

Article

Local-to-Global Wedge Retention in Heavy-Tailed Geometric Random Graphs

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Abstract

A spatially localized vertex set may contain many internal wedges, yet the same vertices can acquire additional external neighbors and form substantially more wedges in the full graph. We quantify this effect in heavy-tailed geometric random graphs through a wedge-retention ratio that compares wedges whose two endpoints remain inside an angular window with all wedges centered at the same window vertices. Unlike conventional triangle-based clustering coefficients, this ratio does not involve triangle counts. Under the scale and dominance conditions studied here, the infinite-variance regime allows high-strength vertices to dominate the wedge counts because each vertex contributes quadratically in its degree. We prove that the ratio is asymptotically a degree-square-weighted average of squared kernel-mass ratios, where each ratio is the expected fraction of a vertex's neighbors lying inside the window. For a standard shrinking window of half-width $\Delta = N^{-s}$, $0 < s < 1$, the ratio remains of order one. Under additional linear-strength assumptions, a dominant set of vertices with strengths of order N yields a ratio of order Δ^2 ; a finite-hub conditional construction shows that these assumptions are mutually compatible. Consequently, order- Δ^2 behavior is not a generic consequence of shrinking the window but requires a specific linear-strength mechanism.

Keywords: random geometric graphs; heavy-tailed distributions; regular variation; wedge counts; wedge retention

MSC: 05C80; 60G70; 60K35; 05C82

1. Introduction

Geometric random graphs with latent variables provide a common framework for networks in which spatial or similarity structure coexists with heterogeneous degrees. Relevant examples include hidden-variable and hidden-metric models [1–3], random hyperbolic graphs [4–7], geometric inhomogeneous random graphs [8], and classical random geometric graphs [9]. A recent overview places these constructions within the broader program of network geometry [10]. In the one-dimensional hidden-metric or $\mathbb{S}^1$ description used here, the angular coordinate θ_i is a similarity coordinate and the latent strength κ_i controls the expected degree or popularity. Under the standard $\mathbb{S}^1$ – $\mathbb{H}^2$ correspondence, θ_i becomes the hyperbolic angular coordinate, while κ_i is logarithmically related to the radial coordinate [6,7,10]. This description is equivalent to the familiar hyperbolic separation between similarity and popularity.

Most clustering statistics are evaluated at a single observation scale. Global transitivity aggregates closed triples across the whole graph, average local clustering averages vertexwise triangle-closure fractions, and degree-dependent clustering $C(k)$ groups vertices



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by degree. Recent analyses of clustering spectra and motif counts in scale-free networks further show how heterogeneous degree classes can control wedge- and subgraph-based statistics [11,12]. These quantities are well suited to triangle closure, yet they do not compare the same set of center vertices before and after the surrounding graph is restored. They therefore do not isolate how external connections enlarge the full-graph wedge count for a spatially localized center set, which is the comparison studied here.

The phenomenon considered here arises when a vertex set contains many internal wedges inside a restricted angular window, while some of its vertices have many neighbors outside that window. Related situations arise in localized network sampling, community-boundary analysis, and spatial or similarity-based subnetworks. Statistical work on sub-sampling and bootstrap methods for network count statistics and latent-space models provides related evidence that partial observation can alter degree- and subgraph-based summaries [13–15]. Our ratio quantifies how much of the wedge structure remains internal when the full graph is revealed, thereby measuring boundary exposure while keeping the center set fixed.

The main aim of this paper is to determine which vertices control the local-to-global wedge-retention ratio and how their geometric connection scales determine its asymptotic order. The general analysis is carried out under the two standing model assumptions stated in Section 2: an integrable nonincreasing connection kernel and a regularly varying latent-strength tail with exponent $\alpha \in (1, 2)$. Additional scale conditions needed for the weighted-ratio reduction are stated explicitly in the main theorem.

Let

$$V_{\Delta} := \{i : |\theta_i| < \Delta\}$$

be an angular window of half-width Δ . For $i \in V_{\Delta}$, let k_i^{int} be the number of neighbors of i inside V_{Δ} , and let k_i be its degree in the full graph. The doubled internal and full wedge counts centered at window vertices are

$$W_{\Delta}^{\text{int}} := \sum_{i \in V_{\Delta}} k_i^{\text{int}}(k_i^{\text{int}} - 1), \quad W_{\Delta}^{\text{full}} := \sum_{i \in V_{\Delta}} k_i(k_i - 1).$$

We study the wedge-retention ratio

$$\Phi_{\Delta} := \begin{cases} W_{\Delta}^{\text{int}} / W_{\Delta}^{\text{full}}, & W_{\Delta}^{\text{full}} > 0, \\ 0, & W_{\Delta}^{\text{full}} = 0. \end{cases}$$

The statistic does not count closed triangles and is distinct from a clustering coefficient. Its connection with clustering comes from the denominator of the usual local clustering coefficient: a vertex of degree k_i contributes $\binom{k_i}{2}$ potential neighbor pairs. External neighbors increase this denominator, even when they contribute nothing to the wedge count retained inside the selected window.

The ratio is mathematically natural in the heavy-tailed regime because wedge counts weight vertices quadratically in their degrees. When the latent-strength tail has exponent $\alpha \in (1, 2)$, the first moment is finite and the second moment is infinite. Under the dominance conditions imposed below, a small number of high-strength vertices can determine the leading orders of both W_{Δ}^{int} and W_{Δ}^{full} . This feature distinguishes wedge retention from statistics whose leading order is governed by an average over typical vertices.

For a window vertex i , let

$$R_i := \frac{\bar{\lambda}_i^{\text{int}}}{\bar{\lambda}_i}$$

denote the ratio between its expected internal degree and expected full degree, conditional on its own latent variables. Thus, R_i is the expected fraction of its connection-kernel mass lying inside the window. We prove that, on a high-strength set B_Δ , the empirical fraction k_i^{int}/k_i concentrates around R_i , and that

$$\Phi_\Delta = (1 + o_{\mathbb{P}}(1)) \frac{\sum_{i \in B_\Delta} R_i^2 k_i^2}{\sum_{i \in B_\Delta} k_i^2}.$$

This formula identifies a degree-square-weighted kernel-mass mechanism behind local-to-global wedge retention.

Existing work on hidden-metric, hyperbolic, and geometric inhomogeneous random graphs has developed degree asymptotics, clustering, triangle counts, and related structural properties [2,4–6,8,10]. Related heavy-tailed network work studies degree-dependent clustering and motif counts [11,12]. The present problem fixes the center set and compares its wedge denominator at two observation scales. To our knowledge, this fixed-center local-to-global wedge-retention quantity has not been isolated in this form for heavy-tailed geometric random graphs.

The distinction between the two settings studied in the paper is consequential. For a standard shrinking window with $\Delta = N^{-s}$, $0 < s < 1$, the largest relevant strengths remain small relative to the geometric window scale $N\Delta$. The high-strength vertices then retain an order-one fraction of their kernel mass inside the window, and

$$\Phi_\Delta = \Theta_{\mathbb{P}}(1).$$

Hence, shrinking the window alone does not produce an order- Δ^2 ratio. A separate conditional theorem considers a dominant set of vertices with strengths of order N . Their full connection scale is of order N , their internal kernel-mass fraction is of order Δ , and the quadratic wedge weighting gives

$$\Phi_\Delta = \Theta_{\mathbb{P}}(\Delta^2).$$

The contrast shows that the Δ^2 behavior reflects a linear-strength mechanism and cannot be inferred from angular-window size alone.

The main contributions are as follows.

- (i) We introduce a fixed-center local-to-global wedge-retention ratio and distinguish it explicitly from triangle-based clustering coefficients.
- (ii) We derive the mixed mean-degree relation

$$\bar{\lambda}_i \sim C_g \kappa_i$$

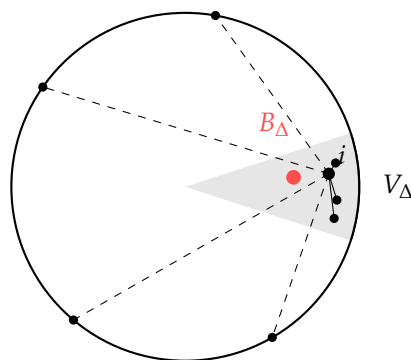
directly from the connection kernel, uniformly over sublinear strengths.

- (iii) We first establish uniform concentration on deterministic high-mean sets and then transfer the estimates to the random high-strength set.
- (iv) We prove the degree-square-weighted kernel-mass formula for Φ_Δ and establish the order-one standard-window result. We also give a conditional linear-strength theorem together with a finite-hub conditional construction showing that its assumptions are mutually compatible.

Organization of the paper. Section 2 introduces the latent geometric graph model, the angular window, the wedge-retention ratio, and the high-strength scale. Section 3 derives the kernel-mass identities, the exact mixed-mean formulas, and the mean-degree regularity relation $\bar{\lambda}_i \sim C_g \kappa_i$. Section 4 proves uniform concentration estimates on high-mean sets.

Section 5 establishes high-strength dominance and the asymptotic weighted-ratio formula. Section 6 proves the standard local-window result, states the conditional linear-strength theorem, and presents the finite-hub construction. Section 7 discusses the scope and limitations of the results, and Section 8 summarizes the asymptotic settings and scales.

Figure 1 schematically illustrates the angular window, internal and external neighbors, and the high-strength set used throughout the paper.



Solid: internal neighbors; dashed: external neighbors.

