

Article

Exact Pollard Solutions for Geophysical Internal Waves: Incorporating Centripetal Forces and Underlying Currents

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Abstract

This paper focuses on geophysical internal waves with centripetal forces in the f -plane approximation, presents Pollard's solution, and constructs the exact solution of the system in different layers. On this basis, this paper further develops the structure of Pollard's solutions that incorporate current terms in the horizontal, meridional, and vertical directions. Findings show that away from the equatorial regions, neither the transverse meridional current nor the vertical current exists, while at the equator, the vertical current vanishes along with the Coriolis parameter f , and the meridional current term is constant.

Keywords: f -plane approximation; centripetal forces; exact solution; meridional current; vertical current

MSC: 34B05; 34C25

1. Introduction

In the field of geophysical fluid dynamics, the study of internal waves in the ocean is of great significance for understanding ocean circulation, material transport, and energy transfer. This paper puts forward an exact, nonlinear, three-dimensional Pollard-like internal wave solution to the governing equations of geophysical flows. The constructed solution models ocean internal waves in the presence of a non-uniform zonal current and is assumed to include meridional and vertical currents. Compared with the ocean model recently proposed in Reference [1–6], which adopted the simplified assumption that particles immediately below the thermocline have no motion, the model in this paper is more refined by integrating a series of transitional layers beneath the thermocline [7].

The exploration of explicit solutions for geophysical ocean flows has a long history. In 1802, Gerstner presented a two-dimensional explicit solution [8]. Later, in 1970, Pollard extended this solution by incorporating the effects of Earth's rotation, thereby constructing a nonlinear, three-dimensional solution to the governing equations of geophysical ocean flows [9]. Derived using the f -plane approximation, this solution is valid for arbitrary but approximately fixed latitudes, excluding the poles. In recent years, a number of papers have applied such explicit solutions to model various ocean flows near the equator [10–18], as well as geophysical internal wave solutions [5,19,20] and azimuthally propagating flows [21–27].

In oceanography, centripetal forces are typically disregarded due to their much smaller magnitude ($\sim O(\Omega^2)$) compared to Coriolis terms ($\sim O(\Omega)$). However, centripetal forces



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can play an important role in specific wave dynamics. For example, Henry et al. found that centripetal forces are crucial for ensuring that the exact solution they present can admit all physically plausible ranges of underlying currents [14]. This phenomenon can also be observed in other geophysical fluid dynamical contexts, such as Rossby waves [28]. To our best knowledge, there are few studies on Pollard's solution for geophysical internal waves with centripetal forces.

An important aspect of Pollard's solutions is that the influence of underlying currents cannot be overlooked, and their role is so significant that incorporating them into the solutions is of substantial significance [29]. In Reference [30], the authors presented Pollard's solutions for geophysical trapped waves with a depth-invariant zonal current under the f -plane approximation. From a geophysical perspective, given the crucial role of underlying currents, it is natural to explore the possibility of integrating meridional and vertical currents into Pollard's solutions. Although the vertical scale of ocean currents is much smaller than their horizontal counterpart, this does not mean that vertical currents are absent in the ocean. As noted in References [31,32], Ekman transport and Ekman pumping phenomena serve as sources of meridional and vertical currents in mid-latitude oceans. More recently, Reference [33] derived Pollard's exact solution for geophysical trapped waves involving meridional and vertical currents, and the authors constructed the exact solution of nonhydrostatic internal waves in the presence of underlying horizontal, meridional, and vertical currents at mid-latitude [29]. Reference [34] formulated the governing equations for internal geophysical flows under the influence of centripetal forces at any latitude in the f -plane approximation and conducts a detailed analysis of the model. However, it is worth emphasizing that there remains a notable scarcity of research findings regarding Pollard's solutions for geophysical internal waves with underlying currents. In this paper, Pollard's exact solutions are presented in the presence of underlying currents, extending Pollard's solution to admit underlying horizontal, meridional, and vertical currents. Unlike Reference [29], this paper incorporates the effects of centripetal forces, resulting in a more complex model. Specifically, in this paper, we present Pollard's solution for geophysical internal waves with centripetal forces in the f -plane approximation and construct the exact solution of the system in the presence of underlying currents in different layers. Against this backdrop, the present paper develops the structure of Pollard's solutions that incorporate current terms in the horizontal, meridional, and vertical directions. It should be noted, though, that away from the equatorial regions, neither the transverse meridional current nor the vertical current exists. At the equator, the vertical current vanishes along with the Coriolis parameter f , while the meridional current term is constant.

