - B_s$ follows a normal distribution with mean 0 and variance $t - s$, i.e., $B_t - B_s \sim N(0, t - s)$.
- (iv) (continuity) The sample paths $t \mapsto B_t$ are almost surely continuous.

Definition 4 (Fractional Brownian Motion) [7].

Let $H \in (0, 1)$ be a constant. A fractional Brownian motion $\{B_t^H\}_{t \geq 0}$ with Hurst exponent H is a continuous Gaussian process satisfying:

- (i) $\mathbb{E}[B_t^H] = 0$ for all $t \geq 0$.
- (ii) (covariance structure) For all $s, t \geq 0$, $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$.

When $H = 0.5$, B_t^H reduces to a standard Brownian motion. For $H > 0.5$, the increments of B_t^H exhibit long-range dependence and are positively correlated; for $H < 0.5$, the increments are negatively correlated.

Definition 5 (Mixed Fractional Brownian Motion) [26].

On a complete probability space $(\Omega, \mathcal{F}, P)$, a mixed fractional Brownian motion $M_t^H(\alpha, \beta)$ is defined as a linear combination of a standard Brownian motion B_t and a fractional Brownian motion B_t^H as follows:

$$M_t^H(\alpha, \beta) = \alpha B_t + \beta B_t^H, (\alpha \neq 0, \beta \neq 0). \quad (3)$$

$M_t^H(\alpha, \beta)$ possesses the following important properties:

- (i) $M_t^H(\alpha, \beta)$ is a centered Gaussian process with $M_0^H(\alpha, \beta) = 0$.
- (ii) For any scaling factor $h > 0$, the process M_t^H satisfies the following scaling relation in finite-dimensional distributions: $\{M_{ht}^H\}_{t \geq 0} \stackrel{f.d.d.}{=} \{\alpha \sqrt{h} B_t + \beta h^H B_t^H\}_{t \geq 0}$, or, equivalently,

$$M_{ht}^H(\alpha, \beta) \stackrel{d}{=} M_t^H(\alpha h^{\frac{1}{2}} + \beta h^H), \quad (4)$$

this property follows directly from the self-similarity of the standard Brownian motion and the fractional Brownian motion. It describes how the statistical behavior of the mixed process changes under temporal scaling: the Brownian component scales with exponent $1/2$, while the fractional component scales with the Hurst exponent H .

- (iii) For $H \in (0.5, 1)$, the increments $\Delta M_k = M_k^H - M_{k-1}^H$ satisfy $\text{Cov}(\Delta M_k, \Delta M_{k+n}) > 0$ for all k, n , and the autocovariance decays so slowly that $\sum_{n=1}^{\infty} \text{Cov}(\Delta M_1, \Delta M_{1+n}) = \infty$. The latter condition is the standard mathematical definition of long-range dependence.

Definition 6 (Triangular Fuzzy Numbers) [27].

Let $\tilde{a} = (a_1, a_2, a_3)$ be a fuzzy set on the real number set R with its membership function defined as:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & 0 \leq x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \leq a_3 \\ 0, & x > a_3 \end{cases} \quad (5)$$

A fuzzy number satisfying the above definition is referred to as a triangular fuzzy number (TFN). The α -cut of the fuzzy variable $\tilde{a}$ is expressed as $\tilde{a}_{\alpha} = [a_{\alpha}^L, a_{\alpha}^U] = [(1-\alpha)a_1 + \alpha a_2, (1-\alpha)a_3 + \alpha a_2]$, where a_1, a_2 and a_3 represent the most pessimistic value, the most likely value, and the most optimistic value of $\tilde{a}$, respectively.

2.2. Construction of the Fuzzy Pricing Model for Two Types of Binary Options

This section proceeds in three stages. First, Theorems 1 and 2 present the pricing formulas for the asset-or-nothing call and the cash-or-nothing call under MFBM. Second, Theorems 3 and 4 establish the general α -cut endpoints by treating stock price and volatility as fuzzy variables and proving the relevant monotonicity properties. Third, Theorem 6 specializes these results to the case of triangular fuzzy numbers, yielding the explicit interval formulas used in the empirical analysis.

Taking binary call options as an example, this paper first derives the pricing formulas for the asset-or-nothing call (AONC) and the cash-or-nothing call (CONC) under mixed fractional Brownian motion (MFBM).

Now consider a mixed fractional Black–Scholes market with two assets. Suppose the bond price M_t satisfies the following equation:

$$dM_t = rM_t dt, \quad M_0 = 1, 0 \leq t \leq T, \quad (6)$$

where r denotes the risk-free interest rate.

The stock price S_t satisfies the following stochastic differential equation under the physical probability measure:

$$dS_t = \mu S_t dt + \sigma S_t d\bar{B}_t + \sigma S_t d\bar{B}_t^H, \quad S_0 = S, 0 \leq t \leq T, \quad (7)$$

where μ and σ represent the expected return rate and volatility, respectively, H is the Hurst exponent with $H \in (0, 1)$. $\bar{B}_t$ is a standard Brownian motion, and $\bar{B}_t^H$ is a fractional Brownian motion independent of $\bar{B}_t$. Cheridito [9] proved that when $H \in (0.75, 1)$, the mixed fractional Brownian motion is equivalent in distribution to a standard Brownian motion and the process becomes a semimartingale. Consequently, an equivalent martingale measure exists and the market is arbitrage-free. To transform the physical measure to the risk-neutral measure, the drift term μ must be replaced by the risk-free rate r through a change of measure. By applying the Girsanov theorem for mixed fractional processes, the standard Brownian motion is shifted while the fractional Brownian motion remains structurally unchanged. Specifically, two new processes are defined under the risk-neutral measure Q as: $B_t = \bar{B}_t + \frac{\mu-r}{\sigma}t$, $B_t^H = \bar{B}_t^H$.

Under Q , B_t becomes a standard Brownian motion and B_t^H remains a fractional Brownian motion, with the two processes still independent. Substituting the inverse relations $\bar{B}_t = B_t - \frac{\mu-r}{\sigma}t$ and $\bar{B}_t^H = B_t^H$ into Equation (7) yields the stock price dynamics under the risk-neutral measure:

$$dS_t = rS_t dt + \sigma S_t dB_t + \sigma S_t dB_t^H. \quad (8)$$

Equation (8) is a linear SDE driven by a mixed fractional process. To obtain its explicit solution, the quasi-conditional expectation method of Sun [28] is employed. Since $H \in (0.75, 1)$, the mixed process $B_t + B_t^H$ shares the same distribution as a standard Brownian motion. Applying the fractional Itô formula to $\ln S_t$ and taking the quasi-conditional expectation gives:

$$S_t = S_0 \cdot \exp \left\{ rt + \sigma B_t + \sigma B_t^H - \frac{1}{2} \sigma^2 t - \frac{1}{2} \sigma^2 t^{2H} \right\} \quad (0 \leq t \leq T) \quad (9)$$

It can be verified directly that the discounted price process $e^{-rt}S_t$ is a martingale under Q , confirming the internal consistency of the risk-neutral pricing framework established above.

First, this paper presents the pricing formula for the AONC at time t :

Theorem 1. Let $V_A(t, S)$ be the price of the AONC at time $t \in [0, T]$, then

$$V_A(t, S) = S_t N(d_1) \quad (10)$$

where:

$$d_1 = [\ln S_t/K + r(T-t) + \frac{1}{2} \sigma^2 (T-t) + \frac{1}{2} \sigma^2 (T^{2H} - t^{2H})] / [\sigma \sqrt{(T-t) + (T^{2H} - t^{2H})}]. \quad (11)$$

Proof. Let $\begin{cases} B_t^* = B_t - \sigma t \\ (B_t^H)^* = B_t^H - \sigma t^{2H} \end{cases} \Rightarrow (M_t^H)^* = M_t^H - (\sigma^2 t + \sigma^2 t^{2H})$, using the method from reference [28], set $X_t = S_t e^{-rt}$, then under the risk-neutral measure we have

$$\begin{aligned} V_A(t, S) &= \mathbb{E}_Q \left\{ e^{-r(T-t)} S_T I_{S_T > K} | \mathcal{F}_t \right\} \\ &= e^{-r(T-t)} \mathbb{E}_Q \left\{ S_T I_{S_T > K} | \mathcal{F}_t \right\} \\ &= e^{-r(T-t)} e^{rT} \mathbb{E}_Q \left\{ X_T I_{S_T > K} | \mathcal{F}_t \right\} \\ &= e^{rt} X_t \mathbb{E}_Q^* \left\{ I_{(M_T^H)^* > \ln K/S_0 - rT - \frac{1}{2} \sigma^2 T - \frac{1}{2} \sigma^2 T^{2H}} | \mathcal{F}_t \right\} \\ &= S_t \int_{\ln K/S_0 - rT - \frac{1}{2} \sigma^2 T - \frac{1}{2} \sigma^2 T^{2H}}^{+\infty} \frac{1}{\sigma \sqrt{2\pi[(T-t) + (T^{2H} - t^{2H})]}} \exp \left\{ \frac{-[x - (M_t^H)^*]^2}{2\sigma^2[(T-t) + (T^{2H} - t^{2H})]} \right\} dx \\ &= S_t \int_{-\infty}^{+\infty} \frac{(M_t^H)^* - (\ln K/S_0 - rT - \frac{1}{2} \sigma^2 T - \frac{1}{2} \sigma^2 T^{2H})}{\sigma \sqrt{[(T-t) + (T^{2H} - t^{2H})]}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y^*)^2}{2}} dy^* \\ &= S_t \int_{-\infty}^{+\infty} \frac{\ln S_t/K + r(T-t) + \frac{1}{2} \sigma^2 (T-t) + \frac{1}{2} \sigma^2 (T^{2H} - t^{2H})}{\sigma \sqrt{[(T-t) + (T^{2H} - t^{2H})]}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y^*)^2}{2}} dy^* \\ &= S_t N(d_1) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + r(T-t) + \frac{1}{2}\sigma^2(T-t) + \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{(T-t) + (T^{2H} - t^{2H})}}.$$

Then, in the risk-neutral pricing of the AONC, replacing the terminal asset price S_T in the expectation $\mathbb{E}_Q\{e^{-r(T-t)}S_T I_{S_T > K} | \mathcal{F}_t\}$ with the fixed cash payment C directly yields the price of the cash-or-nothing call. The derivation is therefore identical in structure to that of Theorem 1, but simpler; the detailed proof is provided in Appendix A.1. $\square$

Theorem 2. Suppose C is a fixed return, and let $V_C(t, S)$ be the price of the CONC at time $t \in [0, T]$, then:

$$V_C(t, S) = Ce^{-r(T-t)}N(d_2), \quad (12)$$

where:

$$d_2 = d_1 - \sigma\sqrt{(T-t) + (T^{2H} - t^{2H})}. \quad (13)$$

After deriving the pricing formulas for the two types of binary call options under MFBM, this paper proceeds to set both the underlying stock price and the volatility as fuzzy variables, thereby beginning the construction of the fuzzy pricing model for binary options. The selection of fuzzy variables is based on the following two aspects:

Stock price and volatility are selected as fuzzy variables because they directly determine the option payoff and its probability distribution, and both are subject to the information incompleteness and expectation divergence discussed in the introduction. Treating them as triangular fuzzy numbers translates input imprecision into an output price interval.

After fuzzifying the stock price and volatility of the binary call options under MFBM, the fuzzy price formulas in Corollary 1 can be obtained.

Corollary 1. Let $\tilde{V}_A(t, \tilde{S})$ and $\tilde{V}_C(t, \tilde{S})$ represent the fuzzy prices of the binary call options AONC and CONC at time $t \in [0, T]$, respectively, then:

$$\tilde{V}_A(t, \tilde{S}) = \tilde{S}_t N(\tilde{d}_1) \quad (14)$$

$$\tilde{V}_C(t, \tilde{S}) = Ce^{-r(T-t)}N(\tilde{d}_2) \quad (15)$$

where:

$$\begin{aligned} \tilde{d}_1 &= [\ln \tilde{S}_t / K + r(T-t) + \frac{1}{2}\tilde{\sigma}^2(T-t) + \frac{1}{2}\tilde{\sigma}^2(T^{2H} - t^{2H})] / [\tilde{\sigma}\sqrt{(T-t) + (T^{2H} - t^{2H})}]; \\ \tilde{d}_2 &= \tilde{d}_1 - \tilde{\sigma}\sqrt{(T-t) + (T^{2H} - t^{2H})}. \end{aligned}$$

Based on the definition of α -cuts in fuzzy set theory and the monotonicity of the fuzzy option price $\tilde{V}_A(t, \tilde{S})$ with respect to the fuzzy variables, the interval endpoints of the fuzzy price for the AONC can be obtained.

