

Article

A Compactness Theorem for Locally Conformally Flat Riemannian Manifolds

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Abstract

In this paper, we prove that if a complete locally conformally flat Riemannian manifold with positive Yamabe invariant has a positive lower bound on its scalar curvature, an upper bound on its Ricci curvature, and the L^q norm ($q \geq 2$) of the traceless Ricci curvature is finite, then it is compact.

Keywords: compactness; locally conformally flat; Riemannian manifolds; integral curvature

MSC: 53C20

1. Introduction

The interplay between curvature and topology has long been a central theme in Riemannian geometry. A landmark result in this direction is Myers' compactness theorem [1], which asserts that a complete Riemannian manifold is compact whenever its Ricci curvature is bounded below by a positive constant. This theorem has since been extended in various directions (see [2–5]); for instance, Ambrose [2] and Galloway [4] showed that compactness still holds if, for some $\sigma \in [0, 1)$, the Ricci curvature Rc along every unit-speed geodesic emanating from a fixed point satisfies $\int_0^{+\infty} s^\sigma Rc(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty$. More recently, Ricci flow techniques have been employed by Chen-Zhu [6], Ni-Wu [7], Brendle-Schoen [8], and others to establish compactness under pinching conditions on the full Riemannian curvature tensor. Gu [9] and Ma-Cheng [10] used the Yamabe flow to prove compactness theorems for locally conformally flat Riemannian manifolds under pinching conditions on the Ricci curvature. Recently, the Hamilton conjecture, which asserts that a complete 3-dimensional Riemannian manifold with $Rc \geq \epsilon Sg \geq 0$ for some positive constant $\epsilon > 0$ is either flat or compact, was verified by Lee and Topping [11] using the Ricci flow. Here S denotes the scalar curvature of a Riemannian manifold. The analogous theorem in all dimensions was proved in [12] for Riemannian manifolds under the PIC1 pinching condition and in [13] for locally conformally flat Riemannian manifolds with pinched Ricci curvature.

In this paper, we consider locally conformally flat Riemannian manifolds with finite integral curvature. For locally conformally flat Riemannian manifolds with constant scalar curvature, Xu-Zhao [14] and Fu-Peng [15] proved the following compactness theorem:

Theorem 1 ([14,15]). *Let (M^n, g) be an n -dimensional complete and simply connected locally conformally flat Riemannian manifold, where $n \geq 3$. Denote by S and $\mathring{R}c$ the scalar curvature and the Ricci curvature tensor of M . Assume S is a positive constant and $\mathring{R}c$ satisfies $\int_M |\mathring{R}c|^q d\mu < \infty$ for some $q \geq n/2$, then M is compact.*



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In Theorem 1, the traceless Ricci curvature tensor $\mathring{R}c$ is defined to be $\mathring{R}c = Rc - \frac{S}{n}g$.

The aim of the present paper is to prove a similar compactness theorem for complete locally conformally flat Riemannian manifolds without requiring S to be constant.

Theorem 2. *Let (M^n, g) be an n -dimensional complete locally conformally flat Riemannian manifold, where $n \geq 3$. Assume $S \geq \lambda_1$ and $Rc \leq \lambda_2$ for some positive constants λ_1 and λ_2 , and the Yamabe invariant $Y_{[g]}$ of the conformal class $[g]$ is positive. If $\int_M |\mathring{R}c|^q d\mu < \infty$ for some $q \geq 2$, then M is compact.*

See Section 2 for the definition of the Yamabe invariant $Y_{[g]}$.

Theorem 2 will be proved by employing the Yamabe flow, as defined in Section 2. An analogous compactness theorem for complete hypersurfaces with finite integral curvature in the Euclidean space was proved by Cao-Zhao [16] with the aid of the mean curvature flow by a similar argument.

The principal difficulty lies in controlling the traceless Ricci curvature in the presence of nonconstant scalar curvature; this is addressed through the Yamabe flow and Moser iteration. The result extends the work of [14,15] to the variable scalar curvature regime and provides a new framework for compactness investigations. Moreover, it clarifies how global curvature constraints force a complete manifold to be closed, with potential applications to the study of singularities in geometric flows and the classification of non-compact manifolds.

The remainder of the paper proceeds as follows. Section 2 provides the necessary background, including evolution equations under the Yamabe flow. In Section 3, the Moser iteration is applied to obtain an L^∞ estimate on the traceless Ricci curvature. Theorem 2 will be proved in Section 4.

2. Notations and Preliminaries

In 1960, Yamabe [17] proposed a famous conjecture that every compact Riemannian manifold (M, g_0) admits a Riemannian metric g conformally equivalent to g_0 such that g has constant scalar curvature. This conjecture was solved by Trudinger [18], Aubin [19], and Schoen [20]. Hamilton [21] introduced the (normalized) Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t} g(t) = (\bar{S}(t) - S(t))g(t), \\ g(0) = g_0, \end{cases} \quad (1)$$

and attempted to use it as a tool to provide a new proof of the Yamabe conjecture. Here $S(t)$ is the scalar curvature of the metric $g(t)$, and

$$\bar{S}(t) = \frac{\int_M S(t) d\mu_t}{\int_M d\mu_t}$$

is the average of $S(t)$. Equation (1) is a modification of the negative gradient flow of the (normalized) total scalar curvature functional $\mathcal{S} = (\int_M d\mu)^{-\frac{n-2}{n}} \int_M S d\mu$, where n is the dimension of M . It is easy to check that the volume of the manifold remains constant along (1). Equation (1) is equivalent to the Yamabe flow

