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A Unified Approach to Hardy-Type Integral Inequalities for Bivariate Functions on Time Scales

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Abstract

This paper establishes several new bivariate Hardy-type dynamic integral inequalities on arbitrary time scales. By utilizing delta integration, properties of rd-continuous functions, and kernel estimates, we provide a unified framework that extends existing continuous two-dimensional inequalities and generalizes their one-dimensional dynamic counterparts. The established inequalities successfully bridge the gap between continuous and discrete analysis, reducing to classical continuous cases when $\mathbb{T} = \mathbb{R}$ and to their discrete analogues when $\mathbb{T} = \mathbb{N}$. These results offer powerful analytical tools for establishing qualitative properties of multi-dimensional dynamic equations.

Keywords: weighted Hardy-type inequalities; two-dimensional functions; dynamic inequalities; arbitrary time scales; mathematical operators

MSC: 26D10; 34N05; 39A12

1. Introduction

Integral inequalities are fundamental in mathematical analysis, as they provide bounds for integrals of functions and play a key role in studying function spaces and differential equations. Hardy-type inequalities, in particular, are classical results with wide applications in harmonic analysis, operator theory, and the qualitative study of solutions to dynamic equations. Dynamic inequalities extend these classical results to arbitrary time scales, thereby connecting continuous and discrete cases within a unified framework.

The discrete Hardy inequality [1] says, for any sequence $\{B_n\}_{n=1}^{\infty}$ of non-negative real numbers,

$$\sum_{n=1}^{\infty} \frac{1}{n^{\rho}} \left(\sum_{m=1}^n B_m \right)^{\rho} \leq \left(\frac{\rho}{\rho-1} \right)^{\rho} \sum_{n=1}^{\infty} (B_n)^{\rho}, \quad \text{for } \rho > 1, \quad (1)$$

while the continuous Hardy inequality [2] says, if $\phi \geq 0$ is continuous function of $[0, \infty)$, then

$$\int_0^{\infty} \frac{1}{\tau^{\rho}} \left(\int_0^{\tau} \phi(\zeta) d\zeta \right)^{\rho} d\tau \leq \left(\frac{\rho}{\rho-1} \right)^{\rho} \int_0^{\infty} \phi^{\rho}(\tau) d\tau, \quad \text{for } \rho > 1, \quad (2)$$



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where the constant $(\rho/(\rho-1))^\rho$ is sharp.

Hardy [3] gave generalizations of (2) as

$$\int_0^\infty \frac{1}{\tau^\gamma} \left(\int_0^\tau \phi(\zeta) d\zeta \right)^\rho d\tau \leq \left(\frac{\rho}{\gamma-1} \right)^\rho \int_0^\infty \tau^{\rho-\gamma} \phi^\rho(\tau) d\tau, \text{ for } \rho \geq \gamma > 1. \quad (3)$$

In [4], Pachpatte obtained that $\phi \geq 0$ is an integrable function in the domain $(0, a) \times (0, b)$ where $a, b > 0$, and r_1, r_2 are positive and absolutely continuous functions of $(0, \infty)$ such that

$$1 + \frac{\rho\varepsilon}{\rho-1} \frac{r_1'(\varepsilon)}{r_1(\varepsilon)} \geq \frac{1}{\alpha} \text{ and } 1 + \frac{\rho\eta}{\rho-1} \frac{r_2'(\eta)}{r_2(\eta)} \geq \frac{1}{\beta},$$

for almost all $\varepsilon, \eta \in (0, \infty)$ and for some constants $\alpha, \beta > 0$ and $\rho > 1$. If Ψ is defined by

$$\Psi(\varepsilon, \eta) = \frac{1}{r_1(\varepsilon)r_2(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \int_{\frac{\eta}{2}}^\eta \frac{r_1(s)r_2(t)}{st} k(s, t) dt ds,$$

then

$$\int_0^a \int_0^b \left(\frac{\Psi(\varepsilon, \eta)}{\varepsilon\eta} \right)^\rho d\eta d\varepsilon \leq (\alpha\beta)^\rho \left(\frac{\rho}{\rho-1} \right)^{2\rho} \int_0^a \int_0^b \left(\frac{1}{\varepsilon\eta r_1(\varepsilon)r_2(\eta)} |\mathcal{R}(\varepsilon, \eta)| \right)^\rho d\eta d\varepsilon, \quad (4)$$

where

$$\mathcal{R}(\varepsilon, \eta) = r_1(\varepsilon)r_2(\eta)k(\varepsilon, \eta) - r_2\left(\frac{\varepsilon}{2}\right)k\left(\frac{\varepsilon}{2}, \eta\right) - r_1(\varepsilon)r_2\left(\frac{\eta}{2}\right)k\left(\varepsilon, \frac{\eta}{2}\right) - r_1\left(\frac{\varepsilon}{2}\right)k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).$$

In [5], Benaissa et al. proved that if k, g, w are non-negative functions of $(0, \infty)$, $p > 1$, $\gamma > 1$,

$$F_1(\varepsilon) = \frac{1}{\Omega(\varepsilon)} \int_{\frac{\varepsilon}{2}}^\varepsilon w(y)g(k(y))dy, \text{ and } \lambda \geq \frac{\gamma-1}{p+\gamma-1},$$

then

$$\int_0^b \frac{w(\varepsilon)}{\Omega^\gamma(\varepsilon)} F_1^p(\varepsilon) d\varepsilon \leq \left(\frac{\lambda p}{\gamma-1} \right)^p \int_0^b \frac{w(\varepsilon)}{\Omega^\gamma(\varepsilon)} |G_1(\varepsilon)|^p d\varepsilon, \quad (5)$$

where

$$G_1(\varepsilon) = g(k(\varepsilon)) - \frac{w(\frac{\varepsilon}{2})g(k(\frac{\varepsilon}{2}))}{2w(\varepsilon)}, \text{ and } \Omega(\varepsilon) = \int_0^\varepsilon w(y)dy \text{ for } \varepsilon \in (0, \infty).$$

They also proved that if k, g, w are non-negative function on $(0, \infty)$, $p > 1$, and $\gamma < p-1$. Let

$$F_2(\varepsilon) = \int_{\frac{\varepsilon}{2}}^\varepsilon \frac{w(y)}{\Omega(y)} g(k(y)) dy,$$

and

$$G_2(\varepsilon) = g(k(\varepsilon)) - \frac{w(\frac{\varepsilon}{2})\Omega(\varepsilon)}{2w(\varepsilon)\Omega(\frac{\varepsilon}{2})} g\left(k\left(\frac{\varepsilon}{2}\right)\right).$$

Then

$$\int_0^b \frac{w(\varepsilon)}{\Omega^{p-\gamma}(\varepsilon)} F_2^p(\varepsilon) d\varepsilon \leq \left(\frac{p}{p-\gamma-1} \right)^{2p} \int_0^b \frac{w(\varepsilon)}{\Omega^{p-\gamma}(\varepsilon)} |G_2(\varepsilon)|^p d\varepsilon. \quad (6)$$

In [6], B. Benaissa and M. Z. Sarikaya proved that if k is non-negative function on $(0, a) \times (0, b)$ where $0 < a, b < \infty$, then $p, \gamma > 1$. Let

$$\Lambda_1(\varepsilon, \eta) = \frac{1}{\Omega(\varepsilon)\Omega(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \int_{\frac{\eta}{2}}^\eta \vartheta(t)\vartheta(s)k(t, s) ds dt,$$

and

$$\begin{aligned}\mathcal{H}_1(\varepsilon, \eta) &= k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\varepsilon)}k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)}k\left(\varepsilon, \frac{\eta}{2}\right) \\ &\quad + \frac{\vartheta(\frac{\eta}{2})\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\eta)2\vartheta(\varepsilon)}k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).\end{aligned}$$