Theorem 3. The interval endpoints of the α -cut $(\tilde{V}_A(t, \tilde{S}))_\alpha$ for the price $\tilde{V}_A(t, \tilde{S})$ of the AONC are:

$$(\tilde{V}_A(t, \tilde{S}))_\alpha^L = (\tilde{S}_t)_\alpha^L N(\tilde{I}_1); \quad (16)$$

$$(\tilde{V}_A(t, \tilde{S}))_\alpha^U = (\tilde{S}_t)_\alpha^U N(\tilde{I}_3). \quad (17)$$

where:

$$\tilde{I}_1 = \frac{\ln \frac{(\tilde{S}_t)_\alpha^L}{K} + r(T-t) + \frac{1}{2}(\tilde{\sigma}_\alpha^L)^2(T-t) + \frac{1}{2}(\tilde{\sigma}_\alpha^L)^2(T^{2H} - t^{2H})}{\tilde{\sigma}_\alpha^L\sqrt{(T-t) + (T^{2H} - t^{2H})}}, \quad (18)$$

$$\tilde{I}_2 = \tilde{I}_1 - \tilde{\sigma}_\alpha^L \sqrt{(T-t) + (T^{2H} - t^{2H})}; \quad (19)$$

$$\tilde{I}_3 = \frac{\ln \frac{(\tilde{S}_t)_\alpha^U}{K} + r(T-t) + \frac{1}{2}(\tilde{\sigma}_\alpha^U)^2(T-t) + \frac{1}{2}(\tilde{\sigma}_\alpha^U)^2(T^{2H} - t^{2H})}{\tilde{\sigma}_\alpha^U \sqrt{(T-t) + (T^{2H} - t^{2H})}}, \quad (20)$$

$$\tilde{I}_4 = \tilde{I}_3 - \tilde{\sigma}_\alpha^U \sqrt{(T-t) + (T^{2H} - t^{2H})}. \quad (21)$$

The proof proceeds by first expressing the α -cut of the fuzzy price in terms of the α -cuts of $\tilde{S}_t$ and $\tilde{\sigma}$, and then determining where the extrema occur. It is shown that $V_A(t, S)$ is always monotonically non-decreasing with respect to the stock price, while its monotonicity with respect to volatility requires the sufficient condition $r \leq \frac{\sigma^2}{2}$. Under this condition, the fuzzy price interval endpoints correspond to the boundaries of the α -cuts of the fuzzy inputs. A detailed proof is given in Appendix A.2.

However, the condition $r \leq \frac{\sigma^2}{2}$, while mathematically sufficient, may be violated in practice—for instance, in high interest rate or low volatility regimes. In such cases, the monotonicity with respect to volatility can fail, potentially distorting the α -cut endpoint mapping and widening the fuzzy price interval in an uncontrolled manner. This limitation should be kept in mind when applying Theorem 3 outside its stated scope. It is worth noting that the subsequent empirical focus of this paper lies on the CONC rather than the AONC, and for the CONC, the monotonicity with respect to volatility is unambiguous and requires no such additional condition. Therefore, the parameter settings used in the empirical section are not tailored to satisfy $r \leq \frac{\sigma^2}{2}$.

Similarly, the interval endpoints of the fuzzy price for the CONC can be obtained.

Theorem 4. The interval endpoints of the α -cut $(\tilde{V}_C(t, \tilde{S}))_\alpha$ for the price $\tilde{V}_C(t, \tilde{S})$ of the CONC are:

$$(\tilde{V}_C(t, \tilde{S}))_\alpha^L = Ce^{-r(T-t)} N(\tilde{I}_5); \quad (22)$$

$$(\tilde{V}_C(t, \tilde{S}))_\alpha^U = Ce^{-r(T-t)} N(\tilde{I}_7). \quad (23)$$

where:

$$\tilde{I}_5 = \frac{\ln \frac{(\tilde{S}_t)_\alpha^L}{K} + r(T-t) - \frac{1}{2}(\tilde{\sigma}_\alpha^U)^2(T-t) - \frac{1}{2}(\tilde{\sigma}_\alpha^U)^2(T^{2H} - t^{2H})}{\tilde{\sigma}_\alpha^U \sqrt{(T-t) + (T^{2H} - t^{2H})}}, \quad (24)$$

$$\tilde{I}_6 = \tilde{I}_5 + \tilde{\sigma}_\alpha^U \sqrt{(T-t) + (T^{2H} - t^{2H})}; \quad (25)$$

$$\tilde{I}_7 = \frac{\ln \frac{(\tilde{S}_t)_\alpha^U}{K} + r(T-t) - \frac{1}{2}(\tilde{\sigma}_\alpha^L)^2(T-t) - \frac{1}{2}(\tilde{\sigma}_\alpha^L)^2(T^{2H} - t^{2H})}{\tilde{\sigma}_\alpha^L \sqrt{(T-t) + (T^{2H} - t^{2H})}}, \quad (26)$$

$$\tilde{I}_8 = \tilde{I}_7 + \tilde{\sigma}_\alpha^L \sqrt{(T-t) + (T^{2H} - t^{2H})}. \quad (27)$$

The proof follows a similar α -cut decomposition. It is shown that $\tilde{V}_C(t, \tilde{S})$ is monotonically increasing with respect to the stock price but monotonically decreasing with respect to the volatility. Hence, the lower price bound comes from the left endpoint of the price α -cut and the right endpoint of the volatility α -cut, while the upper bound comes from the opposite combination. The complete derivation is provided in Appendix A.3.

According to the definitions of possibilistic mean and lower variance, we obtain:

Theorem 5. The possibilistic means and possibilistic lower variances of the two types of binary options $\tilde{V}_A(t, \tilde{S})$ and $\tilde{V}_C(t, \tilde{S})$ are expressed as follows:

$$\overline{M}(\tilde{V}_A(t, \tilde{S})) = \int_0^1 \alpha [(\tilde{V}_A(t, \tilde{S}))_\alpha^L + (\tilde{V}_A(t, \tilde{S}))_\alpha^U] d\alpha, \quad (28)$$

$$LSVar(\tilde{V}_A(t, \tilde{S})) = 2 \int_0^1 \alpha [\overline{M}(\tilde{V}_A(t, \tilde{S})) - (\tilde{V}_A(t, \tilde{S}))_\alpha^L]^2 d\alpha; \quad (29)$$

$$\overline{M}(\tilde{V}_C(t, \tilde{S})) = \int_0^1 \alpha [(\tilde{V}_C(t, \tilde{S}))_\alpha^L + (\tilde{V}_C(t, \tilde{S}))_\alpha^U] d\alpha, \quad (30)$$

$$LSVar(\tilde{V}_C(t, \tilde{S})) = 2 \int_0^1 \alpha [\overline{M}(\tilde{V}_C(t, \tilde{S})) - (\tilde{V}_C(t, \tilde{S}))_\alpha^L]^2 d\alpha. \quad (31)$$

Triangular fuzzy numbers (TFNs) are an extremely typical and commonly used class of fuzzy numbers in fuzzy set theory. Due to their simple structure, computational convenience, and intuitive comprehensibility, they have been widely applied in numerous fields such as fuzzy decision-making, fuzzy optimization, and risk assessment. Since Theorems 3 and 4 provide the fuzzy price cut endpoints for the AONC and CONC, respectively, if the fuzzy stock price $\tilde{S}_t$ and fuzzy volatility $\tilde{\sigma}$ are regarded as triangular fuzzy numbers, then the following conclusion can be drawn.

Theorem 6. The interval endpoints of the α -cuts for the two types of binary call options are respectively:

(1) The interval endpoints of the α -cut for the AONC:

$$(\tilde{V}_A(t, \tilde{S}))_\alpha^L = ((1 - \alpha)S_1 + \alpha S_2)N((\tilde{d}_1)_\alpha^L); \quad (32)$$

$$(\tilde{V}_A(t, \tilde{S}))_\alpha^U = ((1 - \alpha)S_3 + \alpha S_2)N((\tilde{d}_1)_\alpha^U). \quad (33)$$

where:

$$\tilde{I}_1 = \frac{\ln \frac{(1-\alpha)S_1 + \alpha S_2}{K} + \left\{ r + \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} \right\} (T - t) + \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} (T^{2H} - t^{2H})}{[(1 - \alpha)\sigma_1 + \alpha\sigma_2] \sqrt{(T - t) + (T^{2H} - t^{2H})}}, \quad (34)$$

$$\tilde{I}_3 = \frac{\ln \frac{(1-\alpha)S_3 + \alpha S_2}{K} + \left\{ r + \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} \right\} (T - t) + \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} (T^{2H} - t^{2H})}{[(1 - \alpha)\sigma_3 + \alpha\sigma_2] \sqrt{(T - t) + (T^{2H} - t^{2H})}}. \quad (35)$$

(2) The interval endpoints of the α -cut for the CONC:

$$(\tilde{V}_C(t, \tilde{S}))_\alpha^L = Ce^{-[(1-\alpha)r_3 + \alpha r_2](T-t)} N(\tilde{I}_5), \quad (36)$$

$$(\tilde{V}_C(t, \tilde{S}))_\alpha^U = Ce^{-[(1-\alpha)r_1 + \alpha r_2](T-t)} N(\tilde{I}_7). \quad (37)$$

where:

$$\tilde{I}_5 = \frac{\ln \frac{(1-\alpha)S_1 + \alpha S_2}{K} + \left\{ r - \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} \right\} (T - t) - \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} (T^{2H} - t^{2H})}{[(1 - \alpha)\sigma_3 + \alpha\sigma_2] \sqrt{(T - t) + (T^{2H} - t^{2H})}}, \quad (38)$$

$$\tilde{I}_7 = \frac{\ln \frac{(1-\alpha)S_3 + \alpha S_2}{K} + \left\{ r - \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} \right\} (T - t) - \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} (T^{2H} - t^{2H})}{[(1 - \alpha)\sigma_1 + \alpha\sigma_2] \sqrt{(T - t) + (T^{2H} - t^{2H})}}. \quad (39)$$

Proof. Since for the fuzzy variables $\tilde{S}_t$ and $\tilde{\sigma}$, we have $\tilde{S}_t = (S_1, S_2, S_3)$ and $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and from the definition of triangular fuzzy numbers:

$$(\tilde{S}_t)_\alpha = [(1 - \alpha)S_1 + \alpha S_2, (1 - \alpha)S_3 + \alpha S_2],$$

$$\tilde{\sigma}_\alpha = [(1 - \alpha)\sigma_1 + \alpha\sigma_2, (1 - \alpha)\sigma_3 + \alpha\sigma_2];$$

Thus, substituting the above results into Theorem 3 and Theorem 4 completes the proof. $\square$

Since the price of a binary option is closely related to the price of the underlying asset (stock), if the stock price can be predicted with relative accuracy, the price interval of the binary option can also be obtained indirectly. Corollary 2 provides the α -cut of $\tilde{S}_t$.

Corollary 2. *The interval endpoints of the α -cut for the fuzzy stock price are:*

$$(S_t)_\alpha^L = [(1 - \alpha)S_1 + \alpha S_2] \cdot \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\}, \quad (40)$$

$$(S_t)_\alpha^U = [(1 - \alpha)S_3 + \alpha S_2] \cdot \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\}. \quad (41)$$

Proof. Because $S_t = S_0 \cdot \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\}$, $(0 \leq t \leq T)$. Then the derivative of S_t with respect to S_0 holds:

$$\frac{\partial S_t}{\partial S_0} = \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\} \geq 0,$$

therefore S_t is monotonically increasing with respect to S_0 , and Corollary 2 is obtained.