The rest of this paper is organized as follows. In Section 2, we present the governing equations for internal geophysical flows in the presence of centripetal forces and analyze the model. In Section 3, the Pollard exact solution for nonhydrostatic geophysical internal waves traversing diverse layers is expounded. Subsequently, in the $M(t)$ layer, we discuss in detail Pollard's exact solution with underlying current terms, and consider the existence of the transverse meridional current and vertical current under two scenarios: equatorial regions and non-equatorial regions.

2. The Governing Equations

We undertake the study of three-dimensional geophysical internal waves at any latitude. The Earth is considered a perfect sphere with radius R , rotating at a constant angular velocity Ω . A rotating framework serves as the natural analytical setting, where the zonal coordinate x is eastward, the meridional coordinate y is northward, and the coordinate z points upward perpendicular to the local Earth's surface. Under the f -plane

approximation, the Euler equations for geophysical internal water waves with centripetal forces are expressed as

$$\begin{cases} u_t + uu_x + vv_y + ww_z + \hat{f}w - fv = -\frac{P_x}{\rho}, \\ v_t + uv_x + vv_y + ww_z + fu + \frac{f\hat{f}}{4}R = -\frac{P_y}{\rho}, \\ w_t + uw_x + vw_y + ww_z - \hat{f}u - \frac{\hat{f}^2}{4}R = -\frac{P_z}{\rho} - g. \end{cases} \quad (1)$$

where (u, v, w) is the fluid velocity, u_x represents the partial derivative of velocity component u with respect to spatial coordinate x , and all other partial derivative symbols adopt this identical notational convention. P is the pressure, $g = 9.8 \text{ m/s}^2$ is the gravitational acceleration at the surface of the Earth, ρ is the constant density of fluid, $R = 6378 \text{ km}$ is the radius of the Earth, $f = 2\Omega \sin \varphi$ and $\hat{f} = 2\Omega \cos \varphi$, t stands for time, $\Omega = 7.3 \times 10^{-5} \text{ rad s}^{-1}$ is the rotational speed of the Earth, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the latitude, and rotational speed of the Earth also may vary (it is known not to be constant).

Remark 1. Acceleration due to gravity g may vary depending on the surface of the Earth (not less than 0.5 or even 1%) due to tidal effects and the variable density of the Earth deep towards its center in various places on its surface. Even such negligible variability may influence the stability of obtained exact results presented by authors (depending on boundary or initial conditions).

Additionally, the requirement that the fluid be incompressible is expressed by

$$u_x + v_y + w_z = 0, \quad (2)$$

and the mass conservation in the flow takes the form

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \quad (3)$$

In this paper, we assume ρ is constant, so (3) holds trivially.

We expand the hydrostatic model depicted in reference [4] through a succession of transitional layers. These transitional layers signify a transition from a particular region. The geophysical internal geometric sketch of this model for that region is illustrated in Figure 1, and it can be elaborated as follows.