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -S(t)g(t), \\ g(0) = g_0. \end{cases} \quad (2)$$

In fact, assume that $g(t)$ satisfies (2), and let

$$\varphi(t) = \exp \left(\int_0^t \tilde{S}(\sigma) d\sigma \right)$$

and

$$s(t) = \int_0^t \varphi(\sigma) d\sigma.$$

Set $\tilde{g}(s) = \varphi(t(s))g(t(s))$, where $t(s)$ is the inverse function of $s(t)$. Then by a direct computation, $\tilde{g}(s)$ satisfies

$$\begin{cases} \frac{\partial}{\partial s} \tilde{g}(s) = (\tilde{\tilde{S}}(s) - \tilde{S}(s)) \tilde{g}(s), \\ \tilde{g}(0) = g_0, \end{cases}$$

where $\tilde{\tilde{S}}(s) = \frac{\int_M \tilde{S} d\tilde{\mu}_s}{\int_M d\tilde{\mu}_s}$.

The most important property of both (1) and (2) is that they preserve the conformal class of the metric, i.e., $g(t)$ and g_0 belong to the same conformal class. If (M, g_0) is a compact Riemannian manifold, the convergence of the Yamabe flow (1) has been studied successively by Hamilton [21], Chow [22], Ye [23], Schwetlick-Struwe [24], Brendle [25,26], among others. If (M, g_0) is a complete noncompact Riemannian manifold, there are relatively few results on the convergence of the Yamabe flow (1) or (2) [9,10,27–30].

In the sequel, we write S for $S(t)$, Rc for $Rc(t)$ and g for $g(t)$. S and Rc satisfy the following evolution equations along (2):

$$\frac{\partial}{\partial t} S = (n-1)\Delta S + S^2, \quad (3)$$

$$\frac{\partial}{\partial t} R_{ij} = \frac{1}{2}\Delta S g_{ij} + \frac{n-2}{2}\nabla_i \nabla_j S.$$

Here R_{ij} and g_{ij} denote the components of Rc and g under a local frame field $\{e_1, e_2, \dots, e_n\}$. If g_0 is locally conformally flat, then

$$\frac{\partial}{\partial t} R_{ij} = (n-1)\Delta R_{ij} + \frac{1}{n-2}B_{ij}, \quad (4)$$

where B_{ij} is given by

$$B_{ij} = (n-1)|Rc|^2 g_{ij} + nSR_{ij} - n(n-1)R_{ij}^2 - S^2 g_{ij}.$$

Define the traceless Ricci curvature tensor $\mathring{R}c$ by $\mathring{R}c = Rc - \frac{S}{n}g$. Then $|\mathring{R}c|^2 = |Rc|^2 - \frac{S^2}{n}$. From (3) and (4), the evolution equation for $|\mathring{R}c|^2$ can be obtained

$$\frac{\partial}{\partial t} |\mathring{R}c|^2 = (n-1)\Delta |\mathring{R}c|^2 - 2(n-1)|\nabla \mathring{R}c|^2 - \frac{2n(n-1)}{n-2} \text{tr}(\mathring{R}c^3).$$

The following inequality, which follows from the Lagrange multiplier method, is needed:

$$\text{tr}(\mathring{R}c^3) \geq -\frac{n-2}{\sqrt{n(n-1)}} |\mathring{R}c|^3.$$

Therefore,

$$\frac{\partial}{\partial t} |\mathring{R}c|^2 \leq (n-1)\Delta |\mathring{R}c|^2 - 2(n-1)|\nabla \mathring{R}c|^2 + 2\sqrt{n(n-1)} |\mathring{R}c|^3. \quad (5)$$

According to [17–20], the Yamabe invariant $Y_{[g]}$ on a complete Riemannian manifold (M^n, g) is defined by

$$Y_{[g]} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2 d\mu + \frac{n-2}{4(n-1)} \int_M S u^2 d\mu}{\left(\int_M |u|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}}}.$$

According to the definition of the Yamabe invariant, if $Y_{[g]} > 0$, then the following Sobolev-type inequality

$$\left(\int_M |u|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq Y_{[g]}^{-1} \left(\int_M |\nabla u|^2 d\mu + \frac{n-2}{4(n-1)} \int_M S u^2 d\mu \right) \quad (6)$$

holds for any $u \in C_0^\infty(M)$.

3. The L^∞ Estimate of $\mathring{R}c$

This section is devoted to deriving an L^∞ estimate for the traceless Ricci curvature. Here, the L^∞ norm of $\mathring{R}c$ is equal to $\sup_M |\mathring{R}c|$. We show that, after a short time, $|\mathring{R}c|$ becomes arbitrarily small outside some compact set. More precisely, we establish the following lemma:

Lemma 1. *Let (M^n, g_0) be an $n(n \geq 3)$ -dimensional complete non-compact locally conformally flat Riemannian manifold, and let $x_0 \in M$ be a fixed point. Assume that the scalar curvature S satisfies $S \geq \lambda_1$ and the Ricci curvature Rc satisfies $Rc \leq \lambda_2$, where λ_1 and λ_2 are positive constants, and the Yamabe invariant $Y_{[g_0]}$ of the conformal class $[g_0]$ is positive. Let $g(t)$ be the solution of the Yamabe flow with initial value g_0 . If for some $q \geq 2$, $\int_M |\mathring{R}c|^q d\mu < \infty$, then for any $\delta > 0$, there exist a time $T_0 > 0$ and a constant $R > 0$ such that $|\mathring{R}c(x, T_0)| \leq \delta$ for any $x \in M \setminus B_{g_0}(x_0, R)$.*