It is noted that, unlike the fuzzy option prices $\tilde{V}_A(t, \tilde{S})$ and $\tilde{V}_C(t, \tilde{S})$, S_t is not monotonic with respect to σ . Taking the derivative:

$$\frac{\partial S_t}{\partial \sigma} = S_0 \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\} \cdot (B_t + B_t^H - \sigma t - \sigma t^{2H}),$$

$$\frac{\partial^2 S_t}{\partial \sigma^2} = -(t + t^{2H})S_0 \exp\left\{rt + \sigma B_t + \sigma B_t^H - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\} \leq 0,$$

then $\frac{\partial S_t}{\partial \sigma}$ is monotonically decreasing, but the sign of $B_t + B_t^H - \sigma t - \sigma t^{2H}$ remains uncertain. At the same time, the expression for S_t also indicates that in actual financial markets, the greater the volatility, the stronger the uncertainty regarding the short-term fluctuation amplitude and long-term trend of the stock price. $\square$

3. Long-Range Dependence Analysis

The long-range dependence in financial markets can be measured using the Hurst exponent H . Theorem 7 demonstrates the impact of the Hurst exponent on the fuzzy prices of options.

Theorem 7. *The effect of the Hurst exponent H on the prices of the two types of binary call options $\tilde{V}_A(t, \tilde{S})$ and $\tilde{V}_C(t, \tilde{S})$ is:*

$$\frac{\partial \tilde{V}_A(t, \tilde{S})}{\partial H} = -\tilde{S}_t \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \frac{\tilde{d}_2(T^{2H} \ln T - t^{2H} \ln t)}{(T - t) + (T^{2H} - t^{2H})}; \quad (42)$$

$$\frac{\partial \tilde{V}_C(t, \tilde{S})}{\partial H} = -C e^{-\tilde{r}(T-t)} \frac{\partial N(\tilde{d}_2)}{\partial \tilde{d}_2} \frac{\tilde{d}_1(T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})}. \quad (43)$$

The proof computes the partial derivative of the fuzzy pricing formulas given in Corollary 1 with respect to H . Since the expressions for $\tilde{V}_A(t, \tilde{S})$ and $\tilde{V}_C(t, \tilde{S})$ depend on H both directly through $T^{2H} \ln T - t^{2H} \ln t$ and indirectly through $\tilde{d}_1$ and $\tilde{d}_2$, the chain rule is applied to each component. The complete derivation is provided in Appendix A.4.

By substituting the results from Theorem 6 into Theorem 7, the impact of the Hurst exponent H on the α -cut interval endpoints of the binary call options under the specific fuzzy pricing model can be obtained.

Corollary 3. The effects of the Hurst exponent H on the α -cut interval endpoints of the two types of binary call options are as follows:

(1) Effects on $(\tilde{V}_A(t, \tilde{S}))_\alpha^L$ and $(\tilde{V}_A(t, \tilde{S}))_\alpha^U$:

$$\frac{\partial ((\tilde{V}_A(t, \tilde{S}))_\alpha^L)}{\partial H} = -(\tilde{S}_t)_\alpha^L \frac{\partial N(\tilde{I}_1)}{\partial \tilde{I}_1} \frac{\tilde{I}_2(T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})}; \quad (44)$$

$$\frac{\partial ((\tilde{V}_A(t, \tilde{S}))_\alpha^U)}{\partial H} = -(\tilde{S}_t)_\alpha^U \frac{\partial N(\tilde{I}_3)}{\partial \tilde{I}_3} \frac{\tilde{I}_4(T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})}. \quad (45)$$

(2) Effects on $(\tilde{V}_C(t, \tilde{S}))_\alpha^L$ and $(\tilde{V}_C(t, \tilde{S}))_\alpha^U$:

$$\frac{\partial ((\tilde{V}_C(t, \tilde{S}))_\alpha^L)}{\partial H} = -C e^{-r(T-t)} \frac{\partial N(\tilde{I}_5)}{\partial \tilde{I}_5} \frac{\tilde{I}_6(T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})}; \quad (46)$$

$$\frac{\partial ((\tilde{V}_C(t, \tilde{S}))_\alpha^U)}{\partial H} = -C e^{-r(T-t)} \frac{\partial N(\tilde{I}_7)}{\partial \tilde{I}_7} \frac{\tilde{I}_8(T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})}. \quad (47)$$

where $\tilde{I}_1, \dots, \tilde{I}_8$ are as defined in Theorems 3 and 4.

In particular, from Corollary 3, the impact of the Hurst exponent on the price cutsets of binary call options when $t = 0$ can be directly derived. Since this paper primarily studies the CONC, only the relevant conclusions for the CONC are provided here. That is:

$$\frac{\partial ((\tilde{V}_C(\tilde{S}))_\alpha^L)}{\partial H} = -C e^{-rT} \frac{\partial N(\hat{I}_5)}{\partial \hat{I}_5} \frac{\hat{I}_6 T^{2H} \ln T}{T + T^{2H}}; \quad (48)$$

$$\frac{\partial ((\tilde{V}_C(\tilde{S}))_\alpha^U)}{\partial H} = -C e^{-rT} \frac{\partial N(\hat{I}_7)}{\partial \hat{I}_7} \frac{\hat{I}_8 T^{2H} \ln T}{T + T^{2H}}. \quad (49)$$

where:

$$\hat{I}_5 = \frac{\ln \frac{(1-\alpha)S_1 + \alpha S_2}{K} + \left\{ r - \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} \right\} T - \frac{[(1-\alpha)\sigma_3 + \alpha\sigma_2]^2}{2} T^{2H}}{[(1-\alpha)\sigma_3 + \alpha\sigma_2] \sqrt{T + T^{2H}}}, \quad (50)$$

$$\hat{I}_6 = \hat{I}_5 + [(1-\alpha)\sigma_3 + \alpha\sigma_2] \sqrt{T + T^{2H}}; \quad (51)$$

$$\hat{I}_7 = \frac{\ln \frac{(1-\alpha)S_3 + \alpha S_2}{K} + \left\{ r - \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} \right\} T - \frac{[(1-\alpha)\sigma_1 + \alpha\sigma_2]^2}{2} T^{2H}}{[(1-\alpha)\sigma_1 + \alpha\sigma_2] \sqrt{T + T^{2H}}}, \quad (52)$$

$$\hat{I}_8 = \hat{I}_7 + [(1-\alpha)\sigma_1 + \alpha\sigma_2] \sqrt{T + T^{2H}}. \quad (53)$$

It is noted that if we assume $S_1 \approx K$ in equations (51) and (53), i.e., the difference between the two is small, then both $\hat{I}_6$ and $\hat{I}_8$ are greater than 0. Therefore, the signs of $\frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^L)}{\partial H}$ and $\frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^U)}{\partial H}$ are related only to T , specifically:

(1) When $0 < T \leq 1$,

$$\frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^L)}{\partial H} \geq 0, \frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^U)}{\partial H} \geq 0; \quad (54)$$

(2) When $T \geq 1$,

$$\frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^L)}{\partial H} \leq 0, \frac{\partial((\tilde{V}_C(\tilde{S}))_\alpha^U)}{\partial H} \leq 0. \quad (55)$$

To more intuitively illustrate the impact of the execution time T on $\frac{\partial \tilde{V}_C(\tilde{S})}{\partial H}$, Figures 1 and 2 show the influence of the Hurst exponent H on the upper and lower option prices when $T = 0.5$ and $T = 2$, respectively.

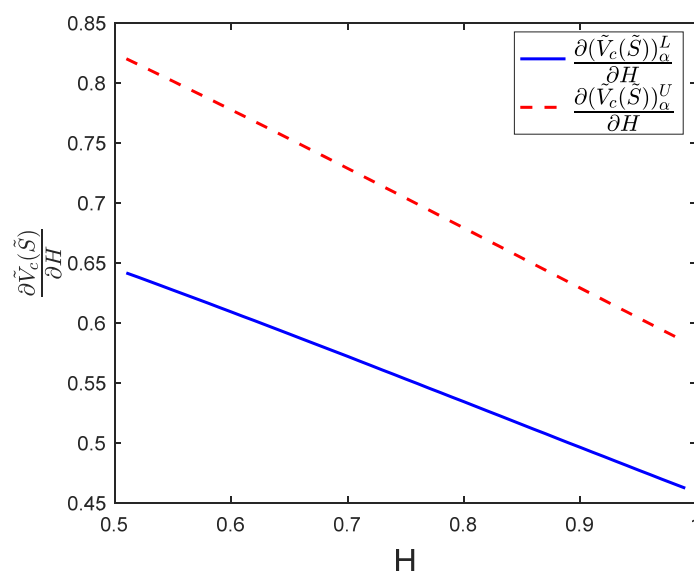


Figure 1. Sign of $\frac{\partial \tilde{V}_C(\tilde{S})}{\partial H}$ when $T = 0.5$.

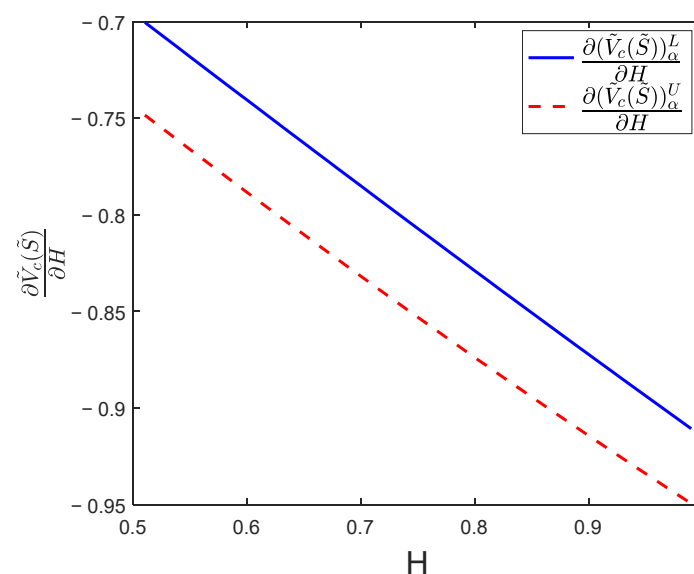


Figure 2. Sign of $\frac{\partial \tilde{V}_C(\tilde{S})}{\partial H}$ when $T = 2$.

In summary, when $0 < T \leq 1$, as H increases, both the upper and lower option prices increase; when $T \geq 1$, the upper and lower option prices gradually decrease as H increases.

4. Empirical Analysis and Numerical Experiments

4.1. Empirical Product Description and Determination of Fixed Payment

Although standardized binary options have not yet been formally introduced in China, a large number of structured wealth management products exist in actual financial markets. These products, by linking to stock indices or asset prices, essentially embed a payoff structure similar to binary options. For example, the Equity Performance-Linked Product under the Focus Link Series (Product Code: 116619), launched by China Merchants Bank in 2018, allows its payoff structure to be separated and simplified so that its return rules can be precisely mapped to a cash-or-nothing binary option, i.e., a CONC. The specifics are as follows:

- (1) Fulfillment condition: The final price of the SSE 50 Index $\geq$ the initial price (barrier price);
- (2) Payoff structure: If the condition is met, a wealth management return rate of 7.68% (annualized) is obtained; otherwise, a wealth management return rate of 1% (annualized) is obtained.

Taking an investment of 200,000 RMB in this wealth management product as an example, and considering that purchasing the product requires paying certain fees, such as sales and custody expenses, we can calculate:

$$\text{Minimum return: } 200000 \times (1.00\% - 0.30\% - 0.02\%) \times \frac{69}{365} \approx 257.10 \text{ RMB;}$$

Maximum return:

$$200000 \times [7.68\% - 0.30\% - 0.02\% - (7.68\% - 6\%) \times 20\%] \times \frac{69}{365} \approx 2656.41 \text{ RMB.}$$

To simplify the model, the risk-free component can be offset against expenses such as sales and custody fees. The fixed payment is then calculated as follows:

$$C = 200000 \times (7.68\% - 1\%) \times \frac{69}{365} \approx 2525.40 \text{ RMB.}$$

4.2. Testing for Long-Range Dependence and Leptokurtic Fat-Tail Characteristics

Although stock price changes are generally considered to exhibit the typical characteristics of long-range dependence and leptokurtic fat tails, whether these properties hold in real markets still needs to be verified through rigorous empirical testing. The following sections will conduct corresponding statistical tests and analyses for the long-range dependence and leptokurtic fat-tail characteristics of the SSE 50 Index return series, respectively.