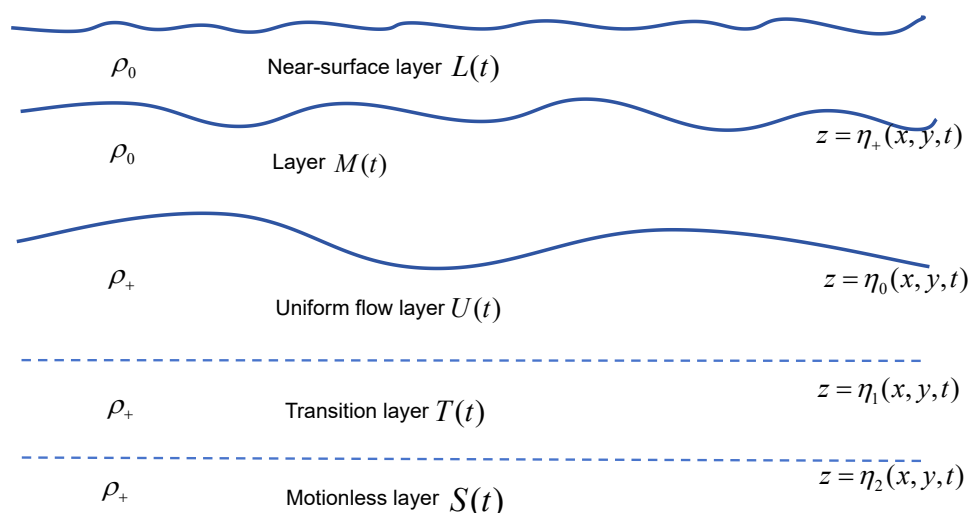


Figure 1. The nonhydrostatic model and its flow regions at a fixed latitude.

Region $L(t)$ corresponds to the upper near-surface oceanic domain where wind forcing plays a dominant role. It is postulated that ocean waves within this region constitute negligible perturbations in the context of ocean dynamics; namely, the motion of geophysical internal waves does not govern the flow characteristics of fluids in $L(t)$. Surface waves in the near-surface layer $L(t)$ can be overlooked, as they primarily represent minor free-surface disturbances induced by wind stress, with geophysical influences being negligible. The thermocline's oscillatory behavior instigates fluid motion in Region $M(t)$, which has a density ρ_0 . This layer is bounded by the thermocline at $z = \eta_0(x, y, t)$ and its upper boundary at $z = \eta_+(x, y, t)$. Geophysical internal water waves manifest as thermocline oscillations propagating above this interface. The thermocline is predominantly observed between 75° N and 75° S, as subpolar regions exhibit isothermal conditions that preclude thermocline formation. Our theoretical solution shows that the amplitude of geophysical internal waves decays exponentially above the thermocline: at a height of half a wavelength, the amplitude diminishes to less than 4% of its value at the thermocline. For modeling consistency, $M(t)$ is reasonably treated as a finite layer with well-defined boundaries. Below the thermocline, the uniform flow layer $U(t)$ with density ρ_+ (where $\rho_+ > \rho_0$) contains a homogeneous horizontal current aligned with the wave propagation direction. Beneath $U(t)$ lies the transitional layer $T(t)$, where horizontal fluid velocity decreases linearly with depth, bounded by $z = \eta_1(x, y, t)$ and $z = \eta_2(x, y, t)$. The motionless abyssal deep-water layer $S(t)$, with density ρ_+ , underlies $T(t)$, featuring stagnant water conditions; we consider the depth of geophysical internal waves as infinite; accordingly, layer $S(t)$ has no lower boundary, and this layer is usually referred to as the motionless layer, where the velocity field $u = v = w = 0$ [29].

The relevant kinematic condition is

$$w = \frac{\partial \eta_i}{\partial t} + u \frac{\partial \eta_i}{\partial x} + v \frac{\partial \eta_i}{\partial y}, \text{ on } z = \eta_i(x, y, t), \text{ for } i = 0, 1, 2$$

and the pressure is continuous across the interface $z = \eta_i(x, y, t)$.

3. Exact Solution

In this section, we shall derive the exact solutions to the governing equations. This endeavor presents a formidable challenge, in that both the normal components of the velocity field and the pressure must remain continuous across every interface. At the outset, our objective is to seek out an appropriate velocity field that satisfies the relevant kinematic conditions within each layer. Subsequently, we shall determine the continuity of pressure at the interfaces to ensure compliance with the governing equations.

