

Article

Orientable Vertex Transitive Embeddings of Complete Graphs K_{p^2}

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Abstract

This paper investigates the enumeration of orientable vertex-transitive embeddings of complete graphs K_{p^2} , where $p \geq 5$ is a prime. Building on the classification framework established by Li, we derive a precise count of these embeddings up to non-isomorphism. Furthermore, we provide explicit examples to demonstrate the realizability of the maps corresponding to our enumeration results.

Keywords: vertex-transitive maps; complete graphs; Frobenius groups

MSC: 05C10; 05C25; 20B25.

1. Introduction

When a graph admits an embedding into a surface, a natural problem arises: how many non-isomorphic embeddings does it have? A fundamental objective of topological graph theory is to enumerate all symmetric embeddings of a given family of graphs into closed surfaces [1–4]. As part of a systematic effort to count the vertex-transitive embeddings of complete graphs, the present work is restricted to the orientable vertex-transitive embeddings of the complete graphs on p^2 vertices, where $p \geq 5$ is a prime.

An *orientable map* $\mathcal{M} = (V, E, F)$ is defined as a two-cell embedding of a finite graph $\Gamma = (V, E)$ into an orientable surface. Here, V , E and F denote the sets of vertices, edges and faces, respectively. An *automorphism* of $\mathcal{M}$ is a permutation on $V \cup E \cup F$ that preserves each of these sets and their incidence relations. Consequently, it corresponds precisely to a graph automorphism of Γ that also preserves the supporting surface. The set of all such automorphisms forms the *automorphism group* $\text{Aut}(\mathcal{M})$ under composition.

Map $\mathcal{M}$ is said to be *G-vertex-transitive* (or a *vertex-transitive embedding* of its underlying graph) if there exists a subgroup $G \leq \text{Aut}(\mathcal{M})$ acting transitively on the vertex set V . Moreover, if G additionally preserves the orientation of the supporting surface, then $\mathcal{M}$ is termed an *orientable vertex-transitive map*. The concept of orientable arc-transitive maps is defined analogously.

A *Cayley map* $\mathcal{M}$ is defined as an embedding of a Cayley graph, Σ , into a surface such that $\text{Aut}(\mathcal{M})$ contains a subgroup N acting regularly on the vertices. In this case, $\mathcal{M}$ is referred to as a Cayley map of N (or a Cayley embedding of Σ with respect to N). Notably, Cayley maps constitute an important and highly interesting class of vertex-transitive maps [5,6].

The latest developments in the theory of maps have been deeply intertwined with the study of map colorings, with highly symmetric maps consistently occupying a central posi-



Academic Editor: Fabio Caldarola

Received: 7 June 2026

Revised: 16 July 2026

Accepted: 24 July 2026

Published: 30 July 2026

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tion in research. The modern systematic investigation of this topic can be traced back to the foundational contributions of Biggs [7,8]. Over the past five decades, a substantial body of results concerning symmetrical maps has accumulated; see [5,6,9,10] and references therein.

Map $\mathcal{M}$ is said to be *complete* precisely when its underlying graph is the complete graph K_n . A classification of arc-transitive complete maps has been obtained by [8,11,12]; edge-transitive complete maps have been established in [9,13]; for edge-transitive complete bipartite maps, please refer to [14–20]; vertex-transitive complete maps $\mathcal{M}$ in four classes are given in [21]:

- (I) $\mathcal{M}$ is a sharp Cayley map, and the $\text{Aut}\mathcal{M}$ acting on vertices is regular;
- (II) $\mathcal{M}$ is a Cayley map of a nilpotent group, and $\text{Aut}\mathcal{M}$ is a Frobenius group;
- (III) $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^d$, where p is a prime and $d \geq 2$, $\text{Aut}\mathcal{M} \cong \mathbb{Z}_p^d : D_{2k}$ has the only minimal normal subgroup $\mathbb{Z}_p^d$, and the index 2 subgroup $\mathbb{Z}_p^d : \mathbb{Z}_k$ is a Frobenius group;
- (IV) $\mathcal{M}$ is a non-orientable non-Cayley map of order $2^{l-1}(2^l - 1)$, and the $\text{Aut}\mathcal{M} \cong \text{PSL}(2, 2^l)$ acting on vertices is primitive with $l \geq 2$.

The problem of determining the number of non-isomorphic vertex-transitive complete maps on a given number n of vertices, originally posed in [21], remains an open problem. Complete counts for maps of type (I) and type (IV) are established in [11,21], respectively. More recently, progress has been made [22–24] by solving the counting problem for certain special families of vertex-transitive embeddings of complete graphs.

The aim of this paper is to determine the number of orientable vertex-transitive complete maps with p^2 vertices, where $p \geq 5$ is a prime. Recall that $\phi(n)$ denotes the Euler phi function, defined as the count of positive integers less than n that are prime relative to n . The principal results of this work are stated in what follows.

Theorem 1. *Let $\mathcal{M}$ be an orientable vertex-transitive embedding of complete graphs K_{p^2} , where $p \geq 5$ is a prime, and let $G = \text{Aut}(\mathcal{M})$. Then, $\mathcal{M}$ is a Cayley map of group $\mathbb{Z}_p^2$, and $G \cong \mathbb{Z}_p^2 : G_\alpha$ is a Frobenius group whenever the vertex stabilizer G_α is cyclic with any vertex α . Further, let p_1 and p_2 be two different primes; the following statements hold:*

- (i) *If $G_\alpha \cong \mathbb{Z}_{p-1}$, then there are exactly $p!(p-1)^p\phi(p-1)$ such different maps. Furthermore, if $p+1 = p_1p_2$, then there are exactly*

$$\frac{|\mathcal{A}_{p+1}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{|\text{SL}(2, p)|}$$

such non-isomorphic maps with $|\mathcal{A}_{p+1}| = p!(p-1)^p\phi(p-1)$ and $|\mathcal{A}_1| = \phi(p^2 - 1)$.

- (ii) *If $G_\alpha \cong \mathbb{Z}_{p+1}$, then there are exactly $(p-2)!(p+1)^{p-2}\phi(p+1)$ such different maps. Furthermore, if $p-1 = p_1p_2$, then there are exactly*

$$\frac{|\mathcal{B}_{p-1}| - |\mathcal{B}_{p_1}| - |\mathcal{B}_{p_2}| + |\mathcal{B}_1|}{2(p-1)}$$

such non-isomorphic maps with $|\mathcal{B}_{p-1}| = (p-2)!(p+1)^{p-2}\phi(p+1)$ and $|\mathcal{B}_1| = \phi(p^2 - 1)$.

Remark 1. (1) *Orientable vertex-transitive complete maps whose vertex stabilizer G_α is a dihedral group only arise as embeddings of K_4 and K_9 [23]. Discussions concerning dihedral stabilizers have been omitted in this paper, as such embeddings do not exist for K_{p^2} with $p \geq 5$ as a prime.*

- (2) *For any divisor k of $p^2 - 1$, distinct structures of automorphism group $\text{Aut}(G)$ are yielded by the varying actions of $G_\alpha \cong \mathbb{Z}_k$ on $\mathbb{Z}_p^2$.*

- (a) *If k is a primitive divisor of $p^2 - 1$, then the action of $\mathbb{Z}_k$ on $\mathbb{Z}_{p^2-1}$ is primitive, and the corresponding enumeration results for such different and non-isomorphic maps are deduced from the assertion in Theorem 1 (ii).*

- (b) If k is not a primitive divisor of $p^2 - 1$, then the action of $\mathbb{Z}_k$ on $\mathbb{Z}_{p^2-1}$ becomes imprimitive, and the corresponding enumeration results for such different maps are deduced from the statement in Theorem 1 (i). However, for $k \neq p - 1$, characterizing the enumeration of non-isomorphic cases requires first determining the structure of $\text{Aut}(G)$. This issue is not resolved within the present paper, and the following open problem is therefore proposed.

Open problem. Let $p \geq 5$ be a prime, and let $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_k$ be an affine group such that $\gcd(k, p) = 1$. Supposing that the linear action of the cyclic subgroup $\mathbb{Z}_k$ on the elementary abelian p -group $\mathbb{Z}_p^2$ is reducible, how can automorphism group $\text{Aut}(G)$ of group G be explicitly characterized and computed?

Furthermore, as a corollary of Theorem 1, we derive the following conclusion.

Corollary 1. Let $p \geq 5$ be a prime, p_1 and p_2 be two different primes.

- (i) The number of non-isomorphic orientable embeddings $\mathcal{M}$ of K_{p^2} , such that $\text{Aut}(\mathcal{M}) \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p-1}$ and $p + 1 = p_1^2 p_2$, equals $\frac{|A_{p+1}| - |A_{p_1 p_2}| - |A_{p_1^2}| + |A_{p_1}|}{|\text{SL}(2, p)|}$.
- (ii) The number of non-isomorphic orientable embeddings $\mathcal{M}$ of K_{p^2} , such that $\text{Aut}(\mathcal{M}) \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p+1}$ and $p - 1 = p_1^2 p_2$, equals $\frac{|B_{p-1}| - |B_{p_1 p_2}| - |B_{p_1^2}| + |B_{p_1}|}{2(p-1)}$.

The structure of this paper is as follows: Following this Introduction Section, preliminary results are given in Section 2. The enumeration of the different and non-isomorphic orientable vertex-transitive complete maps is then presented in Sections 3 and 4, respectively. Finally, Section 5 is devoted to the complete proof of Theorem 1.