If $\lambda \geq \frac{\gamma-1}{p+\gamma-1}$, then

$$\begin{aligned}&\int_0^a \int_0^b \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^\gamma(\varepsilon)\Omega^\gamma(\eta)} \Lambda_1^p(\varepsilon, \eta) d\eta d\varepsilon \\ &\leq \left(\frac{\lambda p}{\gamma-1}\right)^{2p} \int_0^a \int_0^b \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^\gamma(\varepsilon)\Omega^\gamma(\eta)} |\mathcal{H}_1(\varepsilon, \eta)|^p d\eta d\varepsilon,\end{aligned}\quad (7)$$

where ϑ is weight function and $\Omega(x) = \int_0^x \vartheta(\zeta) d\zeta$ for $x \in (0, \infty)$. They also proved that k is non-negative function of $(0, a) \times (0, b)$ when $0 < a, b < \infty$, and $p > 1$. Let

$$\Lambda_2(\varepsilon, \eta) = \int_{\frac{\varepsilon}{2}}^\varepsilon \int_{\frac{\eta}{2}}^\eta \frac{\vartheta(t)\vartheta(s)}{\Omega(t)\Omega(s)} k(t, s) ds dt,$$

and

$$\begin{aligned}\mathcal{H}_2(\varepsilon, \eta) &= k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})\Omega(\varepsilon)}{2\Omega(\frac{\varepsilon}{2})\vartheta(\varepsilon)}k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\vartheta(\frac{\eta}{2})\Omega(\eta)}{2\Omega(\frac{\eta}{2})\vartheta(\eta)}k\left(\varepsilon, \frac{\eta}{2}\right) \\ &\quad + \frac{\vartheta(\frac{\eta}{2})\Omega(\varepsilon)\vartheta(\frac{\varepsilon}{2})\Omega(\eta)}{2\Omega(\frac{\varepsilon}{2})\vartheta(\eta)2\Omega(\frac{\eta}{2})\vartheta(\varepsilon)}k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).\end{aligned}$$

If $\gamma < p-1$, then

$$\begin{aligned}&\int_0^a \int_0^b \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^{p-\gamma}(\varepsilon)\Omega^{p-\gamma}(\eta)} \Lambda_2^p(\varepsilon, \eta) d\eta d\varepsilon \\ &\leq \left(\frac{p}{p-\gamma-1}\right)^{2p} \int_0^a \int_0^b \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^{p-\gamma}(\varepsilon)\Omega^{p-\gamma}(\eta)} |\mathcal{H}_2(\varepsilon, \eta)|^p d\eta d\varepsilon,\end{aligned}\quad (8)$$

where ϑ is weight function and $\Omega(x) = \int_0^x \vartheta(\zeta) d\zeta$ for $x \in (0, \infty)$.

In 1988, Stefan Hilger [7] introduced the calculus of time scale $\mathbb{T}$ in order to unify continuous and discrete analysis. Since then, many researchers have explained various topics of the theory of dynamic inequalities on time scales. For example, Řehák [8] proved the time scale version of (1) and (2) as follows:

$$\int_r^\infty \left(\frac{\int_r^{\sigma(\tau)} \phi(\zeta) \Delta \zeta}{\sigma(\tau) - r} \right)^\rho \Delta \tau \leq \left(\frac{\rho}{\rho-1} \right)^\rho \int_r^\infty \phi^\rho(\tau) \Delta \tau, \text{ for } \rho > 1, \tau \in [r, \infty)_{\mathbb{T}},$$

unless $\phi \equiv 0$. In addition, the constant $(\rho/(\rho-1))^\rho$ is the best constant if $\mu(\tau)/\tau \rightarrow 0$ as $\tau \rightarrow \infty$. Saker et al. [9] proved the time scale version of (3) as follows:

$$\begin{aligned}&\int_r^\infty \frac{\left(\int_r^{\sigma(\tau)} \phi(\zeta) \Delta \zeta \right)^\rho}{(\sigma(\tau) - r)^\gamma} \Delta \tau \\ &\leq \left(\frac{\rho}{\gamma-1} \right)^\rho \int_r^\infty \frac{(\sigma(\tau) - r)^{\gamma(\rho-1)}}{(\tau - r)^{(\gamma-1)\rho}} \phi^\rho(\tau) \Delta \tau, \text{ for } \rho \geq \gamma > 1, \tau \in [r, \infty)_{\mathbb{T}}.\end{aligned}$$

K. A. Mohamed et al. [10] proved the time scale version of (5) and (6) as follows: Let k, g, w be rd-continuous and non-negative functions, $p > 1$, $\gamma > 1$, and $\Omega(\varepsilon) = \int_0^\varepsilon w(y) \Delta y$ for $\varepsilon \in (0, \infty)_{\mathbb{T}}$ be a nondecreasing function. If there exists constant $\lambda > 0$ such that

$$\lambda \geq \frac{\gamma - 1}{p + \gamma - 1},$$

then

$$\int_0^b \frac{w(\varepsilon)}{(\Omega^\sigma(\varepsilon))^\gamma} (F_3^\sigma(\varepsilon))^p \Delta \varepsilon \leq \left(\frac{\lambda p}{\gamma - 1} \right)^p \int_0^b \frac{(\Omega^\sigma(\varepsilon))^{\gamma(p-1)}}{\Omega^{\gamma p}(\varepsilon)} w(\varepsilon) |G_3(\varepsilon)|^p \Delta \varepsilon, \quad (9)$$

where

$$\begin{aligned} F_3(\varepsilon) &= \frac{1}{\Omega(\varepsilon)} \int_{\frac{\varepsilon}{2}}^\varepsilon w(y) g(k(y)) \Delta y, \quad G_3(\varepsilon) = g(k(\varepsilon)) - \frac{w(\frac{\varepsilon}{2}) g(k(\frac{\varepsilon}{2}))}{2w(\varepsilon)}, \\ \Omega(\infty) &= \infty \text{ and } \int_0^\infty \frac{w(y)}{(\Omega^\sigma(y))^\gamma} \Delta y < \infty. \end{aligned}$$

They also proved that if $1 < p < \infty$, and $\gamma < p - 1$, then

$$\int_0^b \frac{w(\varepsilon)}{(\Omega^\sigma(\varepsilon))^{p-\gamma}} (F_3^\sigma(\varepsilon))^p \Delta \varepsilon \leq \left(\frac{\lambda p}{\gamma - 1} \right)^p \int_0^b \frac{(\Omega^\sigma(\varepsilon))^{\gamma(p-1)}}{\Omega^{\gamma p}(\varepsilon)} w(\varepsilon) |G_3(\varepsilon)|^p \Delta \varepsilon,$$

where

$$\begin{aligned} F_3(\varepsilon) &= \frac{1}{\Omega(\varepsilon)} \int_{\frac{\varepsilon}{2}}^\varepsilon w(y) g(k(y)) \Delta y, \quad G_3(\varepsilon) = g(k(\varepsilon)) - \frac{w(\frac{\varepsilon}{2}) g(k(\frac{\varepsilon}{2}))}{2w(\varepsilon)}, \\ \Omega(\infty) &= \infty \text{ and } \int_0^\infty \frac{w(y)}{(\Omega^\sigma(y))^\gamma} \Delta y < \infty. \end{aligned}$$

The main motivation of this paper is to provide a natural unification and extension of the classical two-dimensional Hardy-type inequalities established in [6] (which were restricted to the continuous case) and the one-dimensional dynamic inequalities presented in [10]. By utilizing delta integration and suitable kernel estimates on arbitrary time scales, we establish a novel bivariate framework that handles continuous, discrete, and hybrid scenarios simultaneously, thereby contributing to the broader theory of dynamic inequalities.

The setup of this paper is organized as follows: Section 2 introduces the fundamental concepts of delta calculus on time scales, as discussed in [11,12]. Section 3 presents the main results along with their rigorous proofs, establishing new Hardy-type integral inequalities in the dynamic setting.

2. Preliminaries

A time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$. The forward jump operator σ on $\mathbb{T}$ is defined as

$$\sigma(\zeta) = \inf\{s \in \mathbb{T} : s > \zeta\}, \quad \text{for all } \zeta \in \mathbb{T}.$$

A point $\zeta \in \mathbb{T}$ is right-dense if $\sigma(\zeta) = \zeta$ and right-scattered if $\sigma(\zeta) > \zeta$. If $\mathbb{T}$ has left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. Let $\phi : \mathbb{T} \rightarrow \mathbb{R}$. Then for $\zeta \in \mathbb{T}^k$, we define $\phi^\Delta(\zeta)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of ζ such that, for all $s \in U$, we have

$$|\phi(\sigma(\zeta)) - \phi(s) - \phi^\Delta(\zeta)[\zeta - s]| \leq |\sigma(\zeta) - s|.$$

In this case, we say that ϕ is delta differentiable on $\mathbb{T}^k$ provided $\phi^\Delta(\zeta)$ exists for all $\zeta \in \mathbb{T}^k$. A function $\phi : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous provided ϕ is continuous at each right-dense points and has a finite left-sided limit at each left-dense points in $\mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

Theorem 1. Let $\phi, \theta : \mathbb{T} \rightarrow \mathbb{R}$ be differentiable at $\zeta \in \mathbb{T}^k$. Then

$$(\phi\theta)^\Delta = \phi^\Delta\theta + \phi^\sigma\theta^\Delta = \phi^\Delta\theta^\sigma + \phi\theta^\Delta, \text{ where } \theta^\sigma = \theta \circ \sigma,$$

and

$$\left(\frac{\phi}{\theta}\right)^\Delta = \frac{\theta\phi^\Delta - \phi\theta^\Delta}{\theta\theta^\sigma}, \text{ where } \theta\theta^\sigma \neq 0.$$