3.1. The Motionless Layer $S(t)$

We defined $z = \eta_2(x, y, t) = -d_2 + \tilde{f}y$, where d_2 is a constant and $d_2 > 0$, $\tilde{f} = \frac{f}{f} = \varphi$. We require the flow to be uniform, with $(u, v, w) = (0, 0, 0)$, and then

$$P = p_0 - \rho_1 g z + \rho_1 \frac{\tilde{f}^2 R z}{4} - \frac{f \tilde{f} R y}{4}, \text{ on } z \leq -d_2 + \tilde{f}y,$$

where p_0 is an arbitrary constant.

3.2. The Transitional Layer $T(t)$

The transitional layer $T(t)$ is enclosed by the lower interface $z = \eta_2(x, y, t) = -d_2 + \tilde{f}y$ and the upper interface $z = \eta_1(x, y, t) = -d_1 + \tilde{f}y$, where $d_2 > d_1 > 0$. We demand that the horizontal component of the particle velocity u increases in a linear manner,

starting from $u = 0$ when $z = \eta_2(x, y, t) = -d_2 + \tilde{f}y$ and reaching $u = c - c_0$ when $z = \eta_1(x, y, t) = -d_1 + \tilde{f}y$, that is

$$u(y, z) = \frac{c - c_0}{d_2 - d_1} (z + d_2 - \tilde{f}y),$$

with the meridional and vertical velocities

$$v = w = 0.$$

Then

$$P = p_0 + \rho_1 \left[\frac{\hat{f}}{2} \frac{c - c_0}{d_2 - d_1} (z + d_2 - \tilde{f}y)^2 - \frac{f\hat{f}Ry}{4} + \frac{\hat{f}^2}{4} Rz - gz \right], \text{ on } -d_2 + \tilde{f}y \leq z \leq -d_1 + \tilde{f}y.$$

3.3. The Uniform Flow Layer $U(t)$

In the region $\eta_1(x, y, t) \leq z \leq \eta_0(x, y, t)$, we define the flow to be uniform with $u = c - c_0 > 0$ and $v = w = 0$, and then

$$p = p_0 + \rho_0 \hat{f} (c - c_0) (z - \tilde{f}y) - \rho_0 gz - \rho_0 \frac{f\hat{f}}{4} Ry + \rho_0 \frac{\hat{f}^2}{4} Rz + \rho_0 \frac{\hat{f}}{2} (c - c_0) (d_1 + d_2), \text{ on } \eta_1(x, y, t) \leq z \leq \eta_0(x, y, t).$$

3.4. The Layer $M(t)$

An important aspect of Pollard's solutions is that they should include the effects of the underlying currents. Here, we present an exact solution to the governing equations in the layer $M(t)$. For the explicit description of the flow, it is convenient to use the Lagrangian framework. The Lagrangian position (x, y, z) of a fluid particle is given as functions of the labeling variable (q, r, s) , time t and real parameters a, b, c, d, m, k . Thus we consider

$$\begin{cases} x = q - c_0 t - b e^{-ms} \sin(k(q - ct)), \\ y = r - d e^{-ms} \cos(k(q - ct)) + tV(q, r, s), \\ z = -d_0 + s - a e^{-ms} \cos(k(q - ct)) + tW(q, r, s), \end{cases} \quad (4)$$

where the c_0 term denotes a constant base current such that for $cc_0 > 0$, this current acts as an adverse flow, while for $cc_0 < 0$, the current follows $d_0 > 0$, such that $d_0 < d_1 < d_2$ is introduced to adjust the depth of the thermocline. The parameter $q \in (-\infty, +\infty)$, while $r \in [-r_0, r_0]$, for some r_0 , as the solution is formulated around a fixed latitude φ . For every fixed value of r , we have $s \in [s_0, s_+]$, where $s = s_0 > 0$ defines the thermocline $z = \eta_0(x, y, t)$ at the latitude φ , while $s = s_+ > s_0$ prescribes the interface $z = \eta_+(x, y, t)$ at the same latitude. The term $V(q, r, s)$ denotes a transverse meridional current, while the term $W(q, r, s)$ represents a vertical current. We will show that (4) is the solution to the governing equations (1) and satisfies the incompressibility condition (2).