2. Preliminary Results

Some notations are fixed at the start of this section, followed by a presentation of fundamental results essential for subsequent arguments.

Let $p \geq 5$ be a prime. The cyclic group of order $p - 1$, the elementary Abelian p group of order p^2 , and the dihedral group of order $2p$ are denoted by $\mathbb{Z}_{p-1}$, $\mathbb{Z}_p^2$ and D_{2p} , respectively. The general linear group and the special linear group of field $\mathbb{F}_{p^2}$ are denoted by $\text{GL}(2, p)$ and $\text{SL}(2, p)$, respectively. The centralizer and the normalizer of $\mathbb{Z}_{p-1}$ in $\text{GL}(2, p)$ are denoted by $C_{\text{GL}(2, p)}(\mathbb{Z}_{p-1})$ and $N_{\text{GL}(2, p)}(\mathbb{Z}_{p-1})$, respectively. The order of a and the cyclic group generated by a are denoted by $o(a)$ and $\langle a \rangle$, respectively. Given two groups H and K , let $H:K$ denote the semidirect product of H by K in which H is normal. The center, the automorphism group and the inner automorphism group of H are denoted by $Z(H)$, $\text{Aut}(H)$ and $\text{Inn}(H)$, respectively.

Let $F = \mathbb{F}_{p^2}$ be the field of order p^2 , with additive group $F^+ = \mathbb{F}_{p^2}^+$ and multiplicative group $F^\times = \mathbb{F}_{p^2}^\times$. Then,

$$F^+ \cong \mathbb{Z}_p^2, \quad F^\times \cong \mathbb{Z}_{p^2-1}.$$

The identity of F^+ is 0 , and let $F^\# = F^+ \setminus \{0\}$ be the set of all non-identity elements. It follows that K_{p^2} , the complete graph on p^2 vertices, can be represented as a Cayley graph

$$K_{p^2} = \text{Cay}(F^+, F^\#).$$

A cyclic permutation of its neighbor set $\Gamma(v)$ of vertex v of finite graph Γ , denoted by R_v , is called a *rotation* at v . The set consisting of rotations at all vertices, denoted by $R(\Gamma) = \{R_v\}_{v \in V}$, is called a *rotation system* of Γ . Consequently, each rotation system $R(\Gamma)$ uniquely determines an orientable embedding of Γ ; refer to ([25], pp. 104–108).

Let $\text{Cay}(H, S)$ be a Cayley graph. Since vertex rotations R_v can be regarded as permutations not only of the neighborhood $\Gamma(v)$ but also the generating set S , Cayley maps admit another equivalent description [26]. A map with underlying graph $\text{Cay}(H, S)$ is called a *Cayley map* if the induced local cyclic permutations of S are all identical. Moreover, each cyclic permutation ρ of $F^\#$ gives rise to a unique orientable Cayley embedding of K_{p^2} with underlying graph $\text{Cay}(F^+, F^\#)$. Therefore, if ρ_1 and ρ_2 are different cyclic permutations of $F^\#$, the resulting orientable vertex-transitive complete maps are also considered different.

3. Enumeration of Different Complete Maps

This section is devoted to enumerating the different vertex-transitive embeddings of K_{p^2} . A well-established result concerning vertex-transitive maps is cited first.

Lemma 1 ([21], Lemma 2.2). *Let $\mathcal{M}$ be a vertex-transitive map with $G = \text{Aut}(\mathcal{M})$. Then, for any vertex α , stabilizer G_α is isomorphic to either $\mathbb{Z}_k$ or D_{2k} , where $k \geq 2$ is a positive integer, and each orbit of the action of G_α on the neighborhood of α has a length of exactly k .*

From Lemma 1, a finite planar rotational symmetry group is formed by stabilizer G_α under its natural action on the neighborhood of α . If $G_\alpha \cong \mathbb{Z}_k$, the stabilizer only admits rotational symmetries, namely, it cyclically permutes all neighbors of α with no reflective symmetries. If $G_\alpha \cong D_{2k}$, the stabilizer possesses both rotational and reflective symmetries. In addition to cyclically permuting the neighboring vertices, there exist axial reflective flips that swap pairs of adjacent vertices.

Next, from ([21], Theorem 1.1 and [25], Lemma 5.4.1), we can directly obtain the following lemma.

Lemma 2. *Let $\mathcal{M}$ be an orientable vertex-transitive map with underlying graph K_{p^2} and prime $p \geq 5$, and let $G = \text{Aut}(\mathcal{M})$. Then, $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^2$, $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_k$ is a Frobenius group such that $G_\alpha \cong \mathbb{Z}_k$, and $(k, p) = 1$.*

Assume that $\mathbb{Z}_k := \langle a \rangle$, with $o(a) \geq 2$. From Lemma 2, we know $G \cong F^+ : \langle a \rangle$ is a Frobenius group. Moreover, the action of $\langle a \rangle$ on F^+ is half-transitive, and $|\langle a \rangle|$ is a divisor of $|F^+| - 1$; in other words, $k \mid (p^2 - 1)$. In particular, if $k = p - 1$ or $p + 1$, it is easy to see that $G \cong F^+ : \mathbb{Z}_{p-1}$ or $F^+ : \mathbb{Z}_{p+1}$, respectively.

Lemma 3. *Let $p \geq 5$ be a prime number. Then, there are exactly $p!(p-1)^p \phi(p-1)$ different orientable vertex-transitive embeddings of K_{p^2} whose automorphism group is of the form $G \cong F^+ : \mathbb{Z}_{p-1}$.*

Proof. Since $F^+ : \mathbb{Z}_{p-1}$ is a Frobenius group, and $p - 1$ is not a primitive divisor of $p^2 - 1$, G is a imprimitive permutation group. For convenience, set $\alpha = \mathbf{0}$. Then, the action of G_0 on neighborhood $\Gamma(\mathbf{0})$ of $\mathbf{0}$ generates $\frac{p^2-1}{p-1} = p + 1$ orbits with $p + 1 \geq 6$, and, in view of Lemma 1, the length of each orbit is $p - 1$. Note that $F^\# = F^+ \setminus \{\mathbf{0}\}$, and we have

$$F^\# = \Delta_1 \dot{\cup} \Delta_2 \dot{\cup} \cdots \dot{\cup} \Delta_{p+1},$$

where Δ_i is an orbit of G_0 acting on $\Gamma(\mathbf{0})$, and $|\Delta_i| = p - 1$ with $1 \leq i \leq p + 1$.

Note that vertex $\mathbf{0}$ and the neighbors of $\mathbf{0}$ can be placed on a disk such that $\mathbf{0}$ is at the center and its neighbors are arranged around it. Without loss of generality, we assume that

the $p^2 - 1$ neighbors of $\mathbf{0}$, i.e., all elements of $F^\#$, are arranged in clockwise order around $\mathbf{0}$ and are denoted by $\beta_1, \beta_2, \dots, \beta_{p^2-1}$. Then,

$$\rho := (\beta_1, \beta_2, \dots, \beta_{p^2-1})$$

is a cyclic permutation of $F^\#$ (see Figure 1). It follows that the number of cyclic permutations of $F^\#$ is exactly the number of ways to arrange β_i with $1 \leq i \leq p^2 - 1$. Hence, the problem of counting orientable vertex-transitive complete maps reduces to counting the different possible choices for ordered set β_i . For simplicity, let $\beta_1 = \mathbf{1}$ with $\beta_1 \in \Delta_1$, and let $\mathbf{1}$ be the identity element of $F^\times$.

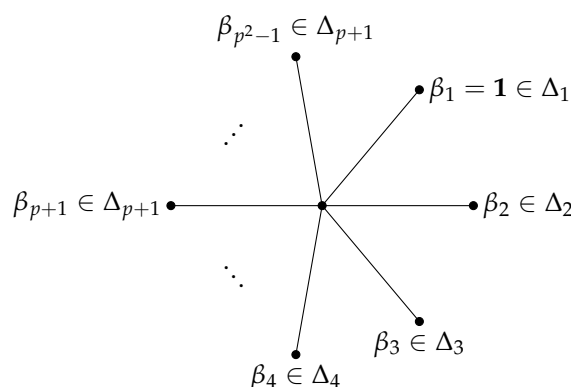


Figure 1. Cyclic permutations $\rho = (\mathbf{1}, \beta_2, \dots, \beta_{p^2-1})$ of $F^\#$.

If $\beta_2 \in \Delta_1$, then $\mathbf{1}^{a^s} = \beta_2$ for some a^s with $0 < s \leq p - 2$. It follows that $a^s : \mathbf{1} \mapsto \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \dots \mapsto \mathbf{1}$. Thus, $(s, p - 1) = 1$, and the action of $G_0 = \langle a^s \rangle$ on $\Gamma(\mathbf{0})$ gives rise to only one orbit, which is a contradiction. So, $\beta_2 \notin \Delta_1$ and $\beta_2 \in \Delta_i$ with $2 \leq i \leq p + 1$. Without loss of generality, set $\beta_2 \in \Delta_2$ for convenience, and then β_2 has $(p + 1 - 1)(p - 1)$ different choices.