Figure 1. A local angular window V_Δ on the circle. The gray sector represents the angular window V_Δ . The filled circles represent vertices, with the red filled circle indicating a vertex in the high-strength set B_Δ . Solid lines indicate connections from the center vertex i to internal neighbors in V_Δ , whereas dashed lines indicate connections to external neighbors outside the window.

2. Model and Notation

2.1. Latent Geometry and Connection Kernel

For each $N \geq 1$, let

$$(\theta_i, \kappa_i)_{1 \leq i \leq N}$$

be independent and identically distributed latent variables with

$$\theta_i \sim \text{Unif}[-\pi, \pi), \quad \kappa_i \sim F,$$

and assume that angles and strengths are independent. Write the circular distance as

$$d(\theta, \theta') := \min_{m \in \mathbb{Z}} |\theta - \theta' + 2\pi m|.$$

Conditional on the latent variables, edges are independent Bernoulli random variables with

$$p_{ij} = g\left(\frac{Nd(\theta_i, \theta_j)}{\mu\kappa_i\kappa_j}\right), \quad 1 \leq i < j \leq N, \quad (1)$$

where $\mu > 0$ is fixed. This kernel form is a one-dimensional hidden-metric model and is closely related to hidden-space, hyperbolic, and geometric inhomogeneous graph models [1,6,8].

Assumption 1. The function $g : [0, \infty) \rightarrow [0, 1]$ is measurable, nonincreasing, and integrable:

$$I_g := \int_0^\infty g(u) \, du \in (0, \infty).$$

Assumption 2. There exist $\alpha \in (1, 2)$, a slowly varying function L , and $\kappa_{\min} > 0$ such that

$$\bar{F}(x) := \mathbb{P}(\kappa > x) = x^{-\alpha}L(x) \quad (x \rightarrow \infty),$$

and

$$\mathbb{P}(\kappa \geq \kappa_{\min}) = 1.$$

Assumptions 1 and 2 are the two standing model assumptions. The kernel assumption supplies a finite connection-mass scale and the monotonicity used in the window bounds. The tail assumption supplies a finite first moment, an infinite second moment, and the extreme-value scale relevant to high-strength dominance. All additional scale, dominance, and concentration conditions are stated explicitly in the corresponding theorems.

We use standard facts from regular variation and heavy-tail extreme value theory throughout; see [16,17]. Assumption 2 places the model in the finite-mean, infinite-variance regime:

$$\mathbb{E}[\kappa] < \infty, \quad \mathbb{E}[\kappa^2] = \infty.$$

In this regime, extreme window vertices can dominate squared-degree-type functionals. The additional conditions in Section 5 make this dominance precise for the wedge counts studied here.

Let

$$\mathcal{F}_N := \sigma((\theta_i, \kappa_i) : 1 \leq i \leq N)$$

be the full latent sigma-field, and let

$$A_{ij} = \mathbf{1}_{\{i \sim j\}}$$

denote the edge indicators.

2.2. Window, Degrees, Wedges, and Local-to-Global Ratio

Fix

$$\Delta = \Delta_N \in (0, \pi/2)$$

and define the local angular window

$$V_\Delta := \{i : |\theta_i| < \Delta\}.$$

Its deterministic size scale is

$$m_N := \frac{N\Delta}{\pi},$$

which equals the expected number of vertices in the window. The actual window size is binomial and, whenever $m_N \rightarrow \infty$, satisfies

$$\frac{|V_\Delta|}{m_N} \xrightarrow{\mathbb{P}} 1.$$

The actual full and internal degrees are

$$k_i := \sum_{j \neq i} A_{ij}, \quad k_i^{\text{int}} := \sum_{\substack{j \in V_\Delta \\ j \neq i}} A_{ij}.$$

Their conditional means given the full latent environment are

$$\lambda_i := \mathbb{E}[k_i \mid \mathcal{F}_N] = \sum_{j \neq i} p_{ij},$$

and

$$\lambda_i^{\text{int}} := \mathbb{E}[k_i^{\text{int}} \mid \mathcal{F}_N] = \sum_{\substack{j \in V_\Delta \\ j \neq i}} p_{ij}.$$

The mixed conditional means, conditioning only on the center vertex, are

$$\bar{\lambda}_i := \mathbb{E}[\lambda_i \mid \theta_i, \kappa_i], \quad \bar{\lambda}_i^{\text{int}} := \mathbb{E}[\lambda_i^{\text{int}} \mid \theta_i, \kappa_i].$$

Definition 1. For $i \in V_\Delta$, define the kernel-mass ratio

$$R_i := \frac{\bar{\lambda}_i^{\text{int}}}{\bar{\lambda}_i}.$$

The empirical internal-degree fraction is

$$q_i := \frac{k_i^{\text{int}}}{k_i},$$

with the convention $q_i = 0$ when $k_i = 0$.

A wedge centered at vertex i is an unordered pair of neighbors of i . The number of such wedges is $\binom{k_i}{2}$, and twice this number is $k_i(k_i - 1)$. Define the doubled internal and full wedge counts by

$$W_\Delta^{\text{int}} := \sum_{i \in V_\Delta} k_i^{\text{int}}(k_i^{\text{int}} - 1), \quad W_\Delta^{\text{full}} := \sum_{i \in V_\Delta} k_i(k_i - 1). \quad (2)$$

We define the local-to-global wedge-retention ratio by

$$\Phi_\Delta := \begin{cases} \frac{W_\Delta^{\text{int}}}{W_\Delta^{\text{full}}}, & W_\Delta^{\text{full}} > 0, \\ 0, & W_\Delta^{\text{full}} = 0. \end{cases} \quad (3)$$

Thus, Φ_Δ is defined for every graph realization. On the event $W_\Delta^{\text{full}} > 0$, the numerator counts twice the wedges that stay inside the window, whereas the denominator counts twice all wedges centered at the same window vertices and measured in the full graph.

2.3. High-Strength Threshold and Asymptotic Scales

Let a_N be the extreme-value scale associated with the window size m_N , defined by

$$m_N \bar{F}(a_N) \rightarrow 1. \quad (4)$$

The high-strength threshold is chosen as

$$K_N := a_N \eta_N, \quad \eta_N \downarrow 0, \quad (5)$$

and the high-strength set is

$$B_\Delta := \{i \in V_\Delta : \kappa_i \geq K_N\}.$$

Remark 1. The later concentration arguments require $K_N = a_N \eta_N \gg \log N$, or more generally $r_N K_N \gg \log N$, where $r_N \in (0, 1]$ denotes a deterministic lower scale for the kernel-mass ratios R_i on the high-strength set. Thus, η_N may tend to zero, with lower-order restriction

$$\eta_N \gg \frac{\log N}{a_N}$$

in the standard kernel-mass scale $r_N = 1$. In the standard local-window regime, a_N grows polynomially up to slowly varying factors, so choices such as $\eta_N = 1/\log N$ are admissible.

Remark 2. Throughout, all asymptotics are taken as $N \rightarrow \infty$, with $m_N \rightarrow \infty$. We write $X_N = \Theta_{\mathbb{P}}(Y_N)$ in the usual positive probabilistic sense.

3. Kernel Mass and Mean Degree Regularity

The bridge between latent strength and expected degree is the kernel profile

$$\Psi_{\kappa}(u) := \mathbb{E}_{\kappa'} \left[g \left(\frac{|u|}{\mu \kappa \kappa'} \right) \right], \quad u \in \mathbb{R}. \quad (6)$$

This is an even, bounded, nonincreasing function of $|u|$.

Lemma 1. Under Assumptions 1 and 2, for every $\kappa > 0$,

$$\int_0^{\infty} \Psi_{\kappa}(u) \, du = \mu \kappa \mathbb{E}[\kappa] I_g. \quad (7)$$

Moreover, if

$$G(t) := \int_t^{\infty} g(v) \, dv,$$

then for every $M > 0$,

$$\frac{1}{\kappa} \int_{M\kappa}^{\infty} \Psi_{\kappa}(u) \, du = \mu \mathbb{E} \left[\kappa' G \left(\frac{M}{\mu \kappa'} \right) \right], \quad (8)$$

and the right-hand side converges to 0 as $M \rightarrow \infty$.

Proof. Since $g \geq 0$, the integrand is non-negative, so Tonelli's theorem applies. By Tonelli's theorem and the substitution $u = \mu \kappa \kappa' v$,

$$\begin{aligned} \int_0^{\infty} \Psi_{\kappa}(u) \, du &= \mathbb{E}_{\kappa'} \left[\int_0^{\infty} g \left(\frac{u}{\mu \kappa \kappa'} \right) \, du \right] \\ &= \mathbb{E}_{\kappa'} \left[\mu \kappa \kappa' \int_0^{\infty} g(v) \, dv \right] = \mu \kappa \mathbb{E}[\kappa] I_g, \end{aligned}$$

which proves (7). The same substitution gives

$$\begin{aligned} \int_{M\kappa}^{\infty} \Psi_{\kappa}(u) \, du &= \mathbb{E}_{\kappa'} \left[\int_{M\kappa}^{\infty} g \left(\frac{u}{\mu \kappa \kappa'} \right) \, du \right] \\ &= \mu \kappa \mathbb{E}_{\kappa'} \left[\kappa' \int_{M/(\mu \kappa')}^{\infty} g(v) \, dv \right] = \mu \kappa \mathbb{E} \left[\kappa' G \left(\frac{M}{\mu \kappa'} \right) \right], \end{aligned}$$

which is (8). Since

$$0 \leq G \left(\frac{M}{\mu \kappa'} \right) \leq G(0) = I_g$$

and $\mathbb{E}[\kappa] < \infty$, dominated convergence gives the tail limit. $\square$

Remark 3. Dividing (8) by (7) gives

$$\frac{\int_{M\kappa}^{\infty} \Psi_{\kappa}(u) \, du}{\int_0^{\infty} \Psi_{\kappa}(u) \, du} = \frac{\mathbb{E}[\kappa' G(M/(\mu \kappa'))]}{\mathbb{E}[\kappa] I_g}.$$

The right-hand side is independent of κ and tends to 0 as $M \rightarrow \infty$.