Figures 3–5 are based on the analysis of the SSE 50 Index closing price series from 12 July 2018 to 19 September 2018. Figure 3 shows the price trend of the index during this period, reflecting the dynamic process of stock price changes; Figure 4 further presents the basic characteristics of the corresponding return series, providing a data foundation for subsequent long-range dependence testing. Based on this, the R/S method was used to test the long-range dependence of the return series, and the results are shown in Figure 5. The calculated Hurst exponent is 0.7744. According to the discriminant criterion ($H > 0.5$ indicates the presence of long-range dependence), this result confirms that the return series exhibits a significant long-range dependence effect, meaning that historical volatility has a persistent impact on future trends. It can be observed from Figure 5 that the scatter points are highly consistent with the fitted line, and the goodness of fit (R^2) reaches 0.9544, indicating that the Hurst exponent estimation results are robust and reliable. Therefore,

the R/S analysis empirically establishes the long-range dependence characteristic of stock prices, providing a solid foundation for using MFBM to construct the option pricing model.

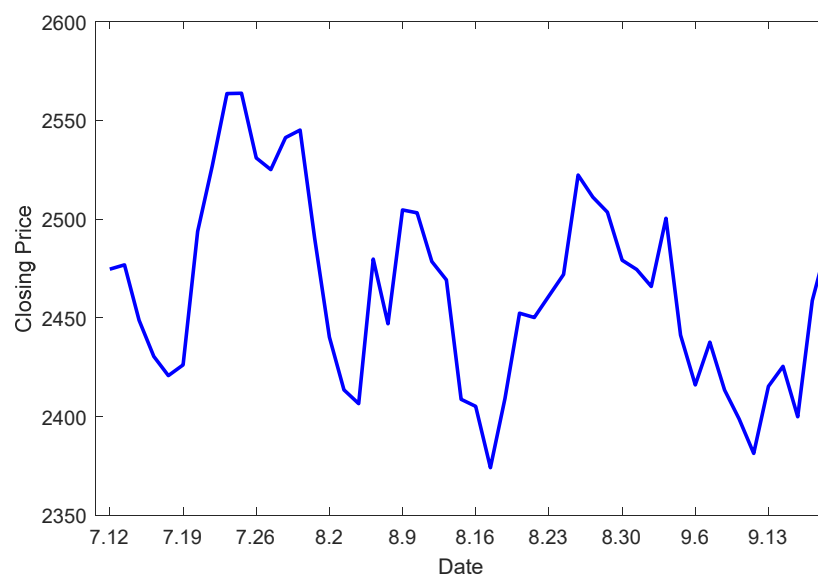


Figure 3. SSE 50 Index Closing Price Series.

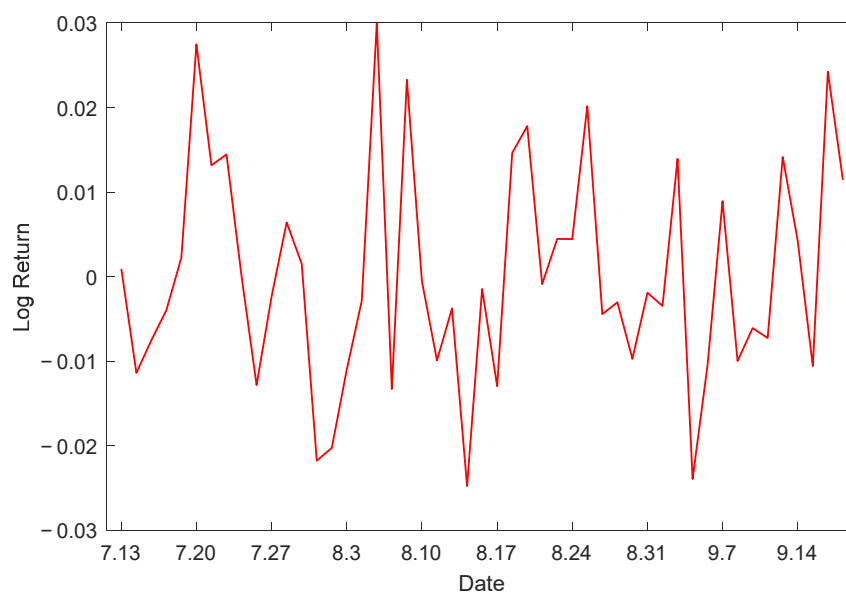


Figure 4. Logarithmic Return Series.

To more clearly test the distribution characteristics of the SSE 50 Index returns, this paper selects data from 1 January 2018 to 19 September 2018 for analysis. Three complementary statistical methods are used to jointly verify the leptokurtic fat-tail characteristics of the returns.

Figure 6, “Comparison of Return Distribution with Normal Distribution,” shows that the actual distribution has a sharper peak and heavier tails compared to the normal curve. Quantitative indicators show a kurtosis of 3.5642 and excess kurtosis of 0.5642, statistically confirming the leptokurtic characteristic. Figure 7, the “Q-Q Plot,” exhibits a typical S-shaped deviation pattern, with scatter points at both ends significantly deviating from the reference line. Key evidence shows that the proportion of observations exceeding ± 3 standard deviations is 0.6%, significantly higher than the normal theoretical value of 0.27%, confirming the fat-tail phenomenon. In Figure 8, “Tail Comparison of Cumulative

Distribution Functions,” the empirical distribution consistently lies above the normal curve in the right tail, further verifying the fat-tail characteristic. Therefore, combining graphical analysis and statistical tests, the SSE 50 Index returns exhibit significant leptokurtic fat-tail characteristics, which also provides reliable empirical evidence for using MFBM to characterize extreme market fluctuations.

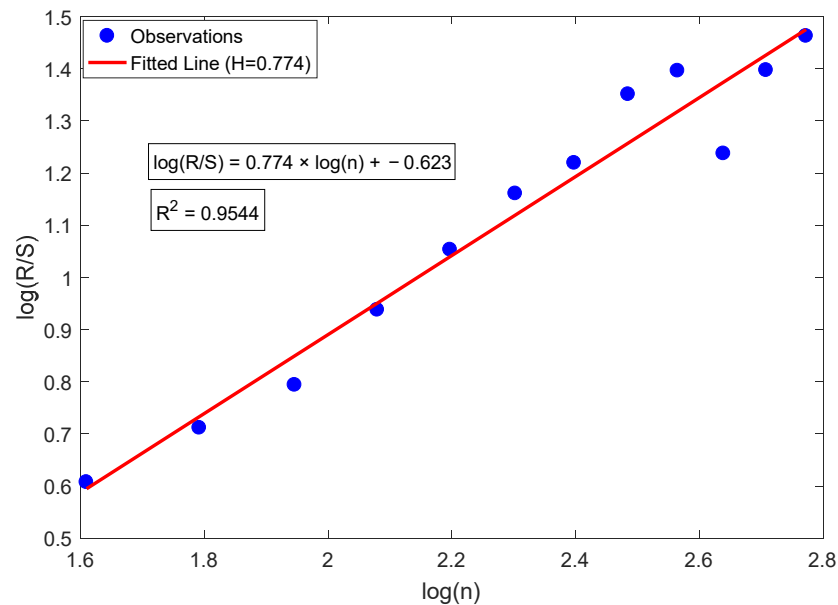


Figure 5. R/S Analysis.

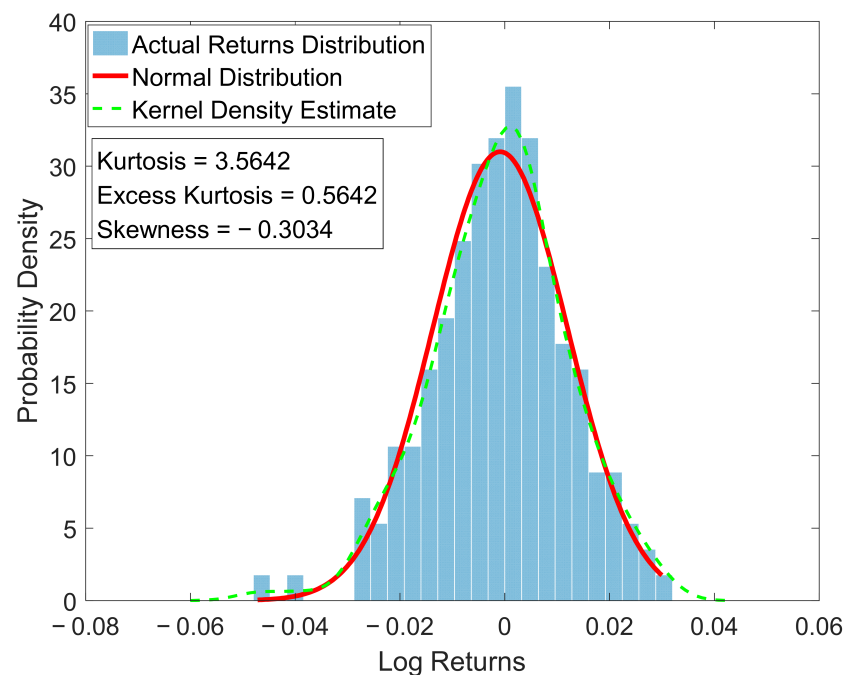


Figure 6. Comparison of SSE 50 Return Distribution with Normal Distribution.

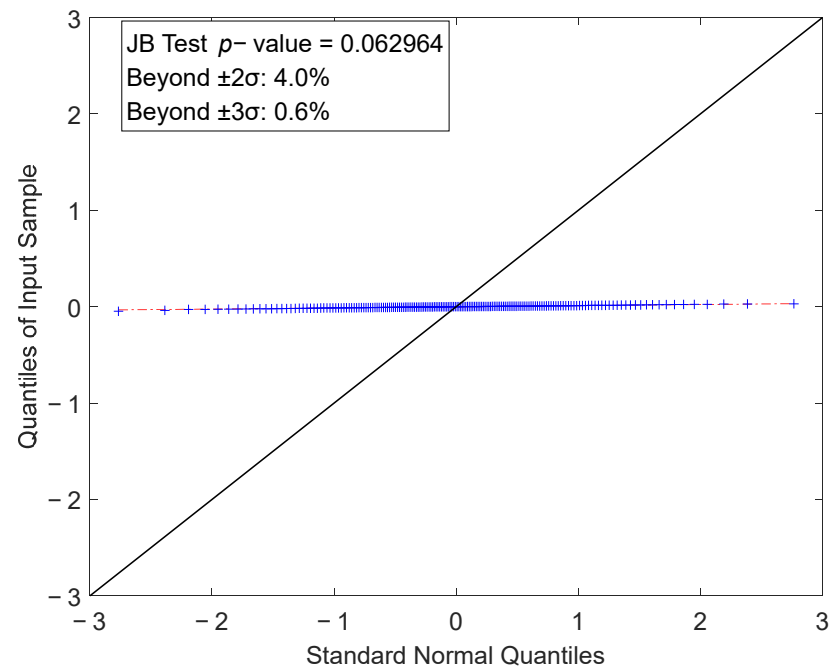


Figure 7. Q-Q Plot of SSE 50 Returns.

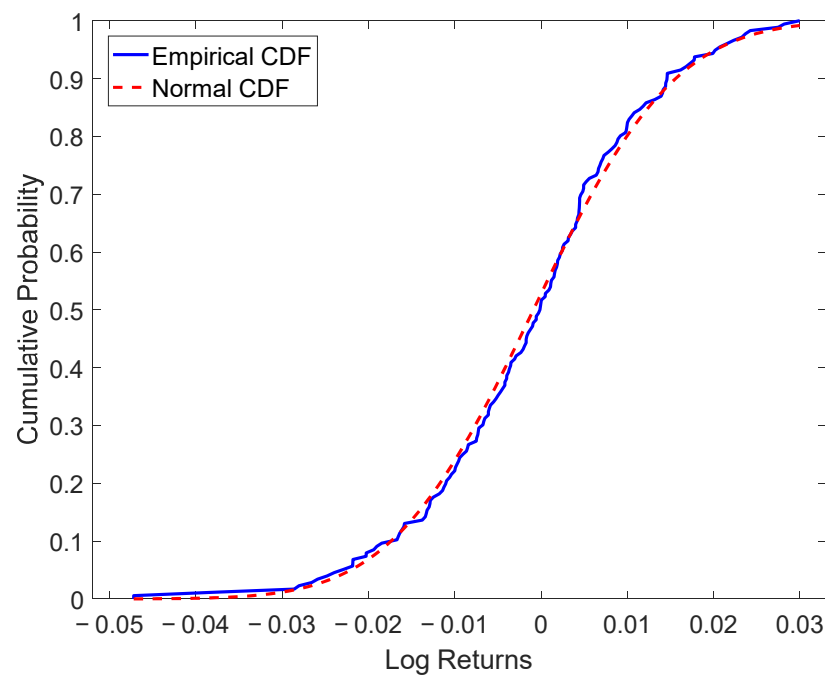


Figure 8. Tail Comparison of Cumulative Distribution Functions.