If $\beta_3 \in \Delta_2$, then $\beta_2^{a^j} = \beta_3$ for some a^j , where $0 < j \leq p - 2$. Furthermore, there is $a^j : \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \dots \mapsto \mathbf{1}$. It follows that $(j, p - 1) = 1$, $\mathbf{1} \in \Delta_2$ and the action of G_0 on $\Gamma(\mathbf{0})$ gives rise to only one orbit, which is a contradiction. Thus, $\beta_3 \notin \Delta_2$. If $\beta_3 \in \Delta_1$, then $\mathbf{1}^{a^{j'}} = \beta_3$ for some $a^{j'}$ with $0 < j' \leq p - 2$. It follows that there are $a^{j'} : \mathbf{1} \mapsto \beta_3 \mapsto \beta_5 \mapsto \dots \mapsto \mathbf{1}$ and $a^{j'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \dots \mapsto \beta_2$. Thus, the action of $G_0 = \langle a^{j'} \rangle$ on $\Gamma(\mathbf{0})$ gives rise to two orbits, which is a contradiction. It follows that $\beta_3 \notin \Delta_1$ and $\beta_3 \in \Delta_i$ with $3 \leq i \leq p + 1$. Without loss of generality, set $\beta_3 \in \Delta_3$ for convenience, so β_3 has $(p + 1 - 2)(p - 1)$ different choices.

If $\beta_4 \in \Delta_3$, then $\beta_3^{a^t} = \beta_4$ for some a^t , where $0 < t \leq p - 2$. Moreover, there is $a^t : \beta_3 \mapsto \beta_4 \mapsto \beta_5 \mapsto \dots \mapsto \mathbf{1}$. Thus, $(t, p - 1) = 1$, $\mathbf{1} \in \Delta_3$ and $G_0 = \langle a^t \rangle$ acting on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction. So, $\beta_4 \notin \Delta_3$. If $\beta_4 \in \Delta_2$, then $\beta_2^{a^{t'}} = \beta_4$ for some $a^{t'}$, where $0 < t' \leq p - 2$. It follows that there are $a^{t'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \dots \mapsto \beta_2$ and $a^{t'} : \beta_3 \mapsto \beta_5 \mapsto \beta_7 \mapsto \dots \mapsto \mathbf{1}$. Thus, $G_0 = \langle a^{t'} \rangle$ acting on $\Gamma(\mathbf{0})$ gives rise to two orbits, which is a contradiction. So, $\beta_4 \notin \Delta_2$. If $\beta_4 \in \Delta_1$, then $\mathbf{1}^{a^{t''}} = \beta_4$ for some $a^{t''}$, where $0 < t'' \leq p - 2$. Further, there are $a^{t''} : \mathbf{1} \mapsto \beta_4 \mapsto \beta_7 \mapsto \dots \mapsto \mathbf{1}$, $a^{t''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \dots \mapsto \beta_2$ and $a^{t''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \dots \mapsto \beta_3$. Thus, the action of $G_0 = \langle a^{t''} \rangle$ on $\Gamma(\mathbf{0})$ gives rise to three orbits, which is a contradiction. So, $\beta_4 \notin \Delta_1$ and $\beta_4 \in \Delta_i$ with $4 \leq i \leq p + 1$. Without loss of generality, set $\beta_4 \in \Delta_4$ for convenience, and we deduce that β_4 has $(p + 1 - 3)(p - 1)$ different choices.

If $\beta_5 \in \Delta_4$, then $\beta_4^{a^k} = \beta_5$ for some a^k with $0 < k \leq p - 2$. It follows that there is $a^k : \beta_4 \mapsto \beta_5 \mapsto \beta_6 \mapsto \dots \mapsto \mathbf{1}$. Thus, $(k, p - 1) = 1$, $\mathbf{1} \in \Delta_4$, and the action of $G_0 = \langle a^k \rangle$ on $\Gamma(\mathbf{0})$ has only one orbit, which is a contradiction. So, $\beta_5 \notin \Delta_4$. If $\beta_5 \in \Delta_3$, then $\beta_3^{a^{k'}} = \beta_5$ for

some $a^{k'}$ with $0 < k' \leq p-2$. Correspondingly, there are $a^{k'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$ and $a^{k'} : 1 \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto 1$. Thus, the action of $G_0 = \langle a^{k'} \rangle$ on $\Gamma(0)$ gives rise to two orbits, which is a contradiction. So, $\beta_5 \notin \Delta_3$. If $\beta_5 \in \Delta_2$, then $\beta_2^{a^{k''}} = \beta_5$ for some $a^{k''}$ with $0 < k'' \leq p-2$. Further, there are $a^{k''} : 1 \mapsto \beta_4 \mapsto \beta_7 \mapsto \cdots \mapsto 1$, $a^{k''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$ and $a^{k''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$. Thus, the action of $G_0 = \langle a^{k''} \rangle$ on $\Gamma(0)$ gives rise to three orbits, which is a contradiction. So, $\beta_5 \notin \Delta_2$. If $\beta_5 \in \Delta_1$, then $1^{a^m} = \beta_5$ for some a^m with $0 < m \leq p-2$. It follows that there are $a^m : 1 \mapsto \beta_5 \mapsto \beta_9 \mapsto \cdots \mapsto 1$, $a^m : \beta_2 \mapsto \beta_6 \mapsto \beta_{10} \mapsto \cdots \mapsto \beta_2$, $a^m : \beta_3 \mapsto \beta_7 \mapsto \beta_{11} \mapsto \cdots \mapsto \beta_3$, and $a^m : \beta_4 \mapsto \beta_8 \mapsto \beta_{12} \mapsto \cdots \mapsto \beta_4$. Thus, the action of $G_0 = \langle a^m \rangle = \langle a \rangle$ on $\Gamma(0)$ has four orbits, which is a contradiction. So, $\beta_5 \notin \Delta_1$ and $\beta_5 \in \Delta_i$ with $5 \leq i \leq p+1$. Without loss of generality, set $\beta_5 \in \Delta_5$ for convenience, and we deduce that β_5 has $(p+1-4)(p-1)$ different choices.

If $\beta_6 \in \Delta_5$, then $\beta_5^{a^k} = \beta_6$ for some a^k with $0 < k \leq p-2$. It follows that there is $a^k : \beta_5 \mapsto \beta_6 \mapsto \beta_7 \mapsto \cdots \mapsto 1$. Thus, $(k, p-1) = 1$, $1 \in \Delta_5$, and $G_0 = \langle a^k \rangle$ acting on $\Gamma(0)$ gives rise to only one orbit, which is a contradiction. So, $\beta_6 \notin \Delta_5$. If $\beta_6 \in \Delta_4$, then $\beta_4^{a^{k'}} = \beta_6$ for some $a^{k'}$ with $0 < k' \leq p-2$. Correspondingly, there are $a^{k'} : \beta_2 \mapsto \beta_4 \mapsto \beta_6 \mapsto \cdots \mapsto \beta_2$ and $a^{k'} : 1 \mapsto \beta_3 \mapsto \beta_5 \mapsto \cdots \mapsto 1$. Thus, the action of $G_0 = \langle a^{k'} \rangle$ on $\Gamma(0)$ gives rise to two orbits, which is a contradiction. So, $\beta_6 \notin \Delta_4$. If $\beta_6 \in \Delta_3$, then $\beta_3^{a^{k''}} = \beta_6$ for some $a^{k''}$ with $0 < k'' \leq p-2$. Further, there are $a^{k''} : 1 \mapsto \beta_4 \mapsto \beta_7 \mapsto \cdots \mapsto 1$, $a^{k''} : \beta_2 \mapsto \beta_5 \mapsto \beta_8 \mapsto \cdots \mapsto \beta_2$ and $a^{k''} : \beta_3 \mapsto \beta_6 \mapsto \beta_9 \mapsto \cdots \mapsto \beta_3$. Thus, the action of $G_0 = \langle a^{k''} \rangle$ on $\Gamma(0)$ gives rise to three orbits, which is a contradiction. So, $\beta_6 \notin \Delta_3$. If $\beta_6 \in \Delta_2$, then $\beta_2^{a^{k'''}} = \beta_6$ for some $a^{k'''}$ with $0 < k''' \leq p-2$. It follows that there are $a^{k'''} : 1 \mapsto \beta_5 \mapsto \beta_9 \mapsto \cdots \mapsto 1$, $a^{k'''} : \beta_2 \mapsto \beta_6 \mapsto \beta_{10} \mapsto \cdots \mapsto \beta_2$, $a^{k'''} : \beta_3 \mapsto \beta_7 \mapsto \beta_{11} \mapsto \cdots \mapsto \beta_3$, and $a^{k'''} : \beta_4 \mapsto \beta_8 \mapsto \beta_{12} \mapsto \cdots \mapsto \beta_4$. Thus, the action of $G_0 = \langle a^{k'''} \rangle$ on $\Gamma(0)$ gives rise to four orbits, which is a contradiction. So, $\beta_6 \notin \Delta_2$. If $\beta_6 \in \Delta_1$, then $1^{a^m} = \beta_6$ for some a^m with $0 < m \leq p-2$. It follows that there are $a^m : 1 \mapsto \beta_6 \mapsto \beta_{11} \mapsto \cdots \mapsto 1$, $a^m : \beta_2 \mapsto \beta_7 \mapsto \beta_{12} \mapsto \cdots \mapsto \beta_2$, $a^m : \beta_3 \mapsto \beta_8 \mapsto \beta_{13} \mapsto \cdots \mapsto \beta_3$, $a^m : \beta_4 \mapsto \beta_9 \mapsto \beta_{14} \mapsto \cdots \mapsto \beta_4$, and $a^m : \beta_5 \mapsto \beta_{10} \mapsto \beta_{15} \mapsto \cdots \mapsto \beta_5$. Thus, the action of $G_0 = \langle a^m \rangle$ on $\Gamma(0)$ gives rise to five orbits, which is a contradiction. So $\beta_6 \notin \Delta_1$ and $\beta_6 \in \Delta_i$ with $6 \leq i \leq p+1$. Without loss of generality, set $\beta_6 \in \Delta_6$ for convenience, and we deduce that β_6 has $(p+1-5)(p-1)$ different choices.