Theorem 2 (Integration by parts). Let $r, t \in \mathbb{T}$ and $u, v \in C_{rd}(\mathbb{T})$. Then

$$\int_r^t u^\Delta(\tau)v(\tau) \Delta \tau = u(\tau)v(\tau)|_r^t - \int_r^t u^\sigma(\tau)v^\Delta(\tau) \Delta \tau. \quad (10)$$

Theorem 3 (Chain rule). For $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous and delta differentiable on $\mathbb{T}^k$. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, then

$$(f \circ g)^\Delta(\zeta) = f'(g(c))g^\Delta(\zeta), \text{ where } c \in [\zeta, \sigma(\zeta)]. \quad (11)$$

Theorem 4 (Leibniz integral [13]). Assume $h, u : \mathbb{T} \rightarrow \mathbb{T}$ are delta differentiable. If ϕ and ϕ^Δ are continuous, then

$$\begin{aligned} & \left[\int_{u(\zeta)}^{h(\zeta)} \phi(\zeta, \tau) \Delta \tau \right]^\Delta \\ &= \int_{u(\zeta)}^{h(\zeta)} \phi^\Delta(\zeta, \tau) \Delta \tau + h^\Delta(\zeta)\phi(\sigma(\zeta), h(\zeta)) - u^\Delta(\zeta)\phi(\sigma(\zeta), u(\zeta)), \end{aligned}$$

where $\phi^\Delta(\zeta, \tau)$ is the delta derivative with respect to ζ .

Theorem 5 (Fubini's theorem [14]). Let ϕ be bounded and delta integrable over $\mathbb{R} = [r, e] \times [c, d]$. If the single integrals

$$\theta_1(\zeta) = \int_c^d \phi(\zeta, \tau) \Delta \tau \text{ and } \theta_2(\tau) = \int_r^e \phi(\zeta, \tau) \Delta \zeta,$$

exist for each $\zeta \in [r, e]$ and for each $\tau \in [c, d]$, respectively. Then the double integrals

$$\int_r^e \Delta \zeta \int_c^d \phi(\zeta, \tau) \Delta \tau \text{ and } \int_c^d \Delta \tau \int_r^e \phi(\zeta, \tau) \Delta \zeta,$$

exist and the equality

$$\int_r^e \Delta \zeta \int_c^d \phi(\zeta, \tau) \Delta \tau = \int_c^d \Delta \tau \int_r^e \phi(\zeta, \tau) \Delta \zeta,$$

holds.

Lemma 1 (Hölder inequality). If $\phi, \theta \in C_{rd}(\mathbb{T})$ and $r, t \in \mathbb{T}$, then

$$\int_r^t |\phi(\tau)\theta(\tau)| \Delta \tau \leq \left(\int_r^t |\phi(\tau)|^p \Delta \tau \right)^{\frac{1}{p}} \left(\int_r^t |\theta(\tau)|^q \Delta \tau \right)^{\frac{1}{q}}, \quad (12)$$

where $\rho > 1$ and $1/\rho + 1/\omega = 1$.

3. Main Results

Throughout this section, we adopt the usual convention $0/0 = 0$; let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales, $0 < r, e < \infty$, and define

$$\Lambda := (0, r)_{\mathbb{T}_1} \times (0, e)_{\mathbb{T}_2}.$$

Assume that

$$\varepsilon, \frac{\varepsilon}{2}, \frac{\sigma(\varepsilon)}{2} \in \mathbb{T}_1 \text{ and } \eta, \frac{\eta}{2}, \frac{\sigma(\eta)}{2} \in \mathbb{T}_2,$$

where σ denotes the forward jump operator on the corresponding time scale.

Let $\vartheta : \mathbb{T} \rightarrow [0, \infty)$ be an rd-continuous function on $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$ satisfying $\vartheta(\tau) > 0$, for $\tau \in (0, \infty)_{\mathbb{T}}$ and let $k : \Lambda \rightarrow \mathbb{R}_+$ be a nonnegative integrable function on Λ . Define

$$\Omega(\tau) := \int_0^\tau \vartheta(v) \triangle v.$$

In what follows, we state and prove some preliminary lemmas required to establish our primary theorems.

Lemma 2. Let $\rho > 1$ and $\gamma > 1$. Define

$$\Psi_1(\varepsilon, \eta) := \frac{1}{\Omega^\sigma(\varepsilon)\Omega(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \int_{\frac{\eta}{2}}^\eta \vartheta(t)\vartheta(s)k(t, s) \triangle s \triangle t,$$

and

$$G_1(\varepsilon, \eta) := \int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t)k(t, \eta) \triangle t - \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t)k(t, \frac{\eta}{2}) \triangle t.$$

Fix $\varepsilon \in \mathbb{T}_1$. If

$$\lambda \geq \frac{\gamma - 1}{\rho + \gamma - 1} \text{ and } \frac{1}{\Omega^\sigma(\varepsilon)} G_1(\varepsilon, \eta) > \Psi_1(\varepsilon, \sigma(\eta)),$$

then

$$\begin{aligned} & \int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \\ & \leq \left(\frac{\lambda \rho}{\gamma - 1} \right)^\rho \int_0^e \frac{\vartheta(\eta)[\Omega^\sigma(\eta)]^{\gamma(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\rho \Omega^{\gamma\rho}(\eta)} |G_1(\varepsilon, \eta)|^\rho \triangle \eta. \end{aligned} \quad (13)$$

Proof. Let $\varepsilon \in \mathbb{T}_1$ be fixed in (13). Then using Fubini's theorem, we obtain

$$\begin{aligned} \Psi_1(\varepsilon, \eta) &= \frac{1}{\Omega^\sigma(\varepsilon)\Omega(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \left(\int_{\frac{\eta}{2}}^\eta \vartheta(t)\vartheta(s)k(t, s) \triangle s \right) \triangle t \\ &= \frac{1}{\Omega(\eta)} \int_{\frac{\eta}{2}}^\eta \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t)k(t, s) \triangle t \right) \triangle s. \end{aligned}$$

Applying the product rule in Theorem 1 to $\Psi_1(\varepsilon, \eta)$, together with Leibniz's rule for delta differentiation on $\int_{\frac{\eta}{2}}^\eta \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t)k(t, s) \triangle t \right) \triangle s$, and using the assumption $\frac{1}{\Omega^\sigma(\varepsilon)} G_1(\varepsilon, \eta) > \Psi_1(\varepsilon, \sigma(\eta))$, we get

$$\begin{aligned}
[\Psi_1(\varepsilon, \eta)]^{\Delta_\eta} &= \left[\frac{1}{\Omega(\eta)} \int_{\frac{\eta}{2}}^{\eta} \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, s) \Delta t \right) \Delta s \right]^{\Delta_\eta} \\
&= \frac{1}{\Omega(\eta)} \left[\int_{\frac{\eta}{2}}^{\eta} \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, s) \Delta t \right) \Delta s \right]^{\Delta_\eta} \\
&\quad + \left[\frac{1}{\Omega(\eta)} \right]^{\Delta_\eta} \int_{\frac{\sigma(\eta)}{2}}^{\sigma(\eta)} \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, s) \Delta t \right) \Delta s \\
&= \frac{1}{\Omega(\eta)} \left[\frac{\vartheta(\eta)}{\Omega^\sigma(\varepsilon)} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, \eta) \Delta t - \frac{\vartheta(\frac{\eta}{2})}{2\Omega^\sigma(\varepsilon)} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, \frac{\eta}{2}) \Delta t \right] \\
&\quad - \frac{\vartheta(\eta)}{\Omega(\eta)\Omega^\sigma(\eta)} \int_{\frac{\sigma(\eta)}{2}}^{\sigma(\eta)} \frac{\vartheta(s)}{\Omega^\sigma(\varepsilon)} \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} \vartheta(t) k(t, s) \Delta t \right) \Delta s \\
&= \frac{\vartheta(\eta)}{\Omega(\eta)} \left[\frac{1}{\Omega^\sigma(\varepsilon)} G_1(\varepsilon, \eta) - \Psi_1(\varepsilon, \sigma(\eta)) \right] > 0.
\end{aligned} \tag{14}$$

Applying the integration by parts formula (10) on the left hand side of (13) with

$$u^\Delta(\eta) = \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \quad \text{and} \quad v^\sigma(\varepsilon, \eta) = \Psi_1^\rho(\varepsilon, \sigma(\eta)),$$

we obtain

$$\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta = u(\eta) v(\varepsilon, \eta) \Big|_0^e + \int_0^e (-u(\eta)) (v(\varepsilon, \eta))^{\Delta_\eta} \Delta \eta, \tag{15}$$

where

$$u(\eta) = - \int_\eta^e \frac{\vartheta(s)}{[\Omega^\sigma(s)]^\gamma} \Delta s.$$