Lemma 2. Under Assumption 1, for every vertex i ,

$$\bar{\lambda}_i = \frac{N-1}{\pi N} \int_0^{N\pi} \Psi_{\kappa_i}(u) \, du. \quad (9)$$

If $i \in V_\Delta$ and $\Delta < \pi/2$, then

$$\bar{\lambda}_i^{\text{int}} = \frac{N-1}{2\pi N} \int_{-N(\Delta+\theta_i)}^{N(\Delta-\theta_i)} \Psi_{\kappa_i}(u) \, du. \quad (10)$$

Consequently,

$$R_i = \frac{\int_{-N(\Delta+\theta_i)}^{N(\Delta-\theta_i)} \Psi_{\kappa_i}(u) \, du}{\int_{-N\pi}^{N\pi} \Psi_{\kappa_i}(u) \, du}, \quad i \in V_\Delta. \quad (11)$$

Proof. Conditional on (θ_i, κ_i) , the remaining latent variables are independent and identically distributed. Hence,

$$\bar{\lambda}_i = (N-1) \mathbb{E} \left[g \left(\frac{Nd(\theta_i, \theta)}{\mu \kappa_i \kappa'} \right) \middle| \theta_i, \kappa_i \right].$$

For a uniform θ , the random distance $d(\theta_i, \theta)$ has density $1/\pi$ on $[0, \pi]$, independent of θ_i . Therefore,

$$\bar{\lambda}_i = \frac{N-1}{\pi} \int_0^\pi \mathbb{E}_{\kappa'} \left[g \left(\frac{Nt}{\mu \kappa_i \kappa'} \right) \right] dt.$$

With the substitution $u = Nt$, this becomes (9).

For the internal mean, if $i \in V_\Delta$ and $\Delta < \pi/2$, then for all $t \in [-\Delta, \Delta]$,

$$d(\theta_i, t) = |t - \theta_i|.$$

Thus,

$$\bar{\lambda}_i^{\text{int}} = (N-1) \frac{1}{2\pi} \int_{-\Delta}^\Delta \mathbb{E}_{\kappa'} \left[g \left(\frac{N|t - \theta_i|}{\mu \kappa_i \kappa'} \right) \right] dt.$$

The substitution $u = N(t - \theta_i)$ yields (10). Taking the ratio of (10) and (9), and using the evenness of Ψ_κ , gives (11). $\square$

Since $\alpha > 1$, Assumption 2 implies $\mathbb{E}[\kappa] < \infty$, so the constant C_g below is finite and positive.

Theorem 1. Under Assumptions 1 and 2, let $\rho_N \downarrow 0$. Then, uniformly for all

$$\kappa \in [\kappa_{\min}, \rho_N N],$$

one has

$$\bar{\lambda}(\kappa) = C_g \kappa (1 + o(1)), \quad C_g := \frac{\mu \mathbb{E}[\kappa] I_g}{\pi}, \quad (12)$$

where $\bar{\lambda}(\kappa)$ denotes the right-hand side of (9). In particular, for all sufficiently large N , there are constants $0 < c_\lambda < C_\lambda < \infty$ such that

$$c_\lambda \kappa \leq \bar{\lambda}(\kappa) \leq C_\lambda \kappa, \quad \kappa \in [\kappa_{\min}, \rho_N N]. \quad (13)$$

Proof. By (9) and (7),

$$\bar{\lambda}(\kappa) = \frac{N-1}{\pi N} \int_0^{N\pi} \Psi_\kappa(u) \, du,$$

whereas

$$C_g \kappa = \frac{1}{\pi} \int_0^\infty \Psi_\kappa(u) \, du.$$

Therefore,

$$\bar{\lambda}(\kappa) - C_g \kappa = -\frac{1}{\pi N} \int_0^{N\pi} \Psi_\kappa(u) \, du - \frac{1}{\pi} \int_{N\pi}^\infty \Psi_\kappa(u) \, du.$$

This is the exact algebraic identity obtained by writing

$$\int_0^\infty = \int_0^{N\pi} + \int_{N\pi}^\infty.$$

For the first term, (7) gives

$$\int_0^{N\pi} \Psi_\kappa(u) \, du \leq \int_0^\infty \Psi_\kappa(u) \, du = \mu \kappa \mathbb{E}[\kappa] I_g.$$

Hence,

$$\frac{1}{\kappa} \frac{1}{\pi N} \int_0^{N\pi} \Psi_\kappa(u) \, du \leq \frac{\mu \mathbb{E}[\kappa] I_g}{\pi N} = O(N^{-1}),$$

uniformly in the stated range.

For the tail term, write

$$M = \frac{N\pi}{\kappa}.$$

If $\kappa \leq \rho_N N$, then

$$M \geq \frac{\pi}{\rho_N} \rightarrow \infty.$$

By (8),

$$\frac{1}{\kappa} \int_{N\pi}^\infty \Psi_\kappa(u) \, du = \mu \mathbb{E} \left[\kappa' G \left(\frac{N\pi}{\mu \kappa \kappa'} \right) \right].$$

For $\kappa \leq \rho_N N$, we have $N\pi/\kappa \geq \pi/\rho_N$. Since G is nonincreasing,

$$G \left(\frac{N\pi}{\mu \kappa \kappa'} \right) \leq G \left(\frac{\pi}{\mu \rho_N \kappa'} \right).$$

Thus, the supremum over $\kappa \leq \rho_N N$ is bounded by

$$\mu \mathbb{E} \left[\kappa' G \left(\frac{\pi}{\mu \rho_N \kappa'} \right) \right],$$

which tends to 0 by dominated convergence, as in Lemma 1. Therefore,

$$\sup_{\kappa \in [\kappa_{\min}, \rho_N N]} \left| \frac{\bar{\lambda}(\kappa)}{\kappa} - C_g \right| \rightarrow 0.$$

This proves (12). The two-sided bounds (13) follow immediately from $C_g > 0$. $\square$

Remark 4. The constant

$$C_g = \frac{\mu \mathbb{E}[\kappa] I_g}{\pi}$$

is the asymptotic conversion factor from latent strength to expected degree. More precisely,

$$\frac{\bar{\lambda}(\kappa)}{\kappa} \longrightarrow C_g$$

uniformly in the sublinear-strength range, so a vertex with latent strength κ has expected degree approximately $C_g \kappa$. Thus, C_g measures the expected number of neighbors generated per unit of

latent strength. Its value combines the connection-density scale μ , the mean latent strength $\mathbb{E}[\kappa]$, the total kernel mass I_g , and the normalization induced by the circle geometry. If

$$\mu = \frac{\pi}{\mathbb{E}[\kappa]I_g},$$

then $C_g = 1$, and latent strength is asymptotically equal to expected degree. The restriction $\kappa \leq \rho_N N$, with $\rho_N \rightarrow 0$, identifies the range in which truncation at distance $N\pi$ captures asymptotically all kernel mass. Vertices with $\kappa \asymp N$ lie outside this uniform approximation, which motivates their separate treatment in the later conditional linear-strength theorem.

4. Uniform Concentration on High-Mean Sets

The concentration estimates below use standard Bernstein, Chernoff, and Hoeffding-type bounds for sums of bounded independent random variables [18–20]. This section addresses concentration on random high-strength sets by first proving estimates on all high-mean vertices and only afterwards restricting to B_Δ .

Let $L_N \rightarrow \infty$, and define the high-mean set

$$S_N := \left\{ 1 \leq i \leq N : \min\{\bar{\lambda}_i, \bar{\lambda}_i^{\text{int}}\} \geq 2L_N \log N \right\}.$$

Although S_N depends on the latent variables, all estimates below are obtained by union bounds over all vertices $1, \dots, N$.