4.3. Determination of Baseline Data

In the empirical analysis and numerical experiments, some baseline data can be directly determined, while others need to be estimated through calculation. According to the wealth management product's prospectus, the initial stock price and the strike price are both taken from the actual market price on 12 July 2018; the exercise time is set as the product's actual holding period. Regarding the risk-free interest rate, referring to the one-year treasury bond yield from the supplementary issuance by the Ministry of Finance on 20 June 2018, which was approximately 3.2% (Source: China Securities Journal), this rate is determined as the one-year treasury bond yield. The long-range dependence indicator, the Hurst exponent H , and the volatility σ , need to be estimated based on the daily closing prices of the SSE 50

Index from 1 January 2018 to 11 July 2018; the confidence level α is calculated using the training set (covering all closing prices from 26 March 2018, to 15 June 2018).

(1) Hurst Exponent H

The Hurst exponent is estimated using 127 daily closing prices of the SSE 50 Index from 2 January 2018 to 11 July 2018 (Figure 9). To ensure the robustness of the estimate, seven methods are applied: standard R/S analysis, optimized R/S analysis with weighted regression, R/S on smoothed returns, R/S on absolute returns, R/S on squared returns, wavelet denoising followed by R/S, and multifractal detrended fluctuation analysis. The results are summarized in Table 1.

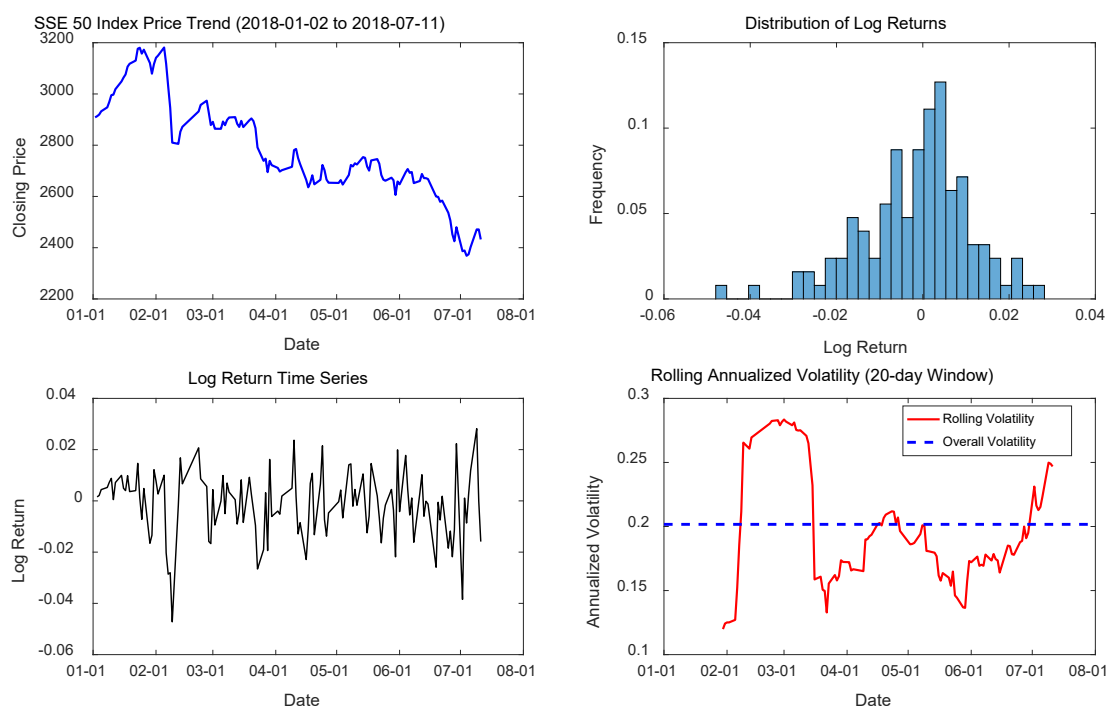


Figure 9. Determination of Hurst Exponent H and Volatility.

Table 1. Hurst exponent estimates from seven R/S analysis methods.

Method	Full Sample H	Sub-Sample 1 H	Sub-Sample 2 H	R^2 (Full)
Standard R/S	0.6697	0.8446	0.6670	0.9845
Optimized R/S	0.7543	0.8307	0.7297	0.9954
Smoothed returns	0.7882	0.9445	0.8250	0.9918
Absolute returns	0.7574	0.8059	0.7563	0.9971
Squared returns	0.7423	0.8193	0.7202	0.9979
Wavelet denoising	0.7882	0.9445	0.8250	0.9918
MF-DFA	0.7704	0.8468	0.7031	0.9320

All seven methods produce Hurst exponent estimates above 0.5, and five of them yield estimates above 0.74. The range of estimates, from 0.6697 to 0.7882, consistently exceeds the threshold $H = 0.5$ that separates long-range dependent series from random walks. The MF-DFA method, which addresses potential non-stationarity through local polynomial detrending, yields an estimate of 0.7704. To further assess the stability of these findings, the return series was evenly divided into two sub-samples of 63 trading days each, and all seven methods were reapplied. Every method yields Hurst estimates above 0.5 in both sub-samples, and all estimates exceed 0.66. The absolute returns method, which achieves the highest goodness of fit in the full sample, gives sub-sample estimates of 0.8059 and 0.7563, both satisfying the theoretical requirement $H \in (0.75, 1)$. The MF-DFA method,

whose estimation framework differs structurally from R/S analysis, produces sub-sample estimates of 0.8468 and 0.7031, remaining well above 0.5 in both windows. The consistency across these different estimation techniques provides strong evidence that the return series exhibits significant long-range dependence.

Among these methods, the absolute returns approach yields the highest coefficient of determination, with $R^2 = 0.9971$, and an estimated H of 0.7574. This value is adopted as the baseline Hurst exponent for all subsequent analyses. The choice is supported by two considerations. First, an H value of 0.7574 satisfies the theoretical condition $H \in (0.75, 1)$ required for the MFBM process to admit a risk-neutral measure, as established by Cheridito [9]. Second, the absolute returns method directly tests the long memory property of volatility, which is particularly relevant for option pricing applications where the persistence of price fluctuations matters as much as the direction of price movements.

(2) Volatility σ

Also based on the daily closing price data of the SSE 50 Index from 1 January 2018, to 11 July 2018, this paper uses standard financial econometric methods to estimate the volatility parameter. Specifically, the logarithmic return series is calculated:

$$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right),$$

where S_t represents the closing price on day t . The daily volatility is estimated by the standard deviation of the logarithmic returns:

$$\sigma_{daily} = \sqrt{\frac{1}{N-1} \sum_{t=1}^N (r_t - \bar{r})^2}.$$

The annualized volatility is calculated using the square root rule:

$$\sigma_{annual} = \sigma_{daily} \times \sqrt{252},$$

where 252 is the number of annual trading days, consistent with the convention in the Chinese A-share market.

The calculated daily volatility of the logarithmic return series during this period is 0.0127, and the annualized volatility is 0.2017, i.e., 20.17%. This value serves as the most likely value of the fuzzy volatility parameter.

(3) Confidence Level α

After measuring the Hurst exponent $H = 0.7574$ and volatility $\sigma = 20.17\%$ for the period from 1 January 2018 to 11 July 2018, it is necessary to further determine the optimal confidence level α based on the training set (26 March 2018, to 15 June 2018). The selection of this training interval is mainly based on the following considerations: In 2018, the SSE 50 Index experienced several significant consecutive declines in closing prices, causing later stock prices to be significantly lower than initial levels. Such abnormal fluctuations are not conducive to effectively estimating α . During the selected period, although stock prices fluctuated, they did not experience such extreme situations, making it more suitable for robust estimation of the confidence level.

The training process for the confidence level is as follows:

1. Data Preparation

Use the data from 26 March 2018 to 15 June 2018 as the training set. Set the initial price range to be $\pm 10\%$ of the initial price, and use the fixed Hurst exponent ($H = 0.7574$) and volatility ($\sigma = 20.17\%$).

2. Parameter Optimization

In the process of determining the optimal confidence level α , a total of 17 candidate α values are set, ranging from 0.1 to 0.9 in steps of 0.05. For each candidate α value, 100 Monte Carlo simulations are performed. During evaluation, MFBM is used as the price generation process. The frequency with which the actual price falls within the simulated price interval is counted, and the coverage rate corresponding to each α value is calculated accordingly.

3. Selection Strategy

When selecting the optimal α value, the primary goal is to choose the largest α value that achieves a target coverage rate of 80%. If none of the candidate α values reach this target, the α value with coverage closest to 80% is selected instead. The rationale for this selection strategy is that a larger α value implies a higher degree of dependence on historical data, resulting in more conservative prediction outcomes.

To further verify the robustness of the selection procedure against potential calibration bias, a rolling window cross-validation was conducted. The training set was partitioned into estimation windows of 40 trading days and validation windows of 5 trading days. For each candidate α value, Monte Carlo simulations were performed within each estimation window, and the coverage rate was evaluated on the subsequent validation window. The average coverage rate across all rolling windows was then computed for each α . The cross-validation results confirmed that the α value selected by the full-sample procedure remained stable and consistent across different window configurations, demonstrating that the selection strategy does not rely on a specific data partition. This additional validation ensures that the chosen α provides reliable out-of-sample coverage and is not subject to overfitting.

This method ensures that the selected α achieves the desired confidence level on the training set while providing a reliable parameter foundation for subsequent predictions.

Based on the above evaluation method, Table 2 presents the coverage statistics corresponding to different confidence levels α . Specifically, when the confidence level α is 0.70, the coverage rate is 83.93%; for confidence levels above this value, the coverage rate falls below the target coverage rate of 80%. Therefore, the optimal confidence level is set to 0.70. Additionally, Figure 10 illustrates the simulated stock price paths under this optimal α value.

Table 2. Coverage Rates of the Fuzzy Stock Price Interval at Different Confidence Levels α .

Confidence Level	Coverage Rate	Confidence Level	Coverage Rate
$\alpha = 0.10$	100%	$\alpha = 0.55$	94.64%
$\alpha = 0.15$	100%	$\alpha = 0.60$	91.07%
$\alpha = 0.20$	100%	$\alpha = 0.65$	89.29%
$\alpha = 0.25$	100%	$\alpha = 0.70$	83.93%
$\alpha = 0.30$	100%	$\alpha = 0.75$	64.29%
$\alpha = 0.35$	100%	$\alpha = 0.80$	42.86%
$\alpha = 0.40$	100%	$\alpha = 0.85$	26.79%
$\alpha = 0.45$	100%	$\alpha = 0.90$	19.64%
$\alpha = 0.50$	98.21%	\	\

At this point, the values for all parameters have been determined, resulting in the baseline data shown in Table 3.