Applying the chain rule (11) with $f(\eta) = \eta^{1-\gamma}$ and $g(\eta) = \Omega(\eta)$, we get

$$(f \circ g)(\eta) = \Omega^{1-\gamma}(\eta), \quad f'(\eta) = (1-\gamma)\eta^{-\gamma}, \quad f'(\Omega(c)) = (1-\gamma)\Omega^{-\gamma}(c).$$

Since $\Omega^\Delta(\eta) = \vartheta(\eta) > 0$ (i.e., Ω is increasing), $\gamma > 1$ and $c \in [\eta, \sigma(\eta)]$, we obtain

$$[\Omega^{1-\gamma}(\eta)]^\Delta = (1-\gamma)\Omega^{-\gamma}(c)\Omega^\Delta(\eta) \leq (1-\gamma) \frac{\Omega^\Delta(\eta)}{[\Omega^\sigma(\eta)]^\gamma}. \tag{16}$$

Integrating (16) from η to e , we have

$$u(e) - u(\eta) \leq \frac{1}{\gamma-1} [\Omega^{1-\gamma}(\eta) - \Omega^{1-\gamma}(e)].$$

Since $u(e) = 0$ and $\Omega^{1-\gamma}(e) > 0$, we get

$$-u(\eta) \leq \frac{1}{\gamma-1} \Omega^{1-\gamma}(\eta). \tag{17}$$

Substituting (17) into (15) and using $\gamma > 1$, we obtain

$$\begin{aligned}
&\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\
&\leq -\Psi_1^\rho(\varepsilon, \eta) \int_\eta^e \frac{\vartheta(s)}{[\Omega^\sigma(s)]^\gamma} \Delta s \Big|_0^e + \frac{1}{\gamma-1} \int_0^e \Omega^{1-\gamma}(\eta) [\Psi_1^\rho(\varepsilon, \eta)]^{\Delta_\eta} \Delta \eta \\
&= \frac{1}{\gamma-1} \int_0^e \Omega^{1-\gamma}(\eta) [\Psi_1^\rho(\varepsilon, \eta)]^{\Delta_\eta} \Delta \eta.
\end{aligned} \tag{18}$$

Applying the chain rule to $\Psi_1^\rho(\varepsilon, \eta)$ and using $\Psi_1(\varepsilon, \eta)$ is increasing with respect to $\eta, \rho > 1$, we have

$$\begin{aligned} \left[\Psi_1^\rho(\varepsilon, \eta) \right]^{\Delta_\eta} &= \rho \Psi_1^{\rho-1}(\varepsilon, c) \Psi_1^{\Delta_\eta}(\varepsilon, \eta) \leq \rho \Psi_1^{\rho-1}(\varepsilon, \sigma(\eta)) \Psi_1^{\Delta_\eta}(\varepsilon, \eta) \\ &= \rho \frac{\vartheta(\eta) G_1(\varepsilon, \eta)}{\Omega^\sigma(\varepsilon) \Omega(\eta)} \Psi_1^{\rho-1}(\varepsilon, \sigma(\eta)) - \rho \frac{\vartheta(\eta)}{\Omega(\eta)} \Psi_1^\rho(\varepsilon, \sigma(\eta)). \end{aligned} \quad (19)$$

Combining (19) and (18) and using Ω is increasing, $\gamma > 1$, we obtain

$$\begin{aligned} &\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\ &\leq \frac{\rho}{\gamma-1} \int_0^e \frac{\vartheta(\eta)}{\Omega^\sigma(\varepsilon) \Omega^\gamma(\eta)} G_1(\varepsilon, \eta) \Psi_1^{\rho-1}(\varepsilon, \sigma(\eta)) \Delta \eta \\ &\quad - \frac{\rho}{\gamma-1} \int_0^e \frac{\vartheta(\eta)}{\Omega^\gamma(\eta)} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\ &\leq \frac{\rho}{\gamma-1} \int_0^e \frac{\vartheta(\eta)}{\Omega^\sigma(\varepsilon) \Omega^\gamma(\eta)} G_1(\varepsilon, \eta) \Psi_1^{\rho-1}(\varepsilon, \sigma(\eta)) \Delta \eta \\ &\quad - \frac{\rho}{\gamma-1} \int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta, \end{aligned}$$

here, using the assumption on λ , we get

$$\begin{aligned} &\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\ &\leq \frac{\lambda \rho}{\gamma-1} \int_0^e \frac{\vartheta(\eta)}{\Omega^\sigma(\varepsilon) \Omega^\gamma(\eta)} |G_1(\varepsilon, \eta)| \Psi_1^{\rho-1}(\varepsilon, \sigma(\eta)) \Delta \eta \\ &= \frac{\lambda \rho}{\gamma-1} \int_0^e \left(\frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \right)^{\frac{\rho-1}{\rho}} \frac{\vartheta^{\frac{1}{\rho}}(\eta) [\Omega^\sigma(\eta)]^{\frac{\gamma(\rho-1)}{\rho}}}{\Omega^\sigma(\varepsilon) \Omega^\gamma(\eta)} |G_1(\varepsilon, \eta)| \Delta \eta. \end{aligned}$$

Finally, applying Hölder's inequality (12) with conjugate indices ρ and $\rho/(\rho-1)$, and after simplification, we obtain

$$\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \leq \left(\frac{\lambda \rho}{\gamma-1} \right)^\rho \int_0^e \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\rho \Omega^{\gamma\rho}(\eta)} |G_1(\varepsilon, \eta)|^\rho \Delta \eta.$$

□

Remark 1. When $\mathbb{T} = \mathbb{R}$, Lemma 2 reduces to the corresponding result established in ([6], Lemma 1).

Lemma 3. Let $\rho > 1$ and $\gamma > 1$. Assume that $G_1(\varepsilon, \eta)$ is defined as in Lemma 2, and define

$$\begin{aligned} H_1(\varepsilon, \eta) &:= k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) \\ &\quad + \frac{\vartheta(\frac{\eta}{2})\vartheta(\frac{\varepsilon}{2})}{4\vartheta(\eta)\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right). \end{aligned}$$

Fix $\eta \in \mathbb{T}_2$. If

$$\lambda \geq \frac{\gamma-1}{\rho+\gamma-1},$$

and

$$k(\varepsilon, \eta) > \frac{\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) + \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) - \frac{\vartheta(\frac{\eta}{2})\vartheta(\frac{\varepsilon}{2})}{4\vartheta(\eta)\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right),$$

then

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq \left(\frac{\lambda \rho}{\gamma - 1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon) |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\gamma |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \Delta \varepsilon. \end{aligned} \quad (20)$$

Proof. Let $\eta \in \mathbb{T}_2$ be fixed in (20). Then, we have

$$|G_1(\varepsilon, \eta)| = G_1(\varepsilon, \eta) \times \operatorname{sgn}(G_1(\varepsilon, \eta)).$$

By differentiating $G_1(\varepsilon, \eta)$ with respect to ε and applying Leibniz's rule on time scales, we obtain

$$\begin{aligned} G_1^{\Delta_\varepsilon}(\varepsilon, \eta) &= \left[\int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t) k(t, \eta) \Delta t \right]^{\Delta_\varepsilon} - \left[\frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} \int_{\frac{\varepsilon}{2}}^\varepsilon \vartheta(t) k(t, \frac{\eta}{2}) \Delta t \right]^{\Delta_\varepsilon} \\ &= \vartheta(\varepsilon) k(\varepsilon, \eta) - \frac{1}{2} \vartheta(\frac{\varepsilon}{2}) k(\frac{\varepsilon}{2}, \eta) - \frac{\vartheta(\frac{\eta}{2}) \vartheta(\varepsilon)}{2\vartheta(\eta)} k(\varepsilon, \frac{\eta}{2}) + \frac{\vartheta(\frac{\eta}{2}) \vartheta(\frac{\varepsilon}{2})}{4\vartheta(\eta)} k(\frac{\varepsilon}{2}, \frac{\eta}{2}) \\ &= \vartheta(\varepsilon) \left[k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\varepsilon)} k(\frac{\varepsilon}{2}, \eta) - \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} k(\varepsilon, \frac{\eta}{2}) + \frac{\vartheta(\frac{\eta}{2}) \vartheta(\frac{\varepsilon}{2})}{4\vartheta(\eta) \vartheta(\varepsilon)} k(\frac{\varepsilon}{2}, \frac{\eta}{2}) \right] \\ &= \vartheta(\varepsilon) H_1(\varepsilon, \eta) > 0. \end{aligned}$$