According to the above derivation, we find that $\beta_7 \in \Delta_1$ and $\beta_7 \notin \Delta_i$ with $2 \leq i \leq 6$. Then, $1^{a^l} = \beta_7$ for some a^l , where $0 < l \leq p-2$. It follows that there are $a^l : 1 \mapsto \beta_7 \mapsto \beta_{13} \mapsto \cdots \mapsto 1$, $a^l : \beta_2 \mapsto \beta_8 \mapsto \beta_{14} \mapsto \cdots \mapsto \beta_2$, $a^l : \beta_3 \mapsto \beta_9 \mapsto \beta_{15} \mapsto \cdots \mapsto \beta_3$, $a^l : \beta_4 \mapsto \beta_{10} \mapsto \beta_{16} \mapsto \cdots \mapsto \beta_4$, $a^l : \beta_5 \mapsto \beta_{11} \mapsto \beta_{17} \mapsto \cdots \mapsto \beta_5$, and $a^l : \beta_6 \mapsto \beta_{12} \mapsto \beta_{18} \mapsto \cdots \mapsto \beta_6$. Thus, the action of $G_0 = \langle a^l \rangle$ on $\Gamma(0)$ gives rise to six orbits, namely, $p+1=6$. Since the number of generators of G_0 is $\phi(p-1)$, we find that a^l has $\phi(p-1)$ different choices. Noting that G_0 is cyclic, then, except for $1, \beta_2, \beta_3, \beta_4, \beta_5$ and β_6 , the remaining vertices of $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ and Δ_6 can be obtained by $1, \beta_2, \beta_3, \beta_4, \beta_5$ and β_6 through the conjugate action of $a, a^2, a^3, \dots, a^{p-2}$, respectively. So,

$$\rho|_{p+1=6} := (1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, 1^a, \beta_2^a, \beta_3^a, \dots, \beta_6^a, \dots, 1^{a^{p-2}}, \beta_2^{a^{p-2}}, \beta_3^{a^{p-2}}, \dots, \beta_6^{a^{p-2}})$$

is a cyclic permutation of $F^\#$. It follows that the number of ρ is determined by the choices of $1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ and a . Further, the number of $\rho|_{p+1=6}$ equals

$$\begin{aligned} & (p+1-1)(p-1) \cdot (p+1-2)(p-1) \cdot (p+1-3)(p-1) \cdot (p+1-4)(p-1) \cdot (p+1-5)(p-1) \cdot \phi(p-1) \\ & = (6-1)!(p-1)^5 \phi(p-1). \end{aligned}$$

Let the corresponding maps generated by $\rho|_{p+1=6}$ be

$$\mathcal{M}_6 := \mathcal{M}_6(\mathbf{1}, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, a).$$

Thus, the number of different orientable vertex-transitive complete maps equals $5!(p-1)^5\phi(p-1)$ if $p+1=6$.

In fact, $p+1$ can be generalized. If $G_\alpha = \langle a \rangle$ acting on $\Gamma(\alpha)$ has $p+1$ orbits, then $\beta_i \in \Delta_i$ such that $\beta_1 = \mathbf{1}$ and $1 \leq i \leq p+1$, and

$$\rho|_{p+1} := (\mathbf{1}, \beta_2, \dots, \beta_{p+1}, \mathbf{1}^a, \beta_2^a, \dots, \beta_{p+1}^a, \dots, \mathbf{1}^{a^{p-2}}, \beta_2^{a^{p-2}}, \dots, \beta_{p+1}^{a^{p-2}})$$

is a cyclic permutation of $F^\#$. Since β_i has $(p+1-i+1)(p-1)$ different choices with $2 \leq i \leq p+1$ and a has $\phi(p-1)$ different choices, it follows that the number of $\rho|_{p+1}$ equals

$$\begin{aligned} & (p+1-1)(p-1) \cdot (p+1-2)(p-1) \cdots (p+1-(p+1)+1)(p-1) \cdot \phi(p-1) \\ & = p!(p-1)^p \phi(p-1). \end{aligned}$$

Let the corresponding maps generated by $\rho|_{p+1}$ be

$$\mathcal{M}_{p+1} := \mathcal{M}_{p+1}(\mathbf{1}, \beta_2, \beta_3, \dots, \beta_{p+1}, a).$$

Hence, if $G_\alpha \cong \mathbb{Z}_{p-1}$, then there are exactly $p!(p-1)^p \phi(p-1)$ different orientable vertex-transitive embeddings of K_{p^2} . $\square$

Next, we apply the same method to determine the number of different orientable vertex-transitive maps with underlying graph K_{p^2} in the case where $G_\alpha \cong \mathbb{Z}_{p+1}$.

Lemma 4. *Let $p \geq 5$ be a prime number. Then, there are exactly $(p-2)!(p+1)^{p-2}\phi(p+1)$ different orientable vertex-transitive embeddings of K_{p^2} whose automorphism group is of the form $G \cong F^+ : \mathbb{Z}_{p+1}$.*

Proof. Use the notations defined in Lemma 3. Note that $F^+ : \mathbb{Z}_{p+1}$ is a Frobenius group, $(p+1)|(p^2-1)$ and $(p+1) \nmid (p-1)$, namely, $p+1$ is a primitive divisor of p^2-1 , and then G is a primitive permutation group, G_0 partitions $\Gamma(\mathbf{0})$ into $\frac{p^2-1}{p+1} = p-1 \geq 4$ orbits and the length of each orbit is $p+1$. It follows that $F^\# = \Theta_1 \dot{\cup} \Theta_2 \dot{\cup} \cdots \dot{\cup} \Theta_{p-1}$, where Θ_i is an orbit of G_0 acting on $\Gamma(\mathbf{0})$, $|\Theta_i| = p+1$ with $1 \leq i \leq p-1$.

From the proof of Lemma 3, we can set that $\mathbf{1} \in \Theta_1$, $\beta_i \in \Theta_i$ with $2 \leq i \leq p-1$. It follows that

$$\rho|_{p-1} = (\mathbf{1}, \beta_2, \dots, \beta_{p-1}, \mathbf{1}^a, \beta_2^a, \dots, \beta_{p-1}^a, \dots, \mathbf{1}^{a^p}, \beta_2^{a^p}, \dots, \beta_{p-1}^{a^p})$$

is a cyclic permutation of $F^\#$. Since β_i has $(p-i)(p+1)$ different choices with $2 \leq i \leq p-1$ and a has $\phi(p+1)$ different choices, the number of $\rho|_{p-1}$ equals

$$\begin{aligned} & (p-2)(p+1) \cdot (p-3)(p+1) \cdots (p-p+1)(p+1) \cdot \phi(p+1) \\ & = (p-2)!(p+1)^{p-2} \phi(p+1). \end{aligned}$$

Let the corresponding maps generated by $\rho|_{p-1}$ be

$$\mathcal{M}_{p-1} := \mathcal{M}_{p-1}(\mathbf{1}, \beta_2, \beta_3, \dots, \beta_{p-1}, a).$$

Hence, if $G_\alpha \cong \mathbb{Z}_{p+1}$, then there are exactly $(p-2)!(p+1)^{p-2}\phi(p+1)$ different orientable vertex-transitive embeddings of K_{p^2} . $\square$

Similarly, we obtain the number of the different orientable vertex-transitive maps with underlying graph K_{p^2} in the case where $G_\alpha \cong \mathbb{Z}_{p^2-1}$.

Lemma 5. *Let $p \geq 5$ be a prime number. Then, there are exactly $\phi(p^2 - 1)$ different orientable arc-transitive embeddings of K_{p^2} whose automorphism group is of the form $G \cong F^+ : \mathbb{Z}_{p^2-1}$.*

Proof. Use the notations defined in Lemma 3. If $G \cong F^+ : \mathbb{Z}_{p^2-1}$, then the action of G_0 on $\Gamma(0)$ has only one orbit. It follows that

$$\rho_1 = (1, 1^a, 1^{a^2}, \dots, 1^{a^{p^2-2}})$$

is a cyclic permutation of $F^\#$. Noting that a has $\phi(p^2 - 1)$ different choices, then the number of ρ_1 equals $\phi(p^2 - 1)$. Let the corresponding maps generated by ρ_1 be $\mathcal{M}_{\rho_1} := \mathcal{M}_{\rho_1}(1, a)$. Hence, if $G_\alpha \cong \mathbb{Z}_{p^2-1}$, then the number of different orientable arc-transitive maps $\mathcal{M}_{\rho_1}$ of K_{p^2} equals $\phi(p^2 - 1)$. $\square$

It is now verified that the automorphism group of the constructed complete map $\mathcal{M}$ satisfies Lemmas 3–5.

Lemma 6. *Let $\mathcal{M}$ be an orientable vertex-transitive embedding of K_{p^2} constructed from the semidirect product $F^+ : \langle a \rangle$ with cyclic generator a . Then, the full automorphism group of $\mathcal{M}$ is exactly given by $\text{Aut}(\mathcal{M}) = F^+ : \langle a \rangle$.*

Proof. From the construction rule of complete map $\mathcal{M}$ in Lemmas 3–5, every element of F^+ acts as a vertex translation on $\mathcal{M}$, and each power of generator a provides rotational symmetry preserving the cyclic adjacency ordering of neighbors at each vertex. Since all such transformations respect the edge incidence and map structure, every element of $F^+ : \langle a \rangle$ belongs to $\text{Aut}(\mathcal{M})$. Therefore, $F^+ : \langle a \rangle \subseteq \text{Aut}(\mathcal{M})$.