For simplicity, we denote $\theta = k(q - ct)$, and we require

$$s > s^* = \min_{r \in [-r_0, r_0]} [s_0] > 0,$$

The Jacobian matrix of the map relating the particle position to the Lagrangian labeling variables is given by

$$J = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 - kbe^{-ms} \cos \theta & kde^{-ms} \sin \theta + V_q t & ake^{-ms} \sin \theta + tW_q \\ 0 & 1 + V_r t & W_r t \\ mbe^{-ms} \sin \theta & mde^{-ms} \cos \theta + V_s t & 1 + mae^{-ms} \cos \theta + W_s t \end{bmatrix}, \quad (5)$$

so we have

$$\begin{aligned} |J| &= 1 - mabke^{-ms} + [V_r + W_s + (mbdkW_r - mabkV_r)e^{-2ms}]t + (V_rW_s - W_rV_s)t^2 \\ &+ [mb(W_rV_q - V_rW_q)t^2 - mbtW_q]e^{-ms} \sin \theta + [(ma - bk)(1 + tV_r) - (bkW_s + mdW_r)t \\ &+ bk(W_rV_s - W_sV_r)t^2]e^{-ms} \cos \theta, \end{aligned}$$

In order to ensure mass conversation, the $|J|$ should be nonzero and time-independent, so any terms involving t and θ must vanish, and then we have following conditions:

- i. $ma = bk$;
- ii. $W_r = -\frac{a}{d}W_s$;
- iii. $W_q = 0$;
- iv. $V_r = -\frac{a}{d}W_s(W_r, W_s \neq 0)$;
- v. $W_rV_q = 0$;
- vi. $V_r + W_s = 0$.

Furthermore, $|J| = 1 - m^2a^2e^{-2ms} \neq 0$, which means the determinant of J is time-independent. By using $s > s^* > 0$, then

$$a^2m^2e^{-2ms^*} < 1,$$

and thus the flow defined by (4) is volume-preserving and the incompressibility condition (2) is satisfied.

From $W_rV_q = 0$, we conclude that either $W_r = 0$ or $V_q = 0$; we discuss both cases separately.

Case I: The case $W_r = 0$.

If $W_r = 0$, the relation *ii* implies $W_s = 0$, so then from *vi*, we get $V_r = 0$. Note that $W_q = 0$, so then we obtain $W_r = W_s = W_q = 0$; this means that the vertical current term W is constant. In order to show V is also constant, we make use of the continuity and compatibility condition of the pressure distribution.

Equation (5) can be written as

$$J = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 - kbe^{-ms} \cos \theta & kde^{-ms} \sin \theta & ake^{-ms} \sin \theta + tW_q \\ 0 & 1 & 0 \\ mbe^{-ms} \sin \theta & mde^{-ms} \cos \theta + V_s t & 1 + mae^{-ms} \cos \theta + W_s t \end{bmatrix}, \quad (6)$$

From (4), we know the velocity of each fluid particle is obtained as follows

$$\begin{cases} u = \frac{Dx}{Dt} = -c_0 + kbce^{-ms} \cos \theta, \\ v = \frac{Dy}{Dt} = -kcde^{-ms} \sin \theta, \\ w = \frac{Dz}{Dt} = -kace^{-ms} \sin \theta, \end{cases} \quad (7)$$

and

$$\begin{cases} \frac{Du}{Dt} = k^2bc^2e^{-ms} \sin \theta, \\ \frac{Dv}{Dt} = k^2c^2de^{-ms} \cos \theta, \\ \frac{Dw}{Dt} = k^2ac^2e^{-ms} \cos \theta, \end{cases}$$