Proof. By the condition, the Ricci curvature of (M, g_0) is bounded in the sense that $|Rc| \leq \tilde{\lambda}$ for a positive constant $\tilde{\lambda}$ that depends only on n, λ_1, λ_2 . To see this, assume Rc is diagonalized as $Rc = \sum_{i=1}^n \mu_i \omega_i$ under an local orthonormal frame $\{e_i\}$, where $\{\omega_i\}$ denotes the dual frame of $\{e_i\}$. Then $S = \sum_{i=1}^n \mu_i$ and $\mu_i \leq \lambda_2$ for any $i = 1, 2, \dots, n$. For any i_0 , one has

$$\lambda_1 \leq S = \sum_{i=1}^n \mu_i = \mu_{i_0} + \sum_{i \neq i_0} \mu_i \leq \mu_{i_0} + (n-1)\lambda_2,$$

which implies

$$\mu_{i_0} \geq \lambda_1 - (n-1)\lambda_2.$$

Therefore, there holds $|Rc| \leq |\lambda_1| + |\lambda_1 - (n-1)\lambda_2|$. Setting $\tilde{\lambda} = |\lambda_1| + |\lambda_1 - (n-1)\lambda_2|$, we get $|Rc| \leq \tilde{\lambda}$.

By [10,27], the Yamabe flow with initial value g_0 has a smooth solution $g(t)$ on $[0, \theta]$, $\theta > 0$. Moreover, the Ricci curvature of $g(t)$ is uniformly bounded on $[0, \theta]$. Since $g(t)$ remains locally conformally flat under the Yamabe flow, the Riemannian curvature tensor is uniformly bounded for $t \in [0, \theta]$. Consequently, the parabolic maximum principle for scalar functions is applicable on this time interval.

By the maximum principle and (3), we have $S \geq \lambda_1$ on $[0, \theta]$. By the maximum principle and (4), we may suppose $|Rc| \leq 2\tilde{\lambda}$ for $t \in [0, \theta]$ by choosing θ small enough. Set $\lambda = 2\tilde{\lambda}$. Then λ is a positive constant depending only on n, λ_1, λ_2 .

By the assumptions of the lemma, we may assume $\int_M |\mathring{R}c|^q d\mu_{g_0} \leq \Lambda$, where Λ is a positive constant and $q \geq 2$. Let $T = \sup\{t \in [0, \min\{\theta, 1\}] : \int_M |\mathring{R}c|^q d\mu \leq 2\Lambda\}$.

Let $f_\varepsilon = |\mathring{R}c_\varepsilon|$, where $\mathring{R}c_\varepsilon = \mathring{R}c + \varepsilon g$. For $\varepsilon > 0$, we have $f_\varepsilon = \sqrt{|\mathring{R}c|^2 + n\varepsilon^2} > 0$. From inequality (5), we obtain

$$\frac{\partial}{\partial t} f_\varepsilon^2 \leq (n-1)\Delta f_\varepsilon^2 - 2(n-1)|\nabla f_\varepsilon|^2 + 2\sqrt{n(n-1)}|\mathring{R}c|f_\varepsilon^2,$$

where we used the inequality $|\nabla \mathring{R}c|^2 \geq |\nabla f_\varepsilon|^2$. Hence,

$$\frac{\partial}{\partial t} f_\varepsilon \leq (n-1)\Delta f_\varepsilon + \beta f_\varepsilon,$$

with β being a positive constant determined only by n and λ . For fixed $x_0 \in M$, let $B_{g(0)}(x_0, R)$ be the geodesic ball of radius R about x_0 under the metric g_0 , which we abbreviate as $B(R)$.

Let η be a Lipschitz function supported in $B(R)$. For any fixed $r, q \in \mathbf{R}$ with $r \geq q \geq 2$, we have

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial t} \int_{B(R)} f_\varepsilon^r \eta^2 d\mu_t &\leq \int_{B(R)} f_\varepsilon^{r-1} ((n-1)\Delta f_\varepsilon + \beta f_\varepsilon) \eta^2 d\mu_t + \frac{1}{r} \int_{B(R)} f_\varepsilon^r \eta^2 \frac{\partial}{\partial t} d\mu_t \\ &= (n-1) \int_{B(R)} \eta^2 f_\varepsilon^{r-1} \Delta f_\varepsilon d\mu_t + \int_{B(R)} \beta \eta^2 f_\varepsilon^r d\mu_t - \frac{n}{2r} \int_{B(R)} \eta^2 S f_\varepsilon^r d\mu_t. \end{aligned}$$

Here, we used the evolution equation for the volume form $d\mu_t$ along the Yamabe flow that $\frac{\partial}{\partial t} d\mu_t = -\frac{1}{2} n S d\mu_t$.