Proposition 1. Let $L_N \rightarrow \infty$ and let S_N be as above. Under Assumptions 1 and 2, for every fixed $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right| > \varepsilon\right) \rightarrow 0,$$

and similarly

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{\lambda_i^{\text{int}}}{\bar{\lambda}_i^{\text{int}}} - 1 \right| > \varepsilon\right) \rightarrow 0.$$

Proof. Fix i . Conditional on (θ_i, κ_i) , the random variables

$$X_{ij} := g\left(\frac{Nd(\theta_i, \theta_j)}{\mu\kappa_i\kappa_j}\right), \quad j \neq i,$$

are independent and bounded by 1. They satisfy

$$\lambda_i = \sum_{j \neq i} X_{ij}, \quad \mathbb{E}[\lambda_i \mid \theta_i, \kappa_i] = \bar{\lambda}_i.$$

For sums of independent random variables bounded by 1, Bernstein's inequality gives

$$\mathbb{P}(|\lambda_i - \bar{\lambda}_i| > \varepsilon \bar{\lambda}_i \mid \theta_i, \kappa_i) \leq 2 \exp\left(-\frac{\varepsilon^2}{2 + 2\varepsilon/3} \bar{\lambda}_i\right).$$

We write

$$c_\varepsilon := \frac{\varepsilon^2}{2 + 2\varepsilon/3}.$$

On S_N , $\bar{\lambda}_i \geq 2L_N \log N$, so the probability is at most

$$2N^{-2c_\varepsilon L_N}.$$

A union bound over all N vertices gives

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right| > \varepsilon\right) \leq 2N^{1-2c_\varepsilon L_N} \rightarrow 0.$$

The proof for the internal means is identical, using

$$X_{ij}^{\text{int}} := \mathbf{1}_{\{|\theta_j| < \Delta\}} \mathcal{G}\left(\frac{Nd(\theta_i, \theta_j)}{\mu \kappa_i \kappa_j}\right),$$

whose sum is λ_i^{int} and whose conditional mean is $\bar{\lambda}_i^{\text{int}}$. $\square$

Proposition 2. Let $L_N \rightarrow \infty$ and let S_N be as above. Under Assumptions 1 and 2, for every fixed $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{k_i}{\lambda_i} - 1 \right| > \varepsilon\right) \rightarrow 0,$$

and similarly

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{k_i^{\text{int}}}{\lambda_i^{\text{int}}} - 1 \right| > \varepsilon\right) \rightarrow 0.$$

Consequently,

$$\mathbb{P}\left(\max_{i \in S_N} |q_i/R_i - 1| > 4\varepsilon\right) \rightarrow 0$$

after replacing 4ε by a harmless constant multiple of ε .

Proof. Let

$$E_N := \left\{ \max_{i \in S_N} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right| \leq \frac{1}{2}, \quad \max_{i \in S_N} \left| \frac{\lambda_i^{\text{int}}}{\bar{\lambda}_i^{\text{int}}} - 1 \right| \leq \frac{1}{2} \right\}.$$

By Proposition 1, $\mathbb{P}(E_N) \rightarrow 1$. On E_N , every $i \in S_N$ satisfies

$$\lambda_i \geq L_N \log N, \quad \lambda_i^{\text{int}} \geq L_N \log N.$$

Conditional on $\mathcal{F}_N$, the degrees k_i and k_i^{int} are sums of independent Bernoulli variables with means λ_i and λ_i^{int} . The multiplicative Chernoff bound gives

$$\mathbb{P}(|k_i - \lambda_i| > \varepsilon \lambda_i \mid \mathcal{F}_N) \leq 2 \exp(-c_\varepsilon \lambda_i),$$

and similarly for k_i^{int} . On E_N ,

$$2 \exp(-c_\varepsilon \lambda_i) \leq 2N^{-c_\varepsilon L_N}.$$

A union bound over all vertices yields

$$\mathbb{P}\left(\max_{i \in S_N} \left| \frac{k_i}{\lambda_i} - 1 \right| > \varepsilon \mid \mathcal{F}_N\right) \leq 2N^{1-c_\varepsilon L_N}$$

on E_N , and the right-hand side tends to 0. Taking expectations and adding $\mathbb{P}(E_N^c)$ proves the first concentration estimate; the proof for k_i^{int} is identical.

Finally, we write the ratio as four separate factors:

$$\frac{q_i}{R_i} = \frac{k_i^{\text{int}}}{\lambda_i^{\text{int}}} \cdot \frac{\lambda_i^{\text{int}}}{\bar{\lambda}_i^{\text{int}}} \cdot \frac{\bar{\lambda}_i}{\lambda_i} \cdot \frac{\lambda_i}{k_i}.$$

Each factor on the right is uniformly $1 + O(\varepsilon)$ on the intersection of the events established above. Their product therefore converges uniformly to one, which gives the stated conclusion. $\square$

Corollary 1. *If*

$$\mathbb{P}(B_\Delta \subseteq S_N) \rightarrow 1,$$

then

$$\max_{i \in B_\Delta} |q_i / R_i - 1| \xrightarrow{\mathbb{P}} 0.$$

Proof. On the event $\{B_\Delta \subseteq S_N\}$,

$$\max_{i \in B_\Delta} |q_i / R_i - 1| \leq \max_{i \in S_N} |q_i / R_i - 1|.$$

The claim follows from Proposition 2. $\square$

Remark 5. *In the standard local-window regime, this containment is verified through the lower bound on R_i and the condition $K_N \gg \log N$. In the conditional linear-strength regime below, the corresponding concentration of q_i / R_i is imposed explicitly.*

5. High-Strength Dominance and the Weighted-Ratio Formula

This section uses regular variation to show that low-strength contributions are negligible at the quadratic wedge scale.

Lemma 3. *Under Assumption 2, let a_N satisfy (4), and define*

$$M_\Delta := \max\{\kappa_i : i \in V_\Delta\},$$

with the convention $M_\Delta = 0$ when $V_\Delta = \emptyset$. Then, for every fixed $x > 0$,

$$\mathbb{P}(M_\Delta \leq xa_N) \rightarrow \exp(-x^{-\alpha}).$$

In particular,

$$M_\Delta = \Theta_{\mathbb{P}}(a_N).$$

Proof. For each vertex,

$$\mathbb{P}(i \in V_\Delta, \kappa_i > xa_N) = \frac{\Delta}{\pi} \bar{F}(xa_N) = \frac{m_N}{N} \bar{F}(xa_N).$$

By independence of the latent variables across vertices,

$$\mathbb{P}(M_\Delta \leq xa_N) = \left(1 - \frac{m_N}{N} \bar{F}(xa_N)\right)^N.$$

Since $m_N \bar{F}(a_N) \rightarrow 1$ and

$$\frac{\bar{F}(xa_N)}{\bar{F}(a_N)} \rightarrow x^{-\alpha}$$

by regular variation [16,17],

$$m_N \bar{F}(xa_N) \rightarrow x^{-\alpha}.$$

Therefore,

$$\left(1 - \frac{m_N}{N} \bar{F}(xa_N)\right)^N \rightarrow \exp(-x^{-\alpha}).$$

The tightness statement $M_\Delta = \Theta_{\mathbb{P}}(a_N)$ follows immediately. $\square$

Proposition 3. Under Assumption 2, fix $\delta \in (0, 2 - \alpha)$ and let $K_N = a_N \eta_N$, where $\eta_N \downarrow 0$. Then,

$$\mathbb{E} \left[\sum_{i \in V_\Delta \setminus B_\Delta} \kappa_i^2 \right] = O \left(a_N^2 \eta_N^{2-\alpha-\delta} \right).$$

Equivalently,

$$\sum_{i \in V_\Delta \setminus B_\Delta} \kappa_i^2 = O_{\mathbb{P}} \left(a_N^2 \eta_N^{2-\alpha-\delta} \right).$$

Moreover, if

$$\eta_N^{2-\alpha-\delta} \rightarrow 0,$$

then

$$\sum_{i \in B_\Delta} \kappa_i^2 = \Omega_{\mathbb{P}}(a_N^2).$$

Proof. By independence between angle and strength,

$$\mathbb{E} \left[\sum_{i \in V_\Delta \setminus B_\Delta} \kappa_i^2 \right] = N \mathbb{P}(|\theta| < \Delta) \mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa < K_N\}}] = m_N \mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa < K_N\}}].$$

By Karamata's theorem for truncated moments ([16], Theorem 1.6.4)

$$\mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa < K_N\}}] \sim \frac{\alpha}{2 - \alpha} K_N^2 \bar{F}(K_N).$$

Since $K_N = a_N \eta_N$ and $\bar{F} \in RV_{-\alpha}$, Potter's bound for regularly varying functions ([16], Theorem 1.5.6) implies that for every fixed $\delta > 0$,

$$\frac{\bar{F}(K_N)}{\bar{F}(a_N)} \leq C_\delta \eta_N^{-\alpha-\delta}$$

for all large N . Using $m_N \bar{F}(a_N) \rightarrow 1$, we obtain

$$m_N K_N^2 \bar{F}(K_N) \leq C_\delta a_N^2 \eta_N^{2-\alpha-\delta}.$$

This proves the expectation bound. Markov's inequality gives the $O_{\mathbb{P}}$ -bound.

For the high-strength lower bound,

$$\sum_{i \in B_\Delta} \kappa_i^2 = \sum_{i \in V_\Delta} \kappa_i^2 - \sum_{i \in V_\Delta \setminus B_\Delta} \kappa_i^2 \geq M_\Delta^2 - \sum_{i \in V_\Delta \setminus B_\Delta} \kappa_i^2.$$

By Lemma 3, $M_\Delta^2 = \Omega_{\mathbb{P}}(a_N^2)$, whereas the second term is $o_{\mathbb{P}}(a_N^2)$ if $\eta_N^{2-\alpha-\delta} \rightarrow 0$. The claim follows. $\square$

Proposition 4. Under Assumptions 1 and 2, suppose that the upper mean-degree bound in (13) holds uniformly for all strengths below K_N . Then, for every fixed $\delta \in (0, 2 - \alpha)$,

$$\sum_{i \in V_\Delta \setminus B_\Delta} (k_i^{\text{int}})^2 = O_{\mathbb{P}} \left(m_N + a_N^2 \eta_N^{2-\alpha-\delta} \right).$$

The same estimate holds with $(k_i^{\text{int}})^2$ replaced by k_i^2 .