so then we have

$$\begin{cases} P_x = \rho_0[(fV - \hat{f}W) - kc(fd - a\hat{f} + kbc)e^{-ms} \sin \theta], \\ P_y = \rho_0[(fc_0 - \frac{f\hat{f}}{4}R) - kc(kcd + bf)e^{-ms} \cos \theta], \\ P_z = \rho_0[(\frac{\hat{f}^2}{4}R - g - c_0\hat{f}) - kc(kac - \hat{f}b)e^{-ms} \cos \theta]. \end{cases} \quad (8)$$

Using the change of variables

$$\begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial P}{\partial r} \\ \frac{\partial P}{\partial s} \end{bmatrix} = J \begin{bmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{bmatrix},$$

we get

$$\begin{cases} \frac{\partial P}{\partial q} = \rho_0(fV - \hat{f}W) + \rho_0(fc_0 - \frac{f\hat{f}}{4}R)V_q t - \rho_0 k[b(fV - \hat{f}W) + (fb + kcd)V_q ct]e^{-ms} \cos \theta \\ \quad + \rho_0 k[(fc_0 - \frac{f\hat{f}}{4}R)d - a(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R) - c(fd - \hat{f}a + kbc)]e^{-ms} \sin \theta \\ \quad + \rho_0 k^3 c^2 (b^2 - d^2 - a^2)e^{-2ms} \sin \theta \cos \theta, \\ \frac{\partial P}{\partial r} = \rho_0(fc_0 - \frac{f\hat{f}}{4}R) - \rho_0 kc(fb + kcd)e^{-ms} \cos \theta, \\ \frac{\partial P}{\partial s} = \rho_0(\frac{\hat{f}^2}{4}R - g - \hat{f}c_0) + \rho_0(fc_0 - \frac{f\hat{f}}{4}R)V_s t + \rho_0 mb(fV - \hat{f}W)e^{-ms} \sin \theta \\ \quad - \rho_0 kmcb(fd - \hat{f}a + kdc)e^{-2ms} + \rho_0[-ak^2 c^2 + \hat{f}bkc - ma(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R) \\ \quad + md(fc_0 - \frac{f\hat{f}}{4}R) - kc(fb + dkc)V_s t]e^{-ms} \cos \theta + \rho_0 k^2 c^2 m(b^2 - d^2 - a^2)e^{-2ms} \cos^2 \theta. \end{cases} \quad (9)$$

Applying the compatibility condition $P_{qr} = P_{rq}$, we obtain $fb + kcd = 0$. $P_{qs} = P_{sq}$ leads to $ak^2 c^2 = mcf d + mbkc^2$; we assume time independence in the pressure distribution, which leads to the relations

$$V_q = 0, V_s = 0, fV = \hat{f}W, b^2 = a^2 + d^2$$

and

$$\hat{f}ac - kbc^2 - a(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R) - fd(c - c_0) - \frac{f\hat{f}Rd}{4} = 0,$$

and thus we have V as constant, and W is constant, so (9) becomes

$$\begin{cases} \frac{\partial p}{\partial q} = 0, \\ \frac{\partial p}{\partial r} = \rho_0(fc_0 - \frac{f\hat{f}}{4}R), \\ \frac{\partial p}{\partial s} = \rho_0(\frac{\hat{f}^2}{4}R - g - \hat{f}c_0) - \rho_0 kmcb(fd - \hat{f}a + kdc)e^{-2ms}, \end{cases} \quad (10)$$

and thus the pressure is prescribed by

$$P = \rho_0(fc_0 - \frac{f\hat{f}}{4}R)r + \rho_0(\frac{\hat{f}^2}{4}R - g - \hat{f}c_0)s + \frac{\rho_0 kcb}{2}(fd - \hat{f}a + kcd)e^{-2ms} + P_0, \quad (11)$$

where P_0 is an arbitrary constant.