Through integration by parts, we obtain

$$\begin{aligned} \int_{B(R)} \eta^2 f_\varepsilon^{r-1} \Delta f_\varepsilon d\mu_t &= - \int_{B(R)} \langle \nabla(\eta^2 f_\varepsilon^{r-1}), \nabla f_\varepsilon \rangle d\mu_t \\ &= - \int_{B(R)} (2\eta f_\varepsilon^{r-1} \langle \nabla \eta, \nabla f_\varepsilon \rangle + (r-1)\eta^2 f_\varepsilon^{r-2} |\nabla f_\varepsilon|^2) d\mu_t \\ &= - \frac{4(r-1)}{r^2} \int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t + \frac{4}{r^2} \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t \\ &\quad + \frac{4(r-2)}{r^2} \int_{B(R)} \nabla(f_\varepsilon^{\frac{r}{2}} \eta) f_\varepsilon^{\frac{r}{2}} \nabla \eta d\mu_t \\ &\leq \int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t + \frac{4}{r^2} \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t \\ &\quad + \frac{2(r-2)}{r^2} \left(\int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t + \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t \right) \\ &= \left(\frac{2(r-2)}{r^2} - \frac{4(r-1)}{r^2} \right) \int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t \\ &\quad + \left(\frac{4}{r^2} + \frac{2(r-2)}{r^2} \right) \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t \\ &= - \frac{2}{r} \int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t + \frac{2}{r} \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t. \end{aligned}$$

Here, for the third equality we have used that

$$\begin{aligned} & - \frac{4(r-1)}{r^2} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 + \frac{4}{r^2} f_\varepsilon^r |\nabla \eta|^2 + \frac{4(r-2)}{r^2} f_\varepsilon^{\frac{r}{2}} \langle \nabla(f_\varepsilon^{\frac{r}{2}} \eta), \nabla \eta \rangle \\ &= - (r-1)\eta^2 f_\varepsilon^{r-2} |\nabla f_\varepsilon|^2 - 2\eta f_\varepsilon^{r-1} \langle \nabla f_\varepsilon, \nabla \eta \rangle, \end{aligned}$$

which can be proved by direct computation. Thus,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B(R)} f_\varepsilon^r \eta^2 d\mu_t + 2(n-1) \int_{B(R)} |\nabla(f_\varepsilon^{\frac{r}{2}} \eta)|^2 d\mu_t \\ \leq 2(n-1) \int_{B(R)} |\nabla \eta|^2 f_\varepsilon^r d\mu_t + \beta r \int_{B(R)} \eta^2 f_\varepsilon^r d\mu_t. \end{aligned} \quad (7)$$

We consider the Yamabe flow on the time interval $[0, T]$, so $\int_M |\mathring{R}c|^q d\mu_t \leq 2\Lambda$ for $t \in [0, T]$. Choose $\eta = \eta_R$ that satisfies $\eta_R = 1$ on $B(\frac{R}{2})$, $\eta_R = 0$ on $M \setminus B(R)$, and $|\nabla \eta_R|_{g_0} \leq \frac{4}{R}$ on $B(R) \setminus B(\frac{R}{2})$. From (7) we obtain

$$\frac{\partial}{\partial t} \int_{B(R)} |\mathring{R}c|^q \eta_R^2 d\mu_t \leq 2(n-1) \int_{B(R)} |\nabla \eta_R|^2 |\mathring{R}c|^q d\mu_t + \beta q \int_{B(R)} \eta_R^2 |\mathring{R}c|^q d\mu_t,$$

where we set $r = q$ and let $\varepsilon \rightarrow 0$. Note that $|\nabla \eta_R|_{g(t)}^2 \leq |\nabla \eta_R|_{g(0)}^2 e^{lt}$, where $l = \max_{0 \leq t \leq T} |\frac{\partial g}{\partial t}|_{g(t)}$. For $t \in [0, T]$, by the evolution equation of $g(t)$ and the choice of η_R , we have

$$\int_{B(R)} |\nabla \eta_R|^2 |\mathring{R}c|^q d\mu_t \leq \int_{B(R)} |\nabla \eta_R|_{g_0}^2 e^{lt} |\mathring{R}c|^q d\mu_t \leq \frac{C_1(n, \lambda)}{R^2} \int_{B(R)} |\mathring{R}c|^q d\mu_t,$$

where the positive constant $C_1(n, \lambda)$ depends only on n and λ , and the $t \leq T \leq 1$ is invoked in the second inequality. For $t \in [0, T]$, since $\int_M |\mathring{R}c|^q d\mu_t \leq 2\Lambda$, for any $\tilde{\varepsilon} > 0$, if $R \geq R_0 := 2\sqrt{\frac{(n-1)\Lambda C_1(n, \lambda)}{\tilde{\varepsilon}}}$, then

$$\int_{B(R)} |\nabla \eta_R|^2 |\mathring{R}c|^q d\mu_t \leq \frac{C_1(n, \lambda)}{R_0^2} \cdot 2\Lambda = \frac{\tilde{\varepsilon}}{2(n-1)}.$$

Hence, for $R \geq R_0$, one has

$$\frac{\partial}{\partial t} \int_{B(R)} |\mathring{R}c|^q \eta_R^2 d\mu_t \leq \tilde{\varepsilon} + \beta q \int_{B(R)} |\mathring{R}c|^q \eta_R^2 d\mu_t.$$

Therefore, for $R \geq R_0$ there holds

$$\int_{B(R)} |\mathring{R}c|^q \eta_R^2 d\mu_t \leq e^{\beta q t} \left(\int_{B(R)} |\mathring{R}c|^q \eta_R^2 d\mu_0 + \tilde{\varepsilon}/(\beta q) \right) - \tilde{\varepsilon}/(\beta q).$$

First sending $R \rightarrow \infty$ and then $\tilde{\varepsilon} \rightarrow 0$, we obtain

$$\int_M |\mathring{R}c|^q d\mu_t \leq e^{\beta q t} \int_M |\mathring{R}c|^q d\mu_0 \leq e^{\beta q t} \Lambda.$$

Let $T_0 = \min\left\{\theta, 1, \frac{\ln \frac{3}{2}}{\beta q}\right\}$. Then for $t \in [0, T_0]$, we have $\int_M |\mathring{R}c|^q d\mu_t \leq \frac{3}{2}\Lambda$. Consequently, $T \geq T_0$.