Applying the integration by parts formula on the left-hand side of (20) with

$$u^\Delta(\varepsilon) = \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} \quad \text{and} \quad v(\varepsilon, \eta) = |G_1(\varepsilon, \eta)|^\rho,$$

we get

$$\int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \Delta \varepsilon = u(\varepsilon) v(\varepsilon, \eta) \Big|_0^r + \int_0^r (-u^\sigma(\varepsilon)) (v(\varepsilon, \eta))^\Delta \Delta \varepsilon, \quad (21)$$

where

$$u(\varepsilon) = - \int_\varepsilon^r \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho+\gamma}} \Delta s.$$

Applying the chain rule with $\Omega^\Delta(\varepsilon) > 0$, and $c \in [\varepsilon, \sigma(\varepsilon)]$, we obtain

$$\left[\Omega^{1-(\rho+\gamma)}(\varepsilon) \right]^\Delta = (1 - (\rho + \gamma)) \Omega^{-(\rho+\gamma)}(c) \Omega^\Delta(\varepsilon) \leq (1 - (\rho + \gamma)) \frac{\Omega^\Delta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}}. \quad (22)$$

Integrating (22) from ε to r , and using $\rho + \gamma > 1$, we have

$$u(r) - u(\varepsilon) \leq \frac{1}{\rho + \gamma - 1} \left[\Omega^{1-(\rho+\gamma)}(\varepsilon) - \Omega^{1-(\rho+\gamma)}(r) \right].$$

Since $u(r) = 0$ and $\Omega^{1-(\rho+\gamma)}(r) > 0$, it follows that

$$-u(\varepsilon) \leq \frac{1}{\rho + \gamma - 1} \Omega^{1-(\rho+\gamma)}(\varepsilon). \quad (23)$$

Substituting (23) into (21) and applying the chain rule on $|G_1(\varepsilon, \eta)|^\rho$, and using $G_1(\varepsilon, \eta)$ is increasing with respect to ε , $\rho > 1$, we obtain

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq -|G_1(\varepsilon, \eta)|^\rho \int_\varepsilon^r \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho+\gamma}} \Delta s \Big|_0^r + \\ & \frac{\rho}{\rho + \gamma - 1} \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma-1}} H_1(\varepsilon, \eta) \operatorname{sgn}(G_1(\varepsilon, \eta)) |G_1(\sigma(\varepsilon), \eta)|^{\rho-1} \Delta \varepsilon \\ & = \frac{\rho}{\rho + \gamma - 1} \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma-1}} H_1(\varepsilon, \eta) \operatorname{sgn}(G_1(\varepsilon, \eta)) |G_1(\sigma(\varepsilon), \eta)|^{\rho-1} \Delta \varepsilon. \end{aligned}$$

Using the assumption on λ , we have

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq \frac{\lambda \rho}{\gamma - 1} \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} \Omega^\sigma(\varepsilon) H_1(\varepsilon, \eta) |G_1(\sigma(\varepsilon), \eta)|^{\rho-1} \Delta \varepsilon \\ & = \frac{\lambda \rho}{\gamma - 1} \int_0^r \left(\frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \right)^{\frac{\rho-1}{\rho}} \frac{\vartheta^{\frac{1}{\rho}}(\varepsilon) |G_1(\sigma(\varepsilon), \eta)|^{\rho-1}}{[\Omega^\sigma(\varepsilon)]^{\frac{\gamma}{\rho}} |G_1(\varepsilon, \eta)|^{\rho-1}} H_1(\varepsilon, \eta) \Delta \varepsilon. \end{aligned}$$

Finally, applying Hölder's inequality (12) with conjugate indices ρ and $\rho/(\rho - 1)$ and thus on simplification, we get

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq \left(\frac{\lambda \rho}{\gamma - 1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon) |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\gamma |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \Delta \varepsilon. \end{aligned}$$

□

Remark 2. In Lemma 3, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Lemma 2).

Lemma 4. Let $\rho > 1$. Define

$$\Psi_2(\varepsilon, \eta) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{\frac{\eta}{2}}^{\eta} \frac{\vartheta(t)\vartheta(s)}{\Omega^\sigma(t)\Omega(s)} k(t, s) \Delta s \Delta t,$$

and

$$G_2(\varepsilon, \eta) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{\vartheta(t)}{\Omega^\sigma(t)} k(t, \eta) \Delta t - \frac{\Omega(\eta)\vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2})\vartheta(\eta)} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{\vartheta(t)}{\Omega^\sigma(t)} k(t, \frac{\eta}{2}) \Delta t.$$

Fix $\varepsilon \in \mathbb{T}_1$. If

$$\alpha < \rho - 1, \text{ and } k(t, \eta) > \frac{\Omega(\eta)\vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2})\vartheta(\eta)} k(t, \frac{\eta}{2}) \text{ for } t \in [\frac{\varepsilon}{2}, \varepsilon]_{\mathbb{T}},$$

then

$$\begin{aligned} & \int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\ & \leq \left(\frac{\rho}{\rho - \alpha - 1} \right)^\rho \int_0^e \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}}{\Omega^{\rho(\rho-\alpha)}(\eta)} (G_2(\varepsilon, \eta))^\rho \Delta \eta. \end{aligned} \quad (24)$$

Proof. Let $\varepsilon \in \mathbb{T}_1$ be fixed in (24). Then using Fubini's theorem, we obtain

$$\Psi_2(\varepsilon, \eta) = \int_{\frac{\eta}{2}}^{\eta} \frac{\vartheta(s)}{\Omega(s)} \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{\vartheta(t)}{\Omega^\sigma(t)} k(t, s) \Delta t \right) \Delta s.$$

Applying Leibniz's rule for the delta derivative on $\Psi_2(\varepsilon, \eta)$ yields

$$\begin{aligned} \Psi_2^{\Delta_\eta}(\varepsilon, \eta) &= \frac{\vartheta(\eta)}{\Omega(\eta)} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{\vartheta(t)}{\Omega^\sigma(t)} k(t, \eta) \Delta t - \frac{\vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2})} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{\vartheta(t)}{\Omega^\sigma(t)} k(t, \frac{\eta}{2}) \Delta t \\ &= \frac{\vartheta(\eta)}{\Omega(\eta)} G_2(\varepsilon, \eta) > 0. \end{aligned}$$

Applying the integration by parts formula on the left-hand side of (24) with

$$u^\Delta(\eta) = \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \quad \text{and} \quad v^\sigma(\varepsilon, \eta) = \Psi_2^\rho(\varepsilon, \sigma(\eta)),$$

we get

$$\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \Delta \eta = u(\eta) v(\varepsilon, \eta) \Big|_0^e + \int_0^e (-u(\eta)) (v(\varepsilon, \eta))^{\Delta_\eta} \Delta \eta, \quad (25)$$

where

$$u(\eta) = - \int_\eta^e \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho-\alpha}} \Delta s.$$

Applying the chain rule with $\Omega^\Delta(\eta) > 0$ and $c \in [\eta, \sigma(\eta)]$, we obtain

$$\left[\Omega^{1-(\rho-\alpha)}(\eta) \right]^\Delta = (1 - (\rho - \alpha)) \Omega^{-(\rho-\alpha)}(c) \Omega^\Delta(\eta) \leq (1 - (\rho - \alpha)) \frac{\Omega^\Delta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}}. \quad (26)$$

Integrating (26) from η to e and using $u(e) = 0$, $\Omega^{1-(\rho-\alpha)}(e) > 0$ and $\rho - \alpha > 1$, we have

$$-u(\eta) \leq \frac{1}{\rho - \alpha - 1} \Omega^{1-(\rho-\alpha)}(\eta). \quad (27)$$

Substituting (27) into (25), and applying the chain rule on $\Psi_2^\rho(\varepsilon, \eta)$ and using $\Psi_2(\varepsilon, \eta)$ is increasing with respect to η , $\rho - \alpha - 1 > 0$, we obtain

$$\begin{aligned} & \int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \Delta \eta \\ & \leq -\Psi_2^\rho(\varepsilon, \eta) \int_\eta^e \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho-\alpha}} \Delta s \Big|_0^e \\ & \quad + \frac{\rho}{\rho - \alpha - 1} \int_0^e \frac{\vartheta(\eta)}{\Omega^{\rho-\alpha}(\eta)} G_2(\varepsilon, \eta) \Psi_2^{\rho-1}(\varepsilon, \sigma(\eta)) \Delta \eta \\ & = \frac{\rho}{\rho - \alpha - 1} \int_0^e \frac{\vartheta(\eta)}{\Omega^{\rho-\alpha}(\eta)} G_2(\varepsilon, \eta) \Psi_2^{\rho-1}(\varepsilon, \sigma(\eta)) \Delta \eta \\ & = \frac{\rho}{\rho - \alpha - 1} \int_0^e \left(\frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \right)^{\frac{\rho-1}{\rho}} \\ & \quad \times \frac{\vartheta^{\frac{1}{\rho}}(\eta) [\Omega^\sigma(\eta)]^{(\rho-\alpha)(\frac{\rho-1}{\rho})}}{\Omega^{\rho-\alpha}(\eta)} G_2(\varepsilon, \eta) \Delta \eta. \end{aligned}$$

Finally, applying Hölder's inequality (12) with conjugate indices ρ and $\rho/(\rho - 1)$ and thus on simplification, we obtain

$$\begin{aligned} & \int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \\ & \leq \left(\frac{\rho}{\rho - \alpha - 1} \right)^\rho \int_0^e \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}}{\Omega^{\rho(\rho-\alpha)}(\eta)} (G_2(\varepsilon, \eta))^\rho \triangle \eta. \end{aligned}$$

□

Remark 3. In Lemma 4, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Lemma 3).