Proof. Condition first on $\mathcal{F}_N$. Since k_i^{int} is a Bernoulli sum with conditional mean λ_i^{int} ,

$$\mathbb{E}[(k_i^{\text{int}})^2 \mid \mathcal{F}_N] = \text{Var}(k_i^{\text{int}} \mid \mathcal{F}_N) + (\lambda_i^{\text{int}})^2 \leq \lambda_i^{\text{int}} + (\lambda_i^{\text{int}})^2.$$

Since $\lambda_i^{\text{int}} \leq \lambda_i$,

$$\mathbb{E}[(k_i^{\text{int}})^2 \mid \theta_i, \kappa_i] \leq \mathbb{E}[\lambda_i + \lambda_i^2 \mid \theta_i, \kappa_i].$$

Now, $\lambda_i = \sum_{j \neq i} X_{ij}$, with independent $0 \leq X_{ij} \leq 1$, conditional on (θ_i, κ_i) . Since $X_{ij}^2 \leq X_{ij}$,

$$\text{Var}(X_{ij} \mid \theta_i, \kappa_i) \leq \mathbb{E}[X_{ij}^2 \mid \theta_i, \kappa_i] \leq \mathbb{E}[X_{ij} \mid \theta_i, \kappa_i].$$

Summing over $j \neq i$ gives

$$\text{Var}(\lambda_i \mid \theta_i, \kappa_i) \leq \bar{\lambda}_i.$$

Consequently,

$$\mathbb{E}[\lambda_i + \lambda_i^2 \mid \theta_i, \kappa_i] = \bar{\lambda}_i + \text{Var}(\lambda_i \mid \theta_i, \kappa_i) + \bar{\lambda}_i^2 \leq 2\bar{\lambda}_i + \bar{\lambda}_i^2.$$

Using the upper bound $\bar{\lambda}_i \leq C_\lambda \kappa_i$, we get

$$\mathbb{E}[(k_i^{\text{int}})^2 \mid \theta_i, \kappa_i] \leq C(\kappa_i + \kappa_i^2).$$

Thus,

$$\mathbb{E} \left[\sum_{i \in V_\Delta \setminus B_\Delta} (k_i^{\text{int}})^2 \right] \leq C m_N \mathbb{E}[\kappa \mathbf{1}_{\{\kappa < K_N\}}] + C m_N \mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa < K_N\}}].$$

The first term is $O(m_N)$ because $\mathbb{E}[\kappa] < \infty$. The second term is

$$O(a_N^2 \eta_N^{2-\alpha-\delta})$$

by Proposition 3. Markov's inequality yields the result. The proof for k_i^2 is identical. $\square$

Before stating the high-strength lower bound, we note that $L_N \rightarrow \infty$ may be chosen arbitrarily slowly, subject to the scale condition below.

Proposition 5. Under Assumptions 1 and 2, suppose that there exist deterministic sequences $r_N \in (0, 1]$, $\rho_N \downarrow 0$, and $L_N \rightarrow \infty$ such that the following properties hold.

(B1) With probability tending to one,

$$\inf_{i \in B_\Delta} R_i \geq c_R r_N$$

for some constant $c_R > 0$.

(B2) With probability tending to one,

$$\max_{i \in V_\Delta} \kappa_i \leq \rho_N N.$$

(B3) The sequences satisfy

$$L_N \log N = o(r_N K_N), \quad \eta_N^{2-\alpha-\delta} \rightarrow 0$$

for some $\delta \in (0, 2 - \alpha)$.

Then,

$$\sum_{i \in B_\Delta} R_i^2 k_i^2 = \Omega_{\mathbb{P}}(r_N^2 a_N^2),$$

and

$$\sum_{i \in B_\Delta} k_i^2 = \Omega_{\mathbb{P}}(a_N^2).$$

Proof. We first establish the inclusion $B_\Delta \subseteq S_N$ using only bounds on the mixed conditional means. Define

$$E_N^{(1)} := \left\{ \inf_{i \in B_\Delta} R_i \geq c_R r_N \right\}, \quad E_N^{(2)} := \left\{ \max_{i \in V_\Delta} \kappa_i \leq \rho_N N \right\}.$$

By (B1) and (B2),

$$\mathbb{P}(E_N^{(1)} \cap E_N^{(2)}) \rightarrow 1.$$

On $E_N^{(2)}$, Theorem 1 gives constants $c_\lambda, C_\lambda > 0$ such that, uniformly over $i \in V_\Delta$,

$$c_\lambda \kappa_i \leq \bar{\lambda}_i \leq C_\lambda \kappa_i.$$

Consequently, on $E_N^{(1)} \cap E_N^{(2)}$, every $i \in B_\Delta$ satisfies

$$\bar{\lambda}_i \geq c_\lambda K_N$$

and

$$\bar{\lambda}_i^{\text{int}} = R_i \bar{\lambda}_i \geq c_R c_\lambda r_N K_N.$$

Since $r_N \leq 1$, it follows that

$$\min\{\bar{\lambda}_i, \bar{\lambda}_i^{\text{int}}\} \geq c_* r_N K_N, \quad c_* := c_\lambda \min\{1, c_R\} > 0.$$

The scale condition $L_N \log N = o(r_N K_N)$ therefore implies that, for all sufficiently large N ,

$$\min\{\bar{\lambda}_i, \bar{\lambda}_i^{\text{int}}\} \geq 2L_N \log N, \quad i \in B_\Delta,$$

on $E_N^{(1)} \cap E_N^{(2)}$. By the definition of S_N , this proves

$$B_\Delta \subseteq S_N$$

on an event whose probability tends to one, and hence

$$\mathbb{P}(B_\Delta \subseteq S_N) \rightarrow 1.$$

No concentration estimate is used in establishing this inclusion.

We now apply the concentration results only after the inclusion has been proved. On the event $\{B_\Delta \subseteq S_N\}$,

$$\max_{i \in B_\Delta} \left| \frac{k_i}{\lambda_i} - 1 \right| \leq \max_{i \in S_N} \left| \frac{k_i}{\lambda_i} - 1 \right|,$$

and

$$\max_{i \in B_\Delta} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right| \leq \max_{i \in S_N} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right|.$$

Propositions 2 and 1, together with $\mathbb{P}(B_\Delta \subseteq S_N) \rightarrow 1$, yield

$$\max_{i \in B_\Delta} \left| \frac{k_i}{\lambda_i} - 1 \right| \xrightarrow{\mathbb{P}} 0$$

and

$$\max_{i \in B_\Delta} \left| \frac{\lambda_i}{\bar{\lambda}_i} - 1 \right| \xrightarrow{\mathbb{P}} 0.$$

Therefore,

$$\max_{i \in B_\Delta} \left| \frac{k_i}{\bar{\lambda}_i} - 1 \right| \xrightarrow{\mathbb{P}} 0.$$

Consequently,

$$\sum_{i \in B_\Delta} k_i^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_\Delta} \bar{\lambda}_i^2 \asymp \sum_{i \in B_\Delta} \kappa_i^2.$$

By Proposition 3,

$$\sum_{i \in B_\Delta} \kappa_i^2 = \Omega_{\mathbb{P}}(a_N^2),$$

and hence

$$\sum_{i \in B_\Delta} k_i^2 = \Omega_{\mathbb{P}}(a_N^2).$$

On $E_N^{(1)}$,

$$\sum_{i \in B_\Delta} R_i^2 k_i^2 \geq c_R^2 r_N^2 \sum_{i \in B_\Delta} k_i^2,$$

which proves the weighted lower bound. $\square$

Lemma 4. Under Assumptions 1 and 2, suppose that the upper mean-degree bound in (13) holds on the relevant strength range. Then,

$$\sum_{i \in V_\Delta} k_i = O_{\mathbb{P}}(m_N), \quad \sum_{i \in V_\Delta} k_i^{\text{int}} = O_{\mathbb{P}}(m_N).$$

Consequently, if

$$m_N = o(r_N^2 a_N^2),$$

then

$$\sum_{i \in V_\Delta} k_i = o_{\mathbb{P}}(r_N^2 a_N^2), \quad \sum_{i \in V_\Delta} k_i^{\text{int}} = o_{\mathbb{P}}(r_N^2 a_N^2).$$

Proof. By iterated expectation,

$$\mathbb{E} \left[\sum_{i \in V_\Delta} k_i \right] = \mathbb{E} \left[\sum_{i \in V_\Delta} \bar{\lambda}_i \right] \leq C_\lambda \mathbb{E} \left[\sum_{i \in V_\Delta} \kappa_i \right].$$

Since angle and strength are independent,

$$\mathbb{E} \left[\sum_{i \in V_\Delta} \kappa_i \right] = m_N \mathbb{E}[\kappa],$$

which is finite. Therefore,

$$\mathbb{E} \left[\sum_{i \in V_\Delta} k_i \right] = O(m_N).$$

Markov's inequality gives the first $O_{\mathbb{P}}$ -bound. Since $k_i^{\text{int}} \leq k_i$, the same estimate holds for the internal sum. The final statement follows from the deterministic condition $m_N = o(r_N^2 a_N^2)$. $\square$

Theorem 2. Under Assumptions 1 and 2, suppose that there exist deterministic sequences

$$r_N \in (0, 1], \quad \rho_N \downarrow 0, \quad L_N \rightarrow \infty,$$

a constant $c_R > 0$, and a number $\delta \in (0, 2 - \alpha)$ such that the following conditions hold.