Reverse inclusion is $\text{Aut}(\mathcal{M}) \subseteq F^+ : \langle a \rangle$. Take an arbitrary automorphism $\sigma \in \text{Aut}(\mathcal{M})$. We decompose σ into a translation and a rotation, and we verify both components lie in F^+ and $\langle a \rangle$, respectively.

Let $0 \in F$ be the base vertex of $\mathcal{M}$, and let $v = \sigma(0)$ be the image of the base vertex under σ . Since F^+ acts regularly on the vertex set of $\mathcal{M}$, there exists a unique translation $t_v \in F^+$ such that $t_v(0) = v$. Define the composition $\tau = t_v^{-1} \circ \sigma$, then $\tau(0) = 0$, i.e., τ fixes the base vertex.

The stabilizer subgroup $(\text{Aut}(\mathcal{M}))_0$ consists of all automorphisms fixing the base vertex, which must preserve the cyclic order of neighbors around 0 . By our construction, the cyclic adjacency around each vertex is exactly induced by the permutation a , so all vertex-stabilizing automorphisms are powers of a . That is,

$$(\text{Aut}(\mathcal{M}))_0 = \langle a \rangle.$$

Since τ fixes 0 , we have $\tau \in \langle a \rangle$.

The two components are combined to express $\sigma = t_v \circ \tau$, where $t_v \in F^+$ and $\tau \in \langle a \rangle$. From the definition of the semidirect product, σ is contained in $F^+ : \langle a \rangle$. Via the two-way subgroup inclusion established above, the equality $\text{Aut}(\mathcal{M}) = F^+ : \langle a \rangle$ is concluded, which verifies that the automorphism group structure adopted for the enumeration is precise. $\square$

4. Enumeration of Non-Isomorphic Complete Maps

Remark that two different orientable vertex-transitive maps with underlying graph K_{p^2} can be isomorphic. For the enumeration of complete maps up to isomorphism, the classical Clifford's theorem is recalled at the outset.

Lemma 7 ([27], Theorem 5.3.2). *Let V be an irreducible FH module, and let N be a normal subgroup of H . Then, the following statements are true:*

(i) V is a completely reducible FN-module, and

$$V = W_1^n \oplus W_2^n \oplus \cdots \oplus W_r^n,$$

where W_i are all non-isomorphic irreducible FN submodules of V with $1 \leq i \leq r$.

(ii) H permutes $\{W_1^n, W_2^n, \dots, W_r^n\}$ transitively.

(iii) If K is the stabilizer of the action of H on W_1^n , then H acting on W_1^n is irreducible.

Let $p \geq 5$ be a prime. From ([28], Theorem 7.3 of Character 2), $GL(2, p)$ has a cyclic subgroup $\mathbb{Z}_{p^2-1}$. Since $\mathbb{Z}_{p^2-1}$ is regular on the finite set Ω of $p^2 - 1$ vertices, from ([29], Exercise 1.4.1), it follows that $GL(2, p) \cong \mathbb{Z}_{p^2-1} : (GL(2, p))_\alpha$ for any $\alpha \in \Omega$. Let $\{\alpha, \beta\}$ be a base of 2-dimensional linear space V on field F . For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, p)$ such that $ad - bc \neq 0$,

$$\{\alpha, \beta\}^g = \{\alpha, \beta\} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{a\alpha + c\beta, b\alpha + d\beta\} = \{\alpha, b\alpha + d\beta\},$$

it follows that $a\alpha + c\beta = \alpha$. Then $a = 1, c = 0$, and $(GL(2, p))_\alpha = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. So, b has p different choices, and d has $p - 1$ different choices. Hence, $(GL(2, p))_\alpha \cong \mathbb{Z}_p : \mathbb{Z}_{p-1}$, and $GL(2, p) \cong \mathbb{Z}_{p^2-1} : (\mathbb{Z}_p : \mathbb{Z}_{p-1})$.

Next, we need to obtain the following lemma, which will determine the normalizer of $\mathbb{Z}_{p-1}$ and $\mathbb{Z}_{p+1}$ in $GL(2, p)$, and further ascertain the structure of $Aut(G)$.

Lemma 8. *Let $p \geq 5$ be a prime number. The following statements holds:*

(i) If $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p-1}$, then $N_{GL(2, p)}(\mathbb{Z}_{p-1}) \cong GL(2, p)$ and $Aut(G) \cong \mathbb{Z}_p^2 : GL(2, p)$.

(ii) If $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p+1}$, then $N_{GL(2, p)}(\mathbb{Z}_{p+1}) \cong \mathbb{Z}_{p^2-1} : \mathbb{Z}_2$ and $Aut(G) \cong \mathbb{Z}_p^2 : (\mathbb{Z}_{p^2-1} : \mathbb{Z}_2)$.

Proof. Let $V = \mathbb{Z}_p^2$, namely, consider $\mathbb{Z}_p^2$ as a 2-dimensional linear space over field F . Since $\mathbb{Z}_{p-1}$ acting on $\mathbb{Z}_p^2$ is reducible, from Lemma 7,

$$\mathbb{Z}_p^2 = V_1 \oplus V_2,$$

where $\dim(V_i) = 1$, and V_i is a faithful irreducible $F_p : \mathbb{Z}_{p-1}$ module with $i = 1, 2$. Fix an F_p -basis $\mathcal{B} = \{1, \alpha\}$ of F_{p^2} , where $\alpha \in F_{p^2} \setminus F_p$ is a root of an irreducible quadratic polynomial over F_p . For any $a \in F_p^*$, define the left multiplication map $\rho_a : F_{p^2} \rightarrow F_{p^2}$ by $\rho_a(x) = a \cdot x$, where the multiplication is the field multiplication in F_{p^2} . Since $a \in F_p$, for any $x = x_1 + x_2\alpha \in F_{p^2}$ with $x_1, x_2 \in F_p$, we have $\rho_a(x) = ax_1 + ax_2\alpha$. The matrix of ρ_a under basis $\mathcal{B}$ is the scalar matrix

$$[\rho_a]_{\mathcal{B}} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI_2.$$

Map $\iota : F_p^* \rightarrow \text{GL}(2, p)$ given by $\iota(a) = aI_2$ is an injective group homomorphism, so its image satisfies $\iota(F_p^*) \cong \mathbb{Z}_{p-1}$. This is the primitive scalar embedding adopted throughout this paper, in contrast to reducible embeddings such as the diagonal embedding.

Let $H = \iota(\mathbb{Z}_{p-1}) = \{aI_2 \mid a \in F_p^*\}$ be the image of the primitive scalar embedding. Then, H is the center of $\text{GL}(2, p)$. Further, from ([30], 27.14(3)), we have $C_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) = \text{GL}(2, p)$. Since $C_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) \leq N_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) \leq \text{GL}(2, p)$, it is easy to see that $N_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) \cong \text{GL}(2, p)$. Furthermore, if $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p-1}$, from ([31], Lemma 4.5), we have

$$\text{Aut}(G) \cong \mathbb{Z}_p^2 : N_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) \cong \mathbb{Z}_p^2 : \text{GL}(2, p).$$

The conclusion of (i) holds.

Next, we prove that the conclusion of (ii) holds. From ([28], Theorem 7.3 of Character 2),

$$N_{\text{GL}(2,p)}(\mathbb{Z}_{p+1}) \cong N_{\text{GL}(2,p)}(\mathbb{Z}_{p^2-1}) \cong \mathbb{Z}_{p^2-1} : \mathbb{Z}_2.$$

Furthermore, if $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p+1}$, from ([31], Lemma 4.5), we have

$$\text{Aut}(G) \cong \mathbb{Z}_p^2 : N_{\text{GL}(2,p)}(\mathbb{Z}_{p+1}) \cong \mathbb{Z}_p^2 : (\mathbb{Z}_{p^2-1} : \mathbb{Z}_2).$$

□

Remark 2. The identity $N_{\text{GL}(2,p)}(\mathbb{Z}_{p-1}) \cong \text{GL}(2, p)$ in Lemma 8(i) holds only for the primitive scalar embedding defined above. For other embeddings of $\mathbb{Z}_{p-1}$ into $\text{GL}(2, p)$, the normalizer may be a proper subgroup. For example, consider the reducible diagonal embedding $\iota' : F_p^* \rightarrow \text{GL}(2, p)$ given by $\iota'(a) = \text{diag}(a, 1)$. Take the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}(2, p),$$

then $P\iota'(a)P^{-1} = \text{diag}(1, a) \notin \iota'(F_p^*)$ for $a \neq 1$, so $P \notin N_{\text{GL}(2,p)}(\iota'(\mathbb{Z}_{p-1}))$. This confirms that the normalizer structure depends critically on the embedding, and all results in this paper are based on the primitive scalar embedding via the right regular representation.

Two maps $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be *isomorphic* if there exists a bijection between their vertex sets that sends flags to flags, denoted by $\mathcal{M}_1 \cong \mathcal{M}_2$. Consequently, if $\mathcal{M}_1 \cong \mathcal{M}_2$, then their automorphism groups are isomorphic as well. A complete map is called a *complete Cayley map* if its automorphism group acts regularly on the vertices. Furthermore, complete Cayley maps derived from non-isomorphic groups are always non-isomorphic.