Case II: The case $V_q = 0$.

When $V_q = 0$, we have

$$J = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 - kbe^{-ms} \cos \theta & kde^{-ms} \sin \theta & ake^{-ms} \sin \theta \\ 0 & 1 + V_r t & W_r t \\ mbe^{-ms} \sin \theta & mde^{-ms} \cos \theta + V_s t & 1 + mae^{-ms} \cos \theta + W_s t \end{bmatrix}, \quad (12)$$

and we obtain

$$\begin{cases} \frac{\partial P}{\partial q} = \rho_0(fV - \hat{f}W) - \rho_0 bk(fV - \hat{f}W)e^{-ms} \cos \theta + \rho_0 k[d(fc_0 - \frac{f\hat{f}}{4}R) - a(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R) \\ \quad - c(fd - \hat{f}a + kbc)]e^{-ms} \sin \theta + \rho_0 k^3 c^2 (b^2 - d^2 - a^2)e^{-2ms} \sin \theta \cos \theta, \\ \frac{\partial P}{\partial r} = \rho_0(fc_0 - \frac{f\hat{f}}{4}R) + \rho_0(fc_0 - \frac{f\hat{f}}{4}R)V_r t - \rho_0(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R)W_r t - \rho_0 kc[(fb + kcd) \\ \quad + (fb + dkc)V_r t - (\hat{f}b - akc)W_r t]e^{-ms} \cos \theta, \\ \frac{\partial P}{\partial s} = \rho_0(\frac{\hat{f}^2 R}{4} - g - \hat{f}c_0) + \rho_0(fc_0 - \frac{f\hat{f}}{4}R)V_s t - \rho_0(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R)W_s t + \rho_0 mb(fV - \hat{f}W)e^{-ms} \sin \theta \\ \quad - \rho_0 kmcb(fd - \hat{f}a + kdc)e^{-2ms} + \rho_0[-ak^2 c^2 + \hat{f}bkc - ma(\hat{f}c_0 + g - \frac{\hat{f}^2 R}{4}) + kc(\hat{f}b - akc)W_s t \\ \quad + md(fc_0 - \frac{f\hat{f}}{4}R) - kc(fb + dkc)V_s t]e^{-ms} \cos \theta + \rho_0 k^2 c^2 m(b^2 - d^2 - a^2)e^{-2ms} \cos^2 \theta. \end{cases} \quad (13)$$

$P_{qr} = P_{rq}$ leads to

$$(fV_r - \hat{f}W_r) - bk(fV_r - \hat{f}W_r)e^{-ms} \cos \theta = k^2 c[(fb + dkc) + (fb + dkc)V_r t - (\hat{f}b - akc)W_r t]e^{-ms} \sin \theta,$$

so then we get

$$fV_r = \hat{f}W_r, \quad fb + dkc = 0, \quad (\hat{f}b - akc)W_r = 0.$$

$P_{qs} = P_{sq}$ leads to

$$fV_s = \hat{f}W_s, \quad ak^2 c^2 = mcf d + mbkc^2, \quad (\hat{f}b - akc)W_s = 0.$$

We assume that the pressure is time-independent, so then

$$V_s = V_r = W_r = W_s = 0, \quad fV = \hat{f}W, \quad b^2 = a^2 + d^2,$$

and

$$\hat{f}ac - kbc^2 - \frac{f\hat{f}}{4}Rd - a(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R) - fd(c - c_0) = 0.$$

From the above relations, we conclude that V and W are constants. In addition,

$$P = \rho_0(fc_0 - \frac{f\hat{f}}{4}R)r + \rho_0(\frac{\hat{f}^2}{4}R - g - \hat{f}c_0)s + \frac{\rho_0 kcb}{2}(fd - \hat{f}a + kcd)e^{-2ms} + P_0,$$

which is the same as (11) in case I.