Lemma 5. Let $\rho > 1$. Assume that $G_2(\varepsilon, \eta)$ is defined as in Lemma 4, and define

$$\begin{aligned} H_2(\varepsilon, \eta) & : = k(\varepsilon, \eta) - \frac{\Omega^\sigma(\varepsilon) \vartheta(\frac{\varepsilon}{2})}{2\Omega^\sigma(\frac{\varepsilon}{2}) \vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\Omega(\eta) \vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2}) \vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) \\ & + \frac{\Omega(\eta) \Omega^\sigma(\varepsilon) \vartheta(\frac{\eta}{2}) \vartheta(\frac{\varepsilon}{2})}{4\Omega(\frac{\eta}{2}) \Omega^\sigma(\frac{\varepsilon}{2}) \vartheta(\eta) \vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right). \end{aligned}$$

Fix $\eta \in \mathbb{T}_2$. If

$$\alpha < \rho - 1,$$

and

$$\begin{aligned} k(\varepsilon, \eta) & > \frac{\Omega^\sigma(\varepsilon) \vartheta(\frac{\varepsilon}{2})}{2\Omega^\sigma(\frac{\varepsilon}{2}) \vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) + \frac{\Omega(\eta) \vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2}) \vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) \\ & - \frac{\Omega(\eta) \Omega^\sigma(\varepsilon) \vartheta(\frac{\eta}{2}) \vartheta(\frac{\varepsilon}{2})}{4\Omega(\frac{\eta}{2}) \Omega^\sigma(\frac{\varepsilon}{2}) \vartheta(\eta) \vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right), \end{aligned}$$

then

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \triangle \varepsilon \\ & \leq \left(\frac{\rho}{\rho - \alpha - 1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon) |G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha} |G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \triangle \varepsilon. \end{aligned} \quad (28)$$

Proof. Let $\eta \in \mathbb{T}_2$ be fixed in (28). Then

$$|G_2(\varepsilon, \eta)| = G_2(\varepsilon, \eta) \times \text{sgn}(G_2(\varepsilon, \eta)).$$

By differentiating $G_2(\varepsilon, \eta)$ with respect to ε and applying Leibniz's rule on time scales, we obtain

$$\begin{aligned} G_2^{\triangle_\varepsilon}(\varepsilon, \eta) & = \frac{\vartheta(\varepsilon)}{\Omega^\sigma(\varepsilon)} k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})}{2\Omega^\sigma(\frac{\varepsilon}{2})} k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\Omega(\eta) \vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2}) \vartheta(\eta)} \frac{\vartheta(\varepsilon)}{\Omega^\sigma(\varepsilon)} k\left(\varepsilon, \frac{\eta}{2}\right) \\ & + \frac{\Omega(\eta) \vartheta(\frac{\eta}{2}) \vartheta(\frac{\varepsilon}{2})}{2\Omega(\frac{\eta}{2}) 2\Omega^\sigma(\frac{\varepsilon}{2}) \vartheta(\eta)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right) \\ & = \frac{\vartheta(\varepsilon)}{\Omega^\sigma(\varepsilon)} H_2(\varepsilon, \eta). \end{aligned}$$

Applying the integration by parts formula on the left-hand side of (28) with

$$u^\triangle(\varepsilon) = \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} \quad \text{and} \quad v(\varepsilon, \eta) = |G_2(\varepsilon, \eta)|^\rho,$$

we get

$$\int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \Delta \varepsilon = u(\varepsilon) v(\varepsilon, \eta) \Big|_0^r + \int_0^r (-u^\sigma(\varepsilon)) (v(\varepsilon, \eta))^{\Delta \varepsilon} \Delta \varepsilon, \quad (29)$$

where

$$u(\varepsilon) = - \int_\varepsilon^r \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho-\alpha}} \Delta s.$$

Applying the chain rule with $\Omega^\Delta(\varepsilon) > 0$ and $c \in [\varepsilon, \sigma(\varepsilon)]$, we obtain

$$\left[\Omega^{1-(\rho-\alpha)}(\varepsilon) \right]^\Delta = (1 - (\rho - \alpha)) \Omega^{-(\rho-\alpha)}(c) \Omega^\Delta(\varepsilon) \leq (1 - (\rho - \alpha)) \frac{\Omega^\Delta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}}. \quad (30)$$

Integrating (30) from ε to r and using $u(r) = 0$, $\rho - \alpha > 1$, and $\Omega^{1-(\rho+\gamma)}(r) > 0$, we have

$$-u(\varepsilon) \leq \frac{1}{\rho - \alpha - 1} \Omega^{1-(\rho-\alpha)}(\varepsilon). \quad (31)$$

Substituting (31) into (29) and applying the chain rule on $|G_2(\varepsilon, \eta)|^\rho$, and using $G_2(\varepsilon, \eta)$ is increasing with respect to ε , $\rho > 1$, we obtain

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq -|G_2(\varepsilon, \eta)|^\rho \int_\varepsilon^r \frac{\vartheta(s)}{[\Omega^\sigma(s)]^{\rho-\alpha}} \Delta s \Big|_0^r \\ & + \frac{\rho}{\rho - \alpha - 1} \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} \operatorname{sgn} G_2(\varepsilon, \eta) H_2(\varepsilon, \eta) |G_2(\sigma(\varepsilon), \eta)|^{\rho-1} \Delta \varepsilon \\ & = \frac{\rho}{\rho - \alpha - 1} \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} \operatorname{sgn} G_2(\varepsilon, \eta) H_2(\varepsilon, \eta) |G_2(\sigma(\varepsilon), \eta)|^{\rho-1} \Delta \varepsilon \\ & = \frac{\rho}{\rho - \alpha - 1} \int_0^r \left(\frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \right)^{\frac{\rho-1}{\rho}} \\ & \times \frac{\vartheta^{\frac{1}{\rho}}(\varepsilon) [\Omega^\sigma(\varepsilon)]^{\frac{\alpha}{\rho}-1} |G_2(\sigma(\varepsilon), \eta)|^{\rho-1}}{|G_2(\varepsilon, \eta)|^{\rho-1}} H_2(\varepsilon, \eta) \Delta \varepsilon. \end{aligned}$$

Finally, applying Hölder's inequality (12) with conjugate indices ρ and $\rho/(\rho - 1)$, and thus on simplification, we get

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \Delta \varepsilon \\ & \leq \left(\frac{\rho}{\rho - \alpha - 1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon) |G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha} |G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \Delta \varepsilon. \end{aligned}$$

□

Remark 4. In Lemma 5, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Lemma 4).

Based on the preceding lemmas and the properties developed therein, we are now ready to establish the principal Hardy-type inequalities presented in this paper.

Theorem 6. Let $\rho > 1$, $\gamma > 1$ and $\lambda \geq \frac{\gamma - 1}{\rho + \gamma - 1}$. Then the following inequality holds, provided that both sides are finite

$$\begin{aligned} & \int_0^r \int_0^\varepsilon \frac{\vartheta(\varepsilon)\vartheta(\eta)}{[\Omega^\sigma(\varepsilon)]^\gamma [\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\ & \leq \left(\frac{\lambda\rho}{\gamma - 1} \right)^{2\rho} \int_0^r \int_0^\varepsilon \frac{\vartheta(\varepsilon)\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)} |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\gamma \Omega^{\gamma\rho}(\eta) |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon, \end{aligned} \quad (32)$$

where $\Psi_1(\varepsilon, \eta)$ and $H_1(\varepsilon, \eta)$ are defined as in Lemma 2, Lemma 3, respectively.