(T1) With probability tending to one,

$$\inf_{i \in B_\Delta} R_i \geq c_R r_N.$$

(T2) With probability tending to one,

$$\max_{i \in V_\Delta} \kappa_i \leq \rho_N N.$$

(T3) The concentration scale satisfies

$$L_N \log N = o(r_N K_N).$$

(T4) The low-strength quadratic contribution and the linear degree contribution satisfy

$$a_N^2 \eta_N^{2-\alpha-\delta} + m_N = o(r_N^2 a_N^2). \quad (14)$$

Then,

$$\mathbb{P}(W_\Delta^{\text{full}} > 0) \rightarrow 1,$$

and

$$\Phi_\Delta = (1 + o_{\mathbb{P}}(1)) \frac{\sum_{i \in B_\Delta} R_i^2 k_i^2}{\sum_{i \in B_\Delta} k_i^2}. \quad (15)$$

Moreover,

$$\sum_{i \in V_\Delta \setminus B_\Delta} (k_i^{\text{int}})^2 = o_{\mathbb{P}} \left(\sum_{i \in B_\Delta} R_i^2 k_i^2 \right),$$

and

$$\sum_{i \in V_\Delta \setminus B_\Delta} k_i^2 = o_{\mathbb{P}} \left(\sum_{i \in B_\Delta} k_i^2 \right).$$

The four assumptions have distinct roles. Condition (T1) gives a uniform lower bound on the expected fraction of neighbors retained inside the window for vertices in the high-strength set. Condition (T2) keeps all window strengths in the sublinear range where Theorem 1 provides a uniform linear approximation of the mixed mean degree. Condition (T3) places the full and internal expected degrees above the logarithmic scale required for the uniform Bernstein and Chernoff estimates. Condition (T4) makes the low-strength quadratic contribution and the linear corrections in the wedge counts negligible relative to the retained high-strength scale. Together, these conditions ensure that both wedge counts are asymptotically determined by the same high-strength set, with the internal count weighted by the squared kernel-mass ratios.

Proof. Since $r_N \leq 1$, condition (T4) implies

$$\eta_N^{2-\alpha-\delta} = o(r_N^2) \rightarrow 0.$$

Thus, conditions (T1)–(T3), together with this consequence of (T4), verify all assumptions of Proposition 5. We use the two lower bounds from that proposition below.

By Proposition 4 and (14),

$$\sum_{i \in V_\Delta \setminus B_\Delta} (k_i^{\text{int}})^2 = o_{\mathbb{P}}(r_N^2 a_N^2).$$

By Proposition 5,

$$\sum_{i \in B_\Delta} R_i^2 k_i^2 = \Omega_{\mathbb{P}}(r_N^2 a_N^2).$$

Dividing the last two relations yields

$$\sum_{i \in V_{\Delta} \setminus B_{\Delta}} (k_i^{\text{int}})^2 = o_{\mathbb{P}} \left(\sum_{i \in B_{\Delta}} R_i^2 k_i^2 \right).$$

The proof for the full square term is the same, using

$$\sum_{i \in B_{\Delta}} k_i^2 = \Omega_{\mathbb{P}}(a_N^2).$$

By Corollary 1,

$$\max_{i \in B_{\Delta}} |q_i / R_i - 1| \xrightarrow{\mathbb{P}} 0.$$

Hence,

$$\sum_{i \in B_{\Delta}} (k_i^{\text{int}})^2 = \sum_{i \in B_{\Delta}} q_i^2 k_i^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}} R_i^2 k_i^2.$$

Adding the negligible low-strength part gives

$$\sum_{i \in V_{\Delta}} (k_i^{\text{int}})^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}} R_i^2 k_i^2.$$

Similarly,

$$\sum_{i \in V_{\Delta}} k_i^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}} k_i^2.$$

By Lemma 4 and (14),

$$\sum_{i \in V_{\Delta}} k_i^{\text{int}} = o_{\mathbb{P}}(r_N^2 a_N^2) = o_{\mathbb{P}} \left(\sum_{i \in B_{\Delta}} R_i^2 k_i^2 \right),$$

and

$$\sum_{i \in V_{\Delta}} k_i = o_{\mathbb{P}}(a_N^2) = o_{\mathbb{P}} \left(\sum_{i \in B_{\Delta}} k_i^2 \right).$$

Therefore,

$$\sum_{i \in V_{\Delta}} k_i^{\text{int}} (k_i^{\text{int}} - 1) = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}} R_i^2 k_i^2,$$

and

$$W_{\Delta}^{\text{full}} = \sum_{i \in V_{\Delta}} k_i (k_i - 1) = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}} k_i^2.$$

Because $m_N \rightarrow \infty$, relation (4) implies $a_N \rightarrow \infty$. Proposition 5 therefore gives

$$W_{\Delta}^{\text{full}} = \Omega_{\mathbb{P}}(a_N^2),$$

and hence

$$\mathbb{P}(W_{\Delta}^{\text{full}} > 0) \rightarrow 1.$$

On this event, dividing the two factorial-moment relations gives (15). Since $\Phi_{\Delta} = 0$ on the complementary event and that event has probability tending to zero, the same asymptotic relation holds in probability under the convention in (3). $\square$

Figure 2 schematically illustrates how external neighbors of a high-strength window vertex enlarge the full wedge count relative to the internal wedge count.

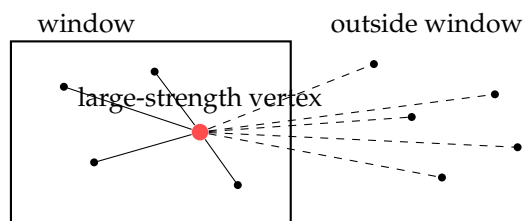


Figure 2. Wedge comparison for a window vertex with many external neighbors. The red filled circle represents the large-strength center vertex, while the black filled circles represent its neighboring vertices. Solid lines indicate connections to internal neighbors inside the window, whereas dashed lines indicate connections to external neighbors outside the window. Thus, the center vertex contributes many full wedges but comparatively fewer internal wedges.

6. Standard and Conditional Regimes

We first prove the standard local-window result. We then state a conditional linear-strength theorem and verify the compatibility of its assumptions in a finite-hub conditional construction.

Theorem 3. Under Assumptions 1 and 2, let

$$\Delta = N^{-s}, \quad 0 < s < 1,$$

so that

$$m_N = \frac{N^{1-s}}{\pi} \rightarrow \infty.$$

Let a_N satisfy (4), and choose any deterministic sequence η_N such that

$$\eta_N \downarrow 0, \quad \frac{a_N \eta_N}{\log N} \rightarrow \infty.$$

Then, the high-strength threshold $K_N = a_N \eta_N$ satisfies the assumptions of Theorem 2 with $r_N \equiv 1$, and

$$\inf_{i \in B_\Delta} R_i = \Omega_{\mathbb{P}}(1), \quad \sup_{i \in B_\Delta} R_i = O_{\mathbb{P}}(1).$$

Consequently,

$$\Phi_\Delta = \Theta_{\mathbb{P}}(1).$$

Proof. By regular variation, the extreme scale has the form

$$a_N = m_N^{1/\alpha} \ell(m_N)$$

for a slowly varying function ℓ . Since $m_N = N\Delta/\pi$,

$$\frac{a_N}{N\Delta} = \frac{a_N}{\pi m_N} = O\left(m_N^{1/\alpha-1} \ell(m_N)\right) \rightarrow 0,$$

because $\alpha > 1$. By Lemma 3,

$$M_\Delta = \Theta_{\mathbb{P}}(a_N),$$

and therefore

$$\frac{M_\Delta}{N\Delta} \xrightarrow{\mathbb{P}} 0.$$

Fix $\varepsilon \in (0, 1/4)$. By the tail-ratio consequence of Lemma 1, choose a fixed truncation constant $T_\varepsilon > 0$ so large that

$$\frac{\int_{T_\varepsilon}^\infty \Psi_\kappa(u) \, du}{\int_0^\infty \Psi_\kappa(u) \, du} \leq \varepsilon$$

for all $\kappa > 0$.

Define

$$A_N := \{T_\varepsilon M_\Delta \leq N\Delta\}.$$

The constant T_ε depends only on ε . Since $M_\Delta/(N\Delta) \xrightarrow{\mathbb{P}} 0$, we have $\mathbb{P}(A_N) \rightarrow 1$. On A_N , every $i \in B_\Delta$ satisfies

$$T_\varepsilon \kappa_i \leq N\Delta.$$

For $i \in V_\Delta$, set

$$I_i := [-N(\Delta + \theta_i), N(\Delta - \theta_i)].$$

The interval I_i contains 0, and at least one of its two sides has length at least $N\Delta$. Hence, I_i contains either $[0, T_\varepsilon \kappa_i]$ or $[-T_\varepsilon \kappa_i, 0]$. By the evenness of Ψ_{κ_i} ,

$$\int_{I_i} \Psi_{\kappa_i}(u) \, du \geq \int_0^{T_\varepsilon \kappa_i} \Psi_{\kappa_i}(u) \, du.$$

Using (11),

$$R_i \geq \frac{\int_0^{T_\varepsilon \kappa_i} \Psi_{\kappa_i}(u) \, du}{2 \int_0^\infty \Psi_{\kappa_i}(u) \, du} \geq \frac{1 - \varepsilon}{2}.$$

The upper bound $R_i \leq 1$ is immediate from the definition. This proves the bounds on R_i .