We now consider the Yamabe flow restricted to $[0, T_0]$.

For a fixed point $x_0 \in M$ and constants $R_1, \dots, R_4$ satisfying $0 < R_1 < R_2 < R_3 < R_4 < \infty$, we define a cutoff function satisfying

$$\tilde{\eta} = \begin{cases} 0, & x \in B(R_1), \\ \tilde{\eta} \in [0, 1], \quad |\nabla \tilde{\eta}|_{g(0)} \leq \frac{1}{R_2 - R_1}, & x \in B(R_2) \setminus B(R_1), \\ 1, & x \in (R_3) \setminus B(R_2), \\ \tilde{\eta} \in [0, 1], \quad |\nabla \tilde{\eta}|_{g(0)} \leq \frac{1}{R_4 - R_3}, & x \in B(R_4) \setminus B(R_3), \\ 0, & x \in M \setminus B(R_4). \end{cases}$$

From (7), for $r \geq q \geq 2$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t + 2(n-1) \int_{B(R_4)} |\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})|^2 d\mu_t \\ \leq 2(n-1) \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_t + \beta r \int_{B(R_4)} \tilde{\eta}^2 f_\varepsilon^r d\mu_t. \end{aligned} \quad (8)$$

For any $0 < \tau < \tau' < T_0$, we define a function ψ on $[0, T_0]$ as follows:

$$\psi(t) = \begin{cases} 0, & 0 \leq t \leq \tau, \\ \frac{t-\tau}{\tau'-\tau}, & \tau \leq t \leq \tau', \\ 1, & \tau' \leq t \leq T_0. \end{cases}$$

Then from (8), we have

$$\begin{aligned} \frac{\partial}{\partial t} \left(\psi \int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t \right) + 2(n-1) \psi \int_{B(R_4)} |\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})|^2 d\mu_t \\ \leq 2(n-1) \psi \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_t + (\beta r \psi + \psi') \int_{B(R_4)} \tilde{\eta}^2 f_\varepsilon^r d\mu_t. \end{aligned} \quad (9)$$

Integrating (9) over $[\tau, t]$ for any $t \in [\tau', T_0]$ gives

$$\begin{aligned} \int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t + 2(n-1) \int_\tau^t \psi \int_{B(R_4)} |\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})|^2 d\mu_s ds \\ \leq 2(n-1) \int_\tau^t \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_s ds + \left(\beta r + \frac{1}{\tau' - \tau} \right) \int_\tau^t \int_{B(R_4)} \tilde{\eta}^2 f_\varepsilon^r d\mu_s ds. \end{aligned}$$

As $\tau < \tau' \leq t \leq T_0$, the above inequality implies

$$\begin{aligned} \int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t + 2(n-1) \int_{\tau'}^t \int_{B(R_4)} |\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})|^2 d\mu_s ds \\ \leq 2(n-1) \int_\tau^{T_0} \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_s ds + \left(\beta r + \frac{1}{\tau' - \tau} \right) \int_\tau^{T_0} \int_{B(R_4)} \tilde{\eta}^2 f_\varepsilon^r d\mu_s ds. \end{aligned}$$

Since $g(t)$ and g_0 are conformally equivalent, one has $Y_{[g(t)]} = Y_{[g_0]}$. By the Sobolev inequality (6), we get

$$\begin{aligned} \left(\int_{B(R_4)} f_\varepsilon^{\frac{rn}{n-2}} \tilde{\eta}^{\frac{2n}{n-2}} d\mu_t \right)^{\frac{n-2}{n}} &= \|f_\varepsilon^{\frac{r}{2}} \tilde{\eta}\|_{\frac{2n}{n-2}}^2 \\ &\leq Y_{[g_0]} \left(\|\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})\|_2^2 + \frac{(n-2)\sqrt{n}\lambda}{4(n-1)} \|f_\varepsilon^{\frac{r}{2}} \tilde{\eta}\|_2^2 \right), \end{aligned}$$

where we have used that $S \leq \sqrt{n}\lambda$ on $[0, T_0]$. Therefore,

$$\begin{aligned} & \int_{\tau'}^{T_0} \int_{B(R_4)} f_\varepsilon^{r(1+\frac{2}{n})} \tilde{\eta}^{2(1+\frac{2}{n})} d\mu_t dt \\ & \leq \int_{\tau'}^{T_0} \left(\int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t \right)^{\frac{2}{n}} \left(\int_{B(R_4)} f_\varepsilon^{\frac{nr}{n-2}} \tilde{\eta}^{\frac{2n}{n-2}} d\mu_t \right)^{\frac{n-2}{n}} dt \\ & \leq \max_{t \in [\tau', T_0]} \left(\int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t \right)^{\frac{2}{n}} \times \int_{\tau}^{T_0} \frac{1}{Y_{[g_0]}} \left(\|\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})\|_2^2 + \frac{(n-2)\sqrt{n}\lambda}{4(n-1)} \|f_\varepsilon^{\frac{r}{2}} \tilde{\eta}\|_2^2 \right) dt \\ & \leq C_4 \max_{t \in [\tau', T_0]} \left(\int_{B(R_4)} f_\varepsilon^r \tilde{\eta}^2 d\mu_t \right)^{\frac{2}{n}} \times \int_{\tau}^{T_0} \left(\|\nabla(f_\varepsilon^{\frac{r}{2}} \tilde{\eta})\|_2^2 + \|f_\varepsilon^{\frac{r}{2}} \tilde{\eta}\|_2^2 \right) dt \\ & \leq \tilde{C}_4 \left(2 \int_{\tau}^{T_0} \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_s ds + \left(\beta r + \frac{1}{\tau' - \tau} \right) \int_{\tau}^{T_0} \int_{B(R_4)} \tilde{\eta}^2 f_\varepsilon^r d\mu_s ds \right)^{1+\frac{2}{n}}, \end{aligned}$$

where $C_4 = Y_{[g_0]}^{-1} \left(1 + \frac{(n-2)\sqrt{n}\lambda}{4(n-1)} \right)$ and $\tilde{C}_4 = (n-1)C_4$.

Note that $|\nabla \tilde{\eta}|_{g(t)}^2 \leq |\nabla \tilde{\eta}|_{g_0}^2 e^{lt}$, where $l = \max_{0 \leq t \leq T} \left| \frac{\partial g}{\partial t} \right|_{g(t)} \leq n\sqrt{n}\lambda$. By the definition of $\tilde{\eta}$, one gets that $|\nabla \tilde{\eta}|_{g_0}^2 \leq \frac{1}{(R_4 - R_3)^2} + \frac{1}{(R_2 - R_1)^2}$ almost everywhere on $B(R_4)$. Thus,

$$\begin{aligned} \int_{\tau}^{T_0} \int_{B(R_4)} |\nabla \tilde{\eta}|^2 f_\varepsilon^r d\mu_t dt & \leq \int_{\tau}^{T_0} \int_{B(R_4) \setminus B(R_1)} \left(|\nabla \tilde{\eta}|_{g_0}^2 e^{\frac{lt}{2}} \right)^2 f_\varepsilon^r d\mu_t dt \\ & \leq \left(\frac{1}{(R_4 - R_3)^2} + \frac{1}{(R_2 - R_1)^2} \right) e^{C_5 T_0} \int_{\tau}^{T_0} \int_{B(R_4)} f_\varepsilon^r d\mu_t dt, \end{aligned}$$

where $C_5 = n\sqrt{n}\lambda$. Hence

$$\begin{aligned} & \int_{\tau'}^{T_0} \int_{B(R_3) \setminus B(R_2)} f_\varepsilon^{r(1+\frac{2}{n})} d\mu_t dt \\ & \leq \tilde{C}_4 \left(\beta r + \frac{1}{\tau' - \tau} + 2 \left(\frac{1}{(R_4 - R_3)^2} + \frac{1}{(R_2 - R_1)^2} \right) e^{C_5 T_0} \right)^{1+\frac{2}{n}} \\ & \quad \times \left(\int_{\tau}^{T_0} \int_{B(R_4) \setminus B(R_1)} f_\varepsilon^r d\mu_t dt \right)^{1+\frac{2}{n}}. \end{aligned}$$

Let $L(r, \tau, \tilde{R}, R) = \int_{\tau}^{T_0} \int_{B(R) \setminus B(\tilde{R})} f_\varepsilon^r d\mu_t dt$. From the above inequality, we obtain

$$\begin{aligned} L(q_2, \tau_2, \tilde{R}_2, R_2) & \leq \tilde{C}_4 \left(\beta q_1 + \frac{1}{\tau_2 - \tau_1} + 2 \left(\frac{1}{(\tilde{R}_2 - \tilde{R}_1)^2} + \frac{1}{(R_2 - R_1)^2} \right) e^{C_5 T_0} \right)^{1+\frac{2}{n}} \\ & \quad \times L(q_1, \tau_1, \tilde{R}_1, R_1)^{1+\frac{2}{n}}. \end{aligned}$$

Let

$$\begin{aligned} \mu & = 1 + \frac{2}{n}, \quad q_k = q\mu^k, \quad \tau_k = \left(1 - \frac{1}{\mu^{k+1}} \right) t, \\ \tilde{R}_k & = R - \frac{r_0}{2} \left(1 + \frac{1}{\mu^{k/2}} \right), \quad R_k = R + \frac{r_0}{2} \left(1 + \frac{1}{\mu^{k/2}} \right) \end{aligned}$$

for $k = 0, 1, 2, 3 \dots$. Then from the inequality above, we get

$$\begin{aligned} & L(q_{k+1}, \tau_{k+1}, \tilde{R}_{k+1}, R_{k+1}) \\ & \leq \tilde{C}_4 \left(\beta q_k + \frac{1}{\tau_{k+1} - \tau_k} + 2e^{C_5 T_0} \left(\frac{1}{(\tilde{R}_{k+1} - \tilde{R}_k)^2} + \frac{1}{(R_{k+1} - R_k)^2} \right) \right)^\mu \\ & \quad \times L(q_k, \tau_k, \tilde{R}_k, R_k)^\mu \\ & = \tilde{C}_4 \left(\beta q \mu^k + \frac{\mu^2}{\mu - 1} \cdot \frac{\mu^k}{t} + 2e^{C_5 T_0} \left(\frac{4\mu}{r_0^2 (\sqrt{\mu} - 1)^2} \cdot \mu^k + \frac{4\mu}{r_0^2 (\sqrt{\mu} - 1)^2} \cdot \mu^k \right) \right)^\mu \\ & \quad \times L(q_k, \tau_k, \tilde{R}_k, R_k)^\mu \\ & = \tilde{C}_4 \left(\beta q + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + 2e^{C_5 T_0} \frac{8}{r_0^2} \frac{\mu}{(\sqrt{\mu} - 1)^2} \right)^\mu \cdot \mu^{k\mu} \cdot L(q_k, \tau_k, \tilde{R}_k, R_k)^\mu. \end{aligned}$$

Hence,

$$\begin{aligned} L(q_{k+1}, \tau_{k+1}, \tilde{R}_{k+1}, R_{k+1})^{\frac{1}{q_{k+1}}} & \leq \tilde{C}_4^{\frac{1}{q_{k+1}}} \left(\beta q + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{16e^{C_5 T_0}}{r_0^2} \frac{\mu}{(\sqrt{\mu} - 1)^2} \right)^{\frac{1}{q_k}} \cdot \mu^{\frac{k}{q_k}} \\ & \quad \times L(q_k, \tau_k, \tilde{R}_k, R_k)^{\frac{1}{q_k}}. \end{aligned}$$

Thus, for every integer $m \geq 1$,

$$\begin{aligned} & L(q_{m+1}, \tau_{m+1}, \tilde{R}_{m+1}, R_{m+1})^{\frac{1}{q_{m+1}}} \\ & \leq \tilde{C}_4^{\sum_{k=0}^m \frac{1}{q_{k+1}}} \left(\beta q + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{16e^{C_5 T_0}}{r_0^2} \frac{\mu}{(\sqrt{\mu} - 1)^2} \right)^{\sum_{k=0}^m \frac{1}{q_k}} \\ & \quad \times \mu^{\sum_{k=0}^m \frac{k}{q_k}} \cdot L(q_0, \tau_0, \tilde{R}_0, R_0)^{\frac{1}{q_0}}. \end{aligned}$$

Letting $m \rightarrow +\infty$, we obtain

$$\begin{aligned} & \sup_{B(R+r_0/2) \setminus B(R-r_0/2) \times [t, T_0]} f_\varepsilon \\ & \leq C_6 \left(C_6 + \frac{1}{t} + \frac{e^{C_5 T_0}}{r_0^2} \right)^{\frac{n+2}{2q}} \left(1 + \frac{2}{n} \right)^{\frac{n(n+2)}{4q}} \left(\int_{2t/(n+2)}^{T_0} \int_{B(R+r_0) \setminus B(R-r_0)} f_\varepsilon^q d\mu_t dt \right)^{\frac{1}{q}} \end{aligned}$$

for a certain positive constant C_6 depending only on n, q, λ and $Y_{[g_0]}$. Sending $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & \sup_{B(R+r_0/2) \setminus B(R-r_0/2) \times [t, T_0]} |\mathring{R}c| \\ & \leq C_6 \left(C_6 + \frac{1}{t} + \frac{e^{C_5 T_0}}{r_0^2} \right)^{\frac{n+2}{2q}} \left(1 + \frac{2}{n} \right)^{\frac{n(n+2)}{4q}} \left(\int_{2t/(n+2)}^{T_0} \int_{B(R+r_0) \setminus B(R-r_0)} |\mathring{R}c|^q d\mu_t dt \right)^{\frac{1}{q}}. \end{aligned}$$

By choosing $r_0 = 2$, we have

$$\sup_{B(R+1) \setminus B(R-1)} |\mathring{R}c|(T_0) \leq C(n, q, \lambda_1, \lambda_2, Y_{[g_0]}) \left(\int_{2T_0/(n+2)}^{T_0} \int_{M \setminus B(R-2)} |\mathring{R}c|^q d\mu_t dt \right)^{\frac{1}{q}}.$$

Since $\int_M |\mathring{R}c|^q d\mu_t \leq 2\Lambda$, we get

$$\int_{2T_0/(n+2)}^{T_0} \int_M |\mathring{R}c|^q d\mu_t dt \leq 2\Lambda T_0 < \infty.$$

This implies

$$\lim_{R \rightarrow \infty} \left(\int_{2T_0/(n+2)}^{T_0} \int_{M \setminus B(R-2)} |\mathring{R}c|^q d\mu_t dt \right)^{\frac{1}{q}} = 0.$$

Consequently, for any given $\delta > 0$, we can choose $\tilde{R}_0 > 0$ large enough such that for all $R \geq \tilde{R}_0$,

$$\left(\int_{2T_0/(n+2)}^{T_0} \int_{M \setminus B(R-2)} |\mathring{R}c|^q d\mu_t dt \right)^{\frac{1}{q}} \leq \frac{\delta}{C(n, q, \lambda_1, \lambda_2, Y_{[g_0]})}.$$

Therefore, for any $R \geq \tilde{R}_0$,

$$\max_{B(R+1) \setminus B(R-1)} |\mathring{R}c|(T_0) \leq \delta.$$

This shows that for any $x \in M \setminus B(\tilde{R}_0 + 1)$,

$$|\mathring{R}c|(x, T_0) \leq \delta.$$

This completes the proof of the Lemma 1. $\square$

4. The Proof of Theorem 2

We are now ready to prove Theorem 2.

Proof. Suppose that M is complete and non-compact.

Under the given conditions, there exist positive constants λ_1 and λ_2 such that the scalar curvature and the Ricci curvature of (M, g) satisfy $S \geq \lambda_1$ and $Rc \leq \lambda_2$. We consider the Yamabe flow with initial value (M, g) . By Lemma 1, there exists a $T_0 > 0$ such that the Yamabe flow has a smooth solution $g(t)$ on $[0, T_0]$ and satisfies $|Rc| \leq \lambda$ for a positive constant λ . By the maximum principle and the evolution of S , we have that $S \geq \lambda_1$ for $t \in [0, T_0]$.

Denote by M_{T_0} the manifold M equipped with the Riemannian metric $g(T_0)$. At every point, we choose a local orthonormal frame $\{e_i\}$ such that $R_{ij} = \mu_i \delta_{ij}$. Then the sectional curvature $K(\pi)$ of the tangent plane π spanned by e_i and e_j satisfies

$$K(\pi) = \frac{\hat{\mu}_i + \hat{\mu}_j}{n-2} + \frac{S}{n(n-1)},$$

where $\hat{\mu}_i = \mu_i - \frac{S}{n}$ are the eigenvalues of $\mathring{R}c$, $i = 1, 2, \dots, n$. For $\delta := \frac{(n-2)\lambda_1}{4n(n-1)}$, by Lemma 1, there is a compact subset Ω in M_{T_0} such that $|Rc| \leq \delta$ on $M_{T_0} \setminus \Omega$. So on $M_{T_0} \setminus \Omega$, we have

$$K(\pi) \geq -\frac{2\delta}{n-2} + \frac{\lambda_1}{n(n-1)} = \frac{\lambda_1}{2n(n-1)}.$$

Setting $\tilde{\delta} = \frac{\lambda_1}{4n(n-1)}$, we have that the sectional curvature $K_{M_{T_0}} > \tilde{\delta}$ on $M_{T_0} \setminus \Omega$.

For a contradiction, by the Hopf–Rinow theorem it suffices to show that M_{T_0} is bounded. Otherwise, there exists a point $p_1 \in M_{T_0}$ such that $d(p_1, \Omega) = \inf_{y \in \Omega} d(p_1, y) > \pi/\sqrt{\tilde{\delta}}$. The distance here and in the following is with respect to the metric $g(T_0)$. Since Ω is compact, there is a point $p_2 \in M_{T_0}$ such that $d(p_1, p_2) = d(p_1, \Omega)$. Let $\gamma : [0, s_1] \rightarrow M_{T_0}$ be a unit-speed minimizing geodesic from p_1 to p_2 , where $s_1 = d(p_1, p_2)$ and $\gamma(0) = p_1$, $\gamma(s_1) = p_2$. Then $\gamma(s) \in M_{T_0} \setminus \Omega$ for $s < s_1$. Pick $p_3 \in \gamma$ such that $\pi/\sqrt{\tilde{\delta}} < d(p_1, p_3) = s_2 < s_1$.

Then $\gamma : [0, s_2] \rightarrow M_{T_0}$ is also a minimizing geodesic with $\gamma(s_2) = p_3$. We take a parallel vector field $E(s)$ along $\gamma : [0, s_2] \rightarrow M_{T_0}$, for $s \in [0, s_2]$, with $E(0) \perp \gamma'(0)$ and $|E(0)| = 1$. Here and in the following, $\gamma'(s)$ denotes the tangent vector of $\gamma(s)$ at s .

By (7.12) of Chapter 7 in [31], there is a piecewise smooth function $\phi : [0, \sqrt{\delta}s_2] \rightarrow \mathbf{R}$ satisfying

$$\phi(0) = \phi(\sqrt{\delta}s_2) = 0 \quad \text{and} \quad \int_0^{\sqrt{\delta}s_2} (\phi')^2 ds < \int_0^{\sqrt{\delta}s_2} \phi^2 ds,$$

where $\sqrt{\delta}s_2 > \pi$. Define $X(s) = \phi(\sqrt{\delta}s)E(s)$. Then the index form satisfies

$$\begin{aligned} I(X, X) &= \int_0^{s_2} \left(\langle X'(s), X'(s) \rangle - \langle R(\gamma'(s), X(s))X(s), \gamma'(s) \rangle \right) ds \\ &= \int_0^{s_2} \left(\delta(\phi'(\sqrt{\delta}s))^2 - K(\gamma'(s), E(s))\phi^2(\sqrt{\delta}s) \right) ds, \end{aligned}$$

where $X'(s) = \nabla_{\gamma'(s)} X(s)$, R is the Riemannian curvature tensor defined by $R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w$ for tangent fields u, v, w on M , and $K(\gamma'(s), E(s))$ is the sectional curvature of the tangent plane spanned by $\gamma'(s)$ and $E(s)$. Since $K_{M_{T_0}} > \tilde{\delta}$ in $M_{T_0} \setminus \Omega$, we have

$$\begin{aligned} I(X, X) &\leq \int_0^{s_2} \tilde{\delta} \left((\phi'(\sqrt{\delta}s))^2 - \phi^2(\sqrt{\delta}s) \right) ds \\ &= \sqrt{\delta} \int_0^{\sqrt{\delta}s_2} \left((\phi')^2 - \phi^2 \right) ds \\ &< 0. \end{aligned}$$

However, as γ is minimizing, one has $I(X, X) \geq 0$, which is contrary to the previous inequality. Therefore, M_{T_0} is bounded, contradicting the assumption. The proof is complete. $\square$

5. Conclusions

We establish a compactness criterion for complete locally conformally flat Riemannian manifolds. Under the hypotheses of positive Yamabe invariant, a positive lower scalar curvature bound, an upper Ricci curvature bound, and finiteness of the L^q -norm ($q \geq 2$) of the traceless Ricci curvature, we prove that the manifold is compact. The proof uses the Yamabe flow together with Moser iteration to obtain a uniform decay of the traceless Ricci curvature outside large balls at a short time. This decay implies a positive lower bound on sectional curvature in the exterior region, which rules out non-compactness by the standard index form comparison. In this way, we extend previous compactness results from the constant scalar curvature setting to the variable curvature case.

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