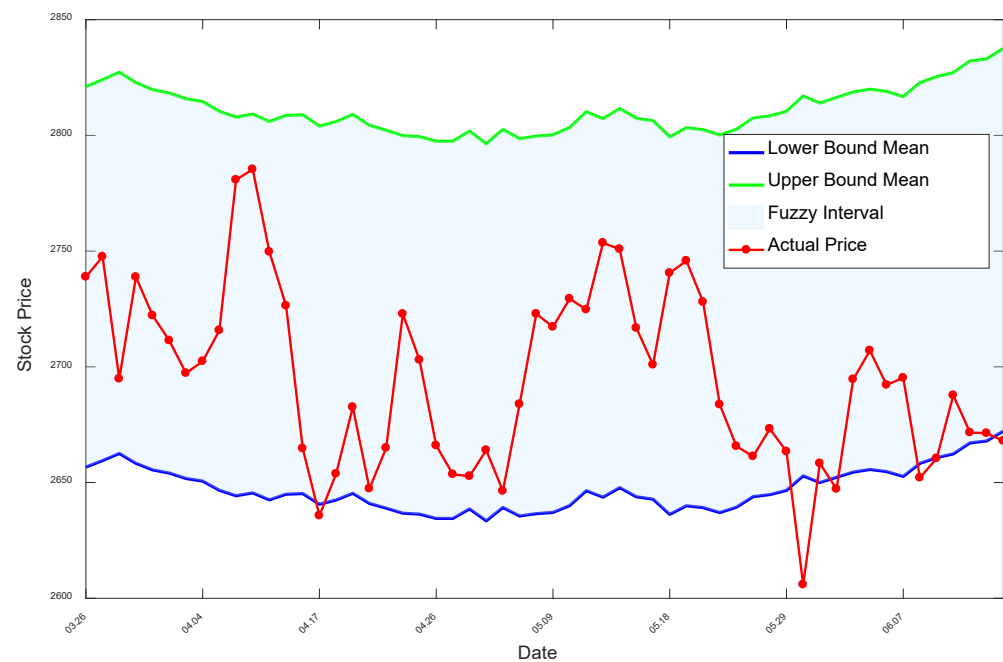


Figure 10. Stock Price Simulation with $H = 0.7574$, $\sigma = 20.17\%$, and $\alpha = 0.70$ (March 26 to June 15).

Table 3. Parameter Values for the Baseline Model.

Parameter Name	Symbol and Value	Parameter Name	Symbol and Value
Initial Stock Price (RMB)	$S_0 = 2474.62$	Volatility	$\sigma = 0.2017$
Strike Price (RMB)	$K = 2474.62$	Hurst Exponent	$H = 0.7574$
Exercise Time (Years)	$T = 50/252$	Confidence Level	$\alpha = 0.70$
Risk-Free Rate	$r = 3.2\%$	Fixed Payment (RMB)	$C = 2525.40$

4.4. Empirical Analysis and Model Comparison

Compared to standard European options, which possess liquidity during their lifespan, binary options are characterized by non-tradability and fixed pricing. This characteristic shares significant commonalities with the structured wealth management product studied in this paper: both contract terms stipulate no early redemption or transfer during the holding period, and the product pricing is locked in at the investment start date. Furthermore, from the perspective of the pricing mechanism, the return of the CONC strictly depends on whether the underlying asset price breaches the preset strike threshold at expiration.

Therefore, to more fully demonstrate the relevance of the pricing model established in this paper to actual financial markets, an empirical test is first conducted to verify whether MFBM and triangular fuzzy numbers can capture the movement characteristics of the SSE 50 Index linked to this wealth management product. The test set spans the period from 12 July 2018 to 17 September 2018, corresponding to the full observation window of the structured product. As shown in Corollary 2, the fuzzy variable is selected as the initial stock price $\tilde{S}_0$, where $\tilde{S}_0 \in [2174.62, 2774.62]$. The central value of the initial stock price is 2474.62. The most likely value of volatility is 0.2017, with the lower and upper bounds at 0.1517 and 0.2517. The remaining parameter settings are taken from the baseline data values in Table 3.

To evaluate whether the choice of price process affects the quality of fuzzy pricing intervals, this paper compares mixed fractional Brownian motion with geometric Brownian motion. Both models apply the same fuzzy interval construction procedure. Stock price and volatility are treated as triangular fuzzy numbers with identical parameter settings.

The only difference lies in the assumed price dynamics: geometric Brownian motion for one model, and mixed fractional Brownian motion with H equal to 0.7574 for the other. This design isolates the effect of the price dynamics specification on interval performance.

Figure 11 displays the fuzzy price intervals generated by the two models during the observation period. The solid red line represents the actual price of the underlying asset. The two solid lines in the middle are the average fuzzy price boundaries from 100 Monte Carlo simulations using MFBM. The remaining two dashed lines are the average boundaries from GBM under the same number of simulations.

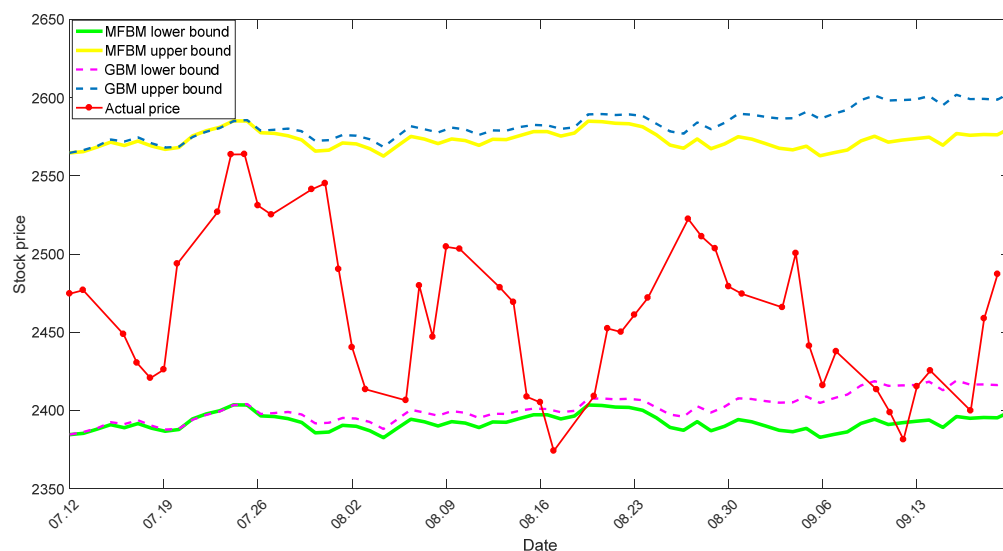


Figure 11. Empirical Analysis and Model Comparison of Stock Prices.

The comparison reveals a clear pattern. Over the observation period from 12 July 2018 to 17 September 2018, the MFBM-based interval covered 96.00 percent of actual price observations, while the GBM-based interval covered 88.00 percent. The average interval widths of the two models were nearly identical, 180.61 index points for MFBM and 181.31 points for GBM. The MFBM model thus attained substantially higher coverage without inflating the interval width. The root mean square error and mean absolute error, computed using the interval midpoint as a point prediction, were 51.54 points and 43.01 points for MFBM, compared with 59.03 points and 49.07 points for GBM. The interval score, which jointly evaluates width and coverage, was 184.97 for MFBM and 194.67 for GBM, a difference of 9.70 in favor of MFBM. Taken together, these five metrics indicate that the MFBM model tracks the actual price path more closely and constructs prediction intervals that are simultaneously narrower and more inclusive than those produced by the standard GBM framework.

Since the wealth management product studied in this paper is not a standard binary option, its actual premium data cannot be obtained. Furthermore, binary options are not tradable during the observation period, thus only providing a final realized price, unlike European options, which have continuous daily price series. Based on these limitations, this paper will adopt numerical simulations and model comparisons to test the superiority of the constructed model.

Figure 12 presents the results of the price dynamics simulation analysis for the wealth management product with a two-tiered payoff structure (Product Code: 116619) from 12 July 2018 to 17 September 2018. The following analysis and explanation are provided for each curve in the figure:

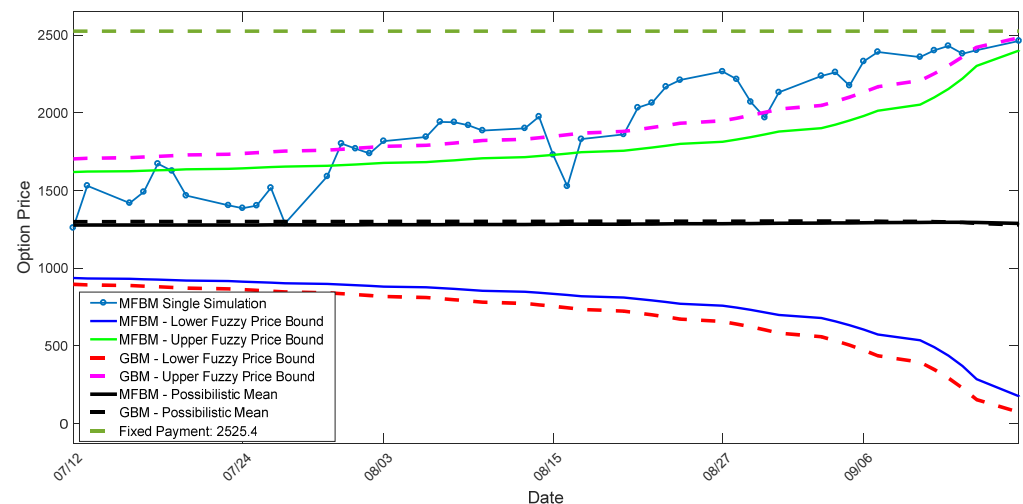


Figure 12. Fuzzy Price Simulation and Model Comparison for the CONC.

(1) The curve labeled “MFBM Single Simulation” in the figure represents the price curve of the binary option formed when fuzziness is ignored and only stock price changes are assumed to follow MFBM, with the Hurst exponent $H = 0.7574$. The path shape exhibits typical long-range dependence characteristics, i.e., reduced volatility clustering and enhanced persistence in sustained growth or decline trends.

(2) The curves labeled “MFBM Fuzzy Price Upper/Lower Bounds” in the figure represent the upper and lower bounds of the option price corresponding to the model established in Theorem 6 of this paper. That is, stock price changes follow MFBM, while triangular fuzzy numbers are used to capture the fuzziness of investment behavior (where $\tilde{S}_0 \in [2174.62, 2774.62]$, $\tilde{\sigma} \in [0.15, 0.25]$). These upper and lower bounds are generated by taking the average after 500 Monte Carlo simulations, reflecting the price boundaries under parameter uncertainty. It is noted that, for a fixed time to maturity, as time progresses toward expiration, the upper bound of the option price gradually increases towards the fixed payment, while the lower bound gradually decreases to 0, causing the fuzzy pricing interval of the option to widen progressively. This indicates that a longer time horizon leads to greater uncertainty and stronger path dependence, meaning that long-range dependence causes historical volatility to have an accumulating impact on future paths, resulting in an increasingly larger difference between the upper and lower price bounds.

(3) Additionally, the two dashed lines located outside the MFBM price interval in Figure 12 represent the upper and lower bounds of the option’s fuzzy price under the classic Black–Scholes (B-S) model. The comparison results show that under identical parameter settings, the model that incorporates long-range dependence and leptokurtic fat-tail characteristics determines fuzzy price intervals with higher precision. The two curves located more centrally and closer together represent the possibilistic mean of the option’s fuzzy price calculated according to Theorem 5, reflecting that the difference in mean predictions between the two models is not significant. However, judging from the possibilistic lower variance of the fuzzy price provided in Figure 13, the lower variance corresponding to MFBM is smaller, indicating that its predictions exhibit lower volatility and higher stability.

Based on the analysis of Figures 12 and 13, although the target wealth management product is not a standard binary call option (CONC) in terms of contract form, its binary payoff structure is similar to that of a binary option. Therefore, the model constructed in this paper remains applicable for the simulation analysis of this product. The results indicate that compared to the more mature and classic B-S model, the fuzzy pricing model based on MFBM demonstrates higher stability and relative accuracy in price prediction. The earlier

measurements of stock price behavior further support the necessity of using MFBM to capture price changes. Furthermore, the research model in this study also holds certain reference and heuristic value for the pricing and analysis of other wealth management products with similar two-tiered payoff structures.

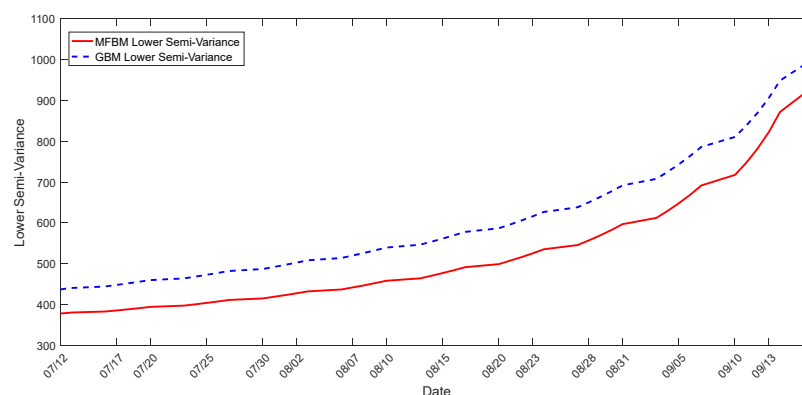


Figure 13. Comparison of Possibilistic Lower Variance of Fuzzy Option Prices.