As z approaches the boundary $z = \eta(x, y, t)$ from $U(t)$, we obtain

$$\begin{aligned} P &= \tilde{P}_0 + \rho_+ \frac{\hat{f}}{2}(c - c_0)(d_1 + d_2) + \rho_+[\hat{f}(c - c_0) - g + \frac{\hat{f}^2}{4}R](-d_0 + s) \\ &\quad + [a\rho_+(g - \frac{\hat{f}^2}{4}R - \hat{f}(c - c_0)) + d\rho_+(f(c - c_0) + \frac{f\hat{f}}{4}R)]e^{-ms} \cos \theta - \rho_+[f(c - c_0) + \frac{f\hat{f}}{4}R]r \\ &\quad + \rho_+[(\hat{f}(c - c_0) - g + \frac{\hat{f}^2}{4}R)W - (f(c - c_0) + \frac{f\hat{f}}{4}R)V]t. \end{aligned} \quad (14)$$

Now, let us discuss the equatorial regions and non-equatorial regions separately.

Case I : Equatorial regions

For the equatorial regions, we take the Coriolis parameters

$$f = 0, \quad \hat{f} = 2\Omega,$$

and from $fV = \hat{f}W$, we have $W = 0$, which means the vertical current vanishes, and (14) becomes

$$P = \tilde{P}_0 + \rho_+ \frac{\hat{f}}{2} (c - c_0)(d_1 + d_2) + \rho_+ [\hat{f}(c - c_0) - g + \frac{\hat{f}^2}{4} R](-d_0 + s) \\ + [a\rho_+(g - \frac{\hat{f}^2}{4} R - \hat{f}(c - c_0)) + d\rho_+(f(c - c_0) + \frac{f\hat{f}}{4} R)]e^{-ms} \cos \theta - \rho_+[f(c - c_0) + \frac{f\hat{f}}{4} R]r,$$

We require the pressure to be continuous across the thermocline; we get

$$kbc^2 = \tilde{g}[a(\hat{f}c_0 + g) + fd(c - c_0) - \hat{f}ac + \frac{f\hat{f}}{4} Rd - \frac{a\hat{f}}{4} R^2],$$

Note that $f = 0, \hat{f} = 2\Omega$, and we have

$$kbc^2 = \tilde{g}[a(2\Omega c_0 + g) - 2\Omega ac - \frac{a\Omega}{2} R^2], \quad (15)$$

where $\tilde{g} = \frac{\rho_+ - \rho_0}{\rho_0}$. Furthermore, we define

$$\tilde{P} - p_0 = \frac{\rho_0 kbc}{2} (fd - \hat{f}a + kcd)e^{-2ms_0} + \rho_0 (\frac{\hat{f}^2}{4} R - \hat{f}c_0 - g)s - \rho_+ [\hat{f}(c - c_0) - g + \frac{\hat{f}^2}{4} R](-d_0 + s) \\ + \rho_0 (fc_0 - \frac{f\hat{f}}{4} R)r + \rho_+ [(f(c - c_0) + \frac{f\hat{f}}{4} R)]r - \rho_+ \frac{\hat{f}}{2} (c - c_0)(d_1 + d_2), \quad (16)$$

where the solution $S_0(r)$ of (16) represents the thermocline; the upper boundary of the layer $M(t)$ is specified by setting $S = S_0(r)$ at a fixed value of $r \in (-r_0, r_0)$ with $S_+(r) > S_0(r)$ in (16).

The continuity of the pressure across the thermocline is possible if and only if (16) and (17) hold for $S = S_0(r)$.

Case II: Non-equatorial regions

For the non-equatorial regions, $\theta \neq 0$, note that $fV = \hat{f}W$, and we have

$$[\hat{f}(c - c_0) - g + \frac{\hat{f}^2}{4}]W - [f(c - c_0) + \frac{f\hat{f}}{4} R]V = 0, \quad (17)$$

which leads to $gW = 0$, and then $W = 0$, so $V = 0$, which means that away from the equatorial regions, neither the transverse meridional current nor the vertical current exists.

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