Now, choose any $\delta \in (0, 2 - \alpha)$. Since $\eta_N \downarrow 0$,

$$\eta_N^{2-\alpha-\delta} \rightarrow 0.$$

Also,

$$\frac{m_N}{a_N^2} = O\left(m_N^{1-2/\alpha} \ell(m_N)^{-2}\right) \rightarrow 0,$$

because $\alpha < 2$. Hence, (14) holds with $r_N \equiv 1$.

The assumptions of Proposition 5 are satisfied. The lower bound on R_i gives (B1). Since $M_\Delta = o_{\mathbb{P}}(N)$, there exists a deterministic sequence $\rho_N \downarrow 0$ such that

$$\mathbb{P}(M_\Delta \leq \rho_N N) \rightarrow 1,$$

which gives (B2). To verify (B3) explicitly, define

$$L_N := \left(\frac{K_N}{\log N}\right)^{1/2}.$$

Since $K_N/\log N \rightarrow \infty$, this choice satisfies

$$L_N \rightarrow \infty$$

and

$$\frac{L_N \log N}{K_N} = \left(\frac{\log N}{K_N}\right)^{1/2} \rightarrow 0.$$

Because $r_N \equiv 1$ in the present regime, it follows that

$$L_N \log N = o(r_N K_N) = o(K_N),$$

which verifies (B3).

Theorem 2 yields

$$\Phi_{\Delta} = (1 + o_{\mathbb{P}}(1)) \frac{\sum_{i \in B_{\Delta}} R_i^2 k_i^2}{\sum_{i \in B_{\Delta}} k_i^2}.$$

Since R_i is uniformly bounded above and below away from 0 on B_{Δ} , the weighted average is $\Theta_{\mathbb{P}}(1)$. Therefore, $\Phi_{\Delta} = \Theta_{\mathbb{P}}(1)$. $\square$

The following theorem is conditional. Its assumptions isolate a linear-strength subset whose full and internal quadratic-degree masses dominate the corresponding contributions from the remaining window vertices. The theorem makes no claim that these assumptions hold typically in the original i.i.d. model. Proposition 6 below gives a finite-hub conditional construction in which all assumptions hold with probability tending to one.

Theorem 4. Under Assumptions 1 and 2, suppose additionally that there exist constants $u_0 > 0$ and $g_0 > 0$ such that

$$g(u) \geq g_0, \quad 0 \leq u \leq u_0.$$

Let

$$B_{\Delta}^{\text{lin}} \subseteq B_{\Delta}$$

be a possibly random subset such that, with probability tending to one, the following conditions hold.

(M1) For fixed constants $0 < c < C < \infty$,

$$|\theta_i| \leq \Delta/2, \quad cN \leq \kappa_i \leq CN, \quad i \in B_{\Delta}^{\text{lin}}.$$

(M2) The squared-degree mass of the linear-strength subset diverges and dominates the full squared-degree contribution outside that subset:

$$\sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2 \xrightarrow{\mathbb{P}} \infty, \quad \sum_{i \in V_{\Delta} \setminus B_{\Delta}^{\text{lin}}} k_i^2 = o_{\mathbb{P}} \left(\sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2 \right).$$

(M3) The internal squared-degree contribution outside the linear-strength subset is negligible at the Δ^2 scale:

$$\sum_{i \in V_{\Delta} \setminus B_{\Delta}^{\text{lin}}} (k_i^{\text{int}})^2 = o_{\mathbb{P}} \left(\Delta^2 \sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2 \right).$$

(M4) The linear degree terms are negligible at their respective quadratic scales:

$$\sum_{i \in V_{\Delta}} k_i = o_{\mathbb{P}} \left(\sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2 \right),$$

and

$$\sum_{i \in V_{\Delta}} k_i^{\text{int}} = o_{\mathbb{P}} \left(\Delta^2 \sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2 \right).$$

(M5) The empirical internal-degree fractions concentrate around the kernel-mass ratios on the linear-strength subset:

$$\max_{i \in B_{\Delta}^{\text{lin}}} \left| \frac{q_i}{R_i} - 1 \right| \xrightarrow{\mathbb{P}} 0.$$

Then,

$$R_i \asymp \Delta$$

uniformly on B_{Δ}^{lin} ,

$$\mathbb{P}(W_{\Delta}^{\text{full}} > 0) \rightarrow 1,$$

and

$$\Phi_{\Delta} = \Theta_{\mathbb{P}}(\Delta^2).$$

Proof. Fix $i \in B_{\Delta}^{\text{lin}}$. Since $|\theta_i| \leq \Delta/2$, the interval in (11) contains

$$[-N\Delta/2, N\Delta/2].$$

For $|u| \leq N\Delta/2$, using $\kappa_i \geq cN$ and $\kappa' \geq \kappa_{\min}$,

$$\frac{|u|}{\mu\kappa_i\kappa'} \leq \frac{N\Delta/2}{\mu cN\kappa_{\min}} = \frac{\Delta}{2\mu c\kappa_{\min}}.$$

For all sufficiently large N , this is at most u_0 . The local positivity assumption gives $\Psi_{\kappa_i}(u) \geq g_0$ on $[-N\Delta/2, N\Delta/2]$. Hence,

$$\int_{-N(\Delta+\theta_i)}^{N(\Delta-\theta_i)} \Psi_{\kappa_i}(u) \, du \geq g_0 N\Delta.$$

Since $\Psi_{\kappa_i} \leq 1$,

$$\int_{-N(\Delta+\theta_i)}^{N(\Delta-\theta_i)} \Psi_{\kappa_i}(u) \, du \leq 2N\Delta.$$

For the denominator, define

$$\beta := \min\left\{\pi, \frac{\mu c\kappa_{\min} u_0}{2}\right\} > 0.$$

If $|u| \leq \beta N$, then

$$\frac{|u|}{\mu\kappa_i\kappa'} \leq \frac{\beta N}{\mu cN\kappa_{\min}} \leq u_0.$$

Thus, $\Psi_{\kappa_i}(u) \geq g_0$ on $[-\beta N, \beta N]$, and

$$2\beta g_0 N \leq \int_{-N\pi}^{N\pi} \Psi_{\kappa_i}(u) \, du \leq 2\pi N.$$

Combining these estimates in (11) gives constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 \Delta \leq R_i \leq c_2 \Delta$$

uniformly on B_{Δ}^{lin} .

By (M5),

$$\sum_{i \in B_{\Delta}^{\text{lin}}} (k_i^{\text{int}})^2 = \sum_{i \in B_{\Delta}^{\text{lin}}} q_i^2 k_i^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}^{\text{lin}}} R_i^2 k_i^2.$$

The uniform relation $R_i \asymp \Delta$ yields

$$\sum_{i \in B_{\Delta}^{\text{lin}}} R_i^2 k_i^2 = \Theta_{\mathbb{P}}\left(\Delta^2 \sum_{i \in B_{\Delta}^{\text{lin}}} k_i^2\right).$$

Conditions (M2) and (M3) therefore give

$$\sum_{i \in V_{\Delta}} (k_i^{\text{int}})^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_{\Delta}^{\text{lin}}} R_i^2 k_i^2$$

and

$$\sum_{i \in V_\Delta} k_i^2 = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_\Delta^{\text{lin}}} k_i^2.$$

Condition (M4) permits the replacement of squared degrees by the corresponding factorial terms. In particular,

$$W_\Delta^{\text{full}} = (1 + o_{\mathbb{P}}(1)) \sum_{i \in B_\Delta^{\text{lin}}} k_i^2.$$

The divergence in (M2) implies

$$\mathbb{P}(W_\Delta^{\text{full}} > 0) \rightarrow 1.$$

On this event,

$$\Phi_\Delta = (1 + o_{\mathbb{P}}(1)) \frac{\sum_{i \in B_\Delta^{\text{lin}}} R_i^2 k_i^2}{\sum_{i \in B_\Delta^{\text{lin}}} k_i^2} = \Theta_{\mathbb{P}}(\Delta^2).$$

□

Proposition 6. Let $\Delta = N^{-s}$ with $0 < s < 1$, and let $J \geq 1$ be fixed. Set $H = \{1, \dots, J\}$. For fixed constants $0 < c < C < \infty$, let $\mathbb{P}_N^{\text{hub}}$ denote the law of the original model conditioned on

$$\mathcal{H}_N := \bigcap_{i \in H} \{|\theta_i| \leq \Delta/2, cN \leq \kappa_i \leq CN\}.$$

All mixed conditional means and all ratios R_i in this proposition are computed under $\mathbb{P}_N^{\text{hub}}$. Assume the local positivity condition in Theorem 4. Choose γ such that

$$\frac{1-s}{\alpha} < \gamma < \min\left\{1, \frac{1-s}{2-\alpha}\right\}. \quad (16)$$

Then, under $\mathbb{P}_N^{\text{hub}}$, the set

$$B_\Delta^{\text{lin}} := H$$

satisfies (M1)–(M5) of Theorem 4 with probability tending to one. In particular,

$$\Phi_\Delta = \Theta_{\mathbb{P}_N^{\text{hub}}}(\Delta^2).$$

Proof. Regular variation implies

$$\mathbb{P}(cN \leq \kappa \leq CN) = \bar{F}(cN) - \bar{F}(CN) > 0$$

for all sufficiently large N . Hence, $\mathcal{H}_N$ has positive probability, and $\mathbb{P}_N^{\text{hub}}$ is well defined. Under this conditional law, the latent variables of vertices outside H retain their original independent distribution. Condition (M1) holds by construction.