4.5. Sensitivity Analysis

To examine the sensitivity of the model to changes in key inputs, this section evaluates how the fuzzy option price interval responds to variations in the initial stock price, risk-free rate, volatility, strike price, exercise time, and confidence level. The analysis is conducted by varying one parameter at a time while holding the others at their baseline values.

Figure 14 depicts the impact of the initial price S_0 on the upper and lower bounds of the fuzzy price of the binary option. An increase in the initial stock price will significantly raise the option price. Simultaneously, the growth rate of the upper bound price is relatively faster, while the growth rate of the lower bound price slows down, which also verifies the asymmetric amplification effect of the fuzzy set on the initial state of the stock price.

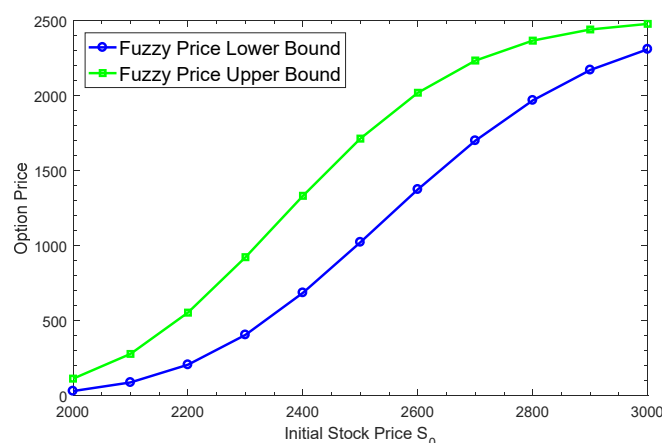


Figure 14. Impact of Initial Stock Price S_0 on Option Price.

Figure 15 indicates that an increase in the risk-free rate r will cause the upper and lower bound prices of the option to decrease. From the pricing formula for the CONC in Theorem 2, an increase in r will lead to a decrease in the discount factor $e^{-r(T-t)}$ and an increase in the probability term $N(d_2)$. However, the decay rate of the discount factor is slightly faster than the growth rate of the probability, thus resulting in an overall decrease in the option price.

Figure 16 illustrates the impact of volatility on the upper and lower bound prices of the CONC. For a standard European call option, an increase in volatility increases both

the possibility of substantial gains and the risk of substantial losses. However, due to the asymmetric payoff, this ultimately leads to an increase in the overall price. In contrast, since the payoff of the CONC is a fixed value, an increase in volatility only brings a negative impact, i.e., no upside gain and full downside loss. Consequently, when volatility increases, the probability term $N(d_2)$ decreases, and the option price also decreases accordingly.

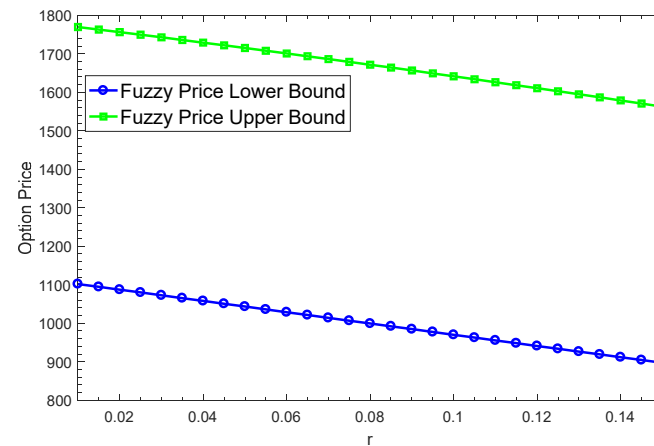


Figure 15. Impact of Risk-Free Rate r on Option Price.

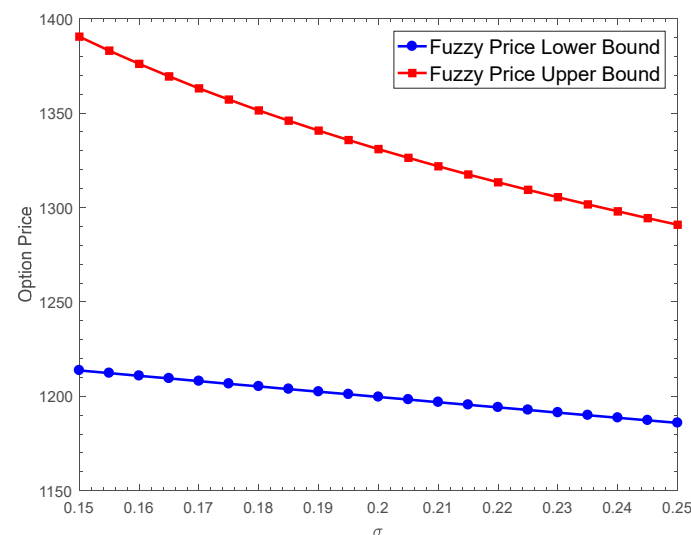


Figure 16. Impact of Volatility σ on Option Price.

Figure 17 reveals the negative correlation between the strike price and the binary option price. The upper and lower bounds of the option price will significantly decrease as the strike price K increases, eventually falling to zero. Under the risk-neutral measure, $N(d_2)$ represents the probability that the stock price exceeds K . Therefore, when only K increases, the stock price needs to reach a higher level to trigger the fixed payment C , reflecting the “strike threshold effect” in option pricing.

Figure 18 examines the comparative static effect of the exercise time parameter T . As T increases, the upper price bound decreases while the lower price bound increases, meaning the fuzzy price interval of the option narrows. This narrowing of the fuzzy interval reflects the filtration of extreme scenarios over time, leading the option pricing to tend towards certainty. It should be stressed that the effect shown here is a cross-sectional comparison across different values of T , and it differs fundamentally from the progressive widening of the interval over the observation horizon under a fixed maturity, as illustrated in Figure 13. The two time-related effects operate through distinct mechanisms and should not be conflated.

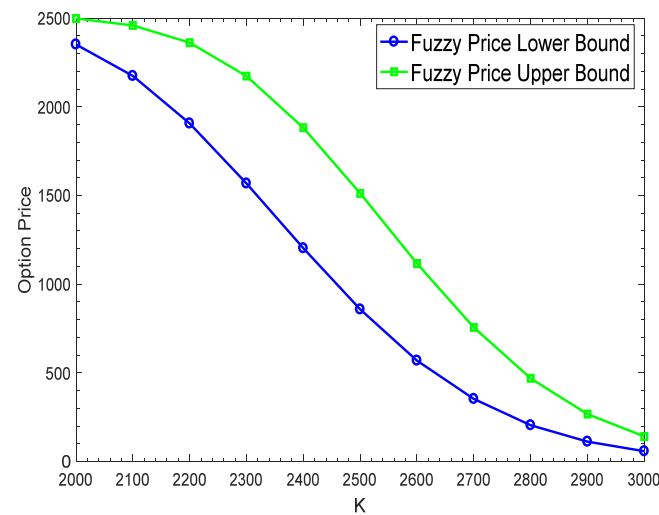


Figure 17. Impact of Strike Price K on Option Price.

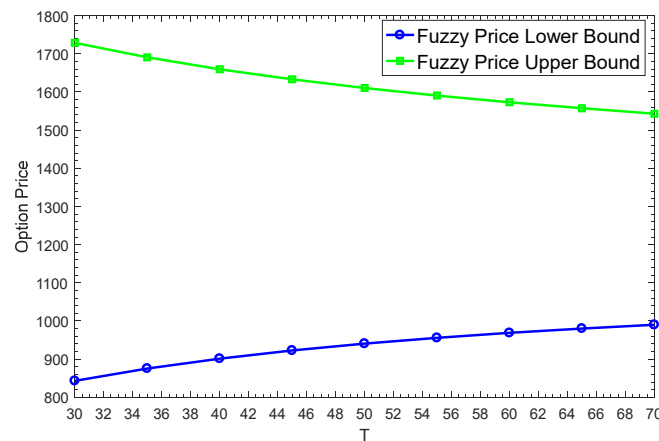


Figure 18. Impact of Exercise Time T on Option Price.

Figure 19 depicts the impact of the confidence level α on the upper and lower bounds of the option price. When $\alpha \rightarrow 0$, the fuzzy set is at its widest, and the difference between the price bounds is significantly amplified. When $\alpha \rightarrow 1$, the fuzzy variables $\tilde{S}_t$ and $\tilde{\sigma}$ degenerate into deterministic constants, and the fuzziness disappears, verifying that the classical pricing model is a special case of the fuzzy pricing model. Additionally, investors can select an appropriate confidence level based on their own circumstances and achieve a dynamic trade-off between risk and return by adjusting the confidence level α .

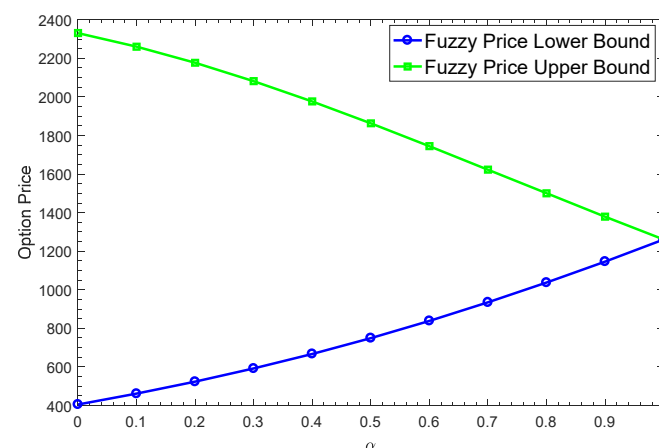


Figure 19. Impact of Confidence Level α on Option Price.

5. Discussion

Binary options are a distinct class of financial derivatives whose payoff depends entirely on whether the underlying asset price breaches a preset threshold at expiration. This all-or-nothing structure provides investors with a straightforward risk management tool. This paper integrates mixed fractional Brownian motion with triangular fuzzy number theory to establish a pricing framework that simultaneously addresses long-range dependence and cognitive fuzziness in binary option valuation.

On the theoretical side, by selecting the underlying stock price and volatility as triangular fuzzy parameters and building on existing results, fuzzy pricing formulas for binary options under MFBM are derived. The monotonic relationship between the fuzzy option price and the fuzzy variables is demonstrated rigorously for the cash-or-nothing call, which yields explicit price interval formulas through α -cuts. Unlike a single-point price estimate, the price cutset offers a flexible tool: investors can determine an option price interval by adjusting the confidence level according to their individual needs and risk preferences. The role of long-range dependence is further explored. Both theoretical analysis and numerical experiments verify that the influence of the Hurst exponent H on the option price is regulated by the exercise time T . A larger H raises both the upper and lower price bounds when the exercise horizon is short, but the direction reverses when the horizon lengthens. This finding indicates that the effect of long-range dependence on option pricing cannot be reduced to a single direction; the time to exercise serves as a key moderating variable.

On the empirical side, statistical tests are first conducted on the SSE 50 Index price linked to the China Merchants Bank Focus Link product to determine whether it exhibits long-range dependence and leptokurtic fat tails. The results confirm both characteristics, justifying the use of MFBM to describe stock price movements. Building on this, the model parameters are estimated objectively to facilitate empirical testing during the observation period. The comparison of stock price coverage between the two models shows a clear pattern. The MFBM-based fuzzy interval covers 96.00 percent of actual price observations, while the GBM-based interval covers 88.00 percent, even though the average widths of the two intervals are nearly identical. The root mean square error and mean absolute error of the interval midpoint are also lower for MFBM, and the interval score, which jointly evaluates width and coverage, favors MFBM by a clear margin. These results indicate that the MFBM-based model tracks the actual price path more faithfully without inflating the interval width. Simulation results for the option price further reveal that the possibilistic means of the fuzzy option price are broadly consistent between the MFBM and Black–Scholes models. However, the possibilistic lower variance of the MFBM-based model is smaller, demonstrating that under identical parameter settings, incorporating long-range dependence yields more stable pricing outputs. The sensitivity analysis investigates the impacts of the risk-free rate, volatility, strike price, exercise time, and confidence level on the option price interval, and the directions of change are consistent with economic reasoning.