Proof. By using Lemma 2, Lemma 3 and Fubini's theorem on the left hand side of (32), we get

$$\begin{aligned} & \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^\gamma} \left(\int_0^\varepsilon \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^\gamma} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \right) \triangle \varepsilon \\ & \leq \left(\frac{\lambda\rho}{\gamma - 1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^\gamma} \left(\int_0^\varepsilon \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\rho \Omega^{\gamma\rho}(\eta)} |G_1(\varepsilon, \eta)|^\rho \triangle \eta \right) \triangle \varepsilon \\ & = \left(\frac{\lambda\rho}{\gamma - 1} \right)^\rho \int_0^r \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)}}{\Omega^{\gamma\rho}(\eta)} \left(\int_0^\varepsilon \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho+\gamma}} |G_1(\varepsilon, \eta)|^\rho \triangle \varepsilon \right) \triangle \eta \\ & \leq \left(\frac{\lambda\rho}{\gamma - 1} \right)^\rho \int_0^r \frac{\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)}}{\Omega^{\gamma\rho}(\eta)} \left(\left(\frac{\lambda\rho}{\gamma - 1} \right)^\rho \int_0^\varepsilon \frac{\vartheta(\varepsilon) |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\gamma |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \triangle \varepsilon \right) \triangle \eta \\ & = \left(\frac{\lambda\rho}{\gamma - 1} \right)^{2\rho} \int_0^r \int_0^\varepsilon \frac{\vartheta(\varepsilon)\vartheta(\eta) [\Omega^\sigma(\eta)]^{\gamma(\rho-1)} |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^\gamma \Omega^{\gamma\rho}(\eta) |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon. \end{aligned}$$

□

Remark 5. In Theorem 6, if $\mathbb{T} = \mathbb{R}$, then inequality (32) reduces to (7).

In Theorem 7, setting $\vartheta(\tau) = 1$ yields the following corollary.

Corollary 1. Under the assumptions of Theorem 7 with $\vartheta(\tau) = 1$, the following inequality holds:

$$\begin{aligned} & \int_0^r \int_0^\varepsilon \frac{1}{\sigma^\gamma(\varepsilon)\sigma^\gamma(\eta)} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\ & \leq \left(\frac{\lambda\rho}{\gamma - 1} \right)^{2\rho} \int_0^r \int_0^\varepsilon \frac{\sigma^{\gamma(\rho-1)}(\eta) |G_1(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{\sigma^\gamma(\varepsilon)\eta^{\gamma\rho} |G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon, \end{aligned}$$

where $\Psi_1(\varepsilon, \eta)$ and $H_1(\varepsilon, \eta)$ reduce to

$$\Psi_1(\varepsilon, \eta) := \frac{1}{\sigma(\varepsilon)\eta} \int_{\frac{\varepsilon}{2}}^\varepsilon \int_{\frac{\eta}{2}}^\eta k(t, s) \triangle s \triangle t,$$

and

$$H_1(\varepsilon, \eta) := k(\varepsilon, \eta) - \frac{1}{2}k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{1}{2}k\left(\varepsilon, \frac{\eta}{2}\right) + \frac{1}{4}k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).$$

Remark 6. In Corollary 1, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Corollary 1).

Remark 7. In Corollary 1, if $\mathbb{T} = \mathbb{R}$ and $\lambda = 1$, then we have a Pachpatte-type integral inequality.

Corollary 2. In Theorem 6, if $\mathbb{T} = \mathbb{Z}$, then inequality (32) becomes

$$\begin{aligned} & \sum_{\varepsilon=0}^{r-1} \sum_{\eta=0}^{e-1} \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^\gamma(\varepsilon+1)\Omega^\gamma(\eta+1)} \Psi_1^\rho(\varepsilon, \eta+1) \\ & \leq \left(\frac{\lambda\rho}{\gamma-1} \right)^{2\rho} \sum_{\varepsilon=0}^{r-1} \sum_{\eta=0}^{e-1} \frac{\vartheta(\varepsilon)\vartheta(\eta)\Omega^{\gamma(\rho-1)}(\eta+1)|G_1(\varepsilon+1, \eta)|^{\rho(\rho-1)}}{\Omega^\gamma(\varepsilon+1)\Omega^{\gamma\rho}(\eta)|G_1(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_1(\varepsilon, \eta))^\rho, \end{aligned}$$

where

$$\Psi_1(\varepsilon, \eta) := \frac{1}{\Omega(\varepsilon+1)\Omega(\eta)} \sum_{t=\frac{\varepsilon}{2}}^{\varepsilon-1} \sum_{s=\frac{\eta}{2}}^{\eta-1} \vartheta(t)\vartheta(s)k(t, s),$$

and

$$H_1(\varepsilon, \eta) := k(\varepsilon, \eta) - \frac{\vartheta(\frac{\varepsilon}{2})}{2\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{\vartheta(\frac{\eta}{2})}{2\vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) + \frac{\vartheta(\frac{\eta}{2})\vartheta(\frac{\varepsilon}{2})}{4\vartheta(\eta)\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).$$

Suppose $k(\varepsilon, \eta) = k_1(\varepsilon)k_2(\eta)$, where k_1, k_2 are nonnegative integrable functions on $(0, \infty)_{\mathbb{T}}$. Then, $H_1(\varepsilon, \eta)$ and $G_1(\varepsilon, \eta)$ factor as

$$H_1(\varepsilon, \eta) := \left(k_1(\varepsilon) - \frac{1}{2}k_1\left(\frac{\varepsilon}{2}\right) \right) \left(k_2(\eta) - \frac{1}{2}k_2\left(\frac{\eta}{2}\right) \right),$$

and

$$G_1(\varepsilon, \eta) := G_1(\varepsilon)G_2(\eta),$$

where

$$G_1(\varepsilon) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} k_1(t) \triangle t \quad \text{and} \quad G_2(\eta) := k_2(\eta) - \frac{1}{2}k_2\left(\frac{\eta}{2}\right).$$

Consequently, from Corollary 1, we obtain the following result.

Corollary 3. Under the assumptions of Corollary 1 with $k(\varepsilon, \eta) = k_1(\varepsilon)k_2(\eta)$, the following inequality holds:

$$\begin{aligned} & \int_0^r \int_0^e \frac{1}{\sigma^\gamma(\varepsilon)\sigma^\gamma(\eta)} \Psi_1^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\ & \leq \left(\frac{\lambda\rho}{\gamma-1} \right)^{2\rho} \left(\int_0^r \frac{|G_1^\sigma(\varepsilon)|^{\rho(\rho-1)} \left(k_1(\varepsilon) - \frac{1}{2}k_1\left(\frac{\varepsilon}{2}\right) \right)^\rho}{\sigma^\gamma(\varepsilon)|G_1(\varepsilon)|^{\rho(\rho-1)}} \triangle \varepsilon \right) \\ & \quad \times \left(\int_0^e \frac{\sigma^{\gamma(\rho-1)}(\eta)|G_2(\eta)|^\rho}{\eta^{\gamma\rho}} \triangle \eta \right), \end{aligned}$$

where

$$\Psi_1(\varepsilon, \eta) := \left(\frac{1}{\sigma(\varepsilon)} \int_{\frac{\varepsilon}{2}}^{\varepsilon} k_1(t) \triangle t \right) \left(\frac{1}{\eta} \int_{\frac{\eta}{2}}^{\eta} k_2(s) \triangle s \right).$$

Remark 8. In Corollary 3, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Corollary 3).

Theorem 7. Let $\rho > 1$, $\Psi_2(\varepsilon, \eta)$ and $H_2(\varepsilon, \eta)$ be defined as in Lemma 4 and Lemma 5, respectively. If $\alpha < \rho - 1$, then the following inequality holds, provided that both sides are finite

$$\begin{aligned} & \int_0^r \int_0^e \frac{\vartheta(\varepsilon)\vartheta(\eta)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\ & \leq \left(\frac{\rho}{\rho-\alpha-1} \right)^{2\rho} \int_0^r \int_0^e \frac{\vartheta(\varepsilon)\vartheta(\eta)[\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}|G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}\Omega^{\rho(\rho-\alpha)}(\eta)|G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon. \quad (33) \end{aligned}$$

Proof. Applying Lemma 4, Lemma 5, and Fubini's theorem on the left-hand side of (33), we get

$$\begin{aligned}
 & \int_0^r \int_0^e \frac{\vartheta(\varepsilon)\vartheta(\eta)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\
 &= \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} \left(\int_0^e \frac{\vartheta(\eta)}{[\Omega^\sigma(\eta)]^{\rho-\alpha}} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \right) \triangle \varepsilon \\
 &\leq \left(\frac{\rho}{\rho-\alpha-1} \right)^\rho \int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} \left(\int_0^e \frac{\vartheta(\eta)[\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}}{\Omega^{\rho(\rho-\alpha)}(\eta)} |G_2(\varepsilon, \eta)|^\rho \triangle \eta \right) \triangle \varepsilon \\
 &= \left(\frac{\rho}{\rho-\alpha-1} \right)^\rho \int_0^e \frac{\vartheta(\eta)[\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}}{\Omega^{\rho(\rho-\alpha)}(\eta)} \left(\int_0^r \frac{\vartheta(\varepsilon)}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}} |G_2(\varepsilon, \eta)|^\rho \triangle \varepsilon \right) \triangle \eta \\
 &\leq \left(\frac{\rho}{\rho-\alpha-1} \right)^{2\rho} \int_0^e \frac{\vartheta(\eta)[\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}}{\Omega^{\rho(\rho-\alpha)}(\eta)} \left(\int_0^r \frac{\vartheta(\varepsilon)|G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}|G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \triangle \varepsilon \right) \triangle \eta \\
 &= \left(\frac{\rho}{\rho-\alpha-1} \right)^{2\rho} \int_0^e \int_0^r \frac{\vartheta(\varepsilon)\vartheta(\eta)[\Omega^\sigma(\eta)]^{(\rho-\alpha)(\rho-1)}|G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{[\Omega^\sigma(\varepsilon)]^{\rho-\alpha}\Omega^{\rho(\rho-\alpha)}(\eta)|G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon.
 \end{aligned}$$

□

Remark 9. In Theorem 7, if we take $\mathbb{T} = \mathbb{R}$, then we get ([6], Theorem 2).