Let $\mathcal{M}$ be an orientable vertex-transitive embedding of complete graphs K_{p^2} , where $p \geq 5$ is a prime. Let $G = \text{Aut}(\mathcal{M}) \cong \mathbb{Z}_p^2 : G_\alpha$, and G_α is a cyclic group for any $\alpha \in V$. From ([31], Lemma 4.5), it follows that

$$\text{Aut}(G) \cong \mathbb{Z}_p^2 : N_{\text{Aut}(\mathbb{Z}_p^2)}(G_\alpha) = \mathbb{Z}_p^2 : N_{\text{GL}(2,p)}(G_\alpha) \leq \text{Sym}(V).$$

Then, the following lemma can be obtained.

Lemma 9. If σ is a permutation on the vertex set V of $\mathcal{M}$, then $\mathcal{M}^\sigma \cong \mathcal{M}$ for each $\sigma \in \text{Aut}(G)$. On the contrary, $\sigma \in \text{Aut}(G)$ if $\mathcal{M}^\sigma \cong \mathcal{M}$.

Proof. Suppose that τ fixes α for some $\tau \in \text{Aut}(G) \leq \text{Sym}(V)$ and $\alpha \in V \setminus \{0\}$. Since $\mathbb{Z}_p^2$ is a regular and normal subgroup of $\text{Aut}(G)$, for any $x \in \mathbb{Z}_p^2 \setminus \{0, 1\}$, $\alpha^x = \alpha x$ or $x^{-1}\alpha$ if x is the right or left multiplication, respectively. So, $\alpha^x \neq \alpha$; namely, x does not fix α . It

follows that $\tau \in N_{\text{GL}(2,p)}(G_\alpha) \leq \text{Sym}(V)$. Since G is a Frobenius group, it is easy to see that $Z(G) = 1$. Thus,

$$\mathbb{Z}_p^2 : G_\alpha \cong G = G/Z(G) \cong \text{Inn}(G) \triangleleft \text{Aut}(G) \cong \mathbb{Z}_p^2 : N_{\text{GL}(2,p)}(G_\alpha).$$

Then, $G_\alpha \triangleleft N_{\text{GL}(2,p)}(G_\alpha)$. In addition, from ([29], Exercise 1.4.1), $\text{Aut}(G) = \mathbb{Z}_p^2 : (\text{Aut}(G))_\alpha$. So, we obtain $G_\alpha \triangleleft (\text{Aut}(G))_\alpha \cong N_{\text{GL}(2,p)}(G_\alpha)$. Then, $G_\alpha^\tau = G_\alpha$ for each $\tau \in N_{\text{GL}(2,p)}(G_\alpha)$.

Further, for each $\sigma \in \text{Aut}(G)$ such that $\sigma = x\tau$, where $x \in \mathbb{Z}_p^2$ and $\tau \in N_{\text{GL}(2,p)}(G_\alpha)$,

$$\mathcal{M}^\sigma = \mathcal{M}^{x\tau} = \mathcal{M}^\tau \cong \mathcal{M}.$$

Hence, $\mathcal{M}^\sigma \cong \mathcal{M}$ for each $\sigma \in \text{Aut}(G)$ by arbitrariness of x .

On the contrary, if $\mathcal{M}^\sigma \cong \mathcal{M}$, then $\text{Aut}(\mathcal{M}^\sigma) \cong \text{Aut}(\mathcal{M})$. It follows that for each $\sigma \in G$,

$$\text{Aut}(\mathcal{M})^\sigma = G^\sigma = G \cong \text{Aut}(\mathcal{M}^\sigma).$$

Note that $Z(G) = 1$, and $G = G/Z(G) \cong \text{Inn}(G) \triangleleft \text{Aut}(G)$, then $\sigma \in \text{Aut}(G)$. $\square$

In view of the above lemmas, we first determine the number of the non-isomorphic embeddings with underlying graph K_{p^2} when $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p-1}$ is an imprimitive permutation group, as stated in the following lemma.

Let

$$\mathcal{A}_{p+1} = \{\mathcal{M}_{p+1}(\mathbf{1}, \beta_2, \beta_3, \dots, \beta_{p+1}, a) \mid \beta_j \in \Delta_j \text{ such that } \beta_1 = \mathbf{1}, \\ o(a) = p-1, \text{ and read the subscripts modulo } p+1, 1 \leq j \leq p+1\}.$$

Then, $\mathcal{A}_{p+1}$ is a finite non-empty set, and $|\mathcal{A}_{p+1}| = p!(p-1)^p \phi(p-1)$.

Lemma 10. If $G \cong F^+ : \mathbb{Z}_{p-1}$, $p+1 = p_1 p_2$ with different primes p_i ($i = 1, 2$), then there are exactly $\frac{|\mathcal{A}_{p+1}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{|\text{SL}(2,p)|}$ non-isomorphic orientable vertex-transitive embeddings of K_{p^2} .

Proof. Let $X = \text{Aut}(G)$. Then, from Lemma 8,

$$X \cong \mathbb{Z}_p^2 : \text{GL}(2, p) \cong \mathbb{Z}_p^2 : (\mathbb{Z}_{p^2-1} : (\mathbb{Z}_p : \mathbb{Z}_{p-1})).$$

Since $\mathbb{Z}_p^2 \triangleleft X$, and $\mathbb{Z}_p^2$ acting on V is regular, from ([29], Exercise 1.4.1), $X_\alpha \cong \mathbb{Z}_{p^2-1} : (\mathbb{Z}_p : \mathbb{Z}_{p-1})$; further, $G \triangleleft X$, and $G_\alpha \triangleleft X_\alpha$. Set $\langle z \rangle := \mathbb{Z}_{p^2-1}$ for convenience.

Since $o(a) = p-1$, it follows that $z^{p+1} = a$. Let

$$(\mathbf{1}, \beta_2, \beta_3, \dots, \beta_{p+1}, \mathbf{1}^a, \beta_2^a, \beta_3^a, \dots, \beta_{p+1}^a, \dots, \mathbf{1}^{a^{p-2}}, \beta_2^{a^{p-2}}, \beta_3^{a^{p-2}}, \dots, \beta_{p+1}^{a^{p-2}})$$

be a cyclic permutation of $F^\#$ such that $\beta_i = \mathbf{1}^{z^{i-1}}$ and $3 \leq i < p+1$. Then, we have

$$z^{i-1} : \mathbf{1} \mapsto \beta_i \mapsto \beta_{2i-1} \mapsto \dots \mapsto \mathbf{1},$$

$$z^{i-1} : \beta_2 \mapsto \beta_{i+1} \mapsto \beta_{2i} \mapsto \dots \mapsto \beta_2,$$

...

$$z^{i-1} : \beta_{i-1} \mapsto \beta_{2i-2} \mapsto \beta_{3i-3} \mapsto \dots \mapsto \beta_{i-1}.$$

That is, z^{i-1} can be identified with the permutation

$$z^{i-1} = (\mathbf{1} \beta_i \beta_{2i-1} \dots \beta_{p^2-i+1})(\beta_2 \beta_{i+1} \beta_{2i} \dots \beta_{p^2-i+2}) \dots (\beta_{i-1} \beta_{2i-2} \beta_{3i-3} \dots \beta_{p^2-1}).$$

Thus, $G_\alpha = \langle z^{i-1} \rangle$, and $a = z^{p+1} \in \langle z^{i-1} \rangle$. Then, $(i-1)|(p+1)$, and $i-1 = p_1$ or p_2 .

Suppose $i-1 = p_1$. Let $\mathcal{M}_{p_1}$ be the only orientable complete map generated by z^{p_1} . It follows that $\text{Aut}(\mathcal{M}_{p_1}) = F^+ : \langle z^{p_1} \rangle = G.p_2 > G$. Thus, $\mathcal{M}_{p_1} \in \mathcal{A}_{p_1}$, $\mathcal{A}_{p_1} \subset \mathcal{A}_{p+1}$. So, $X \setminus G$ contains no element which is an automorphism of $\mathcal{M}'_{p_1}$ for $\mathcal{M}'_{p_1} \in \mathcal{A}_{p+1} \setminus \mathcal{A}_{p_1}$. Since $G \triangleleft X$ and

$$(X/G)_{\mathcal{M}'_{p_1}} = \{xG \in X/G \mid (\mathcal{M}'_{p_1})^{xG} = (\mathcal{M}'_{p_1})^{Gx} = (\mathcal{M}'_{p_1})^x = \mathcal{M}'_{p_1}\} = G,$$

then X/G acting on $\mathcal{A}_{p+1} \setminus \mathcal{A}_{p_1}$ is semiregular. Similarly, X/G acting on $\mathcal{A}_{p+1} \setminus \mathcal{A}_{p_2}$ is semiregular.

Especially, we notice that

$$z = (1, \beta_2, \dots, \beta_{p+1}, 1^a, \beta_2^a, \dots, \beta_{p+1}^a, \dots, 1^{a^{p-2}}, \beta_2^{a^{p-2}}, \dots, \beta_{p+1}^{a^{p-2}})$$

can also be seen as a cyclic permutation of $F^\#$ such that $\beta_2 = 1^z$. Let $\mathcal{M}_1$ be the only orientable complete map generated by z . It follows that $\text{Aut}(\mathcal{M}_1) = F^+ : \langle z \rangle = G.(p+1) > G$, and $\mathcal{M}_1$ is arc-transitive. Thus, $\mathcal{M}_1 \in \mathcal{A}_1$, $\mathcal{A}_1 \subset \mathcal{A}_{p+1}$, and $|\mathcal{A}_1| = \phi(p^2 - 1)$. So, $X \setminus G$ contains no element that is an automorphism of $\mathcal{M}'_1$ for $\mathcal{M}'_1 \in \mathcal{A}_{p+1} \setminus \mathcal{A}_1$. Since $G \triangleleft X$ and

$$(X/G)_{\mathcal{M}'_1} = \{xG \in X/G \mid (\mathcal{M}'_1)^{xG} = (\mathcal{M}'_1)^{Gx} = (\mathcal{M}'_1)^x = \mathcal{M}'_1\} = G,$$

then X/G acting on $\mathcal{A}_{p+1} \setminus \mathcal{A}_1$ is semiregular.

Let X act on $\mathcal{A}_{p+1} \setminus (\mathcal{A}_{p_1} \cup \mathcal{A}_{p_2})$. It follows that for each $\mathcal{M}'_{p+1} \in \mathcal{A}_{p+1} \setminus (\mathcal{A}_{p_1} \cup \mathcal{A}_{p_2})$, $(\mathcal{M}'_{p+1})^X$ is an orbit of this action, and the length of this orbit equals

$$|(\mathcal{M}'_{p+1})^X| = \frac{|X|}{|X_{\mathcal{M}'_{p+1}}|} = \frac{|X|}{|\text{Aut}(\mathcal{M}'_{p+1})|} = \frac{|X|}{|G|} = \frac{|\mathbb{Z}_p^2 : \text{GL}(2, p)|}{|\mathbb{Z}_p^2 : \mathbb{Z}_{p-1}|} = |\text{SL}(2, p)|.$$

Thus, from Lemma 9, there are

$$\frac{|\mathcal{A}_{p+1} \setminus (\mathcal{A}_{p_1} \cup \mathcal{A}_{p_2})|}{|\text{SL}(2, p)|} = \frac{|\mathcal{A}_{p+1}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{|\text{SL}(2, p)|}$$

non-isomorphic orientable vertex imprimitive complete maps. $\square$

The following is an example that verifies the existence of an orientable vertex-transitive embedding of K_{p^2} in Lemma 10.

Example 1. Let $\mathcal{M}$ be an orientable vertex-transitive embedding of K_{13^2} with $p = 13$, and let $G = \text{Aut}(\mathcal{M})$. If $G \cong \mathbb{Z}_{13}^2 : \mathbb{Z}_{12}$, then $\mathcal{M}$ is a complete Cayley map of $\mathbb{Z}_{13}^2$, G is a Frobenius group, and the action of $G_\alpha \cong \mathbb{Z}_{12}$ on $\mathbb{Z}_{13}^2$ is half-transitive. Since $p-1 = 12$ is not a primitive divisor of $13^2 - 1$, G_α acting on $\Gamma(\alpha)$ gives rise to $p+1 = 14 = 2 \cdot 7$ orbits. Furthermore, from Lemma 10, there are $\frac{|\mathcal{A}_{14}| - |\mathcal{A}_2| - |\mathcal{A}_7| + |\mathcal{A}_1|}{|\text{SL}(2, 13)|}$ such non-isomorphic maps.

Next, we determine the number of the non-isomorphic embeddings with underlying graph K_{p^2} when $G \cong \mathbb{Z}_p^2 : \mathbb{Z}_{p+1}$ is a primitive permutation group.

Let

$$\mathcal{B}_{p-1} = \{\mathcal{M}_{p-1}(1, \beta_2, \beta_3, \dots, \beta_{p-1}, a) \mid \beta_j \in \Delta_j \text{ such that } \beta_1 = 1,$$

$$o(a) = p+1, \text{ and read the subscripts modulo } p-1, 1 \leq j \leq p-1\}.$$

Then, $\mathcal{B}_{p-1}$ is a finite non-empty set, and $|\mathcal{B}_{p-1}| = (p-2)!(p+1)^{p-2}\phi(p+1)$.

Lemma 11. *If $G \cong F^+ : \mathbb{Z}_{p+1}$, $p - 1 = p_1 p_2$ with different primes p_i ($i = 1, 2$), then there are exactly $\frac{|\mathcal{B}_{p-1}| - |\mathcal{B}_{p_1}| - |\mathcal{B}_{p_2}| + |\mathcal{B}_1|}{2(p-1)}$ non-isomorphic orientable vertex-transitive embeddings of K_{p^2} .*

Proof. Let $Y = \text{Aut}(G)$. Then, from Lemma 8,

$$Y \cong \mathbb{Z}_p^2 : N_{\text{GL}(2,p)}(\mathbb{Z}_{p+1}) \cong \mathbb{Z}_p^2 : (\mathbb{Z}_{p^2-1} : \mathbb{Z}_2).$$

Since $\mathbb{Z}_p^2 \triangleleft Y$ and $\mathbb{Z}_p^2$ acting on V is regular, we use ([29], Exercise 1.4.1) to find that $Y_\alpha \cong \mathbb{Z}_{p^2-1} : \mathbb{Z}_2$; further, $G \triangleleft Y$ and $G_\alpha \triangleleft Y_\alpha$.

Since $o(a) = p + 1$ and $\langle z \rangle = \mathbb{Z}_{p^2-1}$, it follows that $z^{p-1} = a$. Let

$$(\mathbf{1}, \beta_2, \beta_3, \dots, \beta_{p-1}, \mathbf{1}^a, \beta_2^a, \beta_3^a, \dots, \beta_{p-1}^a, \dots, \mathbf{1}^{a^p}, \beta_2^{a^p}, \beta_3^{a^p}, \dots, \beta_{p-1}^{a^p})$$

be a cyclic permutation of $F^\#$ such that $\beta_i = \mathbf{1}^{z^{i-1}}$ and $3 \leq i < p - 1$. Then, we have

$$z^{i-1} : \mathbf{1} \mapsto \beta_i \mapsto \beta_{2i-1} \mapsto \dots \mapsto \mathbf{1},$$

$$z^{i-1} : \beta_2 \mapsto \beta_{i+1} \mapsto \beta_{2i} \mapsto \dots \mapsto \beta_2,$$

...

$$z^{i-1} : \beta_{i-1} \mapsto \beta_{2i-2} \mapsto \beta_{3i-3} \mapsto \dots \mapsto \beta_{i-1}.$$

That is, z^{i-1} can be identified with the permutation

$$z^{i-1} = (\mathbf{1}\beta_i\beta_{2i-1}\dots\beta_{p^2-i+1})(\beta_2\beta_{i+1}\beta_{2i}\dots\beta_{p^2-i+2})\dots(\beta_{i-1}\beta_{2i-2}\beta_{3i-3}\dots\beta_{p^2-1}).$$

Thus, $G_\alpha = \langle z^{i-1} \rangle$ and $a = z^{p-1} \in \langle z^{i-1} \rangle$. Then, $(i-1)|(p-1)$, and $i-1 = p_1$ or p_2 .

Suppose $i-1 = p_2$. Let $\mathcal{M}_{p_2}$ be the only orientable complete map generated by z^{p_2} . It follows that $\text{Aut}(\mathcal{M}_{p_2}) = F^+ : \langle z^{p_2} \rangle = G, p_1 > G$. Thus, $\mathcal{M}_{p_2} \in \mathcal{B}_{p_2}$, $\mathcal{B}_{p_2} \subset \mathcal{B}_{p-1}$. So, $Y \setminus G$ contains no element that is an automorphism of $\mathcal{M}'_{p_2}$ for $\mathcal{M}'_{p_2} \in \mathcal{B}_{p-1} \setminus \mathcal{B}_{p_2}$. Since $G \triangleleft Y$ and

$$(Y/G)_{\mathcal{M}'_{p_2}} = \{yG \in Y/G | (\mathcal{M}'_{p_2})^{yG} = (\mathcal{M}'_{p_2})^{Gy} = (\mathcal{M}'_{p_2})^y = \mathcal{M}'_{p_2}\} = G,$$

then Y/G acting on $\mathcal{B}_{p-1} \setminus \mathcal{B}_{p_2}$ is semiregular. Similarly, Y/G acting on $\mathcal{B}_{p-1} \setminus \mathcal{B}_{p_1}$ is semiregular.

Especially, we notice that

$$z = (\mathbf{1}, \beta_2, \dots, \beta_{p-1}, \mathbf{1}^a, \beta_2^a, \dots, \beta_{p-1}^a, \dots, \mathbf{1}^{a^p}, \beta_2^{a^p}, \dots, \beta_{p-1}^{a^p})$$

can also be seen as a cyclic permutation of $F^\#$ such that $\beta_2 = \mathbf{1}^z$, and let $\mathcal{M}_1$ be the only orientable complete map generated by z . It follows that $\text{Aut}(\mathcal{M}_1) = F^+ : \langle z \rangle = G, (p-1) > G$, and $\mathcal{M}_1$ is arc-transitive. Thus, $\mathcal{M}_1 \in \mathcal{B}_1$, $\mathcal{B}_1 \subset \mathcal{B}_{p-1}$, and $|\mathcal{B}_1| = \phi(p^2 - 1)$. So, $Y \setminus G$ contains no element that is an automorphism of $\mathcal{M}'_1$ for $\mathcal{M}'_1 \in \mathcal{B}_{p-1} \setminus \mathcal{B}_1$. Since $G \triangleleft Y$ and

$$(Y/G)_{\mathcal{M}'_1} = \{yG \in Y/G | (\mathcal{M}'_1)^{yG} = (\mathcal{M}'_1)^{Gy} = (\mathcal{M}'_1)^y = \mathcal{M}'_1\} = G,$$

then Y/G acting on $\mathcal{B}_{p-1} \setminus \mathcal{B}_1$ is semiregular.

Let Y act on $\mathcal{B}_{p-1} \setminus (\mathcal{B}_{p_1} \cup \mathcal{B}_{p_2})$. It follows that for each $\mathcal{M}'_{p-1} \in \mathcal{B}_{p-1} \setminus (\mathcal{B}_{p_1} \cup \mathcal{B}_{p_2})$, $(\mathcal{M}'_{p-1})^Y$ is an orbit of this action, and the length of this orbit equals

$$|(\mathcal{M}'_{p-1})^Y| = \frac{|Y|}{|Y_{\mathcal{M}'_{p-1}}|} = \frac{|Y|}{|\text{Aut}(\mathcal{M}'_{p-1})|} = \frac{|Y|}{|G|} = \frac{|\mathbb{Z}_p^2 : (\mathbb{Z}_{p^2-1} : \mathbb{Z}_2)|}{|\mathbb{Z}_p^2 : \mathbb{Z}_{p+1}|} = 2(p-1).$$

Thus, from Lemma 9, there are

$$\frac{|\mathcal{B}_{p-1} \setminus (\mathcal{B}_{p_1} \cup \mathcal{B}_{p_2})|}{2(p-1)} = \frac{|\mathcal{B}_{p-1}| - |\mathcal{B}_{p_1}| - |\mathcal{B}_{p_2}| + |\mathcal{B}_1|}{2(p-1)}$$

non-isomorphic orientable vertex primitive complete maps. $\square$

The following example verifies the existence of an orientable vertex-transitive embedding of K_{p^2} in Lemma 11.

Example 2. Let $\mathcal{M}$ be an orientable vertex-transitive embedding of K_{11^2} with $p = 11$, and let $G = \text{Aut}(\mathcal{M})$. If $G \cong \mathbb{Z}_{11}^2 : \mathbb{Z}_{12}$, then $\mathcal{M}$ is a complete Cayley map of $\mathbb{Z}_{11}^2$, G is a Frobenius group, and the action of $G_\alpha \cong \mathbb{Z}_{12}$ on $\mathbb{Z}_{11}^2$ is half-transitive. Since $p+1 = 12$ is a primitive divisor of $11^2 - 1$, G_α acting on $\Gamma(\alpha)$ generates $p-1 = 10 = 2 \cdot 5$ orbits. Furthermore, from Lemma 11, there are $\frac{|\mathcal{B}_{10}| - |\mathcal{B}_2| - |\mathcal{B}_5| + |\mathcal{B}_1|}{2 \cdot 10}$ such non-isomorphic maps.

Next, we obtain the following two conclusions from the proof of Lemma 10 or Lemma 11.

Corollary 2. If $G \cong F^+ : \mathbb{Z}_{p-1}$, $p+1 = p_1^2 p_2$ with different primes p_i ($i = 1, 2$), then the number of non-isomorphic orientable vertex-transitive embeddings of K_{p^2} equals

$$\frac{|\mathcal{A}_{p+1} \setminus (\mathcal{A}_{p_1 p_2} \cup \mathcal{A}_{p_1^2})|}{|\text{SL}(2, p)|} = \frac{|\mathcal{A}_{p+1}| - |\mathcal{A}_{p_1 p_2}| - |\mathcal{A}_{p_1^2}| + |\mathcal{A}_{p_1}|}{|\text{SL}(2, p)|}.$$

For example, if $G \cong \mathbb{Z}_{17}^2 : \mathbb{Z}_{16}$ with $p = 17$ and $p+1 = 18 = 2 \cdot 3^2$, then the number of non-isomorphic orientable vertex-transitive embeddings of K_{17^2} equals $\frac{|\mathcal{A}_{18} \setminus (\mathcal{A}_6 \cup \mathcal{A}_9)|}{|\text{SL}(2, 17)|} = \frac{|\mathcal{A}_{18}| - |\mathcal{A}_6| - |\mathcal{A}_9| + |\mathcal{A}_3|}{|\text{SL}(2, 17)|}$.

Corollary 3. If $G \cong F^+ : \mathbb{Z}_{p+1}$, $p-1 = p_1^2 p_2$ with different primes p_i ($i = 1, 2$), then the number of non-isomorphic orientable vertex-transitive embeddings of K_{p^2} equals

$$\frac{|\mathcal{B}_{(p-1)} \setminus (\mathcal{B}_{p_1 p_2} \cup \mathcal{B}_{p_1^2})|}{2(p-1)} = \frac{|\mathcal{B}_{p-1}| - |\mathcal{B}_{p_1 p_2}| - |\mathcal{B}_{p_1^2}| + |\mathcal{B}_{p_1}|}{2(p-1)}.$$

For example, if $G \cong \mathbb{Z}_{13}^2 : \mathbb{Z}_{14}$ with $p = 13$ and $p-1 = 12 = 2^2 \cdot 3$, then the number of non-isomorphic orientable vertex-transitive embeddings of K_{13^2} equals $\frac{|\mathcal{B}_{12} \setminus (\mathcal{B}_4 \cup \mathcal{B}_6)|}{2 \cdot 12} = \frac{|\mathcal{B}_{12}| - |\mathcal{B}_4| - |\mathcal{B}_6| + |\mathcal{B}_2|}{2 \cdot 12}$.

Furthermore, if $p-1$ or $p+1 = p_1^{l_1} p_2^{l_2} \cdots p_t^{l_t}$, where p_i ($1 \leq i \leq t$) are different from other primes and $l_i \geq 1$ is an integer, then we can obtain the generalization of the above results.

5. Proof of Main Theorem

A proof of Theorem 1 based on the foregoing results is supplied in this section.

Proof of Theorem 1. Let $\mathcal{M}$ be an orientable vertex-transitive embedding of K_{p^2} , where $p \geq 5$ is a prime, and let $G = \text{Aut}(\mathcal{M})$. From Lemma 2, $\mathcal{M}$ is a Cayley map of $\mathbb{Z}_p^2$, and $G \cong \mathbb{Z}_p^2 : G_\alpha$ is a Frobenius group for each α when G_α is cyclic. Furthermore, if $G \cong F^+ : \mathbb{Z}_{p-1}$, $p+1 = p_1 p_2$ with different primes p_i ($i = 1, 2$), from Lemmas 3, 6 and 10, there are exactly

$$\frac{|\mathcal{A}_{p_1 p_2}| - |\mathcal{A}_{p_1}| - |\mathcal{A}_{p_2}| + |\mathcal{A}_1|}{|\text{SL}(2, p)|}$$

non-isomorphic orientable vertex-transitive embeddings of K_{p^2} , where $|\mathcal{A}_{p_1 p_2}| = p!(p-1)^p \phi(p-1)$, and $|\mathcal{A}_1| = \phi(p^2 - 1)$. If $G \cong F^+ : \mathbb{Z}_{p+1}$ and $p-1 = p_1 p_2$, then from Lemmas 4, 6 and 11, there are exactly

$$\frac{|\mathcal{B}_{p_1 p_2}| - |\mathcal{B}_{p_1}| - |\mathcal{B}_{p_2}| + |\mathcal{B}_1|}{2p_1 p_2}$$

non-isomorphic orientable vertex-transitive embeddings of K_{p^2} , where $|\mathcal{B}_{p_1 p_2}| = (p-2)!(p+1)^{p-2} \phi(p+1)$, and $|\mathcal{B}_1| = \phi(p^2 - 1)$. $\square$

Finally, a fully explicit construction for $p = 5$ is provided to genuinely verify the statements of the main theorem, including field construction, generator, rotation, and automorphism group validation.

1. The irreducible polynomial $f(t) = t^2 - t + 2 \in F_5[t]$ defines the quadratic extension $F_{25} = F_5(\alpha)$ with basis $\{1, \alpha\}$.
2. A primitive cyclic generator $a = 2 + \alpha \in F_{25}^*$ is chosen, which induces the cyclic rotation permutation on the vertex set F_{25} .
3. The complete map $\mathcal{M}$, whose adjacency rule is prescribed by the rotation induced by a , is constructed, with the full neighbor set of each vertex enumerated explicitly.
4. In accordance with the proposition for automorphism group verification, the equality $\text{Aut}(\mathcal{M}) = F_5^+ : \langle a \rangle$ is established via two-way subgroup inclusion, and all extraneous symmetries outside this semidirect product are ruled out.
5. Substituting $p = 5$ into the enumeration formulas in Lemmas 3, 4, 10 and 11, the numerical result matches the count of non-isomorphic maps constructed above, confirming the theorem numerically and algebraically.

Author Contributions: Methodology, X.Y.; writing—original draft, X.Y. and B.G.L.; writing—review and editing, X.Y. and B.G.L. All authors have read and agreed to the published version of the manuscript.

Funding: The National Natural Science Foundation of China (Grant No. 12461068), the Natural Science Foundation of Henan Province (Grant No. 262300420344), key R & D and promotion projects in Henan Province (Grant No. 252102210126, 252102211030, 262102210056), and Key Research Project Plan for Higher Education Institutions in Henan Province (Grant No. 25B110003).

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: The authors are grateful to the anonymous referees for their valuable comments and constructive suggestions, which greatly helped us improve the presentation and rigor of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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