Set $b_N = N^\gamma$. Since $m_N = N^{1-s}/\pi$, regular variation and the lower bound in (16) give

$$\mathbb{P}_N^{\text{hub}}\left(\max_{i \in V_\Delta \setminus H} \kappa_i > b_N\right) \leq m_N \bar{F}(b_N) = N^{1-s-\alpha\gamma+o(1)} \rightarrow 0.$$

Let $\hat{k}_i$ be the degree of a non-hub vertex after deleting its at most J incident edges to vertices in H . On the event above, all non-hub window strengths are at most $b_N = o(N)$. The upper

mean-degree estimate from Theorem 1, together with the second-moment estimate used in Proposition 4, yields

$$\sum_{i \in V_\Delta \setminus H} \widehat{k}_i^2 = O_{\mathbb{P}_N^{\text{hub}}} \left(m_N + m_N \mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa \leq b_N\}}] \right).$$

Karamata's theorem gives

$$\mathbb{E}[\kappa^2 \mathbf{1}_{\{\kappa \leq b_N\}}] = b_N^{2-\alpha+o(1)},$$

and therefore

$$\sum_{i \in V_\Delta \setminus H} k_i^2 = O_{\mathbb{P}_N^{\text{hub}}} \left(N^{1-s+\gamma(2-\alpha)+o(1)} \right). \quad (17)$$

Here, we used $k_i \leq \widehat{k}_i + J$ and the fact that J is fixed. Since $k_i^{\text{int}} \leq k_i$, the same upper bound holds for

$$\sum_{i \in V_\Delta \setminus H} (k_i^{\text{int}})^2.$$

For a hub $i \in H$, local positivity on a fixed angular neighborhood gives

$$\bar{\lambda}_i = \Theta(N),$$

while $|\theta_i| \leq \Delta/2$ and local positivity throughout the window give

$$\bar{\lambda}_i^{\text{int}} = \Theta(N\Delta).$$

Bernstein and Chernoff bounds, applied to the fixed set H , imply

$$k_i = \Theta_{\mathbb{P}_N^{\text{hub}}}(N), \quad k_i^{\text{int}} = \Theta_{\mathbb{P}_N^{\text{hub}}}(N\Delta),$$

uniformly for $i \in H$. They also imply

$$\max_{i \in H} \left| \frac{q_i}{R_i} - 1 \right| \rightarrow 0$$

in $\mathbb{P}_N^{\text{hub}}$ -probability. In particular,

$$\sum_{i \in H} k_i^2 = \Theta_{\mathbb{P}_N^{\text{hub}}}(N^2), \quad \Delta^2 \sum_{i \in H} k_i^2 = \Theta_{\mathbb{P}_N^{\text{hub}}}(N^2 \Delta^2).$$

The upper bound in (16) is equivalent to

$$1 - s + \gamma(2 - \alpha) < 2 - 2s.$$

Combining this inequality with (17) proves (M2) and (M3).

Finally, first-moment bounds give

$$\sum_{i \in V_\Delta} k_i = O_{\mathbb{P}_N^{\text{hub}}}(N)$$

and

$$\sum_{i \in V_\Delta} k_i^{\text{int}} = O_{\mathbb{P}_N^{\text{hub}}}(N\Delta).$$

Since $N\Delta \rightarrow \infty$, these estimates imply (M4). The concentration estimate above gives (M5). Thus, all assumptions of Theorem 4 hold under $\mathbb{P}_N^{\text{hub}}$, and the stated conclusion follows. $\square$

Remark 6. The conditioning event $\mathcal{H}_N$ is rare under the original i.i.d. law. Proposition 6 establishes that the assumptions of Theorem 4 are mutually compatible and arise in a precise finite-hub conditional model. The proposition establishes compatibility only; typicality under the standard i.i.d. model remains outside its scope.

Remark 7. Under the standard local-window regime of Theorem 3, the maximum window strength satisfies

$$M_\Delta = \Theta_{\mathbb{P}}(a_N), \quad a_N \ll N\Delta.$$

Thus, the typical high-strength set remains below the linear scale $\kappa_i \asymp N$ used in Theorem 4. The standard and conditional results therefore describe distinct asymptotic settings.

The contrast between these two asymptotic settings is summarized schematically in Figure 3.

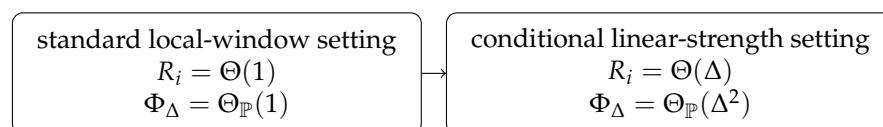


Figure 3. Comparison of the two formally stated settings. The standard local-window theorem yields $\Phi_\Delta = \Theta_{\mathbb{P}}(1)$, whereas the conditional linear-strength theorem yields $\Phi_\Delta = \Theta_{\mathbb{P}}(\Delta^2)$ under its additional assumptions.

7. Discussion

The statistic Φ_Δ compares two doubled wedge counts built from the same set of center vertices. Its numerator equals twice the number of wedges whose two endpoints remain inside V_Δ , while its denominator equals twice the number of all wedges centered at vertices of V_Δ in the full graph. Thus, Φ_Δ measures wedge retention across observation scales. It does not count closed triangles and is related to conventional clustering coefficients only through the shared wedge-counting denominator.

The main theorem identifies the leading mechanism in the infinite-variance range $1 < \alpha < 2$. Since the wedge contribution of vertex i is $k_i(k_i - 1)$, its leading magnitude is quadratic in degree. The high-strength set therefore carries the dominant quadratic mass, and the asymptotic formula

$$\Phi_\Delta = (1 + o_{\mathbb{P}}(1)) \frac{\sum_{i \in B_\Delta} R_i^2 k_i^2}{\sum_{i \in B_\Delta} k_i^2}$$

shows that the retained fraction is determined by a degree-square-weighted average of the squared kernel-mass ratios R_i .

The two results in Section 6 concern distinct asymptotic settings. In the standard shrinking-window model, the dominant strengths satisfy $a_N \ll N\Delta$, and the kernel-mass ratios on the high-strength set remain bounded away from zero. The resulting statement is

$$\Phi_\Delta = \Theta_{\mathbb{P}}(1).$$

The order- Δ^2 statement is conditional on a dominant linear-strength subset satisfying the assumptions of Theorem 4. In that setting, $R_i = \Theta(\Delta)$ on the degree-square-dominant vertices, which yields

$$\Phi_\Delta = \Theta_{\mathbb{P}}(\Delta^2).$$

Proposition 6 verifies that these assumptions are compatible in a finite-hub conditional construction. The construction serves solely as a compatibility example under a conditional law and does not describe typical realizations of the original i.i.d. model.

The finite-variance case $\alpha > 2$ falls outside the present extreme-value argument because $\mathbb{E}[\kappa^2] < \infty$. In the absence of an imposed linear-strength subset, the natural scale of both quadratic degree sums is expected to be m_N , with contributions accumulated from many typical and moderate-strength vertices. A corresponding analysis would therefore use a law-of-large-numbers-type argument with normalization m_N and would seek a bulk-average limit for the numerator and denominator. Establishing such a result requires control of the dependence among degree-square terms in the triangular graph array and is left as an open problem.

Further directions include identifying sufficient conditions for convergence of the weighted ratio to a deterministic constant, extending the analysis to higher-dimensional latent spaces, and studying related wedge-retention statistics for threshold and positive-temperature connection kernels.

8. Summary of Asymptotic Regimes

Table 1 compares the two asymptotic settings considered in this paper in terms of their dominant strength scales, kernel-mass ratios, and resulting orders of the wedge-retention ratio. Table 2 summarizes the principal asymptotic quantities and the sufficient scaling conditions used in the proofs.

Table 1. Comparison of the standard local-window theorem and the conditional linear-strength theorem.

Setting	Dominant Strength Scale	Kernel-Mass Ratio	Consequence
Standard local window	$\kappa_i = \Theta_{\mathbb{P}}(a_N)$, with $a_N \ll N\Delta$	$\Theta_{\mathbb{P}}(1)$	$\Phi_{\Delta} = \Theta_{\mathbb{P}}(1)$
Conditional linear-strength setting	$\kappa_i \asymp N$ on a dominant high-strength subset	$\Theta_{\mathbb{P}}(\Delta)$	$\Phi_{\Delta} = \Theta_{\mathbb{P}}(\Delta^2)$

Table 2. Main asymptotic scales and sufficient conditions.

Quantity	Meaning	Sufficient Scaling Conditions Used in the Proofs
$m_N = N\Delta/\pi$	Expected number of window vertices	$m_N \rightarrow \infty$
a_N	Extreme scale in the window; see (4)	In the local regime, $a_N = m_N^{1/\alpha} \ell(m_N)$ for slowly varying ℓ
$K_N = a_N \eta_N$	High-strength threshold; see (5)	$\eta_N \downarrow 0$, $K_N / \log N \rightarrow \infty$ for concentration when $r_N = 1$
r_N	Lower scale for the kernel-mass ratio on B_{Δ}	$L_N \log N = o(r_N K_N)$, and the scale condition (14)
ρ_N	Upper relative strength scale for mean-degree regularity	$\rho_N \downarrow 0$, and $\max_{i \in V_{\Delta}} \kappa_i \leq \rho_N N$ with high probability

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