Although this paper focuses on binary options, the modeling approach for the long memory property of stock prices, the objective parameter determination process, and the application framework of the fuzzy pricing model also provide a feasible analytical tool for pricing research on existing structured derivatives with layered payoffs. Extending the framework to fractional jump-diffusion processes may further accommodate unexpected market events that trigger significant price anomalies.

6. Limitations and Future Research

The empirical analysis in this study is conducted on a single structured product over a specific observation window. This limitation is partly due to the availability of suitable data. Although structured wealth management products with binary payoff features exist in the market, those with fully disclosed contract terms and continuous, publicly accessible price records are uncommon. Consecutive product issuances that would permit rolling out-of-sample validation are even rarer. It should be noted, however, that the valuation of binary options is fundamentally a study of the underlying asset rather than the product itself. The empirical results have demonstrated the advantages of the MFBM-based fuzzy pricing model in capturing the behavior of the underlying price, which to a meaningful extent compensates for the difficulty in obtaining detailed product-level data. Extending the empirical application to a broader set of products and longer time spans that include different market regimes remains a natural direction for assessing the generalizability of the findings.

On the technical side, the current framework treats the Hurst exponent as constant over the pricing horizon. Long-range dependence structures may evolve over time due to shifts in market conditions, and incorporating a time-varying Hurst parameter would be a useful refinement. The triangular fuzzy number, chosen for its simplicity and interpretability, imposes symmetric membership, which may not fully capture market conditions where downside uncertainty is perceived as disproportionately large. Asymmetric fuzzy specifications could be explored in such cases. Furthermore, extending the model to accommodate dynamic fuzzy intervals whose spreads adapt to evolving market conditions, such as volatility clustering, may broaden the practical applicability of the framework while preserving analytical tractability. The simulation of mixed fractional Brownian motion paths involves greater computational effort than standard Brownian motion, which may deserve attention for applications requiring real-time pricing or large-scale portfolio analysis. These directions represent natural extensions of the current work and are left for future investigation.

Author Contributions: Writing—original draft preparation, T.Y.; software, T.Y.; conceptualization, Z.Y.; methodology, Z.Y.; writing—review and editing, X.L. and Z.H.; visualization, X.L.; supervision, Z.Y. and Z.H.; project administration, T.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Social Science Foundation of China: solution and application of fixed-effect semiparametric variable coefficient panel quantile regression model (21BTJ043) and the Natural Science Foundation of Inner Mongolia (General Program): (2025MS01025).

Data Availability Statement: Empirical datasets—including historical prices, volatility of the SSE 50 Index, and specifications of the structured financial product—were sourced from authenticated Choice Financial Terminal. Numerical experiments employing the ceteris paribus approach were conducted using parameters fully documented in Section 4.5 of this manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GBM	geometric Brownian motion
FBM	fractional Brownian motion
MFBM	mixed fractional Brownian motion
TFNs	triangular fuzzy numbers
B-S	Black–Scholes

AONC asset-or-nothing call
CONC cash-or-nothing call

Appendix A

Appendix A.1. Proof of Theorem 2

Proof. Under the risk-neutral measure, we have:

$$\begin{aligned}
 V_C(t, S) &= \mathbb{E}_Q \left\{ Ce^{-r(T-t)} I_{S(T) > K} | \mathcal{F}_t \right\} \\
 &= Ce^{-r(T-t)} \mathbb{E}_Q \left\{ I_{\ln S(T) > \ln K} | \mathcal{F}_t \right\} \\
 &= Ce^{-r(T-t)} \mathbb{E}_Q \left\{ I_{M_t^H > \ln K/s_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}} | \mathcal{F}_t \right\} \\
 &= Ce^{-r(T-t)} \frac{1}{\sigma \sqrt{2\pi[(T-t) + (T^{2H} - t^{2H})]}} \int_{\ln K/s_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}^{+\infty} \exp \left\{ \frac{(x - M_t^H)^2}{2\sigma^2[(T-t) + (T^{2H} - t^{2H})]} \right\} dx \\
 &= Ce^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{M_t^H - (\ln K/s_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H})}{\sigma \sqrt{(T-t) + (T^{2H} - t^{2H})}}} e^{-\frac{y^2}{2}} dy \\
 &= Ce^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}} dy \\
 &= Ce^{-r(T-t)} N(d_2)
 \end{aligned}$$

where d_2 is as shown in Theorem 2. $\square$

Appendix A.2. Proof of Theorem 3

Proof. By definition, the α -cuts of the fuzzy variables $\tilde{S}_t$ and $\tilde{\sigma}$ are $(\tilde{S}_t)_\alpha = [(\tilde{S}_t)_\alpha^L, (\tilde{S}_t)_\alpha^U]$ and $\tilde{\sigma}_\alpha = [\tilde{\sigma}_\alpha^L, \tilde{\sigma}_\alpha^U]$, respectively. Then, the α -cut of $\tilde{V}_A(t, \tilde{S})$ can be expressed as:

$$(\tilde{V}_A(t, \tilde{S}))_\alpha = [(\tilde{V}_A(t, \tilde{S}))_\alpha^L, (\tilde{V}_A(t, \tilde{S}))_\alpha^U].$$

where:

$$(\tilde{V}_A(t, \tilde{S}))_\alpha^L = \min_{(\tilde{S}_t)_\alpha^L \leq S \leq (\tilde{S}_t)_\alpha^U, \tilde{\sigma}_\alpha^L \leq \sigma \leq \tilde{\sigma}_\alpha^U} V_A(t, S), (\tilde{V}_A(t, \tilde{S}))_\alpha^U = \max_{(\tilde{S}_t)_\alpha^L \leq S \leq (\tilde{S}_t)_\alpha^U, \tilde{\sigma}_\alpha^L \leq \sigma \leq \tilde{\sigma}_\alpha^U} V_A(t, S).$$

From Theorem 1, we have:

$$\begin{aligned}
 \frac{\partial V_A(t, S)}{\partial S_t} &= N(d_1) + \frac{\partial N(d_1)}{\partial d_1} \frac{1}{\sigma \sqrt{(T-t) + (T^{2H} - t^{2H})}} \geq 0; \\
 \frac{\partial V_A(t, S)}{\partial \sigma} &= S_t \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial \sigma} \\
 &= S_t \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\sqrt{(T-t) + (T^{2H} - t^{2H})}}{\sigma^2[(T-t) + (T^{2H} - t^{2H})]} \\
 &\quad [-\ln S_t/K - r(T-t) + \frac{1}{2}\sigma^2(T-t) + \frac{1}{2}\sigma^2(T^{2H} - t^{2H})] \\
 &= -S_t \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{d_2}{\sigma}.
 \end{aligned}$$

Here, for a binary call option, it is commonly assumed that the strike price is not lower than the stock price, i.e., $K \geq S_t$. Under a sufficiently low interest rate (for instance, when the risk-free rate satisfies $r \leq \frac{\sigma^2}{2}$), one can conclude that $d_2 \leq 0$, then $\frac{\partial V_A(t, S)}{\partial \sigma} \geq 0$. It should be noted that this condition serves merely as a relatively crude sufficient condition for guaranteeing $d_2 \leq 0$. Therefore, $V_A(t, S)$ is a monotonically non-decreasing function with respect to S_t and σ , and the theorem is proved. $\square$

Appendix A.3. Proof of Theorem 4

Proof. The α -cut of $\tilde{V}_C(t, \tilde{S})$ can be expressed as

$$(\tilde{V}_C(t, \tilde{S}))_\alpha = [(\tilde{V}_C(t, \tilde{S}))_\alpha^L, (\tilde{V}_C(t, \tilde{S}))_\alpha^U],$$

where:

$$(\tilde{V}_C(t, \tilde{S}))_\alpha^L = \min_{(\tilde{S}_t)_\alpha^L \leq S \leq (\tilde{S}_t)_\alpha^U, \tilde{\sigma}_\alpha^L \leq \sigma \leq \tilde{\sigma}_\alpha^U} V_C(t, S), (\tilde{V}_C(t, \tilde{S}))_\alpha^U = \max_{(\tilde{S}_t)_\alpha^L \leq S \leq (\tilde{S}_t)_\alpha^U, \tilde{\sigma}_\alpha^L \leq \sigma \leq \tilde{\sigma}_\alpha^U} V_C(t, S).$$

(1) Determination of monotonicity with respect to the variable S_t :

$$\begin{aligned} \frac{\partial V_C(t, S)}{\partial S_t} &= Ce^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S_t} \\ &= Ce^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{1}{S_t \sigma \sqrt{(T-t) + (T^{2H} - t^{2H})}} \geq 0 \end{aligned}$$

thus $V_C(t, S)$ is monotonically increasing with respect to the variable S_t .

(2) Determination of monotonicity with respect to the fuzzy variable σ :

$$\begin{aligned} \frac{\partial V_C(t, S)}{\partial \sigma} &= Ce^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial \sigma} \\ &= Ce^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \cdot \frac{\sqrt{(T-t) + (T^{2H} - t^{2H})}}{\sigma^2 [(T-t) + (T^{2H} - t^{2H})]} \\ &\quad \left[-\ln S_t / K - r(T-t) - \frac{1}{2} \sigma^2 (T-t) - \frac{1}{2} \sigma^2 (T^{2H} - t^{2H}) \right] \\ &= -Ce^{-r(T-t)} \frac{\partial N(d_2)}{\partial d_2} \frac{d_1}{\sigma} \leq 0 \end{aligned}$$

i.e., $V_C(t, S)$ is monotonically decreasing with respect to σ . Substituting the fuzzy variables into $\tilde{V}_C(t, \tilde{S})$ and $\tilde{d}_2$ leads to the conclusion. $\square$

Appendix A.4. Proof of Theorem 7

Proof.

$$\begin{aligned} \frac{\partial \tilde{V}_A(t, \tilde{S})}{\partial H} &= \tilde{S}_t \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \frac{\partial \tilde{d}_1}{\partial H} \\ &= \tilde{S}_t \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \cdot \\ &\quad (T^{2H} \ln T - t^{2H} \ln t) \frac{\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})} - \frac{\ln \tilde{S}_t + r(T-t) + \frac{1}{2} \tilde{\sigma}^2 [(T-t) + (T^{2H} - t^{2H})]}{\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})}}}{(T-t) + (T^{2H} - t^{2H})} \\ &= \tilde{S}_t \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} (T^{2H} \ln T - t^{2H} \ln t) \frac{\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})} - \tilde{d}_1}{(T-t) + (T^{2H} - t^{2H})} \\ &= -\tilde{S}_t \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \frac{\tilde{d}_2 (T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})} \\ &\quad \frac{\partial \tilde{V}_C(t, \tilde{S})}{\partial H} \\ &= Ce^{-r(T-t)} \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \frac{\partial \tilde{d}_1}{\partial H} \\ &= Ce^{-r(T-t)} \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} \cdot \\ &\quad (T^{2H} \ln T - t^{2H} \ln t) \frac{-\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})} - \frac{\ln \tilde{S}_t + r(T-t) + \frac{1}{2} \tilde{\sigma}^2 [(T-t) + (T^{2H} - t^{2H})]}{\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})}}}{(T-t) + (T^{2H} - t^{2H})} \\ &= Ce^{-r(T-t)} \frac{\partial N(\tilde{d}_1)}{\partial \tilde{d}_1} (T^{2H} \ln T - t^{2H} \ln t) \frac{-\tilde{\sigma} \sqrt{(T-t) + (T^{2H} - t^{2H})} - \tilde{d}_2}{(T-t) + (T^{2H} - t^{2H})} \\ &= -Ce^{-r(T-t)} \frac{\partial N(\tilde{d}_2)}{\partial \tilde{d}_2} \frac{\tilde{d}_1 (T^{2H} \ln T - t^{2H} \ln t)}{(T-t) + (T^{2H} - t^{2H})} \end{aligned}$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are as shown in Corollary 1. $\square$

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