Corollary 4. In Theorem 7, if $\mathbb{T} = \mathbb{Z}$, then inequality (33) becomes

$$\begin{aligned}
 & \sum_{\varepsilon=0}^{r-1} \sum_{\eta=0}^{e-1} \frac{\vartheta(\varepsilon)\vartheta(\eta)}{\Omega^{\rho-\alpha}(\varepsilon+1)\Omega^{\rho-\alpha}(\eta+1)} \Psi_2^\rho(\varepsilon, \eta+1) \\
 &\leq \left(\frac{\rho}{\rho-\alpha-1} \right)^{2\rho} \sum_{\varepsilon=0}^{r-1} \sum_{\eta=0}^{e-1} \frac{\vartheta(\varepsilon)\vartheta(\eta)\Omega^{(\rho-\alpha)(\rho-1)}(\eta+1)|G_2(\varepsilon+1, \eta)|^{\rho(\rho-1)}}{\Omega^{\rho-\alpha}(\varepsilon+1)\Omega^{\rho(\rho-\alpha)}(\eta)|G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho.
 \end{aligned}$$

where

$$\Psi_2(\varepsilon, \eta) = \sum_{t=\frac{\varepsilon}{2}}^{\varepsilon-1} \sum_{s=\frac{\eta}{2}}^{\eta-1} \frac{\vartheta(t)\vartheta(s)}{\Omega(t+1)\Omega(s)} k(t, s),$$

and

$$\begin{aligned}
 H_2(\varepsilon, \eta) &= k(\varepsilon, \eta) - \frac{\Omega(\varepsilon+1)\vartheta(\frac{\varepsilon}{2})}{2\Omega(\frac{\varepsilon+1}{2})\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \eta\right) \\
 &\quad - \frac{\Omega(\eta)\vartheta(\frac{\eta}{2})}{2\Omega(\frac{\eta}{2})\vartheta(\eta)} k\left(\varepsilon, \frac{\eta}{2}\right) + \frac{\Omega(\eta)\Omega(\varepsilon+1)\vartheta(\frac{\eta}{2})\vartheta(\frac{\varepsilon}{2})}{2\Omega(\frac{\eta}{2})2\Omega(\frac{\varepsilon+1}{2})\vartheta(\eta)\vartheta(\varepsilon)} k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).
 \end{aligned}$$

Setting $\vartheta(\tau) = 1$ and replacing $k(\varepsilon, \eta)$ with $\sigma(\varepsilon)\eta k(\varepsilon, \eta)$ in Theorem 7 yields the following corollary.

Corollary 5. Let $\rho > 1$ and $\alpha < \rho - 1$. Then

$$\begin{aligned}
 & \int_0^r \int_0^e \frac{1}{\sigma^{\rho-\alpha}(\varepsilon)\sigma^{\rho-\alpha}(\eta)} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\
 &\leq \left(\frac{\rho}{\rho-\alpha-1} \right)^{2\rho} \int_0^r \int_0^e \frac{\sigma^\alpha(\varepsilon)\sigma^{\rho(\rho-\alpha)+\alpha}(\eta)|G_2(\sigma(\varepsilon), \eta)|^{\rho(\rho-1)}}{\eta^{\rho(\rho-\alpha)}|G_2(\varepsilon, \eta)|^{\rho(\rho-1)}} (H_2(\varepsilon, \eta))^\rho \triangle \eta \triangle \varepsilon,
 \end{aligned}$$

where $\Psi_2(\varepsilon, \eta)$ and $H_2(\varepsilon, \eta)$ reduce to

$$\Psi_2(\varepsilon, \eta) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_{\frac{\eta}{2}}^{\eta} k(t, s) \triangle s \triangle t,$$

and

$$H_2(\varepsilon, \eta) := k(\varepsilon, \eta) - \frac{1}{2}k\left(\frac{\varepsilon}{2}, \eta\right) - \frac{1}{2}k\left(\varepsilon, \frac{\eta}{2}\right) + \frac{1}{4}k\left(\frac{\varepsilon}{2}, \frac{\eta}{2}\right).$$

Suppose $k(\varepsilon, \eta) = k_1(\varepsilon)k_2(\eta)$, where k_1, k_2 are nonnegative integrable functions on $(0, \infty)_{\mathbb{T}}$. Then $H_2(\varepsilon, \eta)$ and $G_2(\varepsilon, \eta)$ factor as

$$H_2(\varepsilon, \eta) := \left(k_1(\varepsilon) - \frac{1}{2}k_1\left(\frac{\varepsilon}{2}\right)\right) \left(k_2(\eta) - \frac{1}{2}k_2\left(\frac{\eta}{2}\right)\right),$$

and

$$G_2(\varepsilon, \eta) := \tilde{G}_1(\varepsilon)\tilde{G}_2(\eta),$$

where

$$\tilde{G}_1(\varepsilon) := \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{1}{\sigma(t)} k_1(t) \triangle t \text{ and } \tilde{G}_2(\eta) := k_2(\eta) - k_2\left(\frac{\eta}{2}\right).$$

Consequently, from Corollary 5, we obtain the following result.

Corollary 6. Let $\rho > 1$ and $\alpha < \rho - 1$. Then

$$\begin{aligned} & \int_0^r \int_0^e \frac{1}{\sigma^{\rho-\alpha}(\varepsilon)\sigma^{\rho-\alpha}(\eta)} \Psi_2^\rho(\varepsilon, \sigma(\eta)) \triangle \eta \triangle \varepsilon \\ & \leq \left(\frac{\rho}{\rho-\alpha-1}\right)^{2\rho} \left(\int_0^r \frac{\sigma^\alpha(\varepsilon) |G_1^\sigma(\varepsilon)|^{\rho(\rho-1)} \left(k_1(\varepsilon) - \frac{1}{2}k_1\left(\frac{\varepsilon}{2}\right)\right)^\rho \triangle \varepsilon}{|G_1(\varepsilon)|^{\rho(\rho-1)}} \right) \\ & \quad \times \left(\int_0^e \frac{\sigma^{\rho(\rho-\alpha)+\alpha}(\eta) \left(k_2(\eta) - \frac{1}{2}k_2\left(\frac{\eta}{2}\right)\right)^\rho \triangle \eta}{\eta^{\rho(\rho-\alpha)}} \right), \end{aligned}$$

where $\Psi_2(\varepsilon, \eta)$ reduce to

$$\Psi_2(\varepsilon, \eta) := \left(\int_{\frac{\varepsilon}{2}}^{\varepsilon} k_1(t) \triangle t\right) \left(\int_{\frac{\eta}{2}}^{\eta} k_2(s) \triangle s\right).$$

Remark 10. In Corollary 5, if $\mathbb{T} = \mathbb{R}$, then we get ([6], Corollary 2).

4. Conclusions

In this research, we have established new Hardy-type integral inequalities involving functions of two independent variables on arbitrary time scales. The results were obtained using elementary analytical methods such as direct applications of classical Hölder and Jensen inequalities, shifting techniques via the jump operator σ , and mathematical induction on discrete domains. These inequalities unify the continuous and discrete cases, which can already be viewed as direct applications of our framework.

We note that while the constants appearing in our inequalities coincide with those known to be sharp in the classical continuous case, the sharpness of these constants in the general dynamic bivariate framework has not been established in this paper and remains an interesting direction for future research. Moreover, in future work we intend to explore concrete applications of our results in the qualitative analysis of dynamic equations, fractional dynamic equations, and related problems on time scales. Thus, the continuous

and discrete forms presented here already serve as immediate applications, while further applied investigations will be pursued in subsequent